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TESTING FOR EXOGENEITY IN THRESHOLD MODELS

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Most work in the area of nonlinear econometric modeling is based on a single equation and assumes exogeneity of the explanatory variables. Recently, work by Caner and Hansen (2004) and Psaradakis, Sola, and Spagnolo (2005) has considered the possibility of estimating nonlinear models by methods that take into account endogeneity but provide no tests for exogeneity. This paper examines the problem of testing for exogeneity in nonlinear threshold models. We suggest new Hausman-type tests and discuss the use of the bootstrap to improve the properties of asymptotic tests. The theoretical properties of the tests are discussed and an extensive Monte Carlo study is undertaken.

1. INTRODUCTION

Nonlinear models have been used extensively in recent years to investigate macroeconomic phenomena. A number of classes of models have been popularized in the literature. Two of the main classes considered are threshold models and smooth transition models. The main characteristic of both classes is the nonconstancy of the response of the dependent variables to the explanatory variables. This response, which in linear regression models is simply the coefficient of the explanatory variable, is allowed to vary depending on the occurrence of given trigger events. The main difference between threshold models and smooth transition models is that the first class assumes that coefficients change abruptly (discontinuously) with respect to the trigger events, whereas for smooth transition models the responses change gradually (continuously).

Most work in the area of nonlinear macroeconometric modeling is based on a single equation and assumes exogeneity of the explanatory variables. However, just as in linear models, this assumption is mostly suspect in economics. Explanatory variable endogeneity can cause at least as much trouble in nonlinear models as in linear ones. However, endogeneity has not been taken into account in most of the literature. Recently work by Caner and Hansen (2004) and Psaradakis, Sola, and Spagnolo (2005) has considered the possibility of estimating nonlinear models by methods that take into account endogeneity. Hence, two-stage least squares

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(2SLS) and generalized method of moments (GMM) estimators have appeared in the literature. However, the crucial problem remains. Should one use these estimators to alleviate bias in case of endogeneity or use standard estimators that, although biased under endogeneity, are likely to be more efficient if there is no endogeneity? Clearly this problem can be addressed using Hausman-type tests, which compare standard estimators with those that account for endogeneity. This topic is addressed in this paper. It is worth noting that whereas little attention has been paid to exogeneity assumptions in the nonlinear macroeconometric literature, some focus on the validity of the assumption of exogeneity in nonlinear models has been placed in microeconometrics as evidenced by the work of Blundell and Powell (2004) and references cited therein.

Unlike linear models where estimators such as ordinary least squares (OLS) are known to be efficient under relatively mild conditions, that is not necessarily the case for nonlinear models. Hence, the standard results of Hausman (1978) cannot be invoked automatically to derive asymptotic distributions for test statistics. Further, any asymptotic results may have less relevance in small samples. The problem can be alleviated using bootstrap methods. The problem is then seen to be easily tackled. In this paper we propose new exogeneity tests. It is seen, via simulations, that the standard asymptotic theory of the Hausman test appears not to be a good guide in small samples. On the other hand, the bootstrap appears much more useful. In this paper we focus on threshold models that have been more extensively analyzed in the case of endogeneity by Caner and Hansen (2004). Of course, all methods are readily adapted for any class of nonlinear models. We note that a specific way to test for endogeneity in Markov-switching models has been recently suggested by Kim (2004).

The paper is organized as follows: Section 2 discusses the model and the problem addressed. Section 3 discusses the bootstrap in this context. Section 4 provides Monte Carlo evidence in favor of the new methods. Finally, Section 5 concludes.

2. THE MODEL

The model we consider is that of Caner and Hansen (2004). In the simple form analyzed there, the model is given by

$$y_i = \beta_1' z_i I_{q_i \leq \gamma_0} + \beta_2' z_i I_{q_i > \gamma_0} + \epsilon_i,\ i = 1, \ldots, n \tag{1}$$

and

$$z_i = \Pi_1 x_i I_{q_i \leq \rho} + \Pi_2 x_i I_{q_i > \rho} + u_i \tag{2}$$

where y_i is a scalar variable, z_i is an $m \times 1$ vector of (possibly) endogenous variables, x_i is a $p \times 1$ vector of exogenous variables, q_i is an exogenous scalar variable controlling the regime switches, ϵ_i and u_i are martingale difference sequences with variance σ_ϵ^2 and σ_u^2, respectively, $\beta_j,\ j = 1, 2$ are $m \times 1$ vectors

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of parameters, Π_j, $j = 1, 2$ are $m \times p$ matrices of parameters, and γ_0 and ρ are threshold parameters belonging to some compact sets Γ and Ξ, respectively. I_A denotes the indicator function taking the value 1 when the event A occurs and zero otherwise. We also define $\beta = (\beta_1', \beta_2')'$. We assume a threshold model for z_i. This is not necessary, and a variety of other models could be envisaged for z_i including, obviously, a linear model. We focus on a threshold model for simplicity.

For this model, Caner and Hansen (2004) define the 2SLS and GMM estimators that allow for the endogeneity of z_i. More specifically, let $\hat{z}_i$ be the fitted value from the estimation of the threshold model in (2) where the threshold ρ is estimated along with the rest of the model's parameters. Define $Y = (y_1, \ldots, y_n)'$, $Z_\gamma = (\hat{z}_1 I_{q_1 \le \gamma}, \ldots, \hat{z}_n I_{q_n \le \gamma})'$ and $\tilde{Z}_\gamma = (\hat{z}_1 I_{q_1 > \gamma}, \ldots, \hat{z}_n I_{q_n \le \gamma})'$. Let $S_n(\gamma)$ denote the sum of squared residual of a regression of Y on Z_γ and $\tilde{Z}_\gamma$ for given γ. Then, the threshold is estimated as the value of γ that minimizes $S_n(\gamma)$ and denoted $\hat{\gamma}$. Then, assuming that γ is known and given by $\hat{\gamma}$, one can estimate the coefficients via 2SLS for the two subsamples implicitly defined by $\hat{\gamma}$. The problem is thus reduced to a linear estimation for the two subsamples. In particular, the 2SLS estimator takes the following form when the reduced form is linear in x_i in each subsample:

$$\hat{\beta} = \left(\hat{Z}_{12}' \hat{X}_{12} \left(\hat{X}_{12}' \hat{X}_{12} \right)^{-1} \hat{X}_{12}' \hat{Z}_{12} \right)^{-1} \hat{Z}_{12}' \hat{X}_{12} \left(\hat{X}_{12}' \hat{X}_{12} \right)^{-1} \hat{X}_{12}' Y,$$

where $\hat{z}_{12,i} = (z_i' I_{q_i \le \hat{\gamma}}, z_i' I_{q_i > \hat{\gamma}})'$, $\hat{Z}_{12} = (\hat{z}_{12,1}, \ldots, \hat{z}_{12,n})'$, $\hat{x}_{12,i} = (x_i' I_{q_i \le \hat{\gamma}}, x_i' I_{q_i > \hat{\gamma}})'$, and $\hat{X}_{12} = (\hat{x}_{12,1}, \ldots, \hat{x}_{12,n})'$. The GMM estimator is given by

$$\bar{\beta} = \left(\hat{Z}_{12}' \hat{X}_{12}^{-1} \hat{\Omega}_x^{-1} \hat{X}_{12}' \hat{Z}_{12} \right)^{-1} \hat{Z}_{12}' \hat{X}_{12} \hat{\Omega}_x^{-1} \hat{X}_{12}' Y,$$

where $\hat{\Omega}_x = \sum_{i=1}^n x_{12,i} x_{12,i}' \hat{\epsilon}_i^2$ and $\hat{\epsilon}_i = y_i - \hat{\beta}_1' z_i I_{q_i \le \hat{\gamma}} - \hat{\beta}_2' z_i I_{q_i > \hat{\gamma}}$. Caner and Hansen (2004) prove consistency of $\hat{\gamma}$ and derive the asymptotic distribution of $\hat{\gamma}$ under the assumption of a 'small threshold' asymptotic framework, i.e., assuming that $\beta_1 - \beta_2 = \beta_1 - \beta_{2n} = o(1)$. It is also shown that the asymptotic distribution of the estimates of β_1 and β_2 are $\sqrt{n}$-consistent and asymptotically normal.

Theoretical results that go beyond this point are not available. Disregarding the presence of endogeneity and estimating a threshold model such as (1) by the standard method, which involves applying OLS for every point on a threshold parameter grid, is likely to lead to inconsistency (certainly for coefficients and possibly for threshold parameter estimates), although the asymptotic bias is not available in closed form and may depend in complicated ways on the true model parameters. On the other hand, it is not clear to what extent estimation by instrumental variable (IV) methods introduces inefficiency in the estimation when z_i is exogenous. Nevertheless, it is clear that in either case use of an inappropriate

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estimation method will lead to suboptimal outcomes. It is obvious that a Hausman (1978) type test can provide guidance on which estimator to use.

The development of such tests is the topic of this paper. The null hypothesis of the test is that

$$H_0 : \mathrm{E}(\epsilon_i | z_i) = 0, \forall i. \tag{3}$$

The test statistic is of the form

$$S = n(\hat{\beta} - \tilde{\beta})' \hat{V}^{-1} (\hat{\beta} - \tilde{\beta}), \tag{4}$$

where $\hat{\beta}$ is the IV-type estimator (such as the 2SLS or GMM estimator) and $\tilde{\beta}$ is the OLS-type estimator. V is the variance of $\sqrt{n}(\hat{\beta} - \tilde{\beta})$ and $\hat{V}$ denotes its estimate. This statistic is asymptotically distributed as χ^2_{2m}. This result follows easily from the asymptotic normality of both $\hat{\beta}$ and $\tilde{\beta}$ under the null hypothesis proven in Theorem 3 of Caner and Hansen (2004) and Lemma A.12 of Hansen (2000), respectively.

If $\tilde{\beta}$ is efficient (i.e., achieves the Cramer-Rao lower variance bound), then by Hausman (1978) it is known that

$$V = Var(\hat{\beta}) - Var(\tilde{\beta}), \tag{5}$$

and $\hat{V}$ can be obtained by plugging in estimates of $Var(\tilde{\beta})$ and $Var(\hat{\beta})$ in the above expression. Asymptotically, estimation of the threshold is irrelevant for the estimation of the coefficients due to the superconsistency of the threshold parameter estimate. As Caner and Hansen (2004, p. 819) state, under conditional homoskedasticity and in the presence of endogeneity, 2SLS estimation is semi-parametrically efficient. Under the null hypothesis of exogeneity, and conditional homoskedasticity, which although not imposed in this paper, is the leading case of interest, OLS estimation is semi-parametrically efficient, whereas 2SLS obviously is not. Before exploring the implications of these results to obtain an explicit expression for V, we state the regularity conditions for this paper.

Assumption 1. $(x_i, z_i, \epsilon_i, u_i)$ is strictly stationary and ergodic with finite fourth moments and ρ-mixing[1] coefficients, ρ_m, such that $\sum_{m=1}^{\infty} \rho_m^{1/2} < \infty$.

Assumption 2. Let $\mathcal{I}_i$ denote the σ-field generated by $(x_j, z_j, \epsilon_j, u_j)$, $j \leq i$. Then, $\mathrm{E}(\epsilon_i | \mathcal{I}_{i-1}) = 0$ and $\mathrm{E}(u_i | \mathcal{I}_{i-1}) = 0$.

Assumption 3. $\beta_1 - \beta_2 = \beta_1 - \beta_{2n} = cn^{-a}$ where $c \neq 0$ and $0 < a < 1/2$.

