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Effective metric model theory

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This paper is a further investigation of a project carried out in Didehvar and Ghasemloo (2009) to study effective aspects of the metric logic. We prove an effective version of the omitting types theorem. We also present some concrete computable constructions showing that both the separable atomless probability algebra and the rational Urysohn space are computable metric structures.

1. Introduction

This article is a further investigation of a project started in Didehvar and Ghasemloo (2009). In the aforementioned article, the authors prove an effective version of the completeness theorem for the rational Pavelka and the metric logics. Moreover, similar results for the metric logic are given by Yaacov and Pedersen (2010). The effective metric model theory for computable structures in analysis could play the same role as the recursive model theory for computable algebraic structures. The metric model theory is the logic for the study of the metric structures i.e. the metric spaces endowed with some extra operations. In this logic, the set $[0, 1]$ serves as the set of truth values. However, unlike the classical logic, while 0 means the absolute truth, the value 1 corresponds to the absolute falsity.

The main goal of this article is to find an effective version of the omitting types theorem in the metric logic. Of course this cannot be achieved, unless an approach of computability is fixed over the real line and the metric spaces. We choose the type-two theory of effectivity (TTE) in order to utilize the computability techniques (see Weihrauch (2000) for an intensive explanation of this method).

There are other frameworks for defining computability over the real numbers, such as domain theory which is used in Didehvar and Ghasemloo (2009). However, the TTE

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method is easier to implement for mathematical structures from analysis, and in particular, for the results proven in this paper. Once the computability approach is set, the notions of computable and decidable metric structures are defined.

The present article is organized as follows. In the second section, a brief explanation of the metric logic is given based on Yaacov *et al.* (2008). Then, some basic notions of TTE are introduced. Afterwards, the concepts of computable and decidable metric structures are defined and some theorems of Didehvar and Ghasemloo (2009) which are needed throughout this article are quoted.

The third section is devoted to giving an effective form of the omitting types theorem in the metric logic. Since the existing proof of the omitting types theorem is based on the Baire category theorem (Corollary 1.13 in Yaacov and Usvytsov (2007)), its proof is modified by presenting a Henkin construction which makes its effective version possible. It is shown that for a given computably axiomatizable theory T and a computable non-principle type in $S_n(T)$ (for some $n > 0$) there is a K-computable model of T which omits p. Here, K is a Σ_1-complete subset of the natural numbers.

In the last section, two important examples of decidable metric structures are presented. By giving some concrete effective constructions, it is shown that the separable atomless probability algebra and the rational Urysohn space are both decidable structures. It is known that both of these structures have separably categorical and computably axiomatizable theories and thus, are decidable structures (in light of Theorem 3.5 from Didehvar and Ghasemloo (2009)). However, no direct construction is given in Didehvar and Ghasemloo (2009) to show the computability of the structures.

2. Preliminaries

2.1. *Model theory of metric structures*

In the following, some notions and facts which will be used in this paper are stated. Further details can be found in Yaacov and Pedersen (2010), Yaacov and Usvytsov (2007) and Yaacov *et al.* (2008).

Let (M, d) be a complete bounded metric space. A *predicate* on M is a uniformly continuous function from M^n (for some $n > 0$) into the closed interval $[0, 1]$. A *function* on M is also a uniformly continuous map from M^n (for some $n > 0$) into M. In both cases, n is called *arity*. A *metric structure* $\mathcal{M}$ based on (M, d), consists of a family of predicates on M, $\{R_i \mid i \in I\}$, a family of functions on M, $\{F_j \mid j \in J\}$, and a family of distinguished elements in M, $\{a_k \mid k \in K\}$. Such a structure is denoted by

$$\mathcal{M} = (M, R_i, F_j, a_k \mid i \in I, j \in J, k \in K).$$

Some examples of metric structures are stated in the following. For studying some more examples, see Yaacov *et al.* (2008).

Example 2.1.

i. The most obvious example is a complete bounded metric space (M, d) with no additional structure.

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ii. Let $\mathcal{M}$ be a first order structure. Consider the discrete metric on its underlying set. Then, $\mathcal{M}$ is a metric structure in which relation symbols takes values from the set $\{0,1\}$.

iii. Suppose $(\Omega,\mathcal{B},\mu)$ is a probability space. One can construct a metric structure over such spaces based on a metric space (M,d) in which M is the measure algebra on $(\Omega,\mathcal{B},\mu)$ and d is defined as the measure of symmetric distance. The Boolean operators, $\cap,\cup,.^c$ are functions, μ is a predicate and $0,1$ are distinguished elements in M.

One can associate a *signature* to each metric structure $\mathcal{M}$ as follows: for each predicate R, associate a predicate symbol P and $a(P)$ to its arity. So, R is considered as $P^{\mathcal{M}}$ which is called the interpretation of P in $\mathcal{M}$. Respectively, relate to each function F, a function symbol f and to its arity, $a(f)$ and we can also denote F by $f^{\mathcal{M}}$, the interpretation of f in $\mathcal{M}$. Finally, relate to each distinguished element a_k, a constant symbol c and denote a_k by $c^{\mathcal{M}}$, the interpretation of c in $\mathcal{M}$. In addition, a signature should consist of more information. For each predicate symbol P and function symbol f, it should provide two functions $\Delta_P : (0,1] \to (0,1]$ and $\Delta_f : (0,1] \to (0,1]$ as the modulus of uniform continuity of $P^{\mathcal{M}}$ and $f^{\mathcal{M}}$, respectively. For this signature L, $\mathcal{M}$ is called an L-structure.

From now on, we fix a signature L associated to a metric structure $\mathcal{M}$ based on a complete metric space (M,d) with $d : M \times M \to [0,1]$.

L-terms and L-formulas are inductively defined as in first order logic.

Definition 2.1.

i. Each variable and constant symbol is an *L-term*. For a function symbol f with arity n and L-terms $t_1,\ldots,t_n$, $f(t_1,\ldots,t_n)$ is also an L-term.

ii. An *atomic L-formula* is expressed as $P(t_1,\ldots,t_n)$, where P is a predicate symbol and $t_1,\ldots,t_n$ are L-terms. The class of *L-formulas* is the smallest class of expressions satisfying the following:

   a. Atomic L-formulas are L-formulas.

   b. If φ and ψ are L-formulas then so is $\varphi \dot{-} \psi$.

   c. If φ is an L-formula then so is $\neg\varphi$.

   d. If φ is an L-formula then so is $\frac{1}{2}\varphi$.

   e. For every $q \in \mathbb{Q} \cap [0,1]$, q is an L-formula.

   f. If φ is an L-formula and x is a variable then $\sup_x \varphi$ and $\inf_x \varphi$ are L-formulas.

Supremum and infimum have the same roles in this logic as $\forall$ and $\exists$ in first order logic. Free and bound variables are defined as in first order logic. An L-formula φ whose free variables are among $\bar{x} = (x_1,\ldots,x_n)$ is denoted by $\varphi(\bar{x})$. A *closed L-formula (an L-sentence)* is defined as an L-formula which has no free variables.

The semantics are defined as usual. Let $\mathcal{M}$ be a metric L-structure, $t(\bar{x})$ be an L-term and $\varphi(\bar{x})$ be an L-formula. Suppose also $a_1,\ldots,a_m \in M$. The value of $t(\bar{x})$ in $\mathcal{M}$ with respect to $(a_1,\ldots,a_m)$, denoted by $t^{\mathcal{M}}(a_1,\ldots,a_m)$, is given as in first order logic. Respectively, the *value of* φ in $\mathcal{M}$ with respect to $(a_1,\ldots,a_m)$ is denoted by $\varphi^{\mathcal{M}}(a_1,\ldots,a_m)$ and defined by induction as follows:

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i. $(P(t_1, t_2, \ldots, t_n)(a_1, \ldots, a_m))^{\mathcal{M}} = P^{\mathcal{M}}(t_1^{\mathcal{M}}(a_1, \ldots, a_m), \ldots, t_n^{\mathcal{M}}(a_1, \ldots, a_m))$, where $P(t_1, \ldots, t_n)$ is an atomic L-formula.
ii. $((\varphi \dot{-} \psi)(a_1, \ldots, a_m))^{\mathcal{M}} = \max(\varphi^{\mathcal{M}}(a_1, \ldots, a_m) - \psi^{\mathcal{M}}(a_1, \ldots, a_m), 0)$.
iii. $((\neg\varphi)(a_1, \ldots, a_m))^{\mathcal{M}} = 1 - \varphi^{\mathcal{M}}(a_1, \ldots, a_m)$.
iv. $(\frac{1}{2}\varphi(a_1, \ldots, a_m))^{\mathcal{M}} = \frac{1}{2}\varphi^{\mathcal{M}}(a_1, \ldots, a_m)$.
v. $(\sup_x \varphi(x, a_2, \ldots, a_m))^{\mathcal{M}} = \sup_{a_1 \in M} \varphi^{\mathcal{M}}(a_1, a_2, \ldots, a_m)$.
vi. $(\inf_x \varphi(x, a_2, \ldots, a_m))^{\mathcal{M}} = \inf_{a_1 \in M} \varphi^{\mathcal{M}}(a_1, a_2, \ldots, a_m)$.
vii. $q^{\mathcal{M}} = q$ for every $q \in \mathbb{Q} \cap [0, 1]$.

