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The structure of modules over polycyclic groups

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Abstract

The structure of modules over polycyclic group rings RG is investigated using the idea of a link $P \rightsquigarrow Q$ between prime ideals of RG. The representation theory of RG splits into two parts – the part we discuss is determined by the representation theory of certain Noetherian polynomial identity factor rings of subrings of RG.

1. *Introduction*

This paper is a continuation of (1). There is an analogue of the Jordan–Hölder theorem which applies to finitely generated modules over a Noetherian ring S; in this result, quoted as ((1), theorem 5·2), the *critical modules* (in the sense of Krull dimension, ((9), page 9) play the role of the simple modules, and *critical composition series* (c.c.s.) replace composition series. There are two varieties of critical module C: we say that C is *type I* if $k\text{-dim}\,(C) = k\text{-dim}\,(S/r(C))$, and *type II* if $k\text{-dim}\,(C) < k\text{-dim}\,(S/r(C))$. (Here, $k\text{-dim}\,(A)$ denotes the Krull dimension of the *right* module A, and

$$r(C) = \{s \in S : Cs = 0\},$$

the *annihilator* of C.)

We are concerned here with the case where S is the group ring RG of a polycyclic-by-finite group G over a commutative Noetherian Hilbert domain R, all of whose field images are absolute – an *absolutely Hilbert ring*, for short. In this case, if C is a type I critical, $r(C) = P$ is a prime ideal of RG, and C is isomorphic to a uniform right ideal of RG/P, ((1), lemma 5·1) and ((11), proposition 1·1 (b)). Now every injective module over a Noetherian ring is a direct sum of uniform injectives, and every non-zero module contains a critical submodule; thus, if one is willing to enlarge modules when necessary, the study of a finitely generated RG-module M reduces to the case where M has an essential critical submodule. We consider only the case where this submodule is type I. Our starting point is

THEOREM A: ((1), theorem 5·4 and theorem 4·2; (4) theorem 3·13) *Let the finitely generated RG-module M have an essential type I critical submodule U of Krull dimension α. Let*

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$$

be a c.c.s. for M. Then, for $1 \leqslant i \leqslant n$, M_i/M_{i-1} is a type I critical of dimension α.

The critical factors M_i/M_{i-1} in the above are determined up to subisomorphism, and can be listed by listing their (prime) annihilators, $P_1, \ldots, P_n$, ((1), lemma 5·1). The questions that concern us may be summarized as: Given P_1, which primes can occur as $P_2, \ldots, P_n$? Given $P_1, P_2, \ldots, P_n$, what can be said about the possibilities for M? Of course,

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answers to these questions are closely tied up with the structure of the injective hull $E(U)$ of U. We are thus led to the following definitions, adapted in part from (15). Let Spec (RG) be the primes of RG. If $P, Q \in \text{Spec}\,(RG)$, with $k\text{-dim}\,(RG/P) = k\text{-dim}\,(RG/Q)$, we say there is *a (right) link from P to Q*, and write $P \rightsquigarrow Q$, if there is an exact sequence

$$0 \to X \to U \to Y \to 0 \tag{1}$$

of right RG-modules, with U finitely generated and uniform, and X and Y type I critical modules with $r(X) = P$, $r(Y) = Q$. Any pair $X \subseteq U$ of RG-modules with these properties is called *a realization* of the link $P \rightsquigarrow Q$. We write $P \rightsquigarrow Q$, and refer to *a left link from P to Q*, when a left-handed analogue of (1) exists. The directed graph with vertices the members of Spec (RG) and edges representing (right) links is called the *(right) graph of links of RG*. We write $P \rightsquigarrow\rightsquigarrow Q$ if there is a directed path in the right graph of links from P to Q.

From the definition, $P \rightsquigarrow\rightsquigarrow Q$ implies that $k\text{-dim}\,(RG/P) = k\text{-dim}\,(RG/Q)$. In fact, it is easily seen that $P \rightsquigarrow\rightsquigarrow Q$ if and only if there is a finite set of primes $P = P_1, P_2, \ldots, P_n = Q$ with $k\text{-dim}\,(RG/P_i) = \alpha$ for all i, and non-zero homomorphisms $E(RG/P_i) \to E(RG/P_{i+1})$, $1 \leqslant i < n$. One of our main aims is to show that the primes in a connected component of the graph of links are rather closely related. We need first some new concepts.

