99.15 Triangles in which $a^n = b^n + c^n$

In this note we investigate triangles whose side-lengths a, b, c (with $a \geqslant b \geqslant c$), satisfy $a^n = b^n + c^n$ for some $n > 1$: notice that n need not be an integer. We shall use the standard Euler notation for triangle ABC where the side of length a is opposite angle A, etc. For fixed values of a and n, we can use a graph plotter to illustrate the locus of A satisfying $a^n = b^n + c^n$. With origin at the midpoint of BC and x-axis along BC, the Cartesian equation of the locus of A is

$$\left[\left(x + \frac{a}{2}\right)^2 + y^2\right]^{\frac{n}{2}} + \left[\left(x - \frac{a}{2}\right)^2 + y^2\right]^{\frac{n}{2}} = a^n.$$

Figure 1 shows this locus for the case $n = 4$.

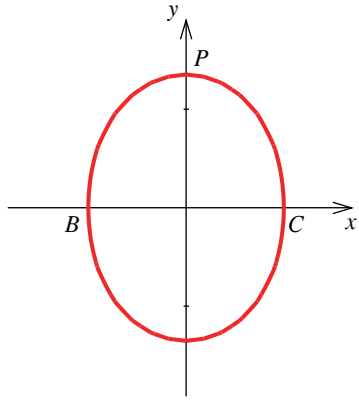


FIGURE 1: The locus of A for the case $n = 4$

If the vertex angle A is fixed, then the locus of A is a circular arc. Considering the intersection of such an arc with the locus in Figure 1 we note that the position P on the y-axis has $PB = PC = 2^{-1/n}a$ and $A = \angle BPC = \cos^{-1}\left(1 - \frac{1}{2}\sqrt[n]{4}\right)$. These observations motivate and make geometrically plausible our main result, which can be summarised as

(a) If $1 < n < 2$ then, for each angle A in the range $\frac{1}{2}\pi < A \leqslant \cos^{-1}\left(1 - \frac{1}{2}\sqrt[n]{4}\right)$, there is a triangle with angle A and side-lengths satisfying $a^n = b^n + c^n$. Moreover, such a triangle is unique up to similarity. There are no such triangles for A outside the given range.

(b) If $n = 2$, then $A = \frac{\pi}{2}$ (and this is the only restriction).

(c) If $n > 2$ then, for each angle A in the range $\cos^{-1}\left(1 - \frac{1}{2}\sqrt[n]{4}\right) \leqslant A < \frac{\pi}{2}$, there is a triangle with angle A and side-lengths satisfying $a^n = b^n + c^n$. Such a triangle is unique up to similarity. There are no such triangles for A outside the given range.

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Of course, (b) is standard, so we concentrate on justifying (a) and (c). The cosine rule gives

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{b^2 + c^2 - \left(b^n + c^n\right)^{\frac{2}{n}}}{2bc}$$

if $a^n = b^n + c^n$. Putting $x = \frac{c}{b}$ with $0 < x \leqslant 1$ yields

$$\cos A = \frac{1 + x^2 - \left(1 + x^n\right)^{2/n}}{2x}. \tag{1}$$

Everything hinges on the behaviour of the function

$$f(x) = \frac{1 + x^2 - \left(1 + x^n\right)^{2/n}}{2x}.$$

In case (a), we will show that $f(x)$ is decreasing on $[0, 1]$ from $f\left(0^+\right) = 0$ to $f(1) = 1 - \frac{1}{2}\sqrt[n]{4} > -1$. Each $0 < x \leqslant 1$ then determines a valid angle A by (1) and the pair (x, A) specifies the resulting triangle up to similarity.

Similarly, in case (c) we will show that $f(x)$ is increasing on $[0, 1]$ from $f\left(0^+\right) = 0$ to $f(1) = 1 - \frac{1}{2}\sqrt[n]{4} < 1$. Each $0 < x \leqslant 1$ then determines A by (1) and the pair (x, A) specifies the resulting triangle up to similarity.

