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# Optimal induced universal graphs for bounded-degree graphs

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*Abstract*

We show that for any constant $\Delta \geqslant 2$, there exists a graph $\Gamma$ with $O(n^{\Delta/2})$ vertices which contains every $n$-vertex graph with maximum degree $\Delta$ as an induced subgraph. For odd $\Delta$ this significantly improves the best-known earlier bound and is optimal up to a constant factor, as it is known that any such graph must have at least $\Omega(n^{\Delta/2})$ vertices.

## 1. *Introduction*

Given a finite family of graphs $\mathcal{H}$, a graph $\Gamma$ is *induced universal* for $\mathcal{H}$ if for every $H \in \mathcal{H}$ it contains an induced subgraph isomorphic to $H$. This notion was introduced by Rado [**21**] in the 1960s. Observe that the maximum possible number of induced subgraphs of $\Gamma$ of a certain size depends only on the number of vertices of $\Gamma$, thus the problem of determining the smallest number of vertices $g_v(\mathcal{H})$ of an $\mathcal{H}$-induced-universal graph arises naturally. Indeed, this problem has received considerable attention in past decades for various families of graphs, including the family of all graphs with $n$ vertices [**3**, **9**, **20**], forests and graphs with bounded arboricity [**8**, **10**], bounded-degree graphs [**11**, **14**], planar graphs [**12**, **15**] and more. (See [**9**] for a detailed summary of the known results.) We briefly mention some of the highlights.

Possibly the most basic family is the family $\mathcal{H}(n)$ of all graphs on $n$ vertices. The problem of estimating $g_v(\mathcal{H}(n))$, first studied by Moon [**20**] and mentioned by Vizing [**22**] in 1968, has been investigated over the years in several papers. In a recent work by Alstrup, Kaplan, Thorup and Zwick [**9**] the authors determined this function up to a constant factor, showing it is at most $16 \cdot 2^{n/2}$. (As observed by Moon [**20**], a simple counting argument implies that $2^{(n-1)/2}$ is a lower bound.) Recently, the first author [**3**] further improved the upper bound on $g_v(\mathcal{H}(n))$ to $(1 + o(1))2^{(n-1)/2}$, thus matching the lower bound up to a lower order additive term. Another family of graphs which has been studied extensively is the family of all trees

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with $n$ vertices. This was first considered in the work of Kannan, Naor and Rudich [**16**], and subsequent improvements were obtained by Chung [**12**] and by Alstrup and Rauhe [**10**]. Finally, a tight bound of $\Theta(n)$ was recently proven by Alstrup, Dahlgaard and Knudsen [**8**].

In this paper we consider the family of bounded-degree graphs with $n$ vertices. Given $\Delta \geqslant 2$ and $n \in \mathbb{N}$, let $\mathcal{H}(n, \Delta)$ denote the family of all graphs with $n$ vertices and maximum degree at most $\Delta$. One should think of $\Delta$ being a constant and $n$ being an arbitrary (large) number. This family has been studied in several papers that discuss the smallest possible number of vertices of an induced universal graph and the smallest possible number of edges in a *universal graph*. (A graph is universal for a family $\mathcal{H}$ if it contains every $H \in \mathcal{H}$ as a subgraph (not necessarily induced)). The latter was studied in a series of papers [**4**, **6**] culminating in the work of the first author and Capalbo [**5**] where it is shown that this minimum is $\Theta(n^{2-2/\Delta})$.

Induced universal graphs for bounded-degree graphs were first studied by Butler [**11**]. Using a simple counting argument he observed that

$$g_v(\mathcal{H}(n, \Delta)) \geqslant cn^{\Delta/2}, \tag{1·1}$$

for some constant $c = c(\Delta) > 0$ depending only on $\Delta$. His main result is that this is indeed the right order of magnitude in the case where $\Delta$ is even. For odd $\Delta$ his result only gives the upper bound $g_v(\mathcal{H}(n, \Delta)) = O(n^{\Delta/2+1/2})$, which simply follows from the bound obtained for the family $\mathcal{H}(n, \Delta + 1)$. Using the reduction of Chung [**12**] which connects universal and induced universal graphs, together with the sparse universal graphs from [**5**], Esperet, Labourel and Ochem [**14**] improved the bound for odd $\Delta$ to $O(n^{\Delta/2+1/2-1/\Delta})$ and mentioned the natural problem of closing the gap between the upper and lower bounds. Here we settle this problem by giving a construction which matches the lower bound (1·1) up to a constant factor.

THEOREM 1. *There exists a constant $c > 1$ such that for every integer $\Delta \geqslant 2$ and $n \in \mathbb{N}$ there exists an $\mathcal{H}(n, \Delta)$-induced-universal graph $\Gamma$ with at most $(c\Delta)^{\Delta} n^{\Delta/2}$ vertices.*

The construction of $\Gamma$ is explicit and the proof supplies a polynomial time deterministic procedure for finding, for any given $H \in \mathcal{H}(n, \Delta)$, an induced subgraph of $\Gamma$ isomorphic to $H$. Note that we allow $\Delta$ to be a function of $n$.

The rest of the paper is organised as follows. In the next section we discuss the main challenges for the case of odd $\Delta$ and present a rough overview of the construction and proof. In Section 3 we introduce the main building block in our construction, high-girth expander graphs, and state some of their properties. In Section 3·2 we state the decomposition result from [**4**], an analogue of Petersen's theorem for $(2k+1)$-regular graphs. The construction of $\Gamma$ is given in Section 4 and in Section 5 we prove that $\Gamma$ is indeed induced universal. Finally, Section 6 summarises further research directions and open problems. Throughout the paper we make no attempts to optimise the constants.

*Notation.* Given a graph $G$ and an integer $k \geqslant 1$, the $k$th power $G^k$ of $G$ is the graph on the same vertex set as $G$ where two distinct vertices are adjacent if and only if they are at distance at most $k$ in $G$. We say that a sequence of vertices $(v_1, v_2, \ldots, v_k)$ forms a *walk* if $\{v_i, v_{i+1}\} \in G$ for every $i < k$. If additionally no two vertices in the sequence are the same, it is a *path*. For two graphs $G$ and $H$, we say that a mapping $f : H \to G$ of the vertices of $H$ into the vertices of $G$ is a *homomorphism* if $\{v, w\} \in H$ implies $\{f(v), f(w)\} \in G$. If $f$ is

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injective we say that it is an *embedding* and furthermore if $\{v, w\} \in H$ iff $\{f(v), f(w)\} \in G$ then we say it is an *induced embedding*.