We call an L-structure $\mathcal{M}$, a *model* of $\varphi(\overline{x})$ with respect to $\overline{a}$ in M if $\varphi^{\mathcal{M}}(\overline{a}) = 0$ and is represented by $\mathcal{M} \models \varphi[\overline{a}/\overline{x}]$. Respectively, $\mathcal{M}$ is a *model* of a closed L-formula σ if $\sigma^{\mathcal{M}} = 0$ and is denoted by $\mathcal{M} \models \sigma$. In this situation, we also say σ is *satisfied* in $\mathcal{M}$.

Consequently, one can also define some more connectives by using the former ones:

$$\begin{aligned} \varphi \wedge \psi &:= \varphi \dot{-} (\varphi \dot{-} \psi), \\ \varphi \vee \psi &:= \neg(\neg\varphi \wedge \neg\psi). \end{aligned}$$

According to the interpretations of $\dot{-}$ and $\neg$, the interpretation of $\wedge$ is given by $x \wedge y = \min(x, y)$ and the interpretation of $\vee$ is given by $x \vee y = \max(x, y)$.

The following axioms hold for all L-formulas φ, ψ and χ.

$(A1)$ $(\varphi \dot{-} \psi) \dot{-} \varphi$,

$(A2)$ $((\chi \dot{-} \varphi) \dot{-} (\chi \dot{-} \psi)) \dot{-} (\psi \dot{-} \varphi)$,

$(A3)$ $(\varphi \dot{-} (\varphi \dot{-} \psi)) \dot{-} (\psi \dot{-} (\psi \dot{-} \varphi))$,

$(A4)$ $(\varphi \dot{-} \psi) \dot{-} (\neg\psi \dot{-} \neg\varphi)$,

$(A5)$ $\frac{1}{2}\varphi \dot{-} (\varphi \dot{-} \frac{1}{2}\varphi)$,

$(A6)$ $(\varphi \dot{-} \frac{1}{2}\varphi) \dot{-} \frac{1}{2}\varphi$,

$(A7)$ $(\sup_x \varphi \dot{-} \sup_x \psi) \dot{-} \sup_x (\varphi \dot{-} \psi)$,

$(A8)$ $\varphi[t/x] \dot{-} \sup_x \varphi$ such that no free variable of t becomes bounded by the substitution of t for x in φ,

$(A9)$ $\sup_x \varphi \dot{-} \varphi$,

$(A10)$ $d(x, x)$,

$(A11)$ $d(x, y) \dot{-} d(y, x)$,

$(A12)$ $(d(x, z) \dot{-} d(x, y)) \dot{-} d(y, z)$,

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(*A*13) $(q \dot{-} d(z,w)) \wedge (d(f(\bar{x},z,\bar{y}), f(\bar{x},w,\bar{y})) \dot{-} r)$ for every function symbol f, $r,q,\varepsilon \in \mathbb{Q} \cap [0,1]$ with $r > \varepsilon$ and $q < \Delta_f(\varepsilon)$,

(*A*14) $(q \dot{-} d(z,w)) \wedge (d(P(\bar{x},z,\bar{y}), P(\bar{x},w,\bar{y})) \dot{-} r)$ for every predicate symbol P, $r,q,\varepsilon \in \mathbb{Q} \cap [0,1]$ with $r > \varepsilon$ and $q < \Delta_P(\varepsilon)$.

Also, we have two rules:

i. $\frac{\varphi \quad \psi \dot{-} \varphi}{\psi}$,
ii. $\frac{\varphi}{\sup_x \varphi}$.

The logic with the axioms (A1)–(A14) and rules (i)–(ii) is called the *metric logic*.

Deduction and consistency are defined as usual. In the metric logic, the soundness and completeness theorems have the following forms, (Yaacov and Pedersen 2010).

Theorem 2.1. Suppose Γ is a set of L-formulas and φ is an L-formula.

i. (Soundness) if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.
ii. (Strong completeness) $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi \dot{-} 2^{-n}$ for all $n \in \mathbb{N}$.

The following definition is from Yaacov and Pedersen (2010) and Yaacov *et al.* (2008).

Definition 2.2. An *L-theory* is a set of closed L-formulas. For an L-theory T, an L-structure $\mathcal{M}$ and a closed L-formula φ, $\mathcal{M}$ is called a *model* of T and write $\mathcal{M} \models T$ if $\mathcal{M} \models \varphi$ for each $\varphi \in T$. φ is also a *logical consequence* of T and write $T \models \varphi$, if $\mathcal{M} \models \varphi$ holds for every model $\mathcal{M}$ of T. A metric L-structure $\mathcal{M}$ is a substructure (respectively elementary substructure) of a metric L-structure $\mathcal{N}$, if $M \subseteq N$ and for every atomic L-formula (respectively every L-formula) $\psi(\bar{x})$ and $a_1,\ldots,a_n \in M$ we have $\psi^{\mathcal{M}}(a_1,\ldots,a_n) = \psi^{\mathcal{N}}(a_1,\ldots,a_n)$. The structure $\mathcal{N}$ is also called an (elementary) extension of $\mathcal{M}$.

Moreover, *the theory of* $\mathcal{M}$ denoted by $Th(\mathcal{M})$, is the set of all closed L-formulas which are satisfied in $\mathcal{M}$. T is called *linear complete* if for every two closed L-formulas φ and ψ, we have $T \vdash \varphi \dot{-} \psi$ or $T \vdash \psi \dot{-} \varphi$. A theory T has *witness property* if for every formula $\varphi(x)$, variable x, and $p,q \in \mathbb{Q}$ with $p < q$ there is a constant symbol c such that

$$(\sup_x \varphi \dot{-} q) \wedge (p \dot{-} \varphi(c)) \in T.$$

Now, we turn to the subject which is essential in this paper. In the following, let T be a linear complete theory.

Definition 2.3. A set $p(\bar{x})$ of L-formulas whose free variables are among $\bar{x}$ is called an *n-type* for some $n > 0$ if there exists a model $\mathcal{M}$ of T and elements $\bar{a} \in M^n$ such that $\mathcal{M} \models \varphi[\bar{a}/\bar{x}]$ for each L-formula $\varphi(\bar{x}) \in p(\bar{x})$. In this situation, $p(\bar{x})$ is denoted by $tp_{\mathcal{M}}(\bar{a})$ or $\bar{a} \models p$ and say $\bar{a}$ *realizes* $p(\bar{x})$ *in* $\mathcal{M}$. Otherwise, we say that $p(\bar{x})$ is *omitted* in $\mathcal{M}$. The space of n-types is denoted by $S_n(T)$.

One can define a metric on the space of types as follows: for $p,q \in S_n(T)$, let

$$d(p,q) = \inf\{d(\bar{b},\bar{c}) \mid \ \mathcal{M} \models p[b_1,\ldots,b_n], \mathcal{M} \models q[c_1,\ldots,c_n]\},$$

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where $d(\bar{b},\bar{c}) = \max_{1\leqslant j\leqslant n} d(b_j, c_j)$. By compactness theorem (Yaacov *et al.* (2008) Theorem 5.8), there exist realizations $(b_1,\ldots,b_n)$ of p and $(c_1,\ldots,c_n)$ of q such that $d(p,q) = d(\bar{b},\bar{c})$. One can verify that d is actually a metric.

The following definition is necessary for Theorem 4.1. (See Yaacov and Usvytsov (2007), Notation 1.2.)

Definition 2.4. Let $p(\bar{x}) \in S_n(T)$ and $\varepsilon \in \mathbb{Q}\cap[0,1]$. Then, the *$\varepsilon$-perturbation* of $p(\bar{x})$ denoted by $p(\bar{x}^\varepsilon)$, is defined by

$$p(\bar{x}^\varepsilon) = \{\inf_{\bar{y}}[(\theta(\bar{y}) \vee d(\bar{x},\bar{y})\dot{-}\varepsilon)] \mid \quad \theta(\bar{y}) \in p(\bar{y})\}.$$

Also, for every $\varepsilon > 0$ we can define

$$[p(\bar{x}^\varepsilon)] = \{q \in S_n(T) \mid \ d(p,q) \leqslant \varepsilon\}.$$

It is known that for a given model $\mathcal{M}$ of T, $p(\bar{x}^\varepsilon)$ is realized by a tuple $\bar{a}$ in $\mathcal{M}$ if and only if in some elementary extension $\mathcal{N}$ of $\mathcal{M}$, there exists a tuple $\bar{b}$ realizing $p(\bar{x})$ and $d(\bar{a},\bar{b}) \leqslant \varepsilon$. If $\mathcal{M}$ is saturated, at least ω_1-saturated, then $\bar{a}$ and $\bar{b}$ can be found in $\mathcal{M}$.