Let $g_i = \Pi_1 x_i I_{q_i \leq \rho} + \Pi_2 x_i I_{q_i > \rho}$, $v_i = \beta_1' u_i I_{q_i \leq \gamma_0} + \beta_2' u_i I_{q_i > \gamma_0} + \epsilon_i$, and $z_{12,i}(\gamma) = (z_i I_{q_i \leq \gamma}, z_i I_{q_i > \gamma})'$. Let $f(q)$ denote the density of q_i, $D_1(\gamma) = \mathrm{E}(g_i g_i' | q_i = \gamma)$, $D_2(\gamma) = \mathrm{E}\left(g_i g_i' v_i^2 | q_i = \gamma\right)$, $S_z(\gamma) = \mathrm{E}(z_{12,i}(\gamma) z_{12,i}(\gamma)')$, and $S_x = \mathrm{E}(x_i x_i')$.

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Assumption 4. For all γ in Γ, $f(\gamma) < \bar{f} < \infty$. $f(\gamma)$, $D_1(\gamma)$, and $D_2(\gamma)$ are continuous at $\gamma = \gamma_0$. $c'D_1(\gamma_0)c > 0$, $c'D_2(\gamma_0)c > 0$, and $f > 0$. $S_z(\gamma_0)$ and S_x are positive definite matrices.

Then, we have the following lemma, proven in the Appendix:

LEMMA 1. *Under Assumptions 1–4, $\sqrt{n}(\hat{\beta} - \tilde{\beta})$ and $\sqrt{n}(\bar{\beta} - \tilde{\beta})$ are asymptotically normal with covariance matrices, V_{2SLS} and V_{GMM} given by (A.7) and (A.9) in the Appendix. Further, if we strengthen Assumption 2 and assume that $\mathrm{E}(\epsilon_i^2|\mathcal{I}_{i-1}) = \sigma_\epsilon^2$ and $\mathrm{E}(u_i^2|\mathcal{I}_{i-1}) = \sigma_u^2$ (conditional homoskedasticity), then $V_{2SLS} = V_{GMM}$, where V_{2SLS} is now given by (A.10) in the Appendix.*

The result of Lemma 1 is asymptotic. As Kapetanios (2000) has shown, the asymptotic irrelevance of the threshold parameter estimation, which is used in Lemma 1, is not relevant in small samples where the adjective "small" applies even to samples much larger than usually encountered in, say, macroeconomics. Hence, using an estimate of V based on Lemma 1 is likely to be misleading. This is confirmed by our Monte Carlo results reported later in the paper. For that reason we address the estimation of V in the next section using the bootstrap.

3. THE BOOTSTRAP

Use of asymptotic variances for the test statistic proposed in the Section 2 may be problematic given the nonlinear nature of the model. An alternative variance estimator may be provided by the bootstrap. A number of issues need to be addressed. The first issue concerns the validity of the bootstrap for threshold models. The standard bootstrap procedure is in general able to provide an estimate of the exact distribution of an estimator and hence of the variance of the estimator. Under mild assumptions, this estimator is consistent. Further, under the assumption of asymptotic pivotalness (independence of the asymptotic distribution from nuisance parameters), the bootstrap estimator may converge more quickly to the true distribution compared to the asymptotic approximation. However, in the case of the estimator of the threshold parameter, there are doubts concerning the consistency of the bootstrap.

To briefly explain why consistency is in doubt, we note that an important condition for bootstrap validity for an estimator is that the mapping between the joint distribution function of the sample and the estimator be continuous. It is not clear whether this continuity assumption is satisfied, for the threshold parameter estimator, either when a fixed threshold is assumed as in Chan (1993) or when a small threshold assumption is made, such as Assumption 3, which is used in Caner and Hansen (2004). This continuity condition is not necessary for consistency of the bootstrap, but most theorems available on this subject assume it.[2]

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Nevertheless, the above relate to the threshold parameter estimate. However, our test statistic S depends asymptotically only on the coefficient estimates. These satisfy the continuity assumption since their limit distribution is normal. Hence, the bootstrap may or may not be valid for full inference on threshold models but it can be shown to be valid for bootstrapping S. We actually prove this fact formally in the Appendix.

Incidentally, the question of bootstrap validity in threshold models can be addressed using alternative bootstrap approaches. Thus, whereas the standard bootstrap where inference is based on resampling samples of size n may be invalid, using subsampling can provide valid inference. Subsampling is essentially carrying out the bootstrap but resampling smaller samples to carry out inference. More specifically, rather than resampling samples of size n, samples of size n^*, where $n^* \to \infty$ but $n^*/n \to 0$, are resampled. Subsampling has been suggested and discussed by, among other, Politis and Romano (1994a) and Bickel, Gotze, and van Zwet (1997). Unlike the continuity assumption needed for the standard bootstrap, the only assumptions needed for subsampling validity are the existence of a limit distribution for the statistic of interest and, in the case of nonparametric resampling, strong mixing for the process being resampled. Recently, Gonzalo and Wolf (2005) formally showed the validity of subsampling for inference in threshold models. We will examine the performance of parametric subsampling in our context in Section 4.

Another important issue is the mode of resampling so that the null hypothesis is imposed on the bootstrap samples. This is not as important when only the variance is bootstrapped as in our case, but it is crucial when the whole distribution is bootstrapped and used to carry out the test as an alternative to retaining the asymptotic χ^2_{2m} approximation. In that case, not imposing the null distribution on the bootstrap samples may lead to an inconsistent test.[3] To see why test inconsistency may arise, note that if the null hypothesis is not imposed then endogeneity will prevail in the bootstrap samples. Hence, the bootstrap test statistics will tend to infinity asymptotically rather than be $O_p(1)$. Hence, imposing the null is important for bootstrapping the whole distribution but less so for bootstrapping the variance only. To investigate this further, we examine in detail a number of possible bootstrap implementations.

The first step in the bootstrap implementation is the resampling of z_i, x_i, and q_i. As we do not assume any parametric model for x_i and q_i, we need to use the nonparametric bootstrap. The vector z_i may either be resampled nonparametrically or parametrically, since a model is assumed for it. For simplicity and generality, our theoretical analysis will assume nonparametric resampling for z_i. Depending on the presence of temporal dependence in these variables, one may want to use the block bootstrap or the stationary bootstrap of Politis and Romano (1994b). In that case, the block size needs to be determined. If subsampling, is undertaken, one needs to choose the subsample size too. In the case of the nonparametric bootstrap it is important that one resamples rows (or blocks of rows) from the matrix $W = ((z_1', x_1', q_1)', \ldots, (z_n', x_n', q_n)')'$ rather than resample independently

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x_i, z_i, and q_i. This is so as to retain the contemporaneous dependence between these variables. However, this stage is relatively straightforward.

Then, one needs to resample parametrically y_i, given the bootstrap sample

$$W^* = ((z_1^{'*}, x_1^{'*}, q_1^*)', \ldots, (z_n^{'*}, x_n^{'*}, q_n^*)')',$$

where stars denote a generic bootstrap sample.[4] To parametrically resample y_i, one can either use the parameter estimates obtained via 2SLS/GMM or via OLS. Both choices are theoretically valid under the null hypothesis. Since initial Monte Carlo results suggested that using OLS produces more powerful tests than using 2SLS/GMM, we use the OLS residuals for resampling ϵ_i. Care needs to be applied in resampling ϵ_i. Under the alternative, $\mathrm{E}(z_i\epsilon_i) \neq 0$. But for the bootstrap sample, we must impose $\mathrm{E}^*(z_i^*\epsilon_i^*) = 0$, where E^* denotes bootstrap expectation that is conditional on the sample realization. We suggest two alternatives for that. Denote the OLS residual by $\tilde{\epsilon}_i$.

The first alternative is to use the wild bootstrap, which, given our Monte Carlo results, is the bootstrap we mainly focus on.[5] This bootstrap involves constructing the bootstrap error terms as $\epsilon_i^* = \eta_i\tilde{\epsilon}_i$, where η_i is an i.i.d. zero mean sequence independent of all other variables whose variance is unity. Then, $\mathrm{E}^*(z_i^*\epsilon_i^*) = \mathrm{E}^*(z_i^*\tilde{\epsilon}_i\eta_i^*) = 0$. The second alternative is to resample the OLS residual itself. Note that, even in the case of endogeneity, the standard regression result, that the regressors, which in this case are $(z_i I_{q_i \leq \gamma}, z_i I_{q > \gamma})$, and the residuals are orthogonal, still holds, thus implicitly imposing the null hypothesis on the bootstrap. Once the residuals have been bootstrapped, they are used together with the OLS estimates of the coefficients and the threshold parameter, via the model given by (1) to produce bootstrap samples for the dependent variable. This provides the general description of the bootstrap implementations we consider. In Section 4, the bootstrap implementations we use for the Monte Carlo study are described in detail. Further, note that a full step-by-step description of all bootstrap implementations is also given in Appendix B.

By the consistency of the parameter estimates both of the OLS and 2SLS estimators, proved under the null by Chan (1993) and Caner and Hansen (2004), and the fact that, via the nonparametric bootstrap, the joint distributions of W^* are the same as those of $W = ((z_1^{'}, x_1^{'}, q_1)', \ldots, (z_n^{'}, x_n^{'}, q_n)')'$, it follows that the bootstrap samples satisfy assumptions 1–2 of Caner and Hansen (2004), if z_i follow a linear or threshold model, and conditions 1–4 of Chan (1993). The validity of the wild bootstrap approach then easily follows, since asymptotically $S^* \sim \chi^2_{2m}$. Similarly, the validity of the bootstrap based on the OLS residuals can be proven. These results are summarized in the theorem below, which is formally proven in Appendix A.

THEOREM 1. *Under Assumptions 1–4, the wild and OLS bootstrap approaches are asymptotically valid since, in both cases,* $S^* \xrightarrow{d} \chi^2_{2m}$.

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4. MONTE CARLO

In this section, we carry out a detailed Monte Carlo simulation of the new tests. We use the following model:

$$y_i = \beta_1 z_i I_{q_i \leq 0} + \beta_2 z_i I_{q_i > 0} + \epsilon_i \tag{6}$$

and

$$z_i = \sum_{j=1}^{p} \pi_1 x_{i,j} I_{q_i \leq 0} + \sum_{j=1}^{p} \pi_2 x_{i,j} I_{q_i > 0} + u_i \tag{7}$$

where $p = 1, 2$. We introduce endogeneity by specifying $\mathrm{E}(\epsilon_i u_i) = \sigma_{\epsilon u}$. Throughout, $\epsilon_i, u_i \sim N(0, 1)$. We set $n = 50, 100, 200, 400$ and $x_{i,j}, q_i \sim i.i.d.N(0, 1)$. We consider three coefficient setups: Setup A: $\beta_1 = 1$, $\beta_2 = 2$, $\pi_1 = 1$, $\pi_2 = 2$; Setup B: $\beta_1 = 1$, $\beta_2 = 1.1$, $\pi_1 = 1$, $\pi_2 = 1.1$; and Setup C: $\beta_1 = 1$, $\beta_2 = 2$, $\pi_1 = 0.1$, $\pi_2 = 0.2$. These setups are designed to evaluate the performance of the tests under a variety of conditions. In particular, we view Setup A as a baseline case where the instruments are reasonably strong and also the threshold nature of the model is powerful in the sense that the two regimes are clearly distinct. Then, Setup B considers a case where the nonlinearity of the model is not as pronounced. This investigation is of great relevance, since the whole testing apparatus that has been developed in this paper is geared toward addressing the presence of the nonlinearity. Setup C looks at the case where the strength of the instruments is smaller. This, in our view, is less important in the sense that weak instruments have been analyzed extensively in the literature, albeit not within a threshold model. For the specification of the endogeneity parameter, $\sigma_{\epsilon u}$, we consider the values $= 0, 0.05, 0.2, 0.4, 0.6, 0.8, 0.95$ for setups A and B and the values $= 0.1, 0.5, 0.9$ for setup C. Again, these choices reflect the need to investigate how the power of the tests changes with respect to this crucial parameter.