Finally, let $\mathcal{P}(S)$ denote the powerset of a finite set $S$ (i.e. the family of all subsets of $S$), and put $[n] := \{0, \ldots, n\}$ for $n \in \mathbb{N}$. Note that this is somewhat different than the usual notation (i.e. we include 0 in $[n]$).

## 2. *Overview of the proof*

In order to demonstrate the main ideas and challenges in our work it is instructive to first review the approach of Butler [**11**] which determines $g_v(\mathcal{H}(n, \Delta))$ up to the constant factor for even $\Delta$. At the heart of his proof lies the classical decomposition result of Petersen (see [**17**]), which states that every $(2k)$-regular graph can be decomposed into $k$ edge-disjoint 2-regular graphs. Since each 2-regular graph is a collection of cycles, it is not too difficult to construct a graph $F$ with $O(n)$ vertices which is induced universal for such graphs with $n$ vertices. Now apply the idea of Chung [**12**] (implicit already in [**16**]): a graph $\Gamma$ is defined on the vertex set $(V(F))^{\Delta/2}$ and two vertices $\mathbf{x} = (x_1, \ldots, x_{\Delta/2})$ and $\mathbf{y} = (y_1, \ldots, y_{\Delta/2})$ are adjacent iff they are adjacent in at least one coordinate (in graph $F$). Such a graph $\Gamma$ has $O(n^{\Delta/2})$ vertices. An induced embedding of $H \in \mathcal{H}(n, \Delta)$ is obtained by decomposing $H$ into $\Delta/2$ 2-regular subgraphs and by embedding each subgraph into the copy of $F$ corresponding to a separate coordinate.

However, this strategy does not work if $\Delta$ is odd. Indeed, in this case one cannot even have $\Delta/2$ coordinates. A simple solution, inspired by the work of Alon and Capalbo [**5**], is to double the number of coordinates: instead of having each coordinate of $\Gamma$ correspond to a graph of size $O(n)$ and be responsible for the existence of an edge, we let each coordinate correspond to a graph $F'$ of size $O(\sqrt{n})$ and be responsible for *half* an edge. In other words, graph $\Gamma$ has $\Delta$ coordinates and in order to have an edge between $\mathbf{x}$ and $\mathbf{y}$ we require that they are adjacent in at least two of them. This gives a graph $\Gamma$ with the desired number of vertices $O(n^{\Delta/2})$, overcoming the divisibility issue. An embedding of $H$ into $\Gamma$ is defined as before (with respect to the new decomposition, which we present shortly).

Several problems in this approach are evident. First, the decomposition result of Petersen no longer holds. Instead, we use the decomposition result of [**4**] which states that every $\Delta$-regular graph can be decomposed into $\Delta$ subgraphs (rather than $\Delta/2$, as in Petersen's theorem) such that each is a collection of 'almost' cycles and each edge of $H$ belongs to exactly two of them. Note that this is exactly what we need since the images of endpoints of each edge in $H$ have to be adjacent in at least two coordinates.

Second, we cannot guarantee that the embedding of each such subgraph into $F'$ is induced simply because the number of vertices of $F'$ is too small. In fact, we have to take a *homomorphism* of each subgraph into $F'$, rather than an injective embedding, which potentially may create halves of some undesired edges. Our aim is to show that by choosing homomorphisms of other subgraphs into $F'$ carefully we can simultaneously avoid creating the other half of any such undesired edge. In order to do that, graph $F'$ has to allow enough flexibility and a simple construction such as the one from [**11**] does not suffice. It turns out that taking $F'$ to be a high-girth expander is helpful here. However, even with such a graph $F'$ we are only able to show that most of the undesired edges will have at most half of an edge present in $\Gamma$, but a few will slip past. Finally, we take care of those by introducing additional 'layers' of coordinates in $\Gamma$ which correspond to constant size structures (with each

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layer dealing with a different type of the remaining undesired edges). These do not have a significant impact on the number of vertices as they have a constant size.

Similarity with the construction of Alon and Capalbo [**5**] is evident, thus we finish this section with a brief overview of the differences. First, note that we aim to achieve somewhat opposite goals: in [**5**] the authors are concerned with $\mathcal{H}(n, \Delta)$-universal graphs which have as few edges as possible, while here we want to make sure that the constructed graph contains every graph from $\mathcal{H}(n, \Delta)$ as an induced subgraph. This allows $F'$ to be much smaller in the first case, in particular of size $\Theta(n^{1/\Delta})$, which results in a universal graph of size $\Theta(n)$. In fact, such a construction coupled with a *concentrator* gives an $\mathcal{H}(n, \Delta)$-universal graph with the optimal number of edges and $(1 + o(1))n$ vertices (see [**6**]). Careful embedding of each of the $\Delta$ subgraphs is required in order to ensure that the resulting composition is injective. On the other hand, due to our choice of $F'$ (having the size of order $\sqrt{n}$) already a careful embedding of the first two subgraphs suffices to make sure that the resulting composition is injective, however, additional effort is needed to rule out undesired edges. The precise details require some careful analysis, as described in the subsequent sections.

## 3. *Preliminaries*

### 3·1. *High-girth expander graphs*

In this section we describe the main building block in our construction, the so-called *Ramanujan* graphs, and state the main lemma (Lemma 2) used to find an induced embedding of a graph $H$.

A $d$-regular graph $F$ is a *Ramanujan* graph if all non-trivial eigenvalues have absolute value at most $2\sqrt{d-1}$. It is wellknown that non-bipartite Ramanujan graphs are good expanders (e.g. see [**7**, Lemma 9·2·4]) and there are known explicit constructions of such graphs with high girth (see [**18**, **19**]). It should be noted that any high-girth expander with constant maximum degree would serve our purpose, thus the latter is the reason why we opt for Ramanujan graphs. See [**13**] for a self-contained account on Ramanujan graphs and the necessary background. The following statement is a simplified version of the theorem of Lubotzky, Phillips and Sarnak [**18**].