Next, we define definable predicates and principle types which are crucial notions in our investigations.

Definition 2.5. A predicate $P : M^n \to [0,1]$ is *definable* in $\mathcal{M}$ if there is a sequence of L-formulas $(\varphi_k(\bar{x}) \mid k \geqslant 1)$ such that

$$\forall\varepsilon > 0\ \exists N\ \forall k \geqslant N\ \forall\bar{x} \in M^n\ (\mid \varphi_k^{\mathcal{M}}(\bar{x}) - P(\bar{x}) \mid \leqslant \varepsilon).$$

Definition 2.6. We say that a type $p(\bar{x})$ is *principle* if for each model $\mathcal{M}$ of T, $dist(\bar{x}, p(\mathcal{M})) = \inf_{\bar{b}\in p(\mathcal{M})} d(\bar{x},\bar{b})$ is a definable predicate in $\mathcal{M}$. Otherwise, $p(\bar{x})$ is called a *non-principle* type.

The next lemma has an important role in the next section and its proof can be found in Yaacov and Usvytsov (2007), Fact 1.8. First, we should define the topology we used for the lemma, (Yaacov *et al.* 2008).

Definition 2.7. The *logic topology* on $S_n(T)$ is defined as follows: If $p(\bar{x})$ is in $S_n(T)$ then the basic open neighbourhoods of $p(\bar{x})$ are the sets of the form

$$[\varphi < \varepsilon] = \{q \in S_n(T) \mid \text{ for some } 0 \leqslant \delta < \varepsilon \text{ the condition } \varphi\dot{-}\delta \text{ is in } q\}$$

where φ is in $p(\bar{x})$ and $\varepsilon > 0$.

Lemma 2.1.

Let $p(\bar{x}) \in S_n(T)$. Then, the following conditions are equivalent:

i. $p(\bar{x})$ is non-principle.
ii. For some positive rational number ε, the set $\{q \in S_n(T) \mid d(q,p) \leqslant \varepsilon\}$ of closed ε-neighbourhood of p is a nowhere dense subset of $S_n(T)$ in logic topology.

Finally, we state the omitting types theorem. In Section 4, an effective form of this theorem is presented.

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Theorem 2.2 (Yaacov *et al.* (2008), Theorem 12.6).
Let $p(\overline{x}) \in S_n(T)$. Then, the following two conditions are equivalent.

i. $p(\overline{x})$ is non-principle.
ii. $p(\overline{x})$ is omitted in a model $\mathcal{M}$ of T.

3. Computable metric structures

In this section, some notions of computability are briefly explained via TTE, (Weihrauch 2000). The authors assume the readers are familiar with the computability notions on natural numbers $\mathbb{N}$, such as c.e. sets, computable sets, computable functions, etc on $\mathbb{N} = \{0, 1, 2, \dots\}$. Define $\mathbb{N}^*$ and $\mathbb{N}^\omega$ to be the set of all finite sequences on $\mathbb{N}$ and the set of all infinite sequences on $\mathbb{N}$, respectively. A naming system on a set M is a surjective partial map $\nu :\subseteq X \to M$ where X is $\mathbb{N}^\omega$ or $\mathbb{N}^*$. $x \in X$ is called a *name* for $m \in M$ if $\nu(x) = m$. While $\nu :\subseteq \mathbb{N}^* \to M$ is called a notation, $\nu :\subseteq \mathbb{N}^\omega \to M$ is named a representation.

Remark 3.1.
We can inductively define standard bijection $\Pi_n : \mathbb{N}^n \to \mathbb{N}$, for each $n \geqslant 1$. Define $\Pi_1 : \mathbb{N} \to \mathbb{N}$ by $\Pi_1(x) = x$ and $\Pi_2 : \mathbb{N}^2 \to \mathbb{N}$ by $(x, y) \mapsto \frac{(x+y)(x+y+1)}{2} + y$. For every $k \geqslant 1$, the bijection Π_k can be defined. For every $a_1, \dots, a_k, a_{k+1} \in \mathbb{N}$,

$$\Pi_{k+1}(a_1, \dots, a_k, a_{k+1}) = \Pi_2(\Pi_k(a_1, \dots, a_k), a_{k+1}).$$

Recall that the tupling function can be defined as follows which is used to obtain a new name of former ones.

Definition 3.1.

i. The *wrapping function* $\iota : \mathbb{N}^* \to \mathbb{N}^*$ is defined by

$$\iota(a_0 a_1, \dots, a_n) = 110 a_0 0 a_1 0 \dots 0 a_n 011$$

for all $n \in \mathbb{N}$ and $a_0, a_1, \dots, a_n \in \mathbb{N}$.

ii. For $x, x_0, x_1, \cdots \in \mathbb{N}^*$, $p, p_0, p_1, \cdots \in \mathbb{N}^\omega$ and $i, j, k \in \mathbb{N}$, define *tupling function* as follows:

$$\begin{aligned}
< x_0, x_1, \dots, x_k > &:= \iota(x_0)\iota(x_1)\dots\iota(x_n) \in \mathbb{N}^*, \\
< x, p > &:= \iota(x)p \in \mathbb{N}^\omega, \\
< p, x > &:= \iota(x)p \in \mathbb{N}^\omega, \\
< p_0, p_1, \dots, p_k > &:= p_0(0)p_1(0)\dots p_k(0)p_0(1)p_1(1)\dots p_k(1)\cdots \in \mathbb{N}^\omega, \\
< x_0, x_1, \cdots > &:= \iota(x_0)\iota(x_1)\cdots \in \mathbb{N}^\omega, \\
< p_0, p_1, \cdots > (\Pi_2(i, j)) &:= p_i(j) \quad (< p_0, p_1, \cdots > \in \mathbb{N}^\omega).
\end{aligned}$$

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In the following, there are some examples of naming systems.

Example 3.1.

i. The notation $\nu_{\mathbb{N}} :\subseteq \mathbb{N}^* \to \mathbb{N}$ is defined by $\nu_{\mathbb{N}}(a_0, \ldots, a_m) = n$, where the decimal representation of n is $n = \Sigma_{n=0}^{m}(10)^n a_n$.

ii. $\nu_{\mathbb{Z}} : \subseteq \mathbb{N}^* \to \mathbb{Z}$, a notation of integers, is $\nu_{\mathbb{Z}}(1w) := \nu_{\mathbb{N}}(w)$ and $\nu_{\mathbb{Z}}(0w) := -\nu_{\mathbb{N}}(w)$ for $w \in dom(\nu_{\mathbb{N}})\backslash\{0\}$.

iii. $\nu_{\mathbb{Q}} : \subseteq \mathbb{N}^* \to \mathbb{Q}$, a notation of rational numbers, is $\nu_{\mathbb{Q}}(\iota(u)0\iota(v)) := \frac{\nu_{\mathbb{Z}}(u)}{\nu_{\mathbb{N}}(v)}$ where $u \in dom(\nu_{\mathbb{Z}})$, $v \in dom(\nu_{\mathbb{N}})$ and $\nu_{\mathbb{N}}(v) \neq 0$.

iv. The *Cauchy representation* $\rho_C : \subseteq \mathbb{N}^\omega \to \mathbb{R}$ is defined as follows: $\rho_C(p) = x$ if and only if there are words $w_0, w_1, \cdots \in dom(\nu_{\mathbb{Q}})$ such that $p = \iota(w_0)\iota(w_1)\ldots$, $\mid \nu_{\mathbb{Q}}(w_i) - \nu_{\mathbb{Q}}(w_k) \mid \leqslant 2^{-i}$ for $i < k$ and $x = \lim_{i\to\infty} \nu_{\mathbb{Q}}(w_i)$.

To define a continuous and so, a computable function, we initially present the Cantor topology on $\mathbb{N}^\omega$. Note that a *prefix* of $p \in \mathbb{N}^\omega$ is a finite word $w \in \mathbb{N}^*$ such that there is a $q \in \mathbb{N}^\omega$ with $p = wq$. This is denoted by $w \sqsubseteq p$.

Remark 3.2. In the metric model theory, the interpretations of predicates and functions are continuous. So, the appropriate computation model for this logic should preserve the continuity. The approach of TTE, which we use in this article, is one of this kind, i.e. every computable function is continuous, (Weihrauch (2000), Theorem 2.2.3). So, in the first step, a topology and then, a continuous and a computable function are defined, (Weihrauch 2000).

From now on, we consider the Cantor topology by taking $\{w\,\mathbb{N}^\omega \mid \ w \in \mathbb{N}^*\}$ where $w\,\mathbb{N}^\omega = \{p \in \mathbb{N}^\omega \mid \ w \sqsubseteq p\}$ as a base. So, just as the topology is fixed, we can consider a continuous function. A function $f : \subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$ is *continuous* if it is continuous with respect to the Cantor topology.