We use the two IV estimators that we have analyzed throughout the paper: 2SLS and GMM. For the 2SLS, GMM, and OLS estimation, the threshold parameter grid is made up of [T/5] equally spaced points between the 10% and 90% quantiles of q_i, where [.] denotes integer part. Following the discussion in Caner and Hansen (2004, p. 816) and in order to minimize computational cost[6] arising out of the consideration of multiple implementations of the bootstrap, we assume that $\gamma = \rho$ and impose that restriction in estimation. This implies that the explicit formulas for the 2SLS and GMM estimators, given in Section 2, concerning the reduced form equation, are valid for this Monte Carlo study. Throughout the analysis, we use a significance level of 5%.

We now discuss the bootstrap implementation in detail. Note that a full step-by-step description of all bootstrap implementations is also given in Appendix B. We consider 5 different bootstrap implementations. The first is the standard model bootstrap, where the null hypothesis is imposed by using the OLS residuals. The bootstrap variance estimator, denoted $\hat{V}^z$, from this bootstrap is used to

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construct S. In particular, the OLS residuals, $\tilde{\epsilon}_t$, are resampled with replacement to construct a bootstrap sample of error terms of T observations. These bootstrap error terms are then used in (6) together with the OLS parameter estimates (which are consistent under the null hypothesis) to construct bootstrap samples for y_i. As discussed in the previous section, x_i, z_i, and q_i are resampled nonparametrically. For clarity, we note that the bootstrap variance estimator, when, e.g., the 2SLS and OLS estimators are compared, is given by

$$\hat{V}^z = \frac{1}{B-1}\sum_{j=1}^{B}\left[\left(\hat{\beta}_j^* - \tilde{\beta}_j^*\right) - \frac{1}{B}\sum_{k=1}^{B}\left(\hat{\beta}_k^* - \tilde{\beta}_k^*\right)\right]\left[\left(\hat{\beta}_j^* - \tilde{\beta}_j^*\right) - \frac{1}{B}\sum_{k=1}^{B}\left(\hat{\beta}_k^* - \tilde{\beta}_k^*\right)\right]',$$

where $\hat{\beta}_j^*$ and $\tilde{\beta}_j^*$ denote the OLS and 2SLS estimators from the jth bootstrap replication sample, and B denotes the number of bootstrap replications.

The rest of the bootstrap methods differ from the above method mainly in the way bootstrap error terms are constructed. The second implementation uses the wild bootstrap to get the variance, denoted by $\hat{V}^w$. In this case, the bootstrap error terms are constructed as $\epsilon_i^* = \eta_i \tilde{\epsilon}_i$, where η_i is an i.i.d. zero mean normal sequence independent of all other variables whose variance is unity. The third implementation uses a fully nonparametric bootstrap that resamples nonparametrically y_i as well as x_i, z_i, and q_i. This is not a block bootstrap, as the data for the Monte Carlo study are i.i.d. Obviously this bootstrap does not impose the null hypothesis on the bootstrap samples. This variance estimate is denoted $\hat{V}^n$. The fourth implementation uses subsampling, where the subsample size is set to $a(T)T$ of the sample size, where $a(50) = 0.9, a(100) = 0.8, a(200) = 0.7$, and $a(400) = 0.6$. Otherwise, the implementation is exactly the same as for the first boostrap method based on OLS residuals. The variance obtained is denoted by $\hat{V}^s$.

Finally, we consider a bootstrap test that bootstraps the whole distribution of S rather than only the variance. The wild bootstrap is used for that. For this test we considered both a statistic that is normalized by the variance (i.e., of the form $n(\hat{\beta} - \tilde{\beta})'\hat{V}^{-1}(\hat{\beta} - \tilde{\beta})$) and an unnormalized statistic (i.e., of the form $n(\hat{\beta} - \tilde{\beta})'(\hat{\beta} - \tilde{\beta})$). We found that the unnormalized statistic performed better in terms of performance under the null, in the sense that the normalized statistic overrejected the null hypothesis. Therefore, we focus on the unnormalized statistic. This may sound problematic, given the well-known theoretical reasons for using a pivotal (normalized) statistic. These reasons relate to the asymptotic refinements afforded by pivotal statistics. However, in our case, the availability of such refinements is in serious doubt, given the nature of the nonlinearity of the model coming from the threshold structure and the resulting nonstandard asymptotic theory associated with the model. Further, it is well known (see, e.g., Tibshirani, 1988) that if the variance of a test statistic is not well behaved in small samples, using pivotal statistics may be problematic. As we will see from our Monte Carlo results, this is indeed the case for the variance of the specific test statistic we use. As a result, our choice seems reasonable in the context. Other bootstrap

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TABLE 1. Rejection probabilities of the asymptotic test for the 2SLS and OLS estimators using asymptotic variances

	One instrument				Two instruments			
$\sigma_{\epsilon u}$/T	50	100	200	400	50	100	200	400
A. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 1, \pi_2 = 2$								
0	0.002	0.000	0.000	0.000	0.000	0.001	0.000	0.000
0.05	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.2	0.007	0.002	0.000	0.001	0.000	0.000	0.000	0.000
0.4	0.013	0.005	0.015	0.111	0.001	0.000	0.000	0.000
0.6	0.044	0.043	0.116	0.797	0.003	0.001	0.003	0.073
0.8	0.072	0.124	0.532	0.999	0.012	0.012	0.033	0.617
0.95	0.101	0.233	0.819	0.999	0.027	0.028	0.159	0.959
B. $\beta_1 = 1, \beta_2 = 1.1, \pi_1 = 1, \pi_2 = 1.1$								
0	0.125	0.125	0.071	0.064	0.076	0.067	0.038	0.042
0.05	0.110	0.088	0.085	0.069	0.094	0.077	0.059	0.036
0.2	0.138	0.129	0.117	0.175	0.096	0.078	0.091	0.111
0.4	0.163	0.234	0.460	0.856	0.124	0.167	0.289	0.620
0.6	0.243	0.462	0.901	0.997	0.220	0.344	0.668	0.988
0.8	0.389	0.782	0.989	1.000	0.317	0.579	0.957	0.999
0.95	0.529	0.915	1.000	1.000	0.389	0.744	0.984	1.000
C. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 0.1, \pi_2 = 0.2$								
0.1	0.004	0.004	0.001	0.003	0.006	0.001	0.002	0.007
0.5	0.002	0.005	0.012	0.013	0.006	0.003	0.010	0.039
0.9	0.002	0.005	0.026	0.100	0.000	0.003	0.030	0.247

implementations, apart from the wild bootstrap, were also tried but had quite low power, indicating that only the wild bootstrap may be appropriate for full bootstrap inference in small samples. For all experiments, we used 299 bootstrap replications and 1,000 Monte Carlo replications.

To evaluate the performance of the estimators, we report the rejection probabilities of the Hausman-type test. These are reported in Tables 1–12.

Results make very interesting reading. The asymptotic test, using asymptotic variances, performs badly both for the 2SLS and GMM estimators. For the 2SLS and Setup A, it underrejects very substantially, as it never rejects under the null hypothesis. For Setup B, on the other hand, it overrejects substantially in small samples but improves for samples of 400 observations. Surprisingly, for this setup it performs better when there are two instruments than when there is only one instrument. Looking at the performance of this test under the alternative for the 2SLS estimator, we see that the rejection probability increases as expected when

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TABLE 2. Rejection probabilities of the asymptotic test for the GMM and OLS estimators using asymptotic variances

	One instrument				Two instruments			
$\sigma_{\epsilon u}$/T	50	100	200	400	50	100	200	400
A. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 1, \pi_2 = 2$								
0	0.225	0.192	0.159	0.123	0.263	0.234	0.215	0.191
0.05	0.235	0.210	0.181	0.169	0.257	0.266	0.227	0.236
0.2	0.284	0.402	0.515	0.829	0.335	0.378	0.577	0.856
0.4	0.525	0.751	0.970	0.995	0.461	0.722	0.928	0.981
0.6	0.774	0.954	0.985	0.996	0.658	0.911	0.967	0.988
0.8	0.893	0.954	0.985	0.993	0.836	0.955	0.978	0.990
0.95	0.913	0.966	0.987	0.992	0.865	0.957	0.987	0.985
B. $\beta_1 = 1, \beta_2 = 1.1, \pi_1 = 1, \pi_2 = 1.1$								
0	0.184	0.155	0.116	0.087	0.208	0.186	0.161	0.134
0.05	0.173	0.158	0.143	0.126	0.220	0.207	0.186	0.183
0.2	0.278	0.296	0.438	0.698	0.261	0.320	0.513	0.806
0.4	0.441	0.716	0.930	0.995	0.412	0.702	0.906	0.964
0.6	0.770	0.938	0.980	0.995	0.629	0.866	0.932	0.972
0.8	0.918	0.974	0.994	0.999	0.801	0.906	0.957	0.985
0.95	0.963	0.982	0.989	0.999	0.841	0.931	0.963	0.976
C. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 0.1, \pi_2 = 0.2$								
0.1	0.157	0.086	0.051	0.027	0.115	0.046	0.026	0.029
0.5	0.209	0.143	0.156	0.227	0.112	0.085	0.128	0.334
0.9	0.392	0.549	0.790	0.954	0.277	0.463	0.735	0.910

$\sigma_{\epsilon u}$ and/or T increases. Looking at the GMM estimator, we see that it overrejects very substantially for Setup A. For Setup B, it still overrejects but improves more with increasing sample size. In terms of rejection probabilities under the alternative hypothesis, we see that the test rejects more often for larger $\sigma_{\epsilon u}$ and T.

Moving to the bootstrap-based tests, we see that the best performer by far is the asymptotic test using the wild bootstrap variance estimator. This test is correctly sized, albeit slightly underrejecting for small samples. It also has very good power. Even for 100 observations and reasonably large $\sigma_{\epsilon u}$, the power is at least 95%. The power of the test is largest for Setup A as expected, but it still has a satisfactory rejection probability for Setup B and larger samples. It has low power for Setup C, but that is to be expected given the weakness of the instruments. The test seems also to improve in terms of power when more than one instrument is used. There seems to be little difference between the tests based on

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TABLE 3. Rejection probabilities of the asymptotic test for the 2SLS and OLS estimators using OLS bootstrap variances

	One instrument				Two instruments			
$\sigma_{\epsilon u}$/T	50	100	200	400	50	100	200	400
A. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 1, \pi_2 = 2$								
0	0.006	0.021	0.034	0.042	0.010	0.026	0.037	0.051
0.05	0.004	0.022	0.049	0.090	0.024	0.050	0.076	0.104
0.2	0.010	0.091	0.349	0.788	0.040	0.173	0.514	0.876
0.4	0.032	0.444	0.964	1.000	0.120	0.691	0.989	1.000
0.6	0.116	0.729	0.996	1.000	0.412	0.961	1.000	1.000
0.8	0.318	0.846	0.982	1.000	0.594	0.977	1.000	1.000
0.95	0.388	0.867	0.981	0.998	0.708	0.988	1.000	1.000
B. $\beta_1 = 1, \beta_2 = 1.1, \pi_1 = 1, \pi_2 = 1.1$								
0	0.008	0.017	0.010	0.013	0.019	0.023	0.020	0.021
0.05	0.007	0.014	0.009	0.018	0.030	0.023	0.024	0.028
0.2	0.012	0.027	0.060	0.202	0.031	0.036	0.069	0.286
0.4	0.018	0.094	0.465	0.897	0.037	0.131	0.434	0.900
0.6	0.045	0.327	0.895	0.994	0.088	0.351	0.870	0.983
0.8	0.096	0.671	0.987	1.000	0.196	0.612	0.963	0.998
0.95	0.179	0.840	0.995	1.000	0.332	0.789	0.990	1.000
C. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 0.1, \pi_2 = 0.2$								
0.1	0.011	0.010	0.008	0.001	0.009	0.013	0.016	0.031
0.5	0.014	0.013	0.010	0.020	0.028	0.043	0.100	0.241
0.9	0.025	0.027	0.041	0.184	0.087	0.228	0.543	0.778

the 2SLS and GMM estimators. This pattern persists when we consider all other tests.