Recall from (19) that a subgroup H of G, an ideal of RH etc., is said to be *(G-)orbital* if it has only finitely many G-conjugates. An orbital subgroup H is *isolated* if it has infinite index in every orbital subgroup properly containing it. (The group G is *orbitally sound* if its isolated orbital subgroups are normal.)

We show in §4 that associated to each prime P of RG there is a conjugacy class $\mathscr{S}_P$ of isolated orbital subgroups of G, called *the inductors of P*, and a G-conjugacy class $\mathscr{T}_P$ of G-orbital prime ideals K of RH (where $H \in \mathscr{S}_P$), called *the subsources of P*, such that $P = \cap_{g \in G} K^g RG$, and such that the members of $\mathscr{S}_P$ and $\mathscr{T}_P$ are, roughly speaking, as small as possible. Write, for an ideal I of RG, $I^+ = \{g \in G : g - 1 \in I\}$. If $K \in \mathscr{T}_P$, and K is an ideal of RH, then H/K^+ is an abelian-by-finite group. The existence of the sets $\mathscr{S}_P$ and $\mathscr{T}_P$ is deduced from work of Lorenz and Passman (14), which is based in turn on (19). In §4, we obtain in passing a result which answers a question raised in (19): *If H is an isolated orbital subgroup of G, and K is a G-orbital prime ideal of RH, then $K^G = \cap_{g \in G} K^g RG$ is prime.*

In §5 we prove the central result of the paper: *If P and Q are primes of RG, then $P \rightsquigarrow Q$ if and only if $\mathscr{S}_P = \mathscr{S}_Q$, and for $H \in \mathscr{S}_P$ there exist ideals K and L of RH with $K \in \mathscr{T}_P$, $L \in \mathscr{T}_Q$, with $K \rightsquigarrow L$ in RH* (Theorem 5·13). Sections 6 and 7 discuss consequences of this result for prime ideals and for modules respectively. In §6 necessary and sufficient conditions for a prime P to belong to a finite component of the graph of links are discussed, and we show that, even if P lies in an infinite component, *there are only a finite number of edges into and out of P* (Theorem 6·4).

In §7 we study the structure of $E = E_{RG}(U)$, where U is a uniform right ideal of RG/P, P prime. Let $H \in \mathscr{S}_P$. *Every finitely generated submodule of E is a submodule of a module X which is induced, in a certain sense, from an RH-module X_0. Moreover, X_0 is finitely generated as a module over $Q(RH/r(X_0))$, the Artinian quotient ring of $RH/r(X_0)$,*

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and $RH/r(X_0)$ *is a P.I. ring.* Let $k\text{-dim}(U) = \alpha$. The *α-socle series* of E, denoted $\text{soc}_n^\alpha(E)$, for $n = 1, 2, \ldots$, is defined in §5. Now $E = \cup_n \text{soc}_n^\alpha(E)$ – in fact, the α-socle series of E is the generalization of the sequence of subgroups of order p^n, $n = 1, 2, \ldots$, in a Prüfer p-group. It follows from Theorem 6·4 and the above result that *the factors* $\text{soc}_n^\alpha(E)/\text{soc}_{n-1}^\alpha(E)$ *all have finite uniform dimension* (Corollary 7·2). Returning finally to links, Corollary 7·2 means that, *if* $P \rightsquigarrow Q$, *then there is a factor ring* $\overline{RG}$ *of* RG/QP, *and a finitely generated injective module* A *over the Artinian quotient ring* $Q(\overline{RG})$ *of* $\overline{RG}$, *such that every realization* $X \subseteq U$ *of* $P \rightsquigarrow Q$ *can be embedded in* A.