Definition 3.2.

i. A function $f :\subseteq \mathbb{N}^* \to \mathbb{N}^*$ is *computable* if $\nu_{\mathbb{N}} \circ f \circ \nu_{\mathbb{N}}^{-1}$ is a computable function from $\mathbb{N}$ into $\mathbb{N}$.

ii. For every partial function $g :\subseteq \mathbb{N}^* \to \mathbb{N}^*$ define $T(g) :\subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$ as follows:

$$q = T(g)(p) \ :\Longleftrightarrow \ (g(w) \sqsubseteq g(w') \text{ if } w \sqsubseteq w' \sqsubseteq p, \text{ and } q = \sup_{w\sqsubseteq p} g(w)).$$

Then, a function $f :\subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$ is computable iff $f = T(g)$ for some computable function g.

iii. An infinite sequence $p \in \mathbb{N}^\omega$ is *computable* if it is computable as a function from $\mathbb{N}$ into $\mathbb{N}$.

Note that, for a given naming system $\delta : X \to M$, $(\delta, \delta, \ldots, \delta) : \subseteq X \to M^n$, where $X \in \{\mathbb{N}^\omega, \mathbb{N}^*\}$ is defined by

$$(\delta, \delta, \ldots, \delta)(< p_1, p_2, \ldots, p_n >) = (\delta(p_1), \delta(p_1), \ldots, \delta(p_n)),$$

for every $n \geqslant 1$.

The notion of a general computable function is established by using the naming systems.

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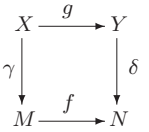


Fig. 1. g is a (γ, δ)-realization of f whenever $f \circ \gamma(x) = \delta \circ g(x)$, for all $x \in X$ such that $f \circ \gamma(x)$ exists.

Definition 3.3. Let $\gamma :\subseteq X \to M$ and $\delta :\subseteq Y \to N$ be two naming systems where $X, Y \in \{\mathbb{N}^*, \mathbb{N}^\omega\}$ and $f :\subseteq M \to N$. A function $g :\subseteq X \to Y$ is called a (γ, δ)-realization of the function f iff $f \circ \gamma(x) = \delta \circ g(x)$, for all $x \in X$ such that $f \circ \gamma(x)$ exists (Figure 1).

The function f is (γ, δ)-computable (continuous) iff it has a continuous (computable) (γ, δ)-realization. The above notion can be naturally generalized to define $(\gamma, \ldots, \gamma, \delta)$-computable (continuous) functions.

In the following, we can explain the concept of the effective and computable metric spaces.

Definition 3.4.

i. An *effective metric space* is a tuple $\mathbf{M} = (M, d, A, \alpha)$ such that

   a. (M, d) is a separable complete metric space.

   b. $\alpha :\subseteq \mathbb{N}^* \to A$ is a notation of a dense and countable subset $A \subseteq M$.

ii. A *computable metric space* is an effective metric space such that

   a. $dom(\alpha)$ is c.e.

   b. $d\,|_{A \times A}$ is a (α, α, ρ_C)-computable function.

Similar to what is defined in example 3.1.3, a generalization of Cauchy representation can be presented for an effective metric space. This representation is defined to study the computability of functions and predicates in a metric structure.

Definition 3.5. For an effective metric space $\mathbf{M} = (M, d, A, \alpha)$, the *Cauchy representation* $\delta_M :\subseteq \mathbb{N}^\omega \to M$ is defined by $\delta_M(p) = x$, where $p = \iota(w_0)\iota(w_1)\ldots$, for $w_0, w_1, \ldots \in dom(\alpha)$, $d(\alpha(w_i), \alpha(w_k)) \leqslant 2^{-i}$ for $i < k$, and $x = \lim_{i \to \infty} \alpha(w_i)$.

For instance, if we let e to be Euclidean metric over $\mathbb{R}$, $(\mathbb{R}, e, \mathbb{Q}, v_{\mathbb{Q}})$ is a computable metric space. In this case, $\delta_{\mathbb{R}}$ is exactly the Cauchy representation ρ_C in the Example 3.1.

In the following, a representation for $F^{\omega\omega}$, the set of all partial continuous functions $f :\subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$ with G_δ-domain, is defined. For more details of this representation, see Grubba *et al.* (2008).

Definition 3.6. Define a representation $\eta :\subseteq \mathbb{N}^\omega \to F^{\omega\omega}$ as follows:for each $p \in \mathbb{N}^\omega$, $\eta(p)$ is denoted by $\eta_p \in F^{\omega\omega}$. For a given $q \in \mathbb{N}^\omega$, the value $\eta_p(q)$ is the supremum of the sequence $z_0 \sqsubset z_1 \sqsubset z_2 \cdots \sqsubset z_n \sqsubset \ldots$ defined inductively by taking $z_0 = \lambda$, the empty word, and let $z_{n-1} \in \mathbb{N}^*$ be defined. Search for the first subword $\iota(< y, z >)$ of p such that $y, z \in \mathbb{N}^*$, $y \sqsubset q$, $z_{n-1} \sqsubset z$. Let $z_n := z$. If there are no such words then the function is

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undefined. It means $p \in \mathbb{N}^\omega$ is a name for a continuous function $\eta_p :\subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$ with a G_δ-domain which on input q returns the value $\eta_p(q)$.

By the above representation, a continuous function $f \in F^{\omega\omega}$ is computable if there is a computable $p \in \mathbb{N}^\omega$ such that $f = \eta_p$.

If there exists a naming system for a set M, a new one can be obtained for M^ω and M^k, for every $k \geqslant 1$.

Notation 3.1. Let $\delta :\subseteq X \to M$ be a naming system for a set M where $X \in \{\mathbb{N}^*, \mathbb{N}^\omega\}$. Then, $[\delta]^\omega$ is a representation of M^ω which is defined by

$$[\delta]^\omega(< p_1, p_2, \cdots >) := (\delta(p_1), \delta(p_2), \dots).$$

Below, by the representation in 3.6, a representation for the set of continuous total functions $f : M_1 \to M_2$ is gained, for two sets M_1 and M_2.

Definition 3.7. Let $\gamma_1 :\subseteq \mathbb{N}^\omega \to M_1$ and $\gamma_2 :\subseteq \mathbb{N}^\omega \to M_2$ be two representations. For the set $C(M_1, M_2)$ of continuous total functions $f : M_1 \to M_2$, define a representation $[\gamma_1 \to \gamma_2] :\subseteq \mathbb{N}^\omega \to C(M_1, M_2)$ as follows:

$$[\gamma_1 \to \gamma_2](p) = f :\Longleftrightarrow (f \circ \gamma_1)(q) = (\gamma_2 \circ \eta_p)(q),$$

for every $q \in \mathbb{N}^\omega$ such that $(f \circ \gamma_1)(q)$ exists.

Next, for an effectively presented language L, the notion of decidable L-structures is established.

Definition 3.8. A countable signature L is *effectively presented* if

i. The sets of variable, predicate, function and constant symbols are computable. It means if $c_V :\subseteq \mathbb{N}^* \to Var$, $c_P :\subseteq \mathbb{N}^* \to \mathcal{P}$, $c_F :\subseteq \mathbb{N}^* \to \mathcal{F}$ and $c_C :\subseteq \mathbb{N}^* \to \mathcal{C}$ are the naming systems for the sets of variables, predicate, function and constant symbols, respectively, then $\nu_{\mathbb{N}}(dom(c_V))$, $\nu_{\mathbb{N}}(dom(c_P))$, $\nu_{\mathbb{N}}(dom(c_F))$ and $\nu_{\mathbb{N}}(dom(c_C))$ are computable subsets of $\mathbb{N}$.

ii. Modulus of uniform continuity of predicate and function symbols are (ρ_C, ρ_C)-computable functions.

Similar to classical first order logic, a notation c for *Form*, the set of L-formulas, exists such that $dom(c)$ is a c.e. set. So, let $\{\varphi_n \mid n \in \mathbb{N}\}$ be an effective list of the the set of all L-formulas.

Now, let (M, d, A, α) be an effective metric space. Put the Cauchy representations δ_M on M and ρ_C on $[0, 1]$. Let $\mathcal{M}$ be a metric L-structure based on (M, d, A, α). Assume

$$Form(\mathcal{M}, L) = \{\varphi^{\mathcal{M}} : M^{n_\varphi} \to [0, 1] \mid \varphi \text{ is an } L\text{-formula with } n_\varphi \text{ free variables}\}.$$

To define a representation on $Form(\mathcal{M}, L)$, take the representation $\beta_n = [[\delta_M]^n \to \rho_C] :\subseteq \mathbb{N}^\omega \to Form(\mathcal{M}, L)_n$, where

$$Form(\mathcal{M}, L)_n = \{\varphi^{\mathcal{M}} : M^n \to [0, 1] \mid \varphi \text{ is an } L\text{-formula with } n \text{ free variables}\},$$

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for any $n \in \mathbb{N}$. Since $Form(\mathcal{M}, L) = \bigcup_{n\in\mathbb{N}} Form(\mathcal{M}, L)_n$ it follows that the function $\beta : \subseteq \mathbb{N}^\omega \to Form(\mathcal{M}, L)$ defined by $\beta(0^n 1p) = \beta_n(p)$ for each $p \in dom(\beta)$, is a representation for $Form(\mathcal{M}, L)$. A similar representation β_{at} can be defined for the set of all interpretations of atomic L-formulas in $\mathcal{M}$, $Form_{at}(\mathcal{M}, L)$, instead of the set $Form(\mathcal{M}, L)$.