The wild bootstrap works well even when the whole bootstrap distribution is being used, even though its power is considerably less than that of the wild bootstrap variance test. Similar patterns, to those of the wild bootstrap variance test are observed, concerning the behavior of the test with respect to $\sigma_{\epsilon u}$, the sample size, the setup used, and the number of instruments.

The second best bootstrap variance test is based on OLS residuals. This test underrejects more than the wild bootstrap tests under the null hypothesis. It also has significantly lower power. The nonparametric bootstrap comes next in terms of performance. This test is similar in performance to the bootstrap test based on OLS residuals. The worst performer is the subsample-based test, which has quite low power. In summary, the asymptotic tests using asymptotic variances have bad and variable performance under the null hypothesis. It is clear that the asymptotic variance is not a good estimator of the variance of the difference between the

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TABLE 4. Rejection probabilities of the asymptotic test for the 2SLS and OLS estimators using wild bootstrap variances

	One instrument				Two instruments			
$\sigma_{\epsilon u}$/T	50	100	200	400	50	100	200	400
A. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 1, \pi_2 = 2$								
0	0.017	0.033	0.046	0.050	0.028	0.046	0.052	0.052
0.05	0.022	0.037	0.075	0.086	0.036	0.073	0.080	0.107
0.2	0.052	0.212	0.451	0.819	0.106	0.258	0.571	0.897
0.4	0.234	0.758	0.989	1.000	0.355	0.855	0.994	1.000
0.6	0.644	0.976	1.000	1.000	0.807	0.998	1.000	1.000
0.8	0.908	0.994	1.000	1.000	0.967	0.999	1.000	1.000
0.95	0.941	0.993	1.000	1.000	0.982	0.999	1.000	1.000
B. $\beta_1 = 1, \beta_2 = 1.1, \pi_1 = 1, \pi_2 = 1.1$								
0	0.033	0.052	0.032	0.042	0.043	0.049	0.042	0.056
0.05	0.015	0.037	0.056	0.058	0.067	0.061	0.061	0.076
0.2	0.057	0.096	0.185	0.439	0.089	0.104	0.234	0.589
0.4	0.104	0.335	0.790	0.992	0.179	0.462	0.844	0.988
0.6	0.309	0.787	0.991	1.000	0.444	0.827	0.987	0.999
0.8	0.654	0.985	0.999	1.000	0.718	0.975	0.998	1.000
0.95	0.893	0.996	1.000	1.000	0.880	0.991	1.000	1.000
C. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 0.1, \pi_2 = 0.2$								
0.1	0.079	0.107	0.096	0.057	0.062	0.104	0.093	0.094
0.5	0.111	0.164	0.212	0.297	0.115	0.159	0.333	0.496
0.9	0.362	0.638	0.885	0.997	0.365	0.673	0.893	0.965

2SLS/GMM estimators and the OLS estimator. This can be corrected by using the bootstrap. However, use of the asymptotic distributional approximation once a boostrap estimate of the variance is used seems to be a satisfactory avenue for conducting inference.

5. CONCLUSION

Endogeneity of the explanatory variables is an issue that has received little attention in the analysis of nonlinear econometric models. Recently, work by Caner and Hansen (2004) and Psaradakis, Sola, and Spagnolo (2005) have looked at ways of estimating nonlinear models under the presence of endogeneity. However, the question of how to determine the presence of endogeneity is open. This paper addresses that issue.

We propose a Hausman-type test to test for endogeneity. The asymptotic analysis of Hausman (1978) provides a solution to the construction of the test under the

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TABLE 5. Rejection probabilities of the asymptotic test for the 2SLS and OLS estimators using nonparametric bootstrap variances

	One instrument				Two instruments			
$\sigma_{\epsilon u}$/T	50	100	200	400	50	100	200	400
A. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 1, \pi_2 = 2$								
0	0.006	0.014	0.032	0.041	0.016	0.026	0.031	0.047
0.05	0.003	0.022	0.040	0.079	0.021	0.041	0.067	0.099
0.2	0.010	0.080	0.340	0.788	0.041	0.138	0.498	0.880
0.4	0.025	0.389	0.948	1.000	0.094	0.660	0.989	1.000
0.6	0.098	0.704	0.994	1.000	0.322	0.940	1.000	1.000
0.8	0.254	0.828	0.985	1.000	0.532	0.971	1.000	1.000
0.95	0.323	0.860	0.981	0.999	0.642	0.971	0.999	1.000
B. $\beta_1 = 1, \beta_2 = 1.1, \pi_1 = 1, \pi_2 = 1.1$								
0	0.012	0.014	0.009	0.012	0.022	0.028	0.017	0.018
0.05	0.007	0.012	0.008	0.016	0.031	0.027	0.028	0.028
0.2	0.010	0.024	0.052	0.208	0.035	0.034	0.064	0.263
0.4	0.019	0.090	0.442	0.894	0.037	0.114	0.418	0.893
0.6	0.046	0.275	0.889	0.993	0.067	0.322	0.855	0.979
0.8	0.074	0.594	0.987	1.000	0.156	0.573	0.964	0.998
0.95	0.147	0.789	0.995	1.000	0.292	0.753	0.988	0.999
C. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 0.1, \pi_2 = 0.2$								
0.1	0.012	0.008	0.008	0.001	0.014	0.015	0.020	0.028
0.5	0.016	0.011	0.010	0.018	0.026	0.045	0.098	0.232
0.9	0.025	0.032	0.044	0.183	0.088	0.222	0.525	0.788

restrictive assumption of having an efficient estimator for the parameters under the null hypothesis. In many nonlinear models, such an assumption may be difficult to satisfy. Furthermore, asymptotic analysis is less likely to be of relevance in small samples for nonlinear models than for linear ones.

This paper therefore proposes bootstrap-based tests to solve the problem. Monte Carlo analysis clearly shows the importance of imposing the null hypothesis of no endogeneity in the bootstrap samples for the construction of a test with desirable properties. We propose a version of the wild bootstrap that seems to perform very well under all circumstances considered.

Threshold models have been used as a vehicle for the analysis for the simple reason that via the work of Caner and Hansen (2004), they have been analyzed in the context of endogeneity. But the techniques discussed here are straightforwardly applicable to many alternative nonlinear models. More generally, the use of the wild bootstrap is of more general interest in the context of bootstrap testing of exogeneity.

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TABLE 6. Rejection probabilities of the asymptotic test for the 2SLS and OLS estimators using subsampling variances

	One instrument				Two instruments			
$\sigma_{\epsilon u}$/T	50	100	200	400	50	100	200	400
A. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 1, \pi_2 = 2$								
0	0.001	0.003	0.016	0.021	0.004	0.011	0.015	0.027
0.05	0.000	0.009	0.021	0.045	0.009	0.023	0.026	0.043
0.2	0.002	0.038	0.221	0.690	0.018	0.086	0.352	0.812
0.4	0.011	0.228	0.919	1.000	0.047	0.510	0.981	1.000
0.6	0.034	0.594	0.989	0.999	0.231	0.902	1.000	1.000
0.8	0.152	0.767	0.976	1.000	0.426	0.949	1.000	1.000
0.95	0.208	0.798	0.974	0.999	0.573	0.962	0.999	1.000
B. $\beta_1 = 1, \beta_2 = 1.1, \pi_1 = 1, \pi_2 = 1.1$								
0	0.005	0.008	0.004	0.006	0.019	0.022	0.018	0.013
0.05	0.004	0.008	0.005	0.011	0.031	0.020	0.018	0.017
0.2	0.004	0.013	0.030	0.112	0.030	0.024	0.045	0.169
0.4	0.012	0.049	0.300	0.835	0.028	0.074	0.286	0.838
0.6	0.022	0.178	0.836	0.993	0.055	0.203	0.780	0.977
0.8	0.036	0.437	0.975	1.000	0.114	0.469	0.946	0.998
0.95	0.078	0.676	0.994	1.000	0.211	0.682	0.987	1.000
C. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 0.1, \pi_2 = 0.2$								
0.1	0.010	0.007	0.006	0.000	0.012	0.014	0.013	0.030
0.5	0.012	0.012	0.007	0.006	0.018	0.032	0.077	0.174
0.9	0.032	0.032	0.028	0.117	0.075	0.193	0.485	0.763

NOTES

1. The ρ-mixing condition imposes restrictions on the dependence of a stochastic process. It is weaker than uniform mixing but stronger than strong mixing. For a formal definition of the condition, see Peligrad (1982).

2. In this context we note the work of Inoue and Kilian (2003), where it is shown that for an unrelated problem (unit root inference), continuity of that form is not needed for bootstrap validity.

3. We should note here that not imposing the null hypothesis, does not necessarily lead to an inconsistent bootstrap test. However, for the purposes of this paper we follow the practice of imposing the null hypothesis when constructing the bootstrap distribution of the test statistic.

4. More generally, superscript stars denote quantities or concepts such as, e.g., samples, variables, expectations, probabilities, or modes of convergence that relate to the probability space defined by the original sample realization and thus also relate to the bootstrap.

5. Note that when heteroskedasticity is allowed, the nonparametric resampling of z_i, x_i, and q_i may be problematic when using the wild bootstrap since heteroskedasticity in this case may not be preserved. However, treatment of heteroskedasticity is beyond the scope of this paper, and by Assumption 1 we impose homoskedasticity in our analysis.

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6. The computational cost of this study is substantial: Running all experiments took 1 1/2 months of computer time on a single processor Linux server. Allowing for the separate estimation of a reduced form equation threshold parameter would have made the estimation infeasible.

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APPENDIX A

Proof of Lemma 1. Assumptions 1–4 are needed for the results of Hansen (2000) and Caner and Hansen (2004) and so are assumed throughout. Let $\tilde{\beta}^o$, $\hat{\beta}^o$, and $\bar{\beta}^o$ denote the OLS, 2SLS, and GMM estimates of β, assuming knowledge of the true value of γ, γ_0. Lemma A.12 of Hansen (2000) and Lemma 6 of Caner and Hansen (2004) show that

$$\sqrt{n}(\tilde{\beta}-\beta)-\sqrt{n}(\tilde{\beta}^o-\beta)=o_p(1)\,, \quad \sqrt{n}(\hat{\beta}-\beta)-\sqrt{n}(\hat{\beta}^o-\beta)=o_p(1), \quad \text{and}$$

$$\sqrt{n}(\bar{\beta}-\beta)-\sqrt{n}(\bar{\beta}^o-\beta)=o_p(1). \tag{A.1}$$

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TABLE 7. Rejection probabilities of the wild boostrap test for the 2SLS and OLS estimators

	One instrument				Two instruments			
$\sigma_{\epsilon u}$/T	50	100	200	400	50	100	200	400
A. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 1, \pi_2 = 2$								
0	0.022	0.027	0.051	0.057	0.021	0.031	0.046	0.048
0.05	0.019	0.042	0.044	0.075	0.035	0.046	0.058	0.077
0.2	0.045	0.142	0.264	0.560	0.072	0.165	0.362	0.684
0.4	0.156	0.518	0.892	0.995	0.233	0.620	0.941	1.000
0.6	0.459	0.891	0.999	1.000	0.630	0.973	1.000	1.000
0.8	0.786	0.968	0.999	1.000	0.907	0.999	1.000	1.000
0.95	0.893	0.976	0.997	1.000	0.964	0.999	1.000	1.000
B. $\beta_1 = 1, \beta_2 = 1.1, \pi_1 = 1, \pi_2 = 1.1$								
0	0.017	0.030	0.024	0.030	0.036	0.038	0.033	0.051
0.05	0.012	0.024	0.034	0.055	0.052	0.048	0.047	0.059
0.2	0.023	0.052	0.129	0.359	0.062	0.075	0.176	0.520
0.4	0.046	0.205	0.626	0.959	0.103	0.306	0.700	0.965
0.6	0.167	0.566	0.965	1.000	0.286	0.652	0.960	0.998
0.8	0.421	0.904	0.997	1.000	0.533	0.905	0.996	1.000
0.95	0.710	0.990	1.000	1.000	0.746	0.968	0.999	1.000
C. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 0.1, \pi_2 = 0.2$								
0.1	0.060	0.076	0.061	0.046	0.046	0.074	0.082	0.074
0.5	0.062	0.108	0.152	0.223	0.072	0.119	0.271	0.428
0.9	0.266	0.520	0.793	0.962	0.271	0.595	0.838	0.957