It remains to outline §§2 and 3. The first contains a characterization (Theorem 2·2) of when a link $P \rightsquigarrow Q$ exists, for a class of Noetherian rings which include the group rings under discussion. In particular, it's shown that $P \rightsquigarrow Q$ *if and only if* $Q \underset{l}{\rightsquigarrow} P$, a fact used frequently in subsequent sections. Section 3 contains a result on families of modules (Corollary 3·2), whose application to group rings in Theorem 3·4 is crucial to the proof of Theorem 5·13. In fact, if the reader is willing to work with an orbitally sound group throughout, the bulk of §§4 and 5 can be ignored, an analogue of Theorem 5·13 being deduced directly from Theorem 3·4.

Throughout, all groups will be polycyclic-by-finite. All modules are unital, and modules, links, etc. will be taken on the right, except when some indication to the contrary is given. We shall use R to denote an absolutely Hilbert domain throughout; see, however, the remark at the end of §4. Indeed, most of our results can be extended to modules over group rings with an arbitrary commutative Noetherian ring as coefficient ring, using fairly routine arguments in the style of Remark 4·8. We leave the details to the interested reader.

2. *Symmetry of the graph of links*

If M is a finitely generated *left* module over the Noetherian ring S, we write $l\text{-dim}_S(M)$ (or just $l\text{-dim}(M)$) for the left Krull dimension of M. The ring S is *weakly K-symmetric* if $k\text{-dim}(S/I) = l\text{-dim}(S/I)$ for all ideals I of S. A right Noetherian ring S is *right smooth* if, for all finitely generated S-modules X with an essential submodule M,

$$k\text{-dim}(S/r(X)) = k\text{-dim}(S/r(M)).$$

(1) Theorem 4·2 states that RG is smooth, and RG is weakly K-symmetric by ((4), 3·14).

LEMMA 2·1. *Let S be a weakly K-symmetric smooth Noetherian ring. Then, for all ideals $T \subseteq I$ of S, $k\text{-dim}(I/T) = l\text{-dim}(I/T)$.*

Proof. Suppose not, and choose a pair of ideals $T \subseteq I$ which furnish a counter-example, with T as large as possible. It is easy to show that if $X \lhd S$ and $T \subsetneqq X$, then $T \subsetneqq X \cap I$. By ((1), theorem 6·3) and its left-hand analogue, $k\text{-dim}(S/T) = k\text{-dim}(I/T)$ and $l\text{-dim}(S/T) = l\text{-dim}(I/T)$. Hence, $k\text{-dim}(S/T) \neq l\text{-dim}(S/T)$, contradicting the weak K-symmetry of S.

A Noetherian ring S is *K-homogeneous* if $k\text{-dim}(I) = k\text{-dim}(S)$ for each non-zero ideal I of S. If P is a prime ideal of S, S is *P-primary* if P is the unique prime right annihilator ideal of S; see ((5), §5). For the definition of a *weakly ideal invariant ring* (a w.i.i. ring for brevity), see ((4), §2). The group ring RG is w.i.i. ((4), theorem 3·13). For a left S-module M, we write $l(M) = \{s \in S : sM = 0\}$.

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THEOREM 2·2. *Let S be a weakly K-symmetric, smooth, w.i.i. Noetherian ring, with prime ideals P and Q such that $k\text{-dim}(S/P) = k\text{-dim}(S/Q)$. Then the following are equivalent:*

(i) $P \rightsquigarrow Q$;

(ii) $k\text{-dim}(Q \cap P/QP) = k\text{-dim}(S/P)$;

(iii) $r(Q \cap P/QP) = P$;

(iv) $Q \underset{l}{\rightsquigarrow} P$;

(v) $l\text{-dim}(Q \cap P/QP) = l\text{-dim}(S/Q)$;

(vi) $l(Q \cap P/QP) = Q$.

Proof. (i) $\Rightarrow$ (ii) Since $P \rightsquigarrow Q$, there exists an exact sequence

$$0 \to X \to U \to Y \to 0$$

of finitely generated right S-modules, with U uniform, and X and Y type I criticals with annihilators P and Q respectively. Let $I = r(U)$, so that S/I is a P/I-primary K-homogeneous ring, ((1), proposition 5·5). Hence, $k\text{-dim}(S/I) = k\text{-dim}(S/P)$. Now $QP \subseteq I \subseteq Q \cap P$. Also, $I \neq Q \cap P$, since otherwise passage to the Artinian quotient ring of S/I shows that $r(U) = P$ or $r(U) = Q$, a contradiction. Thus, since S/I is K-homogeneous, $k\text{-dim}(Q \cap P/I) = k\text{-dim}(S/P)$. Clearly,

$$k\text{-dim}(Q \cap P/QP) \leqslant k\text{-dim}(S/P),$$

and so (ii) is proved.