Therefore, we can say what we mean by a decidable metric structure.

Definition 3.9.

i. With the preceding assumption, a metric structure $\mathcal{M}$ is computable iff the sequence

$$< \varphi_n^{\mathcal{M}} : M^{n_\varphi} \to [0,1] \mid \varphi \text{ is an atomic } L\text{-formula with } n_\varphi \text{ free variables and } n \in \mathbb{N} >$$

has a computable $[\beta_{at}]^\omega$-name.

ii. Respectively, a metric structure $\mathcal{M}$ is decidable iff the sequence

$$< \varphi_n^{\mathcal{M}} : M^{n_\varphi} \to [0,1] \mid \varphi \text{ is an } L\text{-formula with } n_\varphi \text{ free variables and } n \in \mathbb{N} >$$

has a computable $[\beta]^\omega$-name.

Actually, $[\beta]^\omega$ is a naming system for $Form(\mathcal{M}, L)^\omega$, all sequences on $Form(\mathcal{M}, L)$. Hence, for a decidable metric structure $\mathcal{M}$, there is an algorithm such that for a given L-formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in A$, it returns a good approximation of $\varphi^{\mathcal{M}}(a_1, \ldots, a_n)$ in rational numbers. This means that, for each $\varepsilon > 0$, $r, s \in \mathbb{Q}$ is computably found such that $r < \varphi^{\mathcal{M}}(a_1, \ldots, a_n) < s$ and $s - r < \varepsilon$.

Consequently, the following proposition shows the equivalence of the approach of this article with what is in Didehvar and Ghasemloo (2009).

Proposition 3.1. Let $\mathcal{M}$ be a metric structure based on a computable metric space (M, d, A, α). In Didehvar and Ghasemloo (2009), $\mathcal{M}$ is computable iff

$$Diag_{at}(\mathcal{M}, A) = \{(\varphi, r, s) \mid \ r, s \in \mathbb{Q},\ \varphi \text{ is an atomic closed } L_A\text{-formula and } r < \varphi^{\mathcal{M}} < s\}$$

is c.e. Respectively, this metric structure is decidable iff

$$Diag_{el}(\mathcal{M}, A) = \{(\varphi, r, s) \mid \ r, s \in \mathbb{Q},\ \varphi \text{ is a closed } L_A\text{-formula and } r < \varphi^{\mathcal{M}} < s\}$$

is also c.e.

Remark 3.3. One can extend the notion of decidable metric structure to *B-decidable* metric structure where B is in the every higher class than *c.e* in arithmetical hierarchy. A particular example of B which is used in this paper is K, where K is a Σ_1-complete subset of $\mathbb{N}$.

Definition 3.10. We call a theory *computably axiomatizable* if the set of its axioms is computably enumerable. Also, an n-type, for every $n \in \mathbb{N}$, is decidable (computably enumerable), if it is computable (computably enumerable) as a set of L-formulas.

The following theorems in Didehvar and Ghasemloo (2009) are used to study computability of particular theorems in the next sections, especially the effective omitting types theorem.

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Theorem 3.1 (Didehvar and Ghasemloo (2009), Theorem 3.7). Every computably axiomatizable consistent theory in the metric logic, has a K-computably axiomatizable, linear complete and consistent Henkin extension.

Theorem 3.2 (Didehvar and Ghasemloo (2009), Theorem 3.5). Every consistent, linear complete, computably axiomatizable Henkin theory in the metric logic has a decidable model.

Similar to classical first order logic, it is clear that the set of all logical consequences of a computable theory T forms a c.e. set.

Remark 3.4. In Didehvar and Ghasemloo (2009), it is also noted that if a theory T is B-computable then it has a B'-computable linear complete extension and therefore, has a B'-decidable model where B' is the jump of B. Moreover, any linear complete B-computable theory has a B-decidable model.

4. The effective omitting types theorem

In this section, we fix an effectively presented signature L. Suppose T is a computably axiomatizable linear complete consistent theory. We also assume that $\mathbb{C}$ is a universal model of T, i.e. a highly saturated model of T in which all types over small sets are realized. Below, we prove an effective version of the omitting types theorem. While the proof of this theorem which is appeared in Yaacov *et al.* (2008), is stated in terms of Baire category theorem, here, in order to accomplish the effective version of the proof, first, we need to adopt the existing argument to a direct Henkin construction.

Theorem 4.1. Let $p(\overline{x})$ be a non-principle computable type in $S_n(T)$. Then, there is a K-decidable model of T which omits $p(\overline{x})$.

Proof. For simplicity, we assume $p(x)$ is a 1-type. Let $L^* = L \cup \{c_n\}_{n\in\omega}$ where $C = \{c_n\}_{n\in\omega}$ is a new c.e set of constant symbols. Also, suppose that $\{(\varphi_n, \chi_n)\}_{n\geqslant 1}$, $\{\psi_n(x)\}_{n\geqslant 0}$ and $(r_n)_{n\geqslant 0}$ are computable enumerations of pairs of closed L^*-formulas, L^*-formulas with one free variable x, and respectively, rational numbers in $(0, 1)$. Finally, one can also effectively enumerate $p(x)$ by $(\theta_k(x) \mid k \in \omega)$. We build T^* in the following steps, by extending T to a linearly complete theory T^* which has the witness property and $p(x)$ is omitted in a model of T^*. Note that in the following, the rational numbers are chosen from the above effective list of rational numbers.

We inductively define a sequence of computably axiomatizable L^*-theories $(T_i \mid i \in \omega)$ satisfying the following properties:

— $T_0 \subseteq T_1 \subseteq \cdots \subseteq T_n \subseteq T_{n+1} \subseteq \ldots$,
— for any $n \geqslant 1$, T_n is of the form $T \cup \{r_{k_i} \dot{-} \theta_i \mid \ i = 1, \ldots, m, k_i \in \omega\}$, where all r_{k_i}'s are positive rational numbers,
— and moreover, for some positive rational number r, $T_{n,r} = T \cup \{(r_{k_i} + r) \dot{-} \theta_i \mid \ i = 1, \ldots, m, k_i \in \omega\}$ is consistent.

We proceed the construction by induction on n, by setting $T_0 = T$ and assuming that for any $i \leqslant n$, T_i satisfies the above three conditions. Now, we consider the following cases.

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i. Suppose $n = 3k + 1$ and $k \geqslant 0$. In the first step, the theory T_n is extended to be a theory with witness property. If φ_k is of the form $\inf_x \psi(x)$, take c_l, the first constant symbol which is not used in $T_n \cup \{\varphi_n\}$ and put $T_{n+1} = T_n \cup \{(\psi(c_l) \dot{-} \inf_x \psi(x)) \dot{-} r_k\}$. Otherwise, take $T_{n+1} = T_n$. Note that the formula $(\psi(c_l) \dot{-} \inf_x \psi(x)) \dot{-} r_k$, is logically equivalent to $(1 - r_k) \dot{-} \neg\big(\psi(c_l) \dot{-} \inf_x \psi(x)\big)$. It is quite straightforward to check that the inductive conditions are satisfied.

ii. Let $n = 3k + 2$, $k \geqslant 0$. In this step, we want to add an L^*-sentence to T_n in order to omit $p(x)$. Then, by induction step, T_n is of the form

$$T \cup \{r_{k_i} \dot{-} \theta_i(\bar{d}, c_k) \mid \quad i = 1, \ldots, m, k_i \in \omega\},$$

such that for some rational number $r \in (0, 1)$

$$T_{n,r} = T \cup \{(r_{k_i} + r) \dot{-} \theta_i(\bar{d}, c_k) \mid \quad i = 1, \ldots, m, k_i \in \omega\} \qquad (*)$$

is consistent.

Let

$$U = \left\{ q \in S_1(T) \mid \quad \exists \bar{b}, a \in \mathbb{C} \ \ a \models q \text{ and } \ \forall i \leqslant m, \ \theta_i^{\mathbb{C}}(\bar{b}, a) > r_{k_i} \right\}.$$

Then, U is an open set in the logic topology, since

$$U^c = [\sup_{\bar{y}} \left(\bigwedge_{i=1}^{m} \theta_i(\bar{y}, x) \dot{-} r_{k_i} \right)]$$

is a closed set in this logic. Moreover, by $(*)$, U is nonempty. Hence, in the light of Lemma 2.1, $U \nsubseteq [p(x^\varepsilon)]$, for some positive rational number ε. In particular, one may find $\bar{b}, a \in \mathbb{C}$, a formula $\psi(x) \in p(x^\varepsilon)$ and $M \in \mathbb{N}$ such that the following properties hold:

i. $\psi(x)$ is of the form $\inf_z [\theta(z) \vee (d(x, z) \dot{-} \varepsilon))]$, $\theta(z) \in p(z)$.

ii. For each $i \leqslant m$, $\theta_i^{\mathbb{C}}(\bar{b}, a) > r_{k_i}$.

iii. $\psi^{\mathbb{C}}(a) > \frac{1}{M}$.

iv. $T \cup \{\frac{1}{M} \dot{-} \psi(x)\} \models (\varepsilon + \varepsilon_1) \dot{-} d(x, p(z))$ for a rational number $\varepsilon_1 > 0$.

v. For some positive rational number t, $T_{n,t,M} = T_{n,t} \cup \{(\frac{1}{M} + t) \dot{-} \psi(c_k)\}$ is consistent.