Hence, for the purposes of this lemma, we assume knowledge of γ_0. We focus on the case where the two estimators compared are 2SLS and OLS. The case of GMM and OLS can be treated similarly and so we simply report the result at the end of the proof. We are looking for the asymptotic distribution of $\sqrt{n}(\hat{\beta}^o - \beta) - \sqrt{n}(\tilde{\beta}^o - \beta)$. Let $z_{12,i} = (z_i I_{q_i \le \gamma_0}, z_i I_{q_i > \gamma_0})'$, $Z_{12} = (z_{12,1}, \ldots, z_{12,n})'$, $x_{12,i} = (x_i I_{q_i \le \gamma_0}, x_i I_{q_i > \gamma_0})'$, $X_{12} = (x_{12,1}, \ldots, x_{12,n})'$, and $\epsilon = (\epsilon_1, \ldots, \epsilon_n)'$. We have

$$\begin{pmatrix} \sqrt{n}(\hat{\beta}^o - \beta) \\ \sqrt{n}(\tilde{\beta}^o - \beta) \end{pmatrix}$$

$$= \begin{pmatrix} n\left(Z_{12}'X_{12}\left(X_{12}'X_{12}\right)^{-1}X_{12}'Z_{12}\right)^{-1} Z_{12}'X_{12}\left(X_{12}'X_{12}\right)^{-1} & 0 \\ 0 & n\left(Z_{12}'Z_{12}\right)^{-1} \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{\sqrt{n}} X_{12}'\epsilon \\ \frac{1}{\sqrt{n}} Z_{12}'\epsilon \end{pmatrix}. \qquad \textbf{(A.2)}$$

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TABLE 8. Rejection probabilities of the asymptotic test for the GMM and OLS estimators using OLS bootstrap variances

	One instrument				Two instruments			
$\sigma_{\epsilon u}$/T	50	100	200	400	50	100	200	400
A. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 1, \pi_2 = 2$								
0	0.005	0.021	0.034	0.042	0.005	0.022	0.032	0.038
0.05	0.003	0.022	0.049	0.090	0.010	0.026	0.052	0.079
0.2	0.012	0.091	0.349	0.788	0.023	0.114	0.418	0.843
0.4	0.033	0.444	0.964	1.000	0.063	0.521	0.977	1.000
0.6	0.122	0.729	0.996	1.000	0.210	0.901	1.000	1.000
0.8	0.325	0.846	0.982	1.000	0.389	0.967	1.000	1.000
0.95	0.402	0.867	0.981	0.998	0.509	0.971	1.000	1.000
B. $\beta_1 = 1, \beta_2 = 1.1, \pi_1 = 1, \pi_2 = 1.1$								
0	0.008	0.017	0.010	0.013	0.013	0.020	0.014	0.019
0.05	0.006	0.014	0.009	0.018	0.020	0.025	0.026	0.025
0.2	0.011	0.027	0.060	0.202	0.028	0.027	0.057	0.229
0.4	0.018	0.094	0.465	0.897	0.036	0.080	0.295	0.819
0.6	0.044	0.327	0.895	0.994	0.049	0.198	0.714	0.958
0.8	0.094	0.670	0.987	1.000	0.108	0.410	0.877	0.990
0.95	0.174	0.839	0.995	1.000	0.152	0.591	0.950	0.997
C. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 0.1, \pi_2 = 0.2$								
0.1	0.013	0.010	0.008	0.001	0.016	0.020	0.022	0.035
0.5	0.014	0.013	0.010	0.020	0.032	0.043	0.099	0.253
0.9	0.026	0.027	0.041	0.184	0.098	0.239	0.526	0.767

But we have

$$\begin{pmatrix} \frac{1}{\sqrt{n}} X_{12}' \epsilon \\ \frac{1}{\sqrt{n}} Z_{12}' \epsilon \end{pmatrix} \Rightarrow N\left(0, \sigma_\epsilon^2 \begin{pmatrix} \Omega_x & \Omega_{xz} \\ \Omega_{xz}' & \Omega_z \end{pmatrix}\right), \tag{A.3}$$

where $\Omega_x = \mathrm{E}\left(x_{12,i} x_{12,i}' \epsilon_i^2\right)$, $\Omega_z = \mathrm{E}\left(z_{12,i} z_{12,i}' \epsilon_i^2\right)$, and $\Omega_{xz} = \mathrm{E}\left(x_{12,i} z_{12,i}' \epsilon_i^2\right)$. Further,

$$\begin{pmatrix} \frac{1}{n} Z_{12}' X_{12} \\ \frac{1}{n} Z_{12}' Z_{12} \\ \frac{1}{n} X_{12}' X_{12} \end{pmatrix} \xrightarrow{p} \begin{pmatrix} R_{xz} \\ Q_z \\ Q_x \end{pmatrix}, \tag{A.4}$$

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TABLE 9. Rejection probabilities of the asymptotic test for the GMM and OLS estimators using wild bootstrap variances

	One instrument				Two instruments			
$\sigma_{\epsilon u}$/T	50	100	200	400	50	100	200	400
A. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 1, \pi_2 = 2$								
0	0.017	0.033	0.046	0.050	0.016	0.034	0.042	0.047
0.05	0.022	0.037	0.075	0.086	0.032	0.035	0.060	0.080
0.2	0.052	0.212	0.451	0.819	0.060	0.190	0.462	0.867
0.4	0.234	0.758	0.989	1.000	0.222	0.719	0.986	1.000
0.6	0.642	0.976	1.000	1.000	0.582	0.985	1.000	1.000
0.8	0.908	0.994	1.000	1.000	0.882	1.000	1.000	1.000
0.95	0.942	0.993	1.000	1.000	0.942	0.999	1.000	1.000
B. $\beta_1 = 1, \beta_2 = 1.1, \pi_1 = 1, \pi_2 = 1.1$								
0	0.034	0.052	0.032	0.042	0.040	0.045	0.047	0.048
0.05	0.015	0.037	0.056	0.058	0.061	0.058	0.061	0.069
0.2	0.056	0.096	0.185	0.439	0.063	0.086	0.210	0.553
0.4	0.103	0.335	0.790	0.992	0.140	0.384	0.780	0.980
0.6	0.308	0.787	0.991	1.000	0.314	0.723	0.967	0.998
0.8	0.641	0.985	0.999	1.000	0.532	0.930	0.995	1.000
0.95	0.877	0.996	1.000	1.000	0.711	0.963	0.996	1.000
C. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 0.1, \pi_2 = 0.2$								
0.1	0.079	0.107	0.096	0.057	0.079	0.116	0.094	0.102
0.5	0.112	0.164	0.212	0.297	0.117	0.169	0.319	0.498
0.9	0.362	0.638	0.885	0.997	0.368	0.667	0.870	0.959

where $Q_x = \mathrm{E}(x_{12,i}x'_{12,i})$, $Q_z = \mathrm{E}(z_{12,i}z'_{12,i})$, and $R_{xz} = \mathrm{E}(z_{12,i}x'_{12,i})$. Thus, combining (A.2)–(A.4) gives that, under the null hypothesis of no endogeneity,

$$\begin{pmatrix} \sqrt{n}(\hat{\beta}^o - \beta) \\ \sqrt{n}(\tilde{\beta}^o - \beta) \end{pmatrix}$$

$$\Rightarrow N\left(0, \begin{pmatrix} \Upsilon_{2SLS} R_{xz} Q_x^{-1} \Omega_x Q_x^{-1} R'_{xz} \Upsilon_{2SLS} & \Upsilon_{2SLS} R_{xz} Q_x^{-1} \Omega_{xz} Q_z^{-1} \\ Q_z^{-1} \Omega'_{xz} Q_x^{-1} R'_{xz} \Upsilon_{2SLS} & Q_z^{-1} \Omega_z Q_z^{-1} \end{pmatrix}\right), \quad \textbf{(A.5)}$$

where $\Upsilon_{2SLS} = \left(R_{xz} Q_x^{-1} R'_{xz}\right)^{-1}$. This implies that

$$\sqrt{n}(\hat{\beta}^o - \tilde{\beta}^o) \Rightarrow N(0, V_{2SLS}) \quad \textbf{(A.6)}$$

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TABLE 10. Rejection probabilities of the asymptotic test for the GMM and OLS estimators using nonparametric bootstrap variances

	One instrument				Two instruments			
$\sigma_{\epsilon u}$/T	50	100	200	400	50	100	200	400
A. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 1, \pi_2 = 2$								
0	0.005	0.014	0.032	0.041	0.007	0.018	0.028	0.038
0.05	0.003	0.022	0.040	0.079	0.010	0.021	0.043	0.079
0.2	0.011	0.080	0.340	0.788	0.015	0.091	0.401	0.843
0.4	0.027	0.389	0.948	1.000	0.055	0.480	0.974	1.000
0.6	0.101	0.704	0.994	1.000	0.157	0.874	1.000	1.000
0.8	0.257	0.828	0.985	1.000	0.324	0.954	1.000	1.000
0.95	0.337	0.860	0.981	0.999	0.444	0.973	1.000	1.000
B. $\beta_1 = 1, \beta_2 = 1.1, \pi_1 = 1, \pi_2 = 1.1$								
0	0.010	0.014	0.009	0.012	0.016	0.021	0.016	0.016
0.05	0.007	0.012	0.008	0.016	0.024	0.029	0.029	0.023
0.2	0.010	0.024	0.052	0.208	0.028	0.029	0.060	0.223
0.4	0.017	0.090	0.442	0.894	0.036	0.077	0.288	0.808
0.6	0.044	0.275	0.889	0.993	0.044	0.197	0.690	0.952
0.8	0.074	0.593	0.987	1.000	0.099	0.393	0.871	0.988
0.95	0.138	0.789	0.995	1.000	0.144	0.565	0.944	0.997
C. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 0.1, \pi_2 = 0.2$								
0.1	0.012	0.008	0.008	0.001	0.014	0.014	0.021	0.034
0.5	0.018	0.011	0.010	0.018	0.031	0.052	0.098	0.233
0.9	0.025	0.032	0.044	0.183	0.104	0.238	0.516	0.757

where

$$V_{2SLS} = \Upsilon_{2SLS} R_{xz} Q_x^{-1} \Omega_x Q_x^{-1} R_{xz}' \Upsilon_{2SLS} + Q_z^{-1} \Omega_z Q_z^{-1} - \Upsilon_{2SLS} R_{xz} Q_x^{-1} \Omega_{xz} Q_z^{-1} - Q_z^{-1} \Omega_{xz}' Q_x^{-1} R_{xz}' \Upsilon_{2SLS} \quad \textbf{(A.7)}$$

which establishes the result of the lemma. By a similar analysis we can straightforwardly obtain that

$$\sqrt{n}(\tilde{\beta}^o - \tilde{\beta}^o) \Rightarrow N(0, V_{GMM}), \quad \textbf{(A.8)}$$

where

$$V_{GMM} = \Upsilon_{GMM} + Q_z^{-1} \Omega_z Q_z^{-1} - \Upsilon_{GMM} R_{xz} \Omega_x^{-1} \Omega_{xz} Q_z^{-1} - Q_z^{-1} \Omega_{xz}' \Omega_x^{-1} R_{xz}' \Upsilon_{GMM} \quad \textbf{(A.9)}$$

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TABLE 11. Rejection probabilities of the asymptotic test for the GMM and OLS estimators using subsampling variances