THEOREM 2. *Let $p$ and $q > 2\sqrt{p}$ be primes congruent to* 1 *modulo* 4*, such that $p$ is a quadratic residue modulo $q$. Then there is an explicit construction of a non-bipartite $(p+1)$-regular Ramanujan graph with $m = q(q^2-1)/2$ vertices and girth at least $(\log_p m)/2$.*

Next, we discuss expansion properties of graphs given by Theorem 2.

#### 3·1·1. *Expansion properties of Ramanujan graphs*

Given a graph $F$ with $\ell$ vertices and a collection of subsets $S_0, \dots, S_{q-1}$, for some $q \in \mathbb{N}$, we say that a vertex $v \in V(F)$ is *$q$-expanding* with respect to the sets $S_i$ if the following holds: for any path $P$ in $F$ of size at most $q$ (i.e. $P$ has at most $q$ vertices), there are at least $\ell/2$ vertices $w_{q-1} \in V(F)$ such that there exists a path $(v, w_0, ..., w_{q-1})$ in $F$ with $w_i \notin S_i \cup P$ for every $0 \leqslant i \leqslant q-1$. The following lemma shows that if $S_i$ is not too large, there are many $q$-expanding vertices.

LEMMA 1 [5, lemma 4·5]. *Suppose $d > 720$ and let $F$ be a non-bipartite $d$-regular Ramanujan graph with $\ell$ vertices and girth at least $(\log_{d-1} \ell)/2$. Then, for $q = \lceil \log_{10} \ell \rceil$ and every collection of subsets $S_0, \dots, S_{q-1} \subseteq V(F)$ of size $|S_i| \leqslant \ell/20$ for $0 \leqslant i \leqslant q-1$, there are at least $2\ell/3$ $q$-expanding vertices.*

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By repeated application of Lemma 1, we show that such graph $F$ contains arbitrarily long walk which is well distributed, certain pairs of vertices are not close, and 'locally' forms a path. The following statement makes this precise.

LEMMA 2. *Let $F$ be as stated in Lemma 1, and set $q = \lceil \log_{10} \ell \rceil$. Then, for every $n \in \mathbb{N}$ and a function $\sigma : [n-1] \to \mathcal{P}([n-1])$ (recall that $[n-1] = \{0, \dots, n-1\}$) such that*:

(i) $\sigma(t) \subseteq \{0, \dots, (\lfloor t/q \rfloor - 1) \cdot q - 1\}$; *and*
(ii) $|\sigma(t)| \leqslant \ell/(40d^4)$,

*for all $0 \leqslant t \leqslant n-1$, there exists a mapping $f : [n-1] \to V(F)$ with the following properties*:

(F1) $|f^{-1}(v)| \leqslant 40\lceil n/\ell \rceil$ *for every $v \in V(F)$* ;
(F2) $f(t) \neq f(t')$ *and the distance between $f(t)$ and $f(t')$ is at least 5, for every $t \in [n-1]$ and $t' \in \sigma(t)$; and*
(F3) *the sequence*

$$(f(kq), f(kq+1), \dots, f(kq+\hat{q}-1))$$

*forms a path in $F$ for every $0 \leqslant k < n/q$ and $\hat{q} = \min\{2q, n-kq\}$.*

We remark that there is a trade-off between the size of $\sigma(t)$ and the distance in the property (F2). In particular, by requiring that $\sigma(t)$ is smaller we could achieve larger distance. From the property (F3) we conclude that the sequence

$$(f(0), f(1), \dots, f(n-1))$$

forms a walk in $F$.

*Proof.* For simplicity, let us assume $n$ is divisible by $q$. We inductively define

$$f_k : \{0, \dots, kq-1\} \longrightarrow V(F)$$

for $1 \leqslant k \leqslant n/q$, such that each $f_k$ satisfies (F1)–(F3), i.e.

(i) $|f^{-1}(v)| \leqslant 40\lceil n/\ell \rceil$ for every $v \in V(F)$,
(ii) $f(t) \neq f(t')$ and the distance between $f(t)$ and $f(t')$ is at least 5 for every $t \in [kq-1]$ and $t' \in \sigma(t)$, and
(iii) the sequence

$$(f(k'q), f(k'q+1), \dots, f(k'q+\hat{q}-1))$$

forms a path in $F$ for every $0 \leqslant k' \leqslant k-1$ and $\hat{q} = \min\{2q, kq-k'q\}$.

Note that then $f := f_{n/q}$ satisfies the properties of the lemma. Moreover, for each $1 \leqslant k < n/q$ we shall further assume that $f_k(kq-1)$ is $q$-expanding with respect to the sets $S_k(i) = D_k(i) \cup A_k$ $(0 \leqslant i \leqslant q-1)$, where

$$D_k(i) = \left\{ v \in V(F) : \exists t' \in \sigma(kq+i) \text{ such that } \{f_k(t'), v\} \in F^4 \text{ or } f_k(t') = v \right\},$$
$$A_k = \{ v \in V(F) : |f_{k-1}^{-1}(v)| = 40\lceil n/\ell \rceil \}.$$

Let $f_0$ be an empty function and set $A_{n/q} = D_{n/q}(i) = \varnothing$ for $0 \leqslant i \leqslant q-1$.

We start by constructing $f_1$. Let $X_1 \subseteq V(F)$ be the set of $q$-expanding vertices with respect to empty sets (i.e. $S_i = \varnothing$ for $0 \leqslant i \leqslant q-1$). From Lemma 1 we get $|X_1| \geqslant 2\ell/3$. Pick an arbitrary vertex $v \in X_1$. By the definition, there are at least $\ell/2$ vertices $w_{q-1} \in V(F)$ for which there exists a path $(v, w_0, \dots, w_{q-1})$ in $F$ and from $|X_1 \setminus \{v\}| \geqslant \ell/2$ we deduce that at least one such vertex $w_{q-1}$ belongs to $X_1$. Let $(v, w_0, \dots, w_{q-1})$ be the

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corresponding path and set $f_1(i) := w_i$ for $0 \leqslant i \leqslant q-1$. We now verify that $f_1$ has the desired properties. First, since $(f_1(0), \dots, f_1(q-1))$ is a path we have $|f_1^{-1}(v)| \leqslant 1$ for every $v \in V(F)$ and, moreover, (iii) is satisfied. Next, observe that $\sigma(i) = \varnothing$ for $i \in [2q-1]$ and therefore the property (ii) is trivially satisfied. Finally, this also implies that the sets $S_1(i)$ are empty and from $f_1(q-1) \in X_1$ we conclude that $f_1(q-1)$ is $q$-expanding with respect to the sets $S_1(i)$.