Now, if we take $T_{n+1} = T_n \cup \{\frac{1}{M} \dot{-} \psi(c_k)\}$ then the above three clauses imply that for some positive rational number t, $T_{n+1,t}$ is consistent and hence, the set T_{n+1} satisfies the inductive hypothesis. Furthermore, the formula $\frac{1}{M} \dot{-} \psi(c_k)$ is a witness for the fact that any realization of c_k is of at least in ε-distance of any realization of $p(x)$.

iii. Suppose $n = 3k$ and $k \geqslant 1$. In the last step, the theory T_n is extended to a linear complete theory. By induction hypothesis, there are positive rational numbers r and r_{k_i}, $i = 1, \ldots m, k_i \in \omega$ such that $T_n = T \cup \{r_{k_i} \dot{-} \theta_i \mid \ i = 1, \ldots, m\}$ and $T \cup \{(r_{k_i} + r) \dot{-} \theta_i \mid i = 1, \ldots, m\}$ is consistent. Fix $s < r$. Let $T_{n,s} = T \cup \{(r_{k_i} + s) \dot{-} \theta_i \mid \ i = 1, \ldots, m\}$. If $T_{n,s} \vdash \varphi_k \dot{-} \chi_k$ then let $T_{n+1} = T_n \cup T_{n,s}$. Otherwise, there should be a natural number N_k such that $T_{n,s} \cup \{\frac{1}{N_k} \dot{-} (\varphi_k \dot{-} \chi_k)\}$ is consistent. Find a large enough natural number N and a positive rational number $t < s$ such that $\frac{1}{N} + t < \frac{1}{N_k}$ and put

$$T_{n+1} = T_n \cup \left\{ \left(r_{k_i} + \frac{t}{2} \right) \dot{-} \theta_i \mid \quad i = 1, \ldots, m \right\} \cup \left\{ \frac{1}{N} \dot{-} (\varphi_k \dot{-} \chi_k) \right\}.$$

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Moreover, either $\varphi_k \dot{-} \chi_k$ or $\chi_k \dot{-} \varphi_k$ can be deducted from T_{n+1}. Note that by the construction and inductive hypothesis for each $i \leqslant n+1$, T_i satisfies the inductive conditions.

Now, let $T^* = \bigcup_{n=1}^{\infty} T_n$. Then, by the usual Henkin construction, it follows that T^* has a model $\mathcal{M}$, in which the set $A = \{{c_n}^{\mathcal{M}} \mid \ n \in \mathbb{N}\}$ is dense in $\mathcal{M}$. Moreover, by step (2), any type q which is realized in $\mathcal{M}$ is of at least ε-distance of $p(x)$. Therefore, $p(x)$ is omitted in $\mathcal{M}$.

We show that T^* is K-computable. Note that, since T_0 is computably axiomatizable and for any $n \in \mathbb{N}$, T_n is a finite set, it follows that T_n is also a computably axiomatizable L^*-theory.

Obviously, step (1) is computable (and hence K-computable), since the set of L^*-formulas as well as the set of constant symbols C are effectively listed.

Now, to see step (2) is K-computable, first, note that the existence of an L-formula $\psi(x)$ of the form $\inf_z[\theta(z) \vee (d(x,y) \dot{-} \epsilon))]$, $\theta(z) \in p(z)$, and natural number M, for which $T_n \cup \{\frac{1}{M} \dot{-} \psi(c_k)\}$ satisfies the inductive hypothesis, are guaranteed by this step. So, to find these, first of all, we should look for a suitable positive rational number r, for which $T_{n,r}$ is consistent and $(*)$ holds.

However, one could easily see that the set

$$\{(t, \psi(x), M) \mid \quad \psi(x) \text{ satisfies } (a) \text{ and } T_{n,t,M} \text{ is not consistent}\}$$

is a c.e. set and so its complement is K-computable. Moreover, as the set of all positive rational numbers as well as the set of formulas are effectively enumerated, the following set

$$\{(t, \psi(x), M) \mid \quad \psi(x) \text{ satisfies } (a) \text{ and } T_{n,t,M} \text{ is consistent } \},$$

is also K-computable. In particular, constructing T_{n+1} from T_n is K-computable.

The proof of K-computability of step (3), involves a similar argument as in step (2). So we leave it to be verified by the reader. Therefore, since all the steps are K-computable, the whole algorithm is also K-computable. □

The following question remains unsolved.

Question: Is it possible to improve the above theorem by showing that $p(x)$ can be omitted in a decidable model of T?

5. Some examples

In this section, two examples of decidable metric structures both having separably categorical and computably axiomatizable theories are presented. Hence, in the light of Theorem 3.2, their unique separable model is decidable.

In this section, some concrete proofs for the computability of their separable models are given.

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5.1. *Atomless probability algebras*

The theory APA of atomless probability algebras is given in section 16 of Yaacov *et al.* (2008). This theory is finitely axiomatizable and hence, is computable. There, it is shown that APA is separably categorical and therefore, linear complete. Moreover, this theory has quantifier elimination. So, in order to show that the unique separable model $\widehat{\mathcal{B}}$ of APA is decidable, it is enough to prove that it is an computable metric structure.

First, we recall the construction of $\widehat{\mathcal{B}}$. Let $\mathcal{B}$ be the Borel σ-algebra on $[0,1]$ generated by open intervals of the form (a,b), $a < b \in [0,1]$. We also suppose that μ is the restriction of Lebesgue measure on $\mathcal{B}$. For every $A_1, A_2 \in \mathcal{B}$, the equivalence relation $\sim$ over $\mathcal{B}$ can be defined as $A_1 \sim A_2$ if and only if $\mu(A_1 \Delta A_2) = 0$, where Δ represents the symmetric difference of A_1 and A_2. The quotient space $\widehat{\mathcal{B}} = \mathcal{B}/\sim$ can be endowed with a metric and Boolean operations $.^c, \cap, \cup$. The metric is defined by $d([A_1]_\mu, [A_2]_\mu) = \mu(A_1 \Delta A_2)$ for every $[A_1]_\mu, [A_2]_\mu \in \widehat{\mathcal{B}}$. It is known that $\widehat{\mathcal{B}}$ forms a separable complete metric space whose dense subset can be described as

$$\mathcal{A} = \{[\cup_{i\in I} O_i]_\mu \mid \ I \text{ is finite and } O_i\text{'s are disjoint intervals with rational endpoints}\}.$$

This set is obviously countable. We can fix a naming system $c : \subseteq \mathbb{N}^* \to \mathcal{A}$ defined by $c(p) = [\cup_{i\in I} O_i]_\mu$ where $p = \imath(v^{i_1})\imath(w^{i_1}) \ldots \imath(v^{i_n})\imath(w^{i_n})$ and $O_{i_k} = (\nu_{\mathbb{Q}}(v^{i_k}), \nu_{\mathbb{Q}}(w^{i_k}))$ for $1 \leqslant k \leqslant n$ and $i_k \in I$. So, $(\widehat{\mathcal{B}}, d, \mathcal{A}, c)$ is an computable metric space. First, we prove that the metric space $(\widehat{\mathcal{B}}, d, \mathcal{A}, c)$ is computable. Since it is clear that $dom(c)$ is c.e., it is enough to show that $d\mid_{\mathcal{A}\times\mathcal{A}}$ is (c, c, ρ_C)-computable. So, we should find a computable realization $f : \subseteq \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^\omega$ such that

$$\rho_c(f(p,q)) = d\mid_{\mathcal{A}\times\mathcal{A}}(c(p), c(q)) = d\mid_{\mathcal{A}\times\mathcal{A}}([\cup_{i\in I} O_i]_\mu, [\cup_{j\in J} O_j]_\mu).$$

Therefore, we prove

$$d\mid_{\mathcal{A}\times\mathcal{A}}([\cup_{i\in I} O_i]_\mu, [\cup_{j\in J} O_j]_\mu) = \mu(\cup_{i\in I} O_i \triangle \cup_{j\in J} O_j) \quad (1)$$

is computable. Note that the equivalence class $[\cup_{i\in I} O_i]_\mu$ is completely determined by the endpoints of intervals O_i's, $i \in I$. Also, the value of the measure in (1) is a function of endpoints of the corresponding intervals. By comparing the endpoints of intervals of every two elements, $[\cup_{i\in I} O_i]_\mu$ and $[\cup_{j\in J} O_j]_\mu$, we can specify the relative position of endpoints of $[\cup_{i\in I} O_i]_\mu$ and $[\cup_{j\in J} O_j]_\mu$ with respect to each other. So based on the equation

$$\begin{aligned}\mu(\cup_{i\in I} O_i \Delta \cup_{j\in J} O_j) \ &= \ \mu(\cup_{i\in I} O_i - \cup_{j\in J} O_j) \\ &+ \ \mu(\cup_{j\in J} O_j - \cup_{i\in I} O_i),\end{aligned}$$

we need to verify that each component of the summation is computable. But $\cup_{i\in I} O_i - \cup_{j\in J} O_j$ and $\cup_{j\in J} O_j - \cup_{i\in I} O_i$ can be effectively written as a finite union of intervals with rational endpoints, i.e. this union is a computable function of their endpoints which is also a rational number as r. Thus, by taking $f(p,q) = \imath(u)\imath(u)\ldots$, where $\nu_{\mathbb{Q}}(u) = r$, the function d on $\mathcal{A}$ is computable and therefore $(\widehat{\mathcal{B}}, d, \mathcal{A}, c)$ is a computable complete separable metric space.