(ii) $\Rightarrow$ (iii): If $r(Q \cap P/QP) = E \neq P$, then

$$k\text{-dim}(Q \cap P/QP) \leqslant k\text{-dim}(S/E) < k\text{-dim}(S/P),$$

by ((9), proposition 6·1).

(iii) $\Rightarrow$ (i): By ((1), propositions 6·1 and 5·5), there exist ideals $I_1, \ldots, I_n$ of S such that $QP = \cap_i I_i$, and S/I_i is a K-homogeneous primary ring. Since $Q \cap P$ is finitely generated as a left ideal, (iii) implies that $k\text{-dim}(Q \cap P/QP) = k\text{-dim}(S/P)$. Thus, there exists j, $1 \leqslant j \leqslant n$, such that $k\text{-dim}((Q \cap P) + I_j/I_j) = k\text{-dim}(S/P)$. It follows that S/I_j is P-primary, and that $I_j \subsetneqq P$. Since $QP \subseteq I_j$, S/I_j embeds in a finite direct sum T of finitely generated submodules of $E_S(S/P)$, with $T(QP) = 0$. However, $TP \neq 0$, and so T has a critical composition factor subisomorphic to S/Q, by ((1), lemma 5·3). This proves (i).

The equivalence of (iv), (v) and (vi) is proved similarly.

(ii) $\Leftrightarrow$ (v): Since $k\text{-dim}(S/P) = k\text{-dim}(S/Q)$, this follows from Lemma 2·1.

Theorem 2·2 generalizes ((15), proposition 2), which is the corresponding result for FBN rings. As in the FBN case, the image of $\mathscr{C}(Q \cap P)$ in S/QP is an Ore set, and the equivalences of Theorem 2·2 may be lifted to the corresponding quotient ring.

We shall need the following well-known necessary condition for the existence of a link $P \rightsquigarrow Q$. (The *right AR property* is the property discussed in ((17), §11·2), and referred to there as the *weak AR property*.)

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LEMMA 2·3. *Let P and Q be prime ideals of the right Noetherian ring S, with $P \rightsquigarrow Q$. If $I \subseteq P$ is an ideal of S with the right AR property, then $I \subseteq Q$.*

Proof. Let X, U be a pair of S-modules realizing $P \rightsquigarrow Q$. Since I has the AR property, there exists $n \geqslant 1$ such that

$$UI^n \cap X \subseteq XI \subseteq XP = 0.$$

Since X is essential in U, $UI^n = 0$. Thus $I^n \subseteq r(U/X) = Q$, and so $I \subseteq Q$ since Q is prime.

3. *Neat families of modules*

Let S be a Noetherian ring. Our aim in this section is to show that certain families of finitely generated S-modules are closed under essential extensions. The result we obtain will concern families $\mathscr{P}$ of finitely generated right and left S-modules, such that

(i) $0 \in \mathscr{P}$;

(ii) if $X \cong Y \in \mathscr{P}$, then $X \in \mathscr{P}$;

(iii) if $0 \to N \to M \to M/N \to 0$ is an exact sequence of finitely generated modules, then (*a*) $M \in \mathscr{P}$ implies that $N, M/N \in \mathscr{P}$, and (*b*) $N, M/N \in \mathscr{P}$ implies that $M \in \mathscr{P}$;

(iv) if $T \subseteq I$ are ideals of S, and $(I/T)_S \in \mathscr{P}$, then ${}_S(I/T) \in \mathscr{P}$.

We call such a family $\mathscr{P}$ *neat*.