Let us assume we have defined such functions $f_1, \dots, f_k$, for some $1 \leqslant k < n/q$. We aim to construct $f_{k+1}$. Recall that $f_k \colon \{0, \dots, kq-1\} \to V(F)$, and let us set $f_{k+1}(i) = f_k(i)$ for $0 \leqslant i \leqslant kq-1$. If $k+1 = n/q$ then $A_{k+1}$ and $D_{k+1}(i)$ are empty sets. Otherwise, for every $0 \leqslant i \leqslant q-1$ and $t' \in \sigma((k+1)q+i)$ we have $t' \leqslant kq-1$. Importantly, this implies that $D_{k+1}(i)$ is fully defined at this point, that is, it does not depend on how we set $f_{k+1}(kq), \dots, f_{k+1}(kq+q-1)$. Since $F$ is $d$-regular, there are at most $d^4$ vertices at distance less than 5 from any vertex $v \in V(F)$, including the vertex $v$ itself. In particular, we have

$$|D_{k+1}(i)| \leqslant d^4 |\sigma((k+1)q+i)| \leqslant \ell/40.$$

Observe also that

$$|A_{k+1}| \leqslant \frac{n}{40\lceil n/\ell \rceil} \leqslant \ell/40.$$

Therefore, we can apply Lemma 1 with $S_i = S_{k+1}(i)$ and let $X_{k+1} \subseteq V(F)$ be the set of $q$-expanding vertices with respect to these sets. From property (iii) of $f_k$ we have that

$$P_{k-1} = (f_k((k-1)q), \dots, f_k(kq-1))$$

forms a path in $F$. Since $|X_{k+1}| \geqslant 2\ell/3$ and $f_k(kq-1)$ is $q$-expanding with respect to the sets $S_k(i)$, there exists a vertex $w_{q-1} \in X_{k+1}$ and a path

$$P_k = (f_k(kq-1), w_0, \dots, w_{q-1})$$

such that

$$w_i \notin S_k(i) \cup P_{k-1} = D_k(i) \cup A_k \cup P_{k-1} \tag{3·1}$$

for every $0 \leqslant i \leqslant q-1$. We claim that $f_{k+1}(kq+i) := w_i$ satisfies the required properties.

To verify property (i), it suffices to only consider a vertex

$$v \in \{f_{k+1}(kq), \dots, f_{k+1}(kq+q-1)\},$$

since for all other vertices this property is inherited from $f_k$. From (3·1) we have $v \notin P_{k-1}$ and therefore

$$f_k^{-1}(v) = f_{k-1}^{-1}(v).$$

Furthermore, from $v \notin A_k$ and the assumption that $f_{k-1}$ satisfies property (i) we infer

$$|f_{k-1}^{-1}(v)| < 40\lceil n/\ell \rceil.$$

Finally, since $P_k$ is a path there exists exactly one $0 \leqslant i \leqslant q-1$ such that $f_{k+1}(kq+i) = v$. Together with the previous observations, this shows $|f_{k+1}^{-1}| \leqslant 40\lceil n/\ell \rceil$.

Similarly, to verify property (ii) it is enough to only consider $f(t)$ and $f(t')$ for $t = kq+i$ and $t' \in \sigma(t)$, for some $0 \leqslant i \leqslant q-1$. However, this follows immediately from the definition of $D_k(i)$ and (3·1). Next, property (iii) for $k' = k-1$ follows from (3·1) and the fact that $P_{k-1}$ and $P_k$ are paths with a common endpoint. For smaller values of $k'$ property (iii) is, again, inherited from $f_k$. Finally, $f_{k+1}((k+1)q-1) = w_{q-1} \in X_{k+1}$ is $q$-expanding with respect to the sets $S_{k+1}(i)$ by the construction. This completes the proof of the lemma.

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3·2. *Graph-decomposition result*

We use the following graph-decomposition result proven in [**4**]. An *augmentation* of a graph $T$ is any graph obtained from $T$ by choosing an arbitrary subset of vertices $U \subseteq V(T)$ and adding a matching between $U$ and a new set $U'$ (i.e. $U' \cap V(T) = \varnothing$) of $|U'| = |U|$ vertices. We call a graph *thin* if it has maximum degree at most 3 and every connected component is either an augmentation of a cycle or a path, or it has at most two vertices of degree 3.

THEOREM 3 [**4**]. *Let $\Delta \geqslant 2$ be an integer and let $H$ be a graph with maximum degree at most $\Delta$. Then there exist spanning subgraphs $H_1, \ldots, H_\Delta \subseteq H$ such that each $H_i$ is thin and every edge of $H$ lies in precisely two graphs $H_i$.*

In the case where $\Delta$ is odd, Theorem 3 can be seen as a generalisation of Petersen's theorem. It was observed in [**4**] that every thin graph with at most $n$ vertices is a subgraph of $P_n^4$, the 4th power of a path with $n$ vertices.

## 4. *The construction*

For the rest of the paper we set

$$d = 734, \quad z = 160d^5, \quad m = m(n) = 2 \cdot 40\Delta d^8 \sqrt{n}.$$

Note that $d$ is chosen such that we can apply Theorem 2 with $p = d - 1$ and Lemma 2. Our construction relies on the existence of high-girth $d$-regular Ramanujan graphs $R_z$ and $R_m$, with $|V(R_z)| \geqslant z$ being a constant and $m \leqslant |V(R_m)| \leqslant 32m$. In particular, $R_z$ is obtained by applying Theorem 2 with the smallest $q > 2z^{1/3}$ such that $p = d - 1$ and $q$ satisfy the required conditions. Note that the size of $R_z$ does not depend on $\Delta$, and as such is used for every family $\mathcal{H}(n, \Delta)$. On the other hand, it follows from the distribution of primes in arithmetic progressions that there exists a prime $q \in (2m^{1/3}, 4m^{1/3})$ such that $q$ is congruent to 1 modulo $4(d-1)$, provided $m = m(n)$ is sufficiently large. Note that both $p = d-1$ and $q$ are congruent to 1 modulo 4 and by the quadratic reciprocity we have that $p$ is a quadratic residue modulo $q$. Therefore Theorem 2 gives an explicit construction of a desired graph $R_m$ with at most $32m$ vertices.