Now, we have the effectively presented signature

$$L = \{\cap, \cup, .^c, \mu, \Delta_\cap, \Delta_\cup, \Delta_{.^c}, \Delta_\mu, \mathbf{0}, \mathbf{1}\}$$

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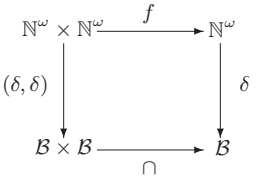


Fig. 2. f is a (δ, δ, δ)-realization of $\cap$.

where $\Delta_\cap(\varepsilon) = \frac{\varepsilon}{2}$, $\Delta_\cup(\varepsilon) = \frac{\varepsilon}{2}$, $\Delta_{.^c}(\varepsilon) = \varepsilon$ and $\Delta_\mu(\varepsilon) = \varepsilon$, for any $\varepsilon \in (0, 1]$, are computable. The interpretations of each function symbol $\cap, \cup, .^c$ and predicate symbol μ in $\widehat{\mathcal{B}}$ are also computable functions. For instance, for intersection, we should prove there is a computable function $f : \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N}^\omega$ such that $\delta(f(p, q)) = \delta(p) \cap \delta(q)$ for $p, q \in \mathbb{N}^\omega$, where δ is the Cauchy representation on the metric space $\widehat{\mathcal{B}}$, Figure 2.

By the definition of Cauchy representation for the metric spaces, Definition 3.5, for every $p, q \in \mathbb{N}^\omega$ such that $\delta(p) \cap \delta(q)$ exists, the following equations holds.

$$\begin{aligned}
\delta(f(p,q)) &= \delta(p) \cap \delta(q) \\
&= \lim_{i \to \infty} c(w_i^p) \cap \lim_{i \to \infty} c(w_i^q) \\
&= \lim_{i \to \infty} [\cup_{j \in J} O_j^p]_\mu^i \cap [\cup_{k \in K} O_k^q]_\mu^i \\
&= \lim_{i \to \infty} [\cup_{l \in L} O_l]_\mu^i.
\end{aligned}$$

So, it is enough to effectively compute $[\cup_{j \in J} O_j^p]^i \cap [\cup_{k \in K} O_k^q]^i$. It means we should compute strict monotonically on every portion of the intersection of $\delta(p)$ and $\delta(q)$. Since, for every i, either $(O_j^p \cap O_k^q)$ is $\varnothing$ or another open interval O_l, we can show $c(w_i^p) \cap c(w_i^q)$ by $[\cup_{l \in L} O_l]_\mu^i$. By the similar argument for (1), this intersection is computable. By taking $f(p, q) = \iota(w_0)\iota(w_1)\ldots$, where $c(w_i) = [\cup_{l \in L} O_l]_\mu^i$, f is a computable function. As the above, $.^c$ and $\cup$ are computable.

The interpretation of an L-formula $\varphi(x_1, \ldots, x_n)$ in $\mathcal{M} = (\widehat{\mathcal{B}}, \cap, \cup, .^c, \mu, [\varnothing]_\mu, [(0, 1)]_\mu)$ with respect to $a_1, \ldots, a_n \in \widehat{\mathcal{B}}$, $\varphi^{\mathcal{M}}(a_1, \ldots, a_n)$, is constructed from a finite number of functions, $\mu, \cap, \cup$ and $.^c$ and constants $\mathbf{0} = [\varnothing]_\mu$ and $\mathbf{1} = [(0, 1)]_\mu$. Since these components are computable, for each $r, s \in \mathbb{Q}$, L-formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in \mathcal{A}$, we can decide whether $r < \varphi^{\mathcal{M}}(a_1, \ldots, a_n) < s$ or not. So, the metric structure $(\widehat{\mathcal{B}}, \cap, \cup, .^c, \mu, [\varnothing]_\mu, [(0, 1)]_\mu)$ is computable and because of the quantifier elimination, it is a decidable metric structure.

5.2. *The rational Urysohn space*

In this section, It is shown that the rational Urysohn space is a computable metric space. Model theoretically, this space can be described as the completion of a countable metric space $\mathbb{U}_0$, where $\mathbb{U}_0$ is the Fraïssé limit of the class of all finite rational metric spaces. We, henceforth, display the rational Urysohn space by $\mathbb{U}_\mathbb{Q}$. In Usvyatsov (2008), it is proved that $Th(\mathbb{U}_\mathbb{Q})$ is separably categorical and has the elimination of quantifiers. Furthermore, a computable axiomatization is given for $Th(\mathbb{U}_\mathbb{Q})$. Hence, on the basis of Theorem 3.2, the rational Uryshon space is a computable metric space. Similar to the first example, we

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give a direct proof for the computability (and by the quantifier elimination, decidability) of this structure.

Let $\mathbb{K}_{\mathbb{Q}}$ be the class of (isomorphic types of) the finite metric spaces (M,d) with rational metrics of diameter 1, i.e. $d_M : M\times M \to \mathbb{Q}\cap[0,1]$. For $(M,d_M),(N,d_N)\in\mathbb{K}_{\mathbb{Q}}$, we say that (M,d_M) is isometrically embeddable in (N,d_N) if there is function $f : M\to N$, such that for any $a,b\in M$, $d_M(a,b)=d_N(f(a),f(b))$. In this situation, we write $(M,d_M)\hookrightarrow(N,d_N)$. Any metric space $(M,d)\in\mathbb{K}_{\mathbb{Q}}$, with n elements and enumeration $M=\{a_1,\ldots,a_n\}$, can be uniquely represented by a distance $n\times n$-matrix (r_{ij}), where $r_{ij}=d(a_i,a_j)$. Call $(r_{ij})\in M_{n\times n}(\mathbb{Q})$ a *distance matrix* if it can be a distance matrix of some finite metric space $(M,d)\in\mathbb{K}_{\mathbb{Q}}$. Let

$$\mathcal{R}_n=\{(r_{ij})\in M_{n\times n}(\mathbb{Q})\mid \quad (r_{ij}) \text{ is a distance matrix }\}.$$

For $m<n$, one may define a binary relation $E_{mn}\subseteq\mathcal{R}_m\times\mathcal{R}_n$ when $(r_{ij})E_{mn}(s_{ij})$ holds if and only if for any $i,j\leqslant m$, $r_{ij}=s_{ij}$. We set $\mathcal{R}_{\mathbb{Q}}=\bigcup_n\mathcal{R}_n$ and $E=\bigcup_{m<n}E_{mn}$. Also, for any $n\in\mathbb{N}$, we let $\sim_n$ be an equivalence relation on $M_{n\times n}(\mathbb{Q})$ defined by $(r_{ij})\sim_n(s_{ij})$ if and only if there is a permutation $\pi : n\to n$ such that $s_{ij}=r_{\pi(i)\pi(j)}$. Note that if $(M,d_M),(N,d_N)\in\mathbb{K}_{\mathbb{Q}}$ are isometric then their corresponding distance matrices are $\sim_n$-equivalent, for some $n\in\mathbb{N}$. Set $\sim=\bigcup_{n\in\mathbb{N}}\sim_n$. So, $\sim$ also defines an equivalence relation on $\mathcal{R}_{\mathbb{Q}}$. Let $\bar{\mathcal{R}}_{\mathbb{Q}}$ be the resulting quotient space. We represent any class of elements $(r_{ij})\in\mathcal{R}$, by $[(r_{ij})]$. One can modify the relation E to another well-defined relation $\bar{E}$ on the set $\bar{\mathcal{R}}$, by defining $[(r_{ij})]\bar{E}[(s_{ij})]$, if there is a $(t_{ij})\in[(r_{ij})]$ such that $(t_{ij})E(s_{ij})$. While the equivalence classes in $\bar{\mathcal{R}}_{\mathbb{Q}}$ exactly describes the isometric types of elements of $\mathbb{K}_{\mathbb{Q}}$, the structure $(\bar{\mathcal{R}}_{\mathbb{Q}},\bar{E})$ naturally identifies isometric embeddings between elements of $\mathbb{K}_{\mathbb{Q}}$. Since any element of $\mathcal{R}_{\mathbb{Q}}$ can be coded by a sequence of rational numbers, by considering a suitable naming system, it can also be endowed with a decidable structure. For such naming system, it can be readily seen that the set $\mathcal{R}_{\mathbb{Q}}$ as well as the structure $(\bar{\mathcal{R}}_{\mathbb{Q}},\bar{E})$ are decidable. Equivalently, the correspondence between isometric types in $\mathbb{K}_{\mathbb{Q}}$ and $\bar{\mathcal{R}}_{\mathbb{Q}}$ induces a naming system for $\mathbb{K}_{\mathbb{Q}}$ and therefore, allows us to transfer the computability notions to $\mathbb{K}_{\mathbb{Q}}$. We represent the corresponding naming system of $\mathbb{K}_{\mathbb{Q}}$ by $c_{\mathbb{K}_{\mathbb{Q}}} :\subseteq\mathbb{N}^*\to\mathbb{K}_{\mathbb{Q}}$. In particular, we can define a naming system on a subclass $\mathcal{K}$ of $\mathbb{K}_{\mathbb{Q}}$ by $c_{\mathbb{K}_{\mathbb{Q}}}|^{\mathcal{K}}:\subseteq\mathbb{N}^*\to\mathcal{K}$. So, this notation system imposes a natural enumeration of elements of $\mathbb{K}_{\mathbb{Q}}$ which is shown by $<M_n\mid n\in dom(c_{\mathbb{K}_{\mathbb{Q}}}\circ\nu_{\mathbb{N}})>$.