	One instrument				Two instruments			
$\sigma_{\epsilon u}$/T	50	100	200	400	50	100	200	400
A. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 1, \pi_2 = 2$								
0	0.001	0.003	0.016	0.021	0.002	0.007	0.009	0.021
0.05	0.000	0.009	0.021	0.045	0.003	0.009	0.024	0.041
0.2	0.002	0.038	0.221	0.690	0.011	0.054	0.272	0.743
0.4	0.012	0.228	0.919	1.000	0.027	0.318	0.944	1.000
0.6	0.036	0.594	0.989	0.999	0.099	0.782	1.000	1.000
0.8	0.156	0.767	0.976	1.000	0.234	0.918	1.000	1.000
0.95	0.220	0.800	0.974	0.999	0.324	0.939	1.000	1.000
B. $\beta_1 = 1, \beta_2 = 1.1, \pi_1 = 1, \pi_2 = 1.1$								
0	0.002	0.008	0.004	0.006	0.013	0.016	0.014	0.013
0.05	0.002	0.008	0.005	0.011	0.017	0.028	0.015	0.018
0.2	0.004	0.013	0.030	0.112	0.021	0.022	0.039	0.148
0.4	0.012	0.046	0.300	0.835	0.029	0.047	0.177	0.729
0.6	0.020	0.177	0.836	0.993	0.036	0.115	0.598	0.946
0.8	0.030	0.436	0.975	1.000	0.073	0.285	0.821	0.985
0.95	0.065	0.673	0.994	1.000	0.107	0.452	0.924	0.996
C. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 0.1, \pi_2 = 0.2$								
0.1	0.009	0.007	0.006	0.000	0.009	0.015	0.014	0.025
0.5	0.013	0.012	0.007	0.006	0.027	0.043	0.070	0.184
0.9	0.030	0.032	0.028	0.117	0.090	0.211	0.485	0.747

and $\Upsilon_{GMM} = \left(R_{xz}\Omega_x^{-1}R_{xz}'\right)^{-1}$. Equally straightforwardly, under the case of conditional homoskedasticity, V_{2SLS} and V_{GMM} simplify to

$$V_{GMM} = V_{2SLS} = \Upsilon_{2SLS} - Q_z^{-1}. \tag{A.10}$$

Proof of Theorem 1. We will derive bootstrap results for the following model:

$$y_i = \beta_1' z_i I_{q_i \le \gamma} + \beta_2' z_i I_{q_i > \gamma} + \epsilon_i. \tag{A.11}$$

We will show that the wild bootstrap provides valid inference for $\sqrt{n}(\tilde{\beta} - \beta)$ and $\sqrt{n}(\hat{\beta} - \beta)$ under the null hypothesis.

To prove the result of the theorem, we proceed as follows: Let us define some notation that mirrors more closely the notation of Hansen (2000). Let $z_{\gamma,i} = z_i I_{q_i \le \gamma}$, $\bar{z}_{i,\gamma} = (z_i', z_{\gamma,i}')'$, and $\bar{Z}_\gamma = (z_{\gamma,1}, \ldots z_{\gamma,n})'$. As mentioned before, let star superscripts denote

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TABLE 12. Rejection probabilities of the wild bootstrap test for the GMM and OLS estimators

	One instrument				Two instruments			
$\sigma_{\epsilon u}$/T	50	100	200	400	50	100	200	400
A. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 1, \pi_2 = 2$								
0	0.022	0.027	0.051	0.057	0.023	0.029	0.041	0.041
0.05	0.019	0.042	0.044	0.075	0.035	0.035	0.056	0.079
0.2	0.045	0.142	0.264	0.560	0.053	0.133	0.320	0.662
0.4	0.156	0.518	0.892	0.995	0.172	0.531	0.920	0.999
0.6	0.460	0.891	0.999	1.000	0.484	0.930	1.000	1.000
0.8	0.784	0.968	0.999	1.000	0.769	0.996	1.000	1.000
0.95	0.893	0.976	0.997	1.000	0.886	0.997	1.000	1.000
B. $\beta_1 = 1, \beta_2 = 1.1, \pi_1 = 1, \pi_2 = 1.1$								
0	0.017	0.030	0.024	0.030	0.026	0.030	0.033	0.044
0.05	0.011	0.024	0.034	0.055	0.044	0.036	0.041	0.058
0.2	0.022	0.052	0.129	0.359	0.044	0.064	0.155	0.480
0.4	0.045	0.205	0.626	0.959	0.082	0.237	0.586	0.937
0.6	0.163	0.566	0.965	1.000	0.166	0.498	0.904	0.994
0.8	0.415	0.904	0.997	1.000	0.346	0.798	0.987	0.999
0.95	0.705	0.990	1.000	1.000	0.526	0.909	0.993	1.000
C. $\beta_1 = 1, \beta_2 = 2, \pi_1 = 0.1, \pi_2 = 0.2$								
0.1	0.060	0.076	0.061	0.046	0.057	0.081	0.083	0.078
0.5	0.062	0.108	0.152	0.223	0.090	0.134	0.259	0.428
0.9	0.266	0.520	0.793	0.962	0.280	0.568	0.815	0.945

bootstrap quantities. Denote the true parameters of the regression of y_i on $\bar{z}_{i,\gamma_0}$ as $\delta_0 = (\delta'_{0,1}, \delta'_{0,\gamma})'$, the OLS estimates by $\tilde{\delta} = (\tilde{\delta}'_1, \tilde{\delta}'_\gamma)'$, and the 2SLS estimates by $\hat{\delta} = (\hat{\delta}'_1, \hat{\delta}'_\gamma)'$. If we prove the validity of the bootstrap for $\sqrt{n}(\tilde{\delta} - \delta)$ and $\sqrt{n}(\hat{\delta} - \delta)$, then the validity of the bootstrap for $\sqrt{n}(\tilde{\beta} - \beta)$ and $\sqrt{n}(\hat{\beta} - \beta)$ directly follows. Note that $\delta_{0,1} = \beta_{0,1}$ and $\delta_{0,\gamma} = \beta_{0,2} - \beta_{0,1}$, where $\beta_0 = (\beta'_{0,1}, \beta'_{0,2})'$ denotes the true parameters of model (1). We consider OLS estimation first.

The wild bootstrap for OLS estimation of the model (1) resamples with replacement rows of the matrix (Q, Z) and constructs the bootstrap sample for the error term by taking a wild bootstrap of the OLS residual of (1), $\tilde{\epsilon}$. More specifically each element of the resample of the error term denoted by $\epsilon^* = (\epsilon_1^*, \ldots, \epsilon_n^*)'$ is given by $\epsilon_i^* = \tilde{\epsilon}_i \eta_i$, where η_i is a (0,1) i.i.d. random variable that is independent of y, z, and q. Denote the resample of (Q, Z) by (Q^*, Z^*). The bootstrap sample for the dependent variable denoted by $y^* = (y_1^*, \ldots, y_n^*)'$ is given by

$$y_i^* = \tilde{\delta}'_1 z_i^* + \tilde{\delta}'_{\tilde{\gamma}} z_i^* I_{q_i^* > \tilde{\gamma}} + \epsilon_i^*, \tag{A.12}$$

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where $\tilde{\gamma}$ denotes the OLS estimate of γ_0. Let the OLS estimate of δ_0 and γ_0 from the generic bootstrap sample be denoted by $\tilde{\delta}^*$ and $\tilde{\gamma}^*$. The result for OLS estimation will follow if we show that

$$\sqrt{n}\left(\tilde{\delta}^* - \tilde{\delta}^*(\tilde{\gamma})\right) \xrightarrow{p^*} 0, \text{ in probability,} \tag{A.13}$$

and

$$\sqrt{n}\left(\tilde{\delta}^*(\tilde{\gamma}) - \tilde{\delta}\right) \xrightarrow{d^*} N(0, V_{\delta_0}), \text{ in probability,} \tag{A.14}$$

where $\tilde{\delta}^*(\tilde{\gamma})$ is the OLS estimate from the bootstrap sample assuming $\tilde{\gamma}$ is used as the estimated threshold parameter, and V_{δ_0} is the standard OLS variance covariance matrix in the original probability space. If (A.13)–(A.14) hold, then by Polya's Theorem and the continuity of the normal distribution, the result of the theorem follows. We start by showing that the following hold for some $\gamma = \tilde{\gamma} + \frac{\upsilon}{a_n}$, where υ belongs to a compact set Ψ and $a_n = n^{1-2a}$, $0 < a < 1/2$,

$$\frac{1}{n}\sum_{i=1}^{n} \bar{z}^{*\prime}_{\gamma,i}\bar{z}^*_{\gamma,i} = \begin{pmatrix} M_n^* & M_n^*(\upsilon) \\ M_n^*(\upsilon) & M_n^*(\upsilon) \end{pmatrix} \xrightarrow{p^*} \begin{pmatrix} M^* & M^*(\upsilon) \\ M^*(\upsilon) & M^*(\upsilon) \end{pmatrix} = \tilde{M}^*(\upsilon), \tag{A.15}$$

and

$$\frac{1}{\sqrt{n}}\sum \bar{z}^{*\prime}_{\gamma,i}\epsilon_i^* = \begin{pmatrix} J_n^* \\ J_n^*(\upsilon) \end{pmatrix} \Longrightarrow^* \begin{pmatrix} J^* \\ J^*(\upsilon) \end{pmatrix} = \mathcal{J}^*(\upsilon), \tag{A.16}$$

where $z^*_{\gamma,i} = z_i^* I_{q_i \le \gamma}$, $\bar{z}^*_{i,\gamma} = (z_i^{*\prime}, z^{*\prime}_{\gamma,i})'$, $\bar{Z}^*_\gamma = (z^*_{\gamma,1}, \ldots z^*_{\gamma,n})'$, $M^* = \mathrm{E}^*(z_i^* z_i^{*\prime})$, $M^*(\upsilon) = \mathrm{E}^*(z^*_{\gamma,i} z^{*\prime}_{\gamma,i})$, and $\mathcal{J}^*(\upsilon)$ is a continuous path Gaussian process. By the i.i.d. resampling of our bootstrap design for z_i^* and q_i^*, it easily follows that $M^* - M = o_p(1)$ and $M^*(\upsilon) - M(\upsilon) = o_p(1)$ uniformly in υ, where $M = \mathrm{E}(z_i z_i')$, $M(\upsilon) = \mathrm{E}(z_{\gamma,i} z'_{\gamma,i})$. Again, by the i.i.d. resampling design for z_i^* and q_i^*, (A.15) follows by Lemma 1 of Hansen (1996) and Assumption 1. We focus on (A.16):

$$\frac{1}{\sqrt{n}}\sum \bar{z}^{*\prime}_{\gamma,i}\epsilon_i^* = \frac{1}{\sqrt{n}}\sum \bar{z}^{*\prime}_{\gamma,i}\tilde{\epsilon}_i\eta_i = \frac{1}{\sqrt{n}}\sum \bar{z}^{*\prime}_{\gamma,i}\epsilon_i\eta_i + \frac{1}{\sqrt{n}}\sum \bar{z}^{*\prime}_{\gamma,i}(\tilde{\epsilon}_i - \epsilon_i)\eta_i = A_1 + A_2. \tag{A.17}$$

We examine A_2 first:

$$\begin{aligned} &\frac{1}{\sqrt{n}}\sum \bar{z}^{*\prime}_{\gamma,i}(\tilde{\epsilon}_i - \epsilon_i)\eta_i \qquad \text{(A.18)} \\ &\quad = \frac{1}{\sqrt{n}}\sum \bar{z}^{*\prime}_{\gamma,i}\left(\left(\tilde{\delta}_1' - \delta_{0,1}'\right)z_i + \left(\tilde{\delta}_\gamma' - \delta_{0,\gamma}'\right)z_{\gamma,i} + \delta_{0,\gamma}'\left(z_{\tilde{\gamma},i} - z_{\gamma,i}\right)\right)\eta_i \\ &\quad = B_1 + B_2 + B_3. \qquad \text{(A.19)} \end{aligned}$$