Owing to the high girth, the degree of every vertex in $R_m^4$, the 4th power of $R_m$, is exactly

$$d + d(d-1) + d(d-1)^2 + d(d-1)^3,$$

which is in turn at most $d^4$. Let $\rho_v \colon N_{R_m^4}(v) \to [d^4]$ be an arbitrary ordering of neighbours of $v$ in $R_m^4$, for every $v \in V(R_m^4)$. We define $\Gamma = \Gamma(\Delta, n)$ to be the graph on the vertex set

$$V(\Gamma) = V(R_m) \times \left(V(R_m) \times \mathcal{P}([d^4]) \times V(R_z)\right)^{\Delta - 1},$$

and vertices $(x_1, x_2, X_2, u_2, \ldots, x_\Delta, X_\Delta, u_\Delta)$ and $(y_1, y_2, Y_2, w_2, \ldots, y_\Delta, Y_\Delta, w_\Delta)$ are adjacent iff there exist two indices $1 \leqslant j < i \leqslant \Delta$ such that the following hold:

(E1) $\{x_j, y_j\}, \{x_i, y_i\} \in R_m^4$;
(E2) $\rho_{x_i}(y_i) \in X_i$ and $\rho_{y_i}(x_i) \in Y_i$; and
(E3) $\{u_i, w_i\} \in R_z^4$.

We leave the discussion on the choice of parameter $m$ and the structure of $\Gamma$ until the next section. Note that $\Gamma$ has at most $(c\Delta)^\Delta n^{\Delta/2}$ vertices, where $c$ is a constant which depends on $d$ and the size of $R_z$.

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## 5. *Proof of Theorem* 1

Consider a graph $H \in \mathcal{H}(n, \Delta)$, for some $\Delta \geqslant 2$ and $n$ sufficiently large. Using the property of high-girth Ramanujan graphs (Lemma 2), we show that there exists an induced embedding of $H$ in $\Gamma = \Gamma(\Delta, n)$. Moreover, we give a deterministic strategy how to find such an embedding.

Let $H_1, \dots, H_\Delta \subseteq H$ be subgraphs given by Theorem 3. As mentioned in Section 3·2, for each $1 \leqslant i \leqslant \Delta$ there exists an embedding $\phi_i \colon H_i \to P_n^4$ of $H_i$ into the 4th power of a path with $n$ vertices. For the rest of the proof we identify $V(P_n^4)$ with $[n-1] = \{0, \dots, n-1\}$, in the natural order. Our plan is to construct homomorphisms $f_i \colon H_i \to R_m^4$ (for $1 \leqslant i \leqslant \Delta$) and $r_i \colon H_i \to R_z^4$ (for $2 \leqslant i \leqslant \Delta$) such that the following holds for $2 \leqslant i \leqslant \Delta$:

(H1) if $f_1(h) = f_1(h')$ then $f_i(h) \neq f_i(h')$, for any distinct $h, h' \in V(H)$;
(H2) if $|\phi_i(h) - \phi_i(h')| \leqslant 8$ then $f_i(h) \neq f_i(h')$, for any distinct $h, h' \in V(H)$;
(H3) for each $h \in V(H)$, the set

$$B_i(h) = \Big\{ h' \in V(H) \colon \{h, h'\} \notin H, |\phi_i(h) - \phi_i(h')| > 4, \text{ and } \exists j < i \text{ such that } \{f_j(h), f_j(h')\}, \{f_i(h), f_i(h')\} \in R_m^4 \Big\}$$

is of size at most $d$ and $|\phi_i(h) - \phi_i(h')| > 2z$ for every $h' \in B_i(h)$;
(H4) $\{r_i(h), r_i(h')\} \notin R_z^4$ for every $h' \in B_i(h)$.

Having such homomorphisms, we define $\gamma \colon H \to \Gamma$ as

$$\gamma(h) = (f_1(h), f_2(h), \Phi_2(h), r_2(h), \dots, f_\Delta(h), \Phi_\Delta(h), r_\Delta(h)),$$

where $\Phi_i(h) = \rho_{f_i(h)}(f_i(N_{H_i}(h)))$ is the set of images of neighbours of $h$ in $H_i$ (or, more precisely, the labels associated with these vertices from the point of view of $f_i(h)$).

Before we prove that $\gamma$ is an induced embedding of $H$ and that such homomorphisms $f_i$ and $r_i$ exist, we briefly spell out properties (H1)–(H4) and discuss roles of different components of $\Gamma$. We say that an edge of $\Gamma$ is *undesired* if it violates the property that $\gamma$ is induced. For brevity, by the *distance* between two vertices of $H$ in $P_n$ we mean the distance of their images in $P_n$ under some $\phi_i(\cdot)$. First, components of $\Gamma$ associated with $R_m^4$ form the 'backbone' of our embedding: the property (H1) ensures that $\gamma$ is injective and the property (H3) further restricts potential undesired edges to be spanned by images of those vertices from $H$ which are either very close or very far apart in $P_n$. This already gives some control over the undesired edges, compared to the construction of Alon and Capalbo [**5**], and is achieved by increasing the size of $R_m$ significantly (in particular, from $n^{1/\Delta}$ in [**5**] to $\sqrt{n}$ here). We then take care of the undesired edges between vertices which are close in $P_n$ using components associated with $\mathcal{P}([d^4])$: from the property (H2) we have that every $f_i$ is 'locally' injective (i.e. no two vertices which are close in $P_n$ are mapped to the same vertex), which together with the choice of $\Phi_i(\cdot)$ and the condition (E2) excludes the possibility that there exists an undesired edge between images of such vertices. Finally, the undesired edges between images of vertices which are far apart in $P_n$ are taken care of by mapping them to non-adjacent vertices in $R_z^4$ (property (H4)). Importantly, the property (H3) also guarantees that for every vertex $h \in V(H)$ there are only constantly many vertices $h' \in V(H)$ such that $\{h, h'\}$ is potentially an undesired edge, which allows us to also take $z$ to be a constant. We make this precise in the rest of the proof.