Now, in order to show that the rational Urysohn space is computable, we give computable versions of the joint embedding and amalgamation properties which are necessary to construct the space $\mathbb{U}_0$. Similar definitions in classical first order logic are appeared in Harizanov *et al.* (2011).

Definition 5.1. Let $\mathcal{K}$ be a computable subclass of the class $\mathbb{K}_{\mathbb{Q}}$.

i. $\mathcal{K}$ has the computable joint embedding property (CJEP) if there is a computable function $h:\mathbb{N}\to\mathbb{N}$ such that for any $i,j\in\mathbb{N}$, both M_i,M_j can be isometrically embeddable in $M_{h(\Pi_2(i,j))}$.

ii. We say that $\mathcal{K}$ has the computable amalgamation property (CAP) if there is a computable function $g :\subseteq\mathbb{N}\to\mathbb{N}$ such that for given $i,j,k\in\mathbb{N}$ and isometric embeddings $f_1 : M_i\to M_j$ and $f_2 : M_i\to M_k$, $g(\Pi_3(i,j,k))$ is defined and there

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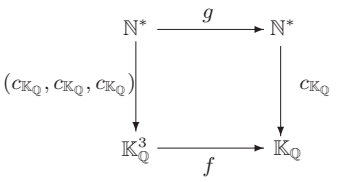


Fig. 3. $c_{K_\mathbb{Q}} \circ g = f \circ (c_{K_\mathbb{Q}}, c_{K_\mathbb{Q}}, c_{K_\mathbb{Q}})$.

are isometric embeddings $h_1 : M_i \to M_{g(\Pi_3(i,j,k))}$ and $h_2 : M_k \to M_{g(\Pi_3(i,j,k))}$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Below, It is shown that the class $\mathbb{K}_\mathbb{Q}$ does satisfy the CJEP and CAP.

Lemma 5.1. $\mathbb{K}_\mathbb{Q}$ has the CAP and CJEP.

Proof. To show the CAP, let $(K, d_K), (N, d_N), (M, d_M) \in \mathbb{K}_\mathbb{Q}$ and $f : K \to M$ and $g : K \to N$ be isometric embeddings. Without loss of generality, we may suppose that f, g are inclusion functions and $M \cap N = K$. Suppose $D = M \cup N$. Define a function $d : D^2 \to \mathbb{Q} \cap [0, 1]$, as follows:

$$d(a,b) = \begin{cases} d_N(a,b) & a,b \in N, \\ d_M(a,b) & a,b \in M, \\ \min(\min_{c \in K}(d_N(a,c) + d_M(c,b)), 1) & a \in N - K, b \in M - K. \end{cases}$$

It is easy to see that d defines a metric on D, in which both M, N are isometrically embeddable. Moreover, the resulting distance matrix of D can be effectively computed from distance matrices of three given metric spaces.

It is left to prove that the corresponding function g in Definition 5.1.2 is computable. According to the definition and the following commutative diagram, it is evident that $g(\Pi_3(i,j,k)) = p$, where $f(K, M, N) = D$ and $c_{\mathbb{K}_\mathbb{Q}}(p) = D$ (Figure 3).

To check the CJEP, we assume that $(N, d_N), (M, d_M) \in \mathbb{K}_\mathbb{Q}$ are given with $M \cap N = \varnothing$. Then, define $D = M \cup N$ and let $d : D^2 \to \mathbb{Q} \cap [0, 1]$ be the following function

$$d(a,b) = \begin{cases} d_N(a,b) & a,b \in N, \\ d_M(a,b) & a,b \in M, \\ \min(diam(M) + diam(N), 1) & \text{otherwise.} \end{cases}$$

As the CAP case, it can be readily seen that (D, d) is a metric, witnessing the CJEP. □

Theorem 5.1. Let $\mathcal{K}$ be a computable subclass of $\mathbb{K}_\mathbb{Q}$ which satisfies CJEP and CAP. Then, the Fraïssé limit of the class $\mathcal{K}$ is computable.

Proof. We find a computable function $\delta : \subseteq \mathbb{N}^* \to \mathbb{N}^*$ such that $c \circ id_*(n) = c \circ \delta(n)$, i.e. $M_n = M_{\delta(n)}$, where $c = c_{\mathbb{K}_\mathbb{Q}} \restriction^{\mathcal{K}}$ (Figure 4).

The above function δ should satisfies in the following properties:

1. For any $n \in \mathbb{N}$, $M_{\delta(n)}$ is a subspace of $M_{\delta(n+1)}$ with the embedding $i_n : M_{\delta(n)} \hookrightarrow M_{\delta(n+1)}$.
2. For any $n, m, l \in \mathbb{N}$ with $< l, m >= n$ and for any embeddings $f : M_l \to M_{\delta(n)}$ and $g : M_l \to M_m$, there is a computable embedding $h : M_n \to M_{\delta(n+1)}$ such that $h \circ g = i_n \circ f$, (Figure 5).

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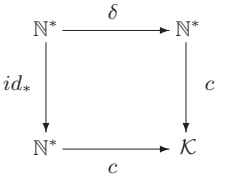


Fig. 4. $c \circ \delta = c \circ id_*$

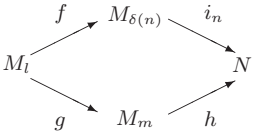


Fig. 5. $h \circ g = i_n \circ f$.

Suppose $\delta(0) = 0$ and assume that for any $s \leqslant n$, $\delta(s)$ is defined and satisfies the above two conditions. Since $\mathcal{K}$ has CJEP, we may effectively take the least number k which both $M_{\delta(n)}$ and M_n are isometrically embeddable in M_k. Let $\mathcal{D}$ be the computable set of triples of the form $(M_n, f : M_l \to M_k, g : M_l \to M_m)$ where $l, m \in \mathbb{N}$ and f, g are isometric embeddings. By CAP, we may effectively find a natural number $t \in \mathbb{N}$ such that for any l, m with $< l, m >= n$, there exist isometric embeddings $h_1 : M_k \to M_t$ and $h_2 : M_m \to M_t$ with $h_1 \circ f = h_2 \circ f_2$. Find the least t satisfying above statement and take $\delta(n+1) = t$. Then, it is clear that $M = \bigcup_{n \geqslant 1} M_{\delta(n)}$ is a homogeneous universal structure and hence, it is the Fraïssé limit of $\mathbb{K}_{\mathbb{Q}}$. We can define a metric d on this space such that (M, d, c) is a computable metric space. For any $n \in \mathbb{N}$, assume the notation $c_{\delta(n)}$ for each $M_{\delta(n)}$ such that $dom(c_{\delta(n)}) \subseteq dom(c_{\delta(n+1)})$. Take $c = \cup_{n \geqslant 0} c_{\delta(n)}$. For any $a, b \in M$, define

$$d(a, b) = d(c(p), c(q)) = d_{\delta(n)}(c_{\delta(n)}(p), c_{\delta(n)}(q)),$$

for some n with $p, q \in dom(c_{\delta(n)})$. So, since every $(M_{\delta(n)}, d_{\delta(n)}, c_{\delta(n)})$, $n \in \mathbb{N}$, is a computable metric space the desired metric space is also computable. □

Since the rational Urysohn structure is the completion of $\mathbb{U}_0$, in the light of Theorem 5.1 and Lemma 5.1, the following corollary is immediate.

Corollary 5.1.
The rational Urysohn space $\mathbb{U}_{\mathbb{Q}}$ is a computable metric space.

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