Let $\mathscr{P}$ be a neat family, and let M be a right S-module such that $S/r(M) \notin \mathscr{P}$. Let

$$\Sigma = \{r(X) \colon X \text{ a submodule of } M,\ S/r(X) \notin \mathscr{P}\},$$

so $\Sigma \neq \Phi$. Since S is Noetherian, Σ has maximal members; any such maximal element P is called a *$\mathscr{P}$-limiting ideal for M*.

THEOREM 3·1. *If S, M and $\mathscr{P}$ are as above, then the $\mathscr{P}$-limiting ideals for M are prime.*

Proof. Let P be a $\mathscr{P}$-limiting ideal, say $P = r(N)$, and suppose that there exist ideals A and B of S with $AB \subseteq P$, while $P \subsetneqq A$, $P \subsetneqq B$. Then $NA \neq 0$, so it may be assumed without loss that $B = r(NA)$. Thus $S/B \in \mathscr{P}$, by maximality of P. Now $(A/AB)_S$ is a finitely generated S/B-module, and so is in $\mathscr{P}$ by property (iii). Therefore ${}_S(A/AB) \in \mathscr{P}$, by (iv).

Let $A/AB = \Sigma_{i=1}^n a_i S + AB/AB$. For $1 \leqslant i \leqslant n$, let $K_i = \{s \in S : sa_i \in AB\}$, so $S/K_i \cong Sa_i + AB/AB$, and $S/K_i \in \mathscr{P}$, by (iii). Let $K = \cap_i K_i$, so that $K = l(A/AB)$, and ${}_S(S/K) \in \mathscr{P}$, again by (iii). Note that K is an ideal, and $(S/K)_S \in \mathscr{P}$, by (iv). Now $N(KA) \subseteq N(AB) \subseteq NP = 0$. Since $A \subseteq r(NK)$, it follows from choice of P that $(S/r(NK))_S \in \mathscr{P}$. Since $(S/K)_S \in \mathscr{P}$, (iii) shows that $(S/Kr(NK))_S \in \mathscr{P}$. However, $Kr(NK) \subseteq P$, so this is a contradiction. Hence, P is prime.

COROLLARY 3·2. *Let S be a smooth Noetherian ring, let $\mathscr{P}$ be a neat family of S-modules, and let U be a type I critical S-module, with $U \in \mathscr{P}$. If $M \subseteq E(U)$, and M is finitely generated, then $M \in \mathscr{P}$.*

Proof. Let $r(U) = Q$; by passing if necessary to a submodule of U, we may assume that Q is prime. Now U is isomorphic to a right ideal of S/Q, ((11), proposition 1·1 (*b*));

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thus, since an essential right ideal of S/Q contains a regular element, S/Q embeds in a finite direct sum of copies of U, ((11), proposition 1·1 (d)). Hence, $S/Q \in \mathscr{P}$. It is enough to show that $S/r(M) \in \mathscr{P}$. Suppose that this is not true, and let P be a $\mathscr{P}$-limiting ideal for M, so P is prime, by the theorem. Since S is smooth,

$$k\text{-dim}\,(S/P) = k\text{-dim}\,(S/Q);$$

but $P \subseteq Q$, so we must in fact have $P = Q$. Thus $S/P \in \mathscr{P}$, a contradiction.

In this paper S will usually be a group ring RG of a polycyclic-by-finite group G, with absolutely Hilbert coefficient ring R. The neat families with which we shall be most concerned are defined as follows. Let α be an ordinal, and H a normal subgroup of the group G. Define $\mathscr{P}_H^\alpha$ to be the union of the family of finitely generated right RG-modules M such that, if $m_1, \ldots, m_t \in M$, then $k\text{-dim}_{RH}(\Sigma_i m_i RH) \leqslant \alpha$, and the corresponding family of left modules. To see that $\mathscr{P}_H^\alpha$ is neat, we need

LEMMA 3·3. *Let $A \subseteq I$ be ideals of RG. If $(I/A)_{RG} \in \mathscr{P}_H^\alpha$, then ${}_{RG}(I/A) \in \mathscr{P}_H^\alpha$.*

Proof. Suppose $(I/A)_{RG} \in \mathscr{P}_H^\alpha$, and let $a_1, \ldots, a_t \in I$. For $1 \leqslant i \leqslant t$, let

$$\{g_{ij} : j = 1, 2, \ldots\} = \text{supp}\,(a_i).$$

Put $Y = \Sigma_i RHa_i RH$, so that

$$Y + A/A \subseteq \sum_{i,j} (g_{ij} RH + A/A) = \sum_{i,j} (RHg_{ij} + A/A).$$

Thus, since RH is Noetherian, $Y + A/A$ is finitely generated both as a right and as a left RH-module, and $a_i + A \in Y + A/A$, $1 \leqslant i \leqslant t$.