PROPOSITION 1. *$\gamma$ is an induced embedding.*

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*Proof.* Since each edge $\{h, h'\} \in H$ belongs to exactly two graphs $H_j$, $H_i$ $(j < i)$ and $f_j$, $f_i$ and $r_i$ are homomorphisms, conditions (E1) and (E3) from the definition of $\Gamma$ are satisfied for this particular choice of $j, i$. From $\rho_{f_i(h)}(f_i(h')) \in \Phi_i(h)$ (and similarly the other way around) we have that (E2) holds as well. This implies that $\gamma$ is a homomorphism of $H$ into $\Gamma$. From property (H1) we infer that $\gamma$ is injective, thus it is an embedding.

Let us assume, towards a contradiction, that there exists $h, h' \in V(H)$ such that $\{h, h'\} \notin H$ and $\{\gamma(h), \gamma(h')\} \in \Gamma$. Let $j, i \in [\Delta]$ be witnesses for $\{\gamma(h), \gamma(h')\} \in \Gamma$, for some $j < i$. Suppose $|\phi_i(h) - \phi_i(h')| \leqslant 4$. Since $\phi_i$ is an embedding of $H_i$ into $P_n^4$, for every $u \in N_{H_i}(h)$ we have $|\phi_i(u) - \phi_i(h')| \leqslant 8$. From property (H2) we then obtain $f_i(h') \neq f_i(u)$ and, more generally, $f_i(h') \notin f_i(N_{H_i}(h))$. Therefore, we have $\rho_{f_i(h)}(f_i(h')) \notin \rho_{f_i(h)}(f_i(N_{H_i}(h))) = \Phi_i(h)$, which contradicts condition (E2). Suppose now that $|\phi_i(h) - \phi_i(h')| > 4$. However, as $\{f_j(h), f_j(h')\}, \{f_i(h), f_i(h')\} \in R_m^4$ we deduce $h' \in B_i(h)$ and so $\{r_i(h), r_i(h')\} \notin R_z^4$, by property (H4). In any case, we get a contradiction with the assumption that $j, i$ is a witness for the edge $\{\gamma(h), \gamma(h')\}$. This finishes the proof that $\gamma$ is an induced embedding.

It remains to show that we can find such homomorphisms. Consider the ordering $h_0, \ldots, h_{n-1}$ of $V(H_i)$ such that $\phi_i(h_t) = t$ for every $t \in [n-1]$. Our construction of $f_i$ and $r_i$ relies on the following observation: suppose we are given a graph $G$ and a function $f\colon V(H_i) \to V(G)$ such that $W = (f(h_0), \ldots, f(h_{n-1}))$ is a walk in $G$ and $f(h_t) \neq f(h_{t'})$ for every distinct $t, t' \in [n-1]$ with $|t - t'| \leqslant 4$. Then, since $H_i$ is a subgraph of $P_n^4$, $f$ is also a homomorphism of $H_i$ into $G^4$. In other words, if the sequence $W$ forms a walk in $G$ which is 'locally' a path, then $f$ is a homomorphism of $H_i$ into $G^4$.

Let $\ell_m = |V(R_m)|$ and $q_m = \lceil \log \ell_m / \log 10 \rceil$. We say that $f_i$ is *well-distributed* if

$$|f_i^{-1}(v)| \leqslant \sqrt{n}, \qquad \text{for every } v \in V(R_m).$$

With the previous observation in mind, the existence of a well-distributed homomorphism $f_1$ follows from Lemma 2 applied with $\sigma(t) = \varnothing$ for every $t \in [n-1]$. In particular, let $f\colon [n-1] \to V(R_m)$ be the mapping obtained by Lemma 2 and set $f_1(h_t) := f(t)$. From property (F1) we get

$$|f_1^{-1}(v)| \leqslant 40\lceil n/\ell_m \rceil \leqslant 50n/m \leqslant \sqrt{n},$$

for every $v \in V(R_m)$. From property (F3) we infer that $(f_1(h_0), \ldots, f_1(h_{n-1}))$ forms a desired walk.

Note that we could achieve the same by replacing $R_m$ corresponding to the first coordinate by the 4th power of a cycle on $\sqrt{n}$ vertices, in which case $f_1$ is simply defined by going around such a cycle. As this would slightly complicate the condition (E1) and some of the arguments, and only has negligible effect on the number of vertices, we use $R_m$ as stated.

Next, we inductively construct $f_i$ and $r_i$ for $2 \leqslant i \leqslant \Delta$.

PROPOSITION 2. *Suppose $2 \leqslant i \leqslant \Delta$ and $f_1, \ldots, f_{i-1}$ are well-distributed. Then there exists a well-distributed homomorphism $f_i\colon H_i \to R_m^4$ which satisfies properties* (H1)–(H3).

*Proof.* Consider the ordering $h_0, \ldots, h_{n-1}$ of $V(H)$ such that $\phi_i(h_t) = t$ for every $t \in [n-1]$. Similarly as in the case of $f_1$, we aim to deduce the existence of $f_i$ by applying Lemma 2 with suitably defined $\sigma(\cdot)$. In particular, for each $t \in [n-1]$ we set $\sigma(t) = D_1(t) \cup D_3(t)$, where $D_1(t)$ and $D_3(t)$ are chosen such that having $f_i(h_t) \notin D_1(t) \cup D_3(t)$ maintains the properties (H1) and (H3), respectively (thus the subscripts 1 and 3 in $D_1$ and $D_3$).