To prove these assertions, consider

$$2x^2 f'(x) = \left(1 - x^n\right)\left(1 + x^n\right)^{\frac{2}{n} - 1} - \left(1 - x^2\right). \tag{2}$$

We aim to show that, for $0 < x < 1$, $f'(x) < 0$ if $1 < n < 2$ and $f'(x) > 0$ if $n > 2$. Setting $t = \dfrac{x^n}{1 + x^n}$ with $0 < t < \frac{1}{2}$ and $\alpha = \frac{2}{n}$, we can recast the right-hand side of (2) as

$$\frac{(1-2t)}{(1-t)^\alpha} - 1 + \frac{t^\alpha}{(1-t)^\alpha} = \frac{t^\alpha - (1-t)^\alpha + 1 - 2t}{(1-t)^\alpha} = \frac{g(t) - g(1-t)}{(1-t)^\alpha},$$

where $g(t) = t^\alpha - t$. Our statements above about $f'(x)$ will then follow from the following Lemma.

Lemma: For $0 < t < \frac{1}{2}$, the function $g(t) = t^\alpha - t$ satisfies:

$$\begin{cases} g(t) > g(1 - t) \text{ if } 0 < \alpha < 1 \text{ or } \alpha > 2; \\ g(t) < g(1 - t) \text{ if } 1 < \alpha < 2. \end{cases}$$

To show this, let $h(t) = g(t) - g(1 - t)$. Then

$$h'(t) = g'(t) + g'(1 - t) = \alpha t^{\alpha - 1} + \alpha(1 - t)^{\alpha - 1} - 2$$

and

$$h''(t) = g''(t) - g''(1 - t) = \alpha(\alpha - 1)\left[t^{\alpha - 2} - (1 - t)^{\alpha - 2}\right].$$

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From these we see that:

(i) for all $\alpha > 0, h(0) = h(\frac{1}{2}) = 0$;

(ii) for small positive t, $h'(t) > 0$ if $0 < \alpha < 1$ or $\alpha > 2$ while $h'(t) < 0$ if $1 < \alpha < 2$;

(iii) for $0 < t < \frac{1}{2}$, $h''(t) < 0$ if $0 < \alpha < 1$ or $\alpha > 2$ while $h''(t) > 0$ if $1 < \alpha < 2$.

Between them, these suffice to establish the Lemma. For example, if $0 < \alpha < 1$, (i) and (ii) together show that $h(t)$ is positive for small positive values of t. Subsequently, $h(t)$ cannot take negative values otherwise there would be a local minimum, contradicting (iii).

Finally, it is worth noting that the special case $n = 4$ is an attractive one which can be dealt with algebraically. For in that case, (1) becomes

$$\cos A = \frac{1 + x^2 - \sqrt{1 + x^4}}{2x} = \frac{(1 + x^2)^2 - (1 + x^4)}{2x\left(1 + x^2 + \sqrt{1 + x^4}\right)}$$

$$= \frac{x}{1 + x^2 + \sqrt{1 + x^4}} = \frac{1}{x + \frac{1}{x} + \sqrt{\left(x + \frac{1}{x}\right)^2 - 2}}.$$

Since, $x + \frac{1}{x}$ is a decreasing function for $0 < x \leqslant 1$, we see that A increases from $\cos^{-1}\left(\frac{1}{2 + \sqrt{2}}\right) = \cos^{-1}\left(1 - \frac{\sqrt[4]{4}}{2}\right)$ to $\frac{\pi}{2}$. Noting that $\cos^{-1}\left(1 - \frac{\sqrt[4]{4}}{2}\right) \approx 72.97°$, a nice (but cruel!) variation on our opening theme is to ask: Is there a triangle with $A = 72°$ and $a^4 = b^4 + c^4$?

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99.16 Conditionally convergent series

Introduction

In 1984, Rao and Subbarao [1] established the following identity

$$\sum_{j=1}^{\infty} a_j \sum_{k=1}^{j} b_k + \sum_{k=1}^{\infty} b_k \sum_{j=1}^{k} a_j = \left(\sum_{j=1}^{\infty} a_j\right)\left(\sum_{k=1}^{\infty} b_k\right) + \sum_{k=1}^{\infty} a_k b_k, \qquad (1)$$

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