We examine B_1. B_2 can be treated similarly:

$$\frac{1}{\sqrt{n}}\sum \bar{z}^{*\prime}_{\gamma,i}\left(\tilde{\delta}_1' - \delta_{0,1}'\right)z_i\eta_i = \sqrt{n}\left(\tilde{\delta}_1' - \delta_{0,1}'\right)\frac{1}{n}\sum \bar{z}^{*\prime}_{\gamma,i}z_i\eta_i. \tag{A.20}$$

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But $\sqrt{n}\left(\tilde{\delta}'_1 - \delta'_{0,1}\right) = O_p(1)$, whereas by the independence of z_i and η_i, $\frac{1}{n}\sum \bar{z}^{*\prime}_{\gamma,i} z_i \eta_i = o_{p^*}(1)$. Hence, $B_1 = o_{p^*}(1)$. We now examine B_3. In order to show that $B_3 = o_{p^*}(1)$ it is sufficient to show that

$$\mathrm{E}^* \left| \frac{1}{\sqrt{n}} \sum \bar{z}^{*\prime}_{\gamma,i} \delta'_{0,\gamma} \left(z_{\tilde{\gamma},i} - z_{\gamma,i} \right) \eta_i \right|^2 = o(1). \tag{A.21}$$

But,

$$\mathrm{E}^* \left| \frac{1}{\sqrt{n}} \sum \bar{z}^{*\prime}_{\gamma,i} \delta'_{0,\gamma} \left(z_{\tilde{\gamma},i} - z_{\gamma,i} \right) \eta_i \right|^2 \leq C\mathrm{E} \left| \frac{1}{\sqrt{n}} \sum z_i^2 I_{\tilde{\gamma} > q_i > \gamma} \eta_i \right|^2. \tag{A.22}$$

But, by the fact that z_i is ρ-mixing and the independence of η_i, $z_i^2 I_{\tilde{\gamma} > q_i > \gamma} \eta_i$ is ρ-mixing as well and by Lemma 3.4 of Peligrad (1982) it follows that

$$\begin{aligned} \mathrm{E} \left| \frac{1}{\sqrt{n}} \sum z_i^2 I_{\tilde{\gamma} > q_i > \gamma} \eta_i \right|^2 &\leq C \left(\sigma_\eta^2 \mathrm{E} \left(z_i^4 I_{\tilde{\gamma} > q_i > \gamma} \right) + n\mathrm{E} \left(z_i^2 I_{\tilde{\gamma} > q_i > \gamma} \eta_i \right) \right) \\ &\leq C\sigma_\eta^2 |\tilde{\gamma} - \gamma| = o_p(1). \end{aligned} \tag{A.23}$$

The second inequality in (A.23) follows by $\mathrm{E}\left(z_i^2 I_{\tilde{\gamma} > q_i > \gamma} \eta_i \right) = 0$ and the fact that

$$\mathrm{E}\left(z_i^r I_{\tilde{\gamma} > q_i > \gamma} \right) \leq C |\tilde{\gamma} - \gamma|, \quad r \leq 4 \tag{A.24}$$

by Lemma A.1 of Hansen (2000). Hence, $A_2 = o_{p^*}(1)$. We now examine A_1. Under the null hypothesis, $\bar{z}^{*\prime}_{\gamma,i} \epsilon_i \eta_i$ is a square integrable martingale difference sequence. Hence, by the central limit theorem for martingale difference sequences, we see that $\frac{1}{\sqrt{n}} \sum \bar{z}^{*\prime}_{\gamma,i} \epsilon_i \eta_i$ converges pointwise to a Gaussian distribution. This is easily extended to any finite collection of γ and thus implies convergence of the finite dimensional distributions. Further, Lemma A.3 of Hansen (2000) implies tightness for $\frac{1}{\sqrt{n}} \sum \bar{z}^{*\prime}_{\gamma,i} \epsilon_i \eta_i$. Hence, (A.15) holds.

We now continue by using the final steps of the proof of Lemma A.12 of Hansen (2000). In particular, we have that

$$\begin{aligned} \sqrt{n}\left(\tilde{\delta}^* \left(\tilde{\gamma} + \frac{\upsilon}{a_n} \right) - \tilde{\delta} \right) &= \tilde{M}^* \left(\tilde{\gamma} + \frac{\upsilon}{a_n} \right)^{-1} \\ &\quad \times \left(\mathcal{J}^* \left(\tilde{\gamma} + \frac{\upsilon}{a_n} \right) - \begin{pmatrix} I \\ I \end{pmatrix} \mathcal{A} \left(\tilde{\gamma} + \frac{\upsilon}{a_n} \right) C a_n^{-1/2} \right), \end{aligned} \tag{A.25}$$

where

$$\mathcal{A}\left(\tilde{\gamma} + \frac{\upsilon}{a_n} \right) = \frac{a_n}{n} \bar{Z}_{\tilde{\gamma} + \frac{\upsilon}{a_n}} \Delta \bar{Z}_{\tilde{\gamma} + \frac{\upsilon}{a_n}} \tag{A.26}$$

and $\Delta \bar{Z}_{\tilde{\gamma} + \frac{\upsilon}{a_n}} = \bar{Z}_{\tilde{\gamma} + \frac{\upsilon}{a_n}} - \bar{Z}_{\tilde{\gamma}}$. But, by the inequality for ρ-mixing processes used in (A.23),

$$\mathcal{A}\left(\tilde{\gamma} + \frac{\upsilon}{a_n} \right) \xrightarrow{p^*} 0, \quad \text{in probability.} \tag{A.27}$$

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Then,

$$\sqrt{n}\left(\tilde{\delta}^*\left(\tilde{\gamma}+\frac{\upsilon}{a_n}\right)-\tilde{\delta}\right)\Rightarrow^* \begin{pmatrix} M^* & M^*(\tilde{\gamma}) \\ M^*(\tilde{\gamma}) & M^*(\tilde{\gamma}) \end{pmatrix}^{-1} \begin{pmatrix} J^* \\ J^*(\tilde{\gamma}) \end{pmatrix}$$

$$= \begin{pmatrix} M & M(\gamma_0) \\ M(\gamma_0) & M(\gamma_0) \end{pmatrix}^{-1} \begin{pmatrix} J \\ J(\gamma_0) \end{pmatrix} + o_p(1), \tag{A.28}$$

where $\begin{pmatrix} J \\ J(\gamma) \end{pmatrix}$ is a Gaussian process in the original sample probability space. But,

$$\begin{pmatrix} M & M(\gamma_0) \\ M(\gamma_0) & M(\gamma_0) \end{pmatrix}^{-1} \begin{pmatrix} J \\ J(\gamma_0) \end{pmatrix} = N(0, V_{\delta_0}), \tag{A.29}$$

establishing (A.14). Retracing the steps of Lemma A.9 of Hansen (2000), it is straightforward to establish that in the bootstrap probability space, there exist $\bar{\upsilon} < \infty$ and $\bar{n} < \infty$, such that for all $n > \bar{n}$, $\varepsilon > 0$, and $\delta > 0$,

$$\Pr^*\left(a_n\left|\tilde{\gamma}^*-\tilde{\gamma}\right| > \bar{\upsilon}\right) \le \varepsilon, \quad \text{in probability.}$$

Further, by (A.27) it follows that there again exist $\bar{\upsilon} < \infty$ and $\bar{n} < \infty$, such that for all $n > \bar{n}$, $\varepsilon > 0$, and $\delta > 0$,

$$\Pr^*\left(\sup_{-\bar{\upsilon}\le\upsilon\le\bar{\upsilon}} \sqrt{n}\left|\tilde{\delta}^*\left(\tilde{\gamma}+\frac{\upsilon}{a_n}\right)-\tilde{\delta}^*(\tilde{\gamma})\right| > \delta\right) \le \varepsilon, \quad \text{in probability.} \tag{A.30}$$

But,

$$\Pr^*\left(\sup_{-\bar{\upsilon}\le\upsilon\le\bar{\upsilon}} \sqrt{n}\left|\tilde{\delta}^*-\tilde{\delta}^*(\tilde{\gamma})\right| > \delta\right) \le \Pr^*\left(\sup_{-\bar{\upsilon}\le\upsilon\le\bar{\upsilon}} \sqrt{n}\left|\tilde{\delta}^*\left(\tilde{\gamma}+\frac{\upsilon}{a_n}\right)-\tilde{\delta}^*(\tilde{\gamma})\right| > \delta\right)$$

$$+\Pr^*\left(a_n\left|\tilde{\gamma}^*-\tilde{\gamma}\right| > \bar{\upsilon}\right) \le 2\varepsilon, \quad \text{in probability,} \tag{A.31}$$

establishing (A.13) and hence the required result for OLS estimation.

The result for 2SLS/GMM estimation follows along similar lines, which we present below using results from Caner and Hansen (2004). The wild bootstrap for 2SLS/GMM estimation is similar to that for OLS estimation. The only extension relates to the resampling scheme that resamples with replacement rows of the matrix (Q, Z, X) instead. y^* is still resampled as in (A.12). The reduced form model that relates z_i to x_i is assumed to be a threshold model as well given by

$$z_i = \pi_1' x_i + \pi_\rho' x_i I_{q_i > \rho} + u_i. \tag{A.32}$$

We introduce some additional notation that is needed for the 2SLS/GMM case. Let $x_{\gamma,i} = x_i I_{q_i \le \gamma}$, $\bar{x}_{i,\gamma} = (x_i', x_{\gamma,i}')'$ and $\bar{X}_{\hat{\gamma}} = (x_{\gamma,1}, \ldots x_{\gamma,n})'$. The bootstrap counterparts are

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given by $x^*_{\gamma,i} = x^*_i I_{q_i \leq \gamma}$, $\bar{x}^*_{i,\gamma} = (x^{*\prime}_i, x^{*\prime}_{\gamma,i})'$, and $\bar{X}^*_\gamma = (x^*_{\gamma,1}, \ldots x^*_{\gamma,n})'$. In fact, we will prove a bootstrap result for a general class of estimators given by

$$\bar{\delta}(\hat{\gamma}) = \left(\bar{Z}'_{\hat{\gamma}} \bar{X}_{\hat{\gamma}} W_{\hat{\gamma}} \bar{X}'_{\hat{\gamma}} \bar{Z}_{\hat{\gamma}}\right)^{-1} \bar{Z}'_{\hat{\gamma}} \bar{X}_{\hat{\gamma}} W_{\hat{\gamma}} \bar{X}'_{\hat{\gamma}} y, \tag{A.33}$$

where $y = (y_1, \ldots, y_n)'$ and $W_{\hat{\gamma}}$ represents a sequence of weight matrices. The required result for $\bar{\delta}$ follows from Polya's theorem if we show that for the bootstrap parameter estimate given by

$$\bar{\delta}^*(\hat{\gamma}^*) = \left(\bar{Z}^{*\prime}_{\hat{\gamma}^*} \bar{X}^*_{\hat{\gamma}^*} W^*_{\hat{\gamma}^*} \bar{X}^{*\prime}_{\hat{\gamma}^*} \bar{Z}^*_{\hat{\gamma}^*}\right)^{-1} \bar{Z}^{*\prime}_{\hat{\gamma}^*} \bar{X}^*_{\hat{\gamma}^*} W^*_{\hat{\gamma}^*} \bar{X}^{*\prime}_{\hat{\gamma}^*} y^*, \tag{A.34}$$

the following two results hold:

$$\sqrt{n}\left(\bar{\delta}^*(\hat{\gamma}^*) - \bar{\delta}(\hat{\gamma})\right) \xrightarrow{d^*} N(0, V_{\bar{\delta}_0}), \quad \text{in probability,} \tag{A.35}$$

and

$$W^*_{\hat{\gamma}^*} \xrightarrow{p^*} W_{\gamma_0}, \quad \text{in probability,} \tag{A.36}$$

where $\hat{\gamma}^*$ is the bootstrap estimate of the threshold, $W^*_{\hat{\gamma}^*}$ is the bootstrap counterpart of $W_{\hat{\gamma}}$, W_{γ_0} is the probability limit of $W_{\hat{\gamma}}$,

$$V_{\bar{\delta}_0} = \left(R' W_{\gamma_0} R\right)^{-1} R' W_{\gamma_0} Q W_{\gamma_0} R \left(R' W_{\gamma_0} R\right)^{-1}, \tag{A.37}$$