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Let

$$t_0(t) := (\lfloor t/q_m \rfloor - 1)q_m - 1$$

and, for each $t \in [n-1]$, let $D_1(t) \subseteq [n-1]$ be defined as follows,

$$D_1(t) = \{0 \leqslant t' \leqslant t_0(t) \colon f_1(h_t) = f_1(h_{t'})\}.$$

Note that $D_1(t)$ is a subset of indices $t'$ for which we require $f_i(h_t) \neq f_i(h_{t'})$, as otherwise property (H1) becomes violated. From the assumption $|f_1^{-1}(v)| \leqslant \sqrt{n}$ for every $v \in V(R_m)$, we obtain $|D_1(t)| \leqslant \sqrt{n}$. Next, we define a subset $D_3(t) \subseteq [n]$,

$$D_3(t) = \left\{0 \leqslant t' \leqslant t_0(t) \colon \{h_t, h_{t'}\} \notin H \text{ and } \exists j < i \text{ such that } \{f_j(h_t), f_j(h_{t'})\} \in R_m^4\right\}.$$

The role of $D_3(t)$ is to restrict $B_i(h_t)$ to vertices $h_{t'}$ with $|t - t'| \leqslant 2q_m$. As we will see shortly, this suffices for property (H3) to hold. From the assumption $|f_j^{-1}(v)| \leqslant \sqrt{n}$ for $1 \leqslant j < i$ we obtain $|D_3(t)| \leqslant \Delta d^4 \sqrt{n}$ as follows: there are at most $\Delta$ choices for $j < i$, at most $d^4$ choices for a neighbour $v'$ of $f_j(h_t)$ in $R_m^4$, and then at most $\sqrt{n}$ choices for $h_{t'} \in f_j^{-1}(v')$. Finally, by the choice of $m \leqslant \ell_m$ we get

$$|\sigma(t)| = |D_1(t) \cup D_3(t)| \leqslant 2\Delta d^4 \sqrt{n} \leqslant \ell_m/(40d^4).$$

Therefore, we can apply Lemma 2 with $\sigma$ (note that the first condition is satisfied by the definition of $D_1$ and $D_3$ and the choice of $t_0(t)$). Let $f$ be the obtained function and set $f_i(h_t) := f(t)$ for $t \in [n-1]$. We claim that $f_i$ satisfies the required properties.

First, it follows from property (F3) that $f_i(h_0), \ldots, f_i(h_{n-1})$ is a walk in $R_m$ with $f_i(h_t) \neq f_i(h_{t'})$ for every distinct $t, t' \in [n-1]$ with $|t - t'| \leqslant 8$ (thus property (H2) holds). Therefore, $f_i$ is a homomorphism of $H_i$ into $R_m^4$. Moreover, from property (F1) we obtain that $f_i$ is well balanced.

Next, consider some $t' < t$ such that $f_1(h_t) = f_1(h_{t'})$. If $t' \leqslant t_0(t)$ then $t' \in \sigma(t)$ and from (F2) we get $f_i(h_t) = f(t) \neq f(t') = f_i(h_{t'})$. Otherwise, from property (F3) we have that

$$(f(t_0 + 1), \ldots, f(t'), \ldots, f(t))$$

is a path and thus $f_i(h_t) \neq f_i(h_{t'})$. This proves property (H1).

It remains to show that property (H3) holds. Consider some $t \in [n-1]$. Observe that for any $h_{t'} \in B_i(h_t)$ there exist two distinct walks from $f_i(h_{t'})$ to $f_i(h_t)$ in $R_m$, one of length at most 4 and the other of length at most $|t - t'|$ (and at least 5). In particular, this implies that $R_m$ contains a cycle of length at most $|t - t'| + 4$ and from the assumption on the girth of $R_m$ we conclude

$$|t - t'| > \frac{1}{3} \log_{d-1} \ell_m > 2z, \tag{5·1}$$

for sufficiently large $m = m(n)$. This proves the second part of (H3).

To prove the first part of the property (H3), we first observe that property (F2) implies $\{f_i(h_t), f_i(h_{t'})\} \notin R_m^4$ for every $t' \in D_3(t)$. In particular, if $h_{t'} \in B_i(h)$ and $t' < t$, then $t' \geqslant t_0(t) + 1$. On the other hand, if $h_{t'} \in B_i(h_t)$ then also $h_t \in B_i(h_{t'})$, and if $t' > t$ then by the same argument we obtain $t \geqslant t_0(t')$. This further implies $t' < \lfloor t/q_m \rfloor + 2q_m$. To summarise, for every $h_{t'} \in B_i(h_t)$ we have

$$t_0(t) < t' < t_0(t) + 3q_m. \tag{5·2}$$

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Consider two distinct vertices $h_{t'}, h_{t''} \in B_i(h_t)$. If they are on the 'opposite' sides of $h_t$ (i.e. $t' < t < t''$ or $t'' < t < t'$) then from (5·1) we have $|t' - t''| > (\log_{d-1} \ell_m)/3$, with room to spare. Otherwise, from the property (F3) we infer that $f_i(h_t) \neq f_i(h_j)$ for every $t_0(t) < j' < t_0(t) + 3q_m$. Therefore, there exist two distinct walks between $f_i(h_{t'})$ and $f_i(h_{t''})$ in $R_m$, one of length at most 8 (going through $f_i(h_t)$) and the other 'direct' walk of length at most $|t' - t''|$. By the previous observation, the second walk avoids the vertex $f_i(h_t)$. As in the previous case, it is easy to see that this implies $R_m$ contains a cycle of length at most $|t' - t''| + 8$. Therefore, by the assumption on the girth of $R_m$ we conclude $|t' - t''| > (\log_{d-1} \ell_m)/3$. Together with (5·2), this gives the following upper bound on the size of $B_i(h_t)$,

$$|B_i(h_t)| < \frac{3q_m}{\log_{d-1} \ell_m/3} < d,$$

as required. This completes the proof of the claim.

Finally, assuming we have already defined $f_1, \ldots, f_i$ such that (H1)–(H3) hold, we construct a homomorphism $r_i$.

PROPOSITION 3. *Suppose* $2 \leqslant i \leqslant \Delta$ *and* $f_1, \ldots, f_i$ *satisfy properties* (H1)–(H3). *Then there exists a homomorphism* $r_i : H_i \to R_z^4$ *which satisfies property* (H4).

*Proof.* Let $\ell_z = |V(R_z)|$ and $q_z = \lceil \log \ell_z / \log 10 \rceil$, and consider the ordering $h_0, \ldots, h_{n-1}$ of $V(H)$ such that $\phi_i(h_t) = t$ for every $t \in [n-1]$.