Let $Y + A/A \subseteq \Sigma_{l=1}^m (g_l RH) + A/A$, where the elements g_l of G belong to a transversal to H, and m is minimal. We prove by induction on m that

$$k\text{-dim}_{RH}(Y + A/A) = l\text{-dim}_{RH}(Y + A/A).$$

If $m = 1$, then

$$Y + A/A \subseteq g_1 RH + A/A \cong g_1 RH/g_1 RH \cap A,$$

where the isoiorphism is of RH-bimodules. Also, the lattices of right submodules of $g_1 RH/g_1 RH \cap A$ and of $RH/RH \cap g_1^{-1} A$ are clearly isomorphic; similarly with the left submodule lattices. Therefore, the case $m = 1$ follows from ((2), lemma 3·3 (ii)).

Suppose now that $m > 1$, and let $X = \Sigma_{l=1}^{m-1} g_l RH$, $U = g_m RH$. Let

$$T = \left(\sum_{l=1}^{m} g_l RH \right) \cap A,$$

so that $Y + A/A$ may be viewed as a sub-bimodule I/T of $(\Sigma_{l=1}^m g_l RH)/T$. Let $\pi : \Sigma_{l=1}^m g_l RH \to U$ be the projection map. Then there is an exact sequence of right and left RH-modules

$$0 \to (I \cap X) + T/T \to I/T \to \pi(I)/\pi(T) \to 0.$$

Since

$$(I \cap X) + T/T \cong I \cap X/(I \cap X) \cap T = I \cap X/T \cap X,$$

the induction step can be completed.

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Hence,

$$\begin{aligned} l\text{-dim}_{RH}(\sum_i RHa_i + A/A) &\leqslant l\text{-dim}_{RH}(Y + A/A) \\ &= k\text{-dim}_{RH}(Y + A/A) \\ &\leqslant \alpha, \end{aligned}$$

and the proof is complete.

Since hypotheses (i), (ii) and (iii) are trivially satisfied by the families $\mathscr{P}_H^\alpha$, Lemma 3·3 shows that they are neat. In particular, Corollary 3·2 may be applied; from it we deduce

THEOREM 3·4. *Suppose that G is orbitally sound. Let U be a type I critical RG-module, with $r(U) = P$, and let Q be the annihilator of a critical composition factor of $E_{RG}(U)$. Let H be an isolated normal subgroup of G such that $P = (P \cap RH)RG$. Then*

$$Q = (Q \cap RH)RG.$$

Proof. Let $k\text{-dim}\,(RH/P \cap RH) = \alpha$. Since U embeds in RG/P, $U \in \mathscr{P}_H^\alpha$. There exist finitely generated submodules $T \subseteq M$ of $E_{RG}(U)$ such that M/T is a critical composition factor, with $r(M/T) = Q$. By Corollary 3·2, $M/T \in \mathscr{P}_H^\alpha$. Since M/T is a type I critical, by Theorem A, RG/Q embeds in a finite direct sum of copies of M/T, so $RG/Q \in \mathscr{P}_H^\alpha$. Hence, $k\text{-dim}_{RH}(RH/Q \cap RH) \leqslant \alpha$.

Let n be the Hirsch number of G/H. By ((22), lemma 10), $k\text{-dim}\,(RG/P) = \alpha + n$. Hence, by Theorem A, $k\text{-dim}\,(RG/Q) = \alpha + n$. However, ((22), lemma 10) also gives $k\text{-dim}\,(RG/(Q \cap RH)RG) \leqslant \alpha + n$. Because H is isolated and G is orbitally sound, $(Q \cap RH)RG$ is prime, ((19), corollary 20). In particular, every proper factor ring of $RG/(Q \cap RH)RG$ has strictly smaller Krull dimension, ((9), proposition 6·1). Hence, $Q = (Q \cap RH)RG$.