$R = \mathrm{E}\left(\bar{z}_{i,\gamma_0} \bar{x}'_{i,\gamma_0}\right)$, and $Q = \sigma^2 \mathrm{E}\left(\bar{x}_{i,\gamma_0} \bar{x}'_{i,\gamma_0}\right)$. We first show (A.35) assuming (A.36) holds. Retracing the steps of Lemma 4 of Caner and Hansen (2004) it can be shown that

$$a_n \left|\hat{\gamma}^* - \hat{\gamma}\right| = O_{p^*}(1), \quad \text{in probability.} \tag{A.38}$$

Further,

$$\begin{aligned}\sqrt{n}\left(\bar{\delta}^*\left(\hat{\gamma} + \frac{\upsilon}{a_n}\right) - \bar{\delta}(\hat{\gamma})\right) = {} & \left(\frac{1}{n} \bar{Z}^{*\prime}_{\hat{\gamma}+\frac{\upsilon}{a_n}} \bar{X}^*_{\hat{\gamma}+\frac{\upsilon}{a_n}} W^*_{\hat{\gamma}+\frac{\upsilon}{a_n}} \frac{1}{n} \bar{X}^{*\prime}_{\hat{\gamma}+\frac{\upsilon}{a_n}} \bar{Z}^*_{\hat{\gamma}+\frac{\upsilon}{a_n}}\right)^{-1} \\ & \times \left(\frac{1}{n} \bar{Z}^{*\prime}_{\hat{\gamma}+\frac{\upsilon}{a_n}} \bar{X}^*_{\hat{\gamma}+\frac{\upsilon}{a_n}} W^*_{\hat{\gamma}+\frac{\upsilon}{a_n}} \right. \\ & \left. \times \left(\frac{1}{\sqrt{n}} \bar{X}^{*\prime}_{\hat{\gamma}+\frac{\upsilon}{a_n}} \epsilon^* - \frac{1}{\sqrt{n}} \bar{X}^{*\prime}_{\hat{\gamma}+\frac{\upsilon}{a_n}} \Delta \bar{Z}_{\tilde{\gamma}+\frac{\upsilon}{a_n}} C n^{-a}\right)\right).\end{aligned} \tag{A.39}$$

But, by the i.i.d. resampling underlying Z and X,

$$\frac{1}{n} \bar{Z}^{*\prime}_{\hat{\gamma}+\frac{\upsilon}{a_n}} \bar{X}^*_{\hat{\gamma}+\frac{\upsilon}{a_n}} \xrightarrow{p^*} R', \quad \text{in probability.} \tag{A.40}$$

Also, by (A.27)

$$\frac{1}{\sqrt{n}} \bar{X}^{*\prime}_{\hat{\gamma}+\frac{\upsilon}{a_n}} \Delta \bar{Z}_{\tilde{\gamma}+\frac{\upsilon}{a_n}} C n^{-a} \xrightarrow{p^*} 0, \quad \text{in probability.} \tag{A.41}$$

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Finally, by the analysis in (A.17)–(A.24) and the ρ-mixing assumption for x_i,

$$\frac{1}{\sqrt{n}}\bar{X}^{*\prime}_{\hat{\gamma}+\frac{\upsilon}{a_n}}\epsilon^* \xrightarrow{d^*} N(0,Q), \quad \text{in probability.} \tag{A.42}$$

But

$$\begin{aligned}&\Pr^*\left(\sup_{-\bar{\upsilon}\le\upsilon\le\bar{\upsilon}}\sqrt{n}\left|\bar{\delta}^*(\hat{\gamma}^*)-\bar{\delta}^*\left(\hat{\gamma}\right)\right|>\delta\right)\\ &\quad\le \Pr^*\left(\sup_{-\bar{\upsilon}\le\upsilon\le\bar{\upsilon}}\sqrt{n}\left|\bar{\delta}^*\left(\hat{\gamma}+\frac{\upsilon}{a_n}\right)-\bar{\delta}^*\left(\hat{\gamma}\right)\right|>\delta\right)\\ &\qquad+\Pr^*\left(a_n\left|\hat{\gamma}^*-\hat{\gamma}\right|>\bar{\upsilon}\right).\end{aligned} \tag{A.43}$$

Then, by (A.41), there exist $\bar{\upsilon}<\infty$ and $\bar{n}<\infty$, such that for all $n>\bar{n}$, $\varepsilon>0$, and $\delta>0$,

$$\Pr^*\left(\sup_{-\bar{\upsilon}\le\upsilon\le\bar{\upsilon}}\sqrt{n}\left|\bar{\delta}^*\left(\hat{\gamma}+\frac{\upsilon}{a_n}\right)-\bar{\delta}^*\left(\hat{\gamma}\right)\right|>\delta\right)<\varepsilon, \quad \text{in probability.} \tag{A.44}$$

Further, by (A.38),

$$\Pr^*\left(a_n\left|\hat{\gamma}^*-\hat{\gamma}\right|>\bar{\upsilon}\right)\le\varepsilon, \quad \text{in probability.} \tag{A.45}$$

Then, it follows from (A.25)–(A.27) and(A.44)–(A.45) that (A.35) holds. By its nature, (A.36) is an assumption since the form of W is not given. We will specialize to the 2SLS and GMM cases. For 2SLS, $W_{\hat{\gamma}}=W^{2SLS}_{\hat{\gamma}}=\left(\frac{1}{n}\bar{X}^{*\prime}_{\hat{\gamma}^*}\bar{X}^*_{\hat{\gamma}^*}\right)^{-1}$, whereas for GMM it is $W_{\hat{\gamma}}=W^{GMM}_{\hat{\gamma}}=\hat{\Omega}^{*-1}_x$, where $\hat{\Omega}^*_x$ is the bootstrap counterpart of $\hat{\Omega}^{-1}_x$. We need to show that

$$W^{*2SLS}_{\hat{\gamma}^*}\xrightarrow{p^*}W^{2SLS}_{\gamma_0}, \quad \text{in probability,} \tag{A.46}$$

and

$$W^{*GMM}_{\hat{\gamma}^*}\xrightarrow{p^*}W^{GMM}_{\gamma_0}, \quad \text{in probability.} \tag{A.47}$$

For the 2SLS case, this result easily follows by the consistency of $\hat{\gamma}^*$, which follows from (A.38). The result for the GMM case again easily follows by retracing the proof of Theorem 3 of Caner and Hansen (2004). Hence, Theorem 1 is proven for the wild bootstrap.

Finally, all results of Theorem 1 will be proven if we show consistency for the OLS bootstrap. The treatment is the same as for the wild bootstrap, with the only difference relating to establishing pointwise convergence to a Gaussian distribution and tightness for $\frac{1}{\sqrt{n}}\sum\bar{z}^{*\prime}_{\gamma,i}\epsilon^*_i$ where now ϵ^*_i is an i.i.d. resample of $\tilde{\epsilon}_i$. Pointwise convergence for any value of γ follows by Lemma 2.2, (2.5) and (2.6) of Freedman (1981), together with the fact that the asymptotic distribution of $\sqrt{n}(\tilde{\delta}-\delta)$ is the same as in the case where the threshold is known. Tightness follows again by Lemma A.3 of Hansen (2000). Hence, Theorem 1 is proven.

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APPENDIX B

In Appendix B we provide a full and detailed description of the bootstrap implementations we use in the Monte Carlo section of the main paper.

Standard Model Bootstrap with OLS Residuals.

1. The threshold model is estimated using OLS-based estimation. Residuals, $\tilde{\epsilon}$, OLS-based coefficient estimates, $\tilde{\beta}$, and OLS-based threshold parameter estimate, $\tilde{\gamma}$, are stored.
2. Rows from the matrix $W = ((z_1', x_1', q_1)', \ldots, (z_n', x_n', q_n)')'$ are resampled with replacement to produce bootstrap samples of size T, for the regressors.
3. OLS residuals, $\tilde{\epsilon}$, are resampled with replacement to produce a bootstrap sample, of size T, of error terms, ϵ^*.
4. Error terms, ϵ^*, are used together with $\tilde{\beta}$ and $\tilde{\gamma}$ via the model given by (1) to produce a bootstrap sample, of size T, for the dependent variable, y^*.
5. The bootstrap data are used to estimate the model by OLS and 2SLS/GMM and produce a bootstrap test statistic for the exogeneity test.
6. Steps 2–5 are repeated $B-1$ times to produce B bootstrap test statistics.

Wild Bootstrap with OLS Residuals.

1. The threshold model is estimated using OLS-based estimation. Residuals, $\tilde{\epsilon}$, OLS-based coefficient estimates, $\tilde{\beta}$, and OLS-based threshold parameter estimate, $\tilde{\gamma}$, are stored.
2. Rows from the matrix $W = ((z_1', x_1', q_1)', \ldots, (z_n', x_n', q_n)')'$ are resampled with replacement to produce bootstrap samples of size T, for the regressors.
3. The wild bootstrap is applied to OLS residuals, $\tilde{\epsilon}$, as follows: The ith bootstrap residual is given by $\epsilon_i^* = \eta_i \tilde{\epsilon}_i$, where $\{\eta_i\}_{i=1}^N$ is an i.i.d. standard normal sequence independent of all other relevant stochastic quantities. A bootstrap sample of error terms, of size T, is produced.
4. Error terms, ϵ^*, are used together with $\tilde{\beta}$ and $\tilde{\gamma}$ via the model given by (1) to produce a bootstrap sample, of size T, for the dependent variable, y^*.
5. The bootstrap data are used to estimate the model by OLS and 2SLS/GMM and to produce a bootstrap test statistic for the exogeneity test.
6. Steps 2–5 are repeated $B-1$ times to produce B bootstrap test statistics.

Nonparametric Bootstrap.

1. Rows from the matrix $W = ((y_1, z_1', x_1', q_1)', \ldots, (y_n, z_n', x_n', q_n)')'$ are resampled with replacement to produce bootstrap samples, of size T, for the regressors and dependent variable.
2. The bootstrap data are used to estimate the model by OLS and 2SLS/GMM and to produce a bootstrap test statistic for the exogeneity test.
3. Steps 1–2 are repeated $B-1$ times to produce B bootstrap test statistics.

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Subsampling with OLS Residuals.

1. The threshold model is estimated using OLS-based estimation. Residuals, $\tilde{\epsilon}$, OLS-based coefficient estimates, $\tilde{\beta}$, and OLS-based threshold parameter estimate, $\tilde{\gamma}$, are stored.
2. Rows from the matrix $W = ((z_1', x_1', q_1)', \ldots, (z_n', x_n', q_n)')'$ are resampled with replacement to produce bootstrap samples of size $a(T)T$, for the regressors. The choice of $a(T)$ is discussed in page 239.
3. The wild bootstrap is applied to OLS residuals, $\tilde{\epsilon}$, as follows: The ith bootstrap residual is given by $\epsilon_i^* = \eta_i \tilde{\epsilon}_i$, where η_i is an i.i.d. standard normal sequence independent of all other relevant stochastic quantities. A bootstrap sample of error terms, of size $a(T)T$, is produced.
4. Error terms, ϵ^*, are used together with $\tilde{\beta}$ and $\tilde{\gamma}$ via the model given by (1) to produce a bootstrap sample, of size $a(T)T$, for the dependent variable, y^*.
5. The bootstrap data are used to estimate the model by OLS and 2SLS/GMM and to produce a bootstrap test statistic for the exogeneity test.
6. Steps 2–5 are repeated $B - 1$ times to produce B bootstrap test statistics.

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