For each $t \in [n]$, we define $\sigma(t)$ as follows,

$$\sigma(t) = \{t' \in B_i(h_t) \ : \ t' < t\}.$$

From the choice of $z$ and the first part of property (H3) of $f_i$ we get $|\sigma(t)| \leqslant d \leqslant \ell_z/(160d^4)$. From the second part of (H3) we have

$$t' < t - 2z < t - 2q_z < (\lfloor t/q_z \rfloor - 1)q_z,$$

for every $t' \in \sigma(t)$. Therefore, we can apply Lemma 2 with $F = R_z$ and $\sigma$ to obtain $f : [n-1] \to V(R_z)$. Set $r_i(h_t) := f(t)$ for $t \in [n-1]$. The same argument as in the proof of the previous claim shows that $r_i$ is a homomorphism. Moreover, since for each $h_{t'} \in B_i(h_t)$ we also have $h_t \in B_i(h_{t'})$, we can assume $t' < t$, and by (F2) we have that $f(t)$ and $f(t')$ are at distance at least 5 in $R_z$. In other words, we conclude $\{r_i(h_t), r_i(h_{t'})\} \notin R_z^4$ as required. This proves the claim and finishes the proof of Theorem 1.

## 6. *Concluding remarks*

### 6·1. *The algorithmic aspects of the proof*

The proof of Theorem 1 relies on a decomposition guaranteed by Theorem 3 and the existence of a mapping given by Lemma 2. Note that all other steps needed to construct the homomorphisms $f_i$ and $r_i$, and therefore an induced embedding of $H$, can be implemented efficiently (i.e. with polynomial running time) as they only involve computing sets of vertices which satisfy certain simple conditions.

The proof of Theorem 3 relies on the Edmonds–Gallai decomposition (e.g. see [**17**]) and Hall's criteria for matchings in bipartite graphs. Since the Edmonds–Gallai decomposition can be obtained using Edmonds' algorithm for finding a maximum matching in a general graph and a careful inspection of the proof of Theorem 3 shows that all other steps can

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be performed efficiently, computing the desired decomposition can also be done efficiently. Similarly, the proof of Lemma 2 is constructive and, in turn, relies on sets of $q$-expanding vertices given by Lemma 1. However, the proof of Lemma 1 is algorithmic and gives an efficient deterministic procedure for computing both a set of sufficiently many $q$-expanding vertices and a set of reachable vertices from each vertex in it. We omit further details.

### 6·2. *Adjacency labelling scheme*

Induced universal graphs are closely related to *adjacency labelling schemes*. Given a family of graphs $\mathcal{H}$ and some $H \in \mathcal{H}$, the goal is to assign labels to vertices of $V(H)$ (the *encoding*) such that the existence of an edge between two vertices can be deduced only from their labels, without knowing which graph from the family is under consideration (the *decoding*). In their paper on implicit graph representation and adjacency labelling schemes, Kannan, Naor and Rudich [**16**] observed that a family of graphs $\mathcal{H}$ admits a labelling scheme in which every label can be described with $L$ bits if and only if $g_v(\mathcal{H}) \leqslant 2^L$. In other words, the two problems are equivalent from the existential point of view. In particular, this approach was used in [**8**, **9**, **10**] to deduce the existence of the desired induced universal graphs.

After proving that a labelling scheme of certain size exists, the next step is to investigate the efficiency of the encoding and the decoding algorithms. For graphs with maximum degree at most some constant $\Delta$, Alstrup and Rauhe [**10**] first obtained a scheme which uses $\lceil \Delta/2 \rceil \log n + O(\log^* n)$ bits with running time of order $O(n)$ for encoding and $O(\log^* n)$ for decoding. This was subsequently improved by Adjiashvili and Rotbart [**2**] to $\lceil \Delta/2 \rceil \log n + O(1)$ bits with running time $O(n \log n)$ for encoding and $O(\log \log n)$ for decoding. Recently, Alstrup et al. [**8**] designed a labelling scheme for the family of all forests (without any restriction on the maximum degree) which uses $\log n + O(1)$ bits and has $O(n)$ encoding and $O(1)$ decoding running time. Since every graph with maximum degree $\Delta$ has arboricity at most $\lceil \Delta/2 \rceil$, this gives a scheme for graphs of bounded degree which uses the same number of bits as the one of Adjiashvili and Rotbart but the encoding and decoding algorithms are faster.

The previously known results give a satisfactory scheme in case $\Delta$ is even, however for odd $\Delta$ they require extra $(\log n)/2$ bits compared to the best possible size. Our construction gives an optimal number of bits, $(\Delta \log n)/2 + O(1)$, on the expense of much worse, but still polynomial, running time for the encoder. Let us assume that the vertices of $\Gamma$ are labelled with $\{1, \dots, |V(\Gamma)|\}$. The encoder first constructs $\Gamma$ and then, given a graph $H \in \mathcal{H}(n, \Delta)$, finds in polynomial time an induced subgraph of $\Gamma$ isomorphic to $H$, following the proof given in the paper. The labels assigned to the vertices of $H$ correspond to the labels of such a subgraph in $\Gamma$. For constant $\Delta$ the decoder works in constant time, as adjacency in the known explicit high girth expanders can be checked with a constant number of modular additions and multiplications (required for computing the products of matrices in $SL_2(Z_p)$). It may be interesting to design a more efficient encoding scheme which uses the optimal number of bits.

### 6·3. *Large maximum degree*

Our main theorem settles the case where $\Delta$ is a constant, thus it is natural to ask what happens when we allow $\Delta$ to grow with $n$. This was explicitly mentioned in [**2**], where the authors established an upper bound of order $\binom{n}{\lceil \Delta/2 \rceil} n^{O(1)}$ on $g_v(\mathcal{H}(n, \Delta))$, for any $\Delta = \Delta(n)$. Compared to the bound from Theorem 1, this is already better for $\Delta = \Omega(\log n / \log \log n)$.

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Alstrup et al. [**1**] have recently improved this for slowly growing $\Delta$ to $\binom{n}{\lceil \Delta/2 \rceil} 2^{O(\Delta)}$, and for larger $\Delta$ to

$$\binom{\lfloor n/2 \rfloor}{\lfloor \Delta/2 \rfloor} 2^{O(\sqrt{\Delta \log n} \cdot \log(n/\Delta))}.$$

Moreover, they showed a lower bound of order

$$\binom{\lfloor n/2 \rfloor}{\lfloor \Delta/2 \rfloor} 2^{-O(\sqrt{\Delta \log n} \cdot \log(n/\Delta))}.$$

It remains an interesting problem to determine the right order of $g_v(\mathcal{H}(n, \Delta))$ for various ranges of $\Delta = \Delta(n)$.

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