Theorem 3·4 is in fact true for an arbitrary polycyclic-by-finite group G, and H need only be assumed to be orbital; these improvements will be obtained in §5.

We end this section by noting other applications of Theorem 3·1 which will not be needed in the rest of the paper. The argument of Corollary 3·2 may be adapted to show that *if K is a commutative Noetherian ring and U is a KG-module which is finitely generated as a K-module, then so is every finitely generated submodule of $E_{RG}(U)$.* Another observation along similar lines is the following: *If, in the situation of Theorem 3·4, $RH/P \cap RH$ is finitely generated as an R-module, then so is $RH/Q \cap RH$.* To prove this apply Corollary 3·2 with $\mathscr{P}$ the family of RG-modules whose finitely generated RH-submodules are finitely generated over R. It is clear that $\mathscr{P}$ is neat.

4. *The inductors of a prime ideal*

Throughout this section we shall use the notation of (19) and (14), and the reader is assumed to have some familiarity with these papers. We continue to assume that R is an absolutely Hilbert domain; but see Remark 4·8.

Let P be a prime ideal of RG. *An inductor for P* is a subgroup H of G such that

(i) H is isolated and orbital;

(ii) there exists a G-orbital prime ideal L of RH such that $P = L^G$;

(iii) $H \subseteq i_G \delta_G(L^+)$;

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(iv) if $E \subseteq H$ satisfies (i), (ii) and (iii), then $E = H$. We see below that the set $\mathscr{S}_P$ of inductors for P is non-empty. If $H \in \mathscr{S}_P$ and L is a prime of RH such that (i), ..., (iv) hold, we say that L is *a subsource for* P, and write $\mathscr{T}_P$ for the set of subsources.

The main aim of this section is to prove Theorem 4·7, which characterizes $\mathscr{S}_P$ and $\mathscr{T}_P$ as non-empty finite sets of G-conjugate subgroups and ideals. The proof of this result requires the following lemmas. If $I \lhd S$, we write $\mathscr{C}_S(I) = \{c \in S : c + I \text{ is regular in } S/I\}$.

LEMMA 4·1. *Let $J_1, \ldots, J_n$ be G-orbital ideals of RH, where H is an orbital subgroup of G. Then*

$$\prod_i J_i^G \subseteq (\prod_i J_i)^G.$$

Proof. When $n = 2$, we have $J_1^G J_2^G \subseteq J_1 RG J_2^G = J_1 J_2^G \subseteq J_1 J_2 RG$. Since $J_1^G J_2^G$ is an ideal, $J_1^G J_2^G \subseteq (J_1 J_2)^G$. The general case follows by induction.

LEMMA 4·2. *Let H be a subgroup of finite index in G.*

(i) *If K is a semiprime ideal of RG with RG/K K-homogeneous, then $RH/K \cap RH$ is K-homogeneous, and $k\text{-dim}_{RH}(RH/K \cap RH) = k\text{-dim}_{RG}(RG/K)$.*

(ii) *Let K be a semiprime ideal of RH, with RH/K K-homogeneous. Then*

$$k\text{-dim}_{RG}(RG/L) = k\text{-dim}_{RH}(RH/K)$$

for every prime ideal L of RG minimal over K^G.

Proof. Let $T = \cap_g H^g$, a normal subgroup of finite index in G.

(i) Let $K \cap RH \subseteq I$, a right ideal of RH. By ((22), lemma 8),

$$k\text{-dim}_{RH}(I/K \cap RH) = k\text{-dim}_{RT}(I/K \cap RH).$$

Now $I/K \cap RH$ is isomorphic to a non-zero RT-submodule of RG/K. Again using ((22), lemma 8),

$$k\text{-dim}_{RT}(RG/K) = k\text{-dim}_{RG}(RG/K), \quad \text{so} \quad k\text{-dim}_{RT}(I/K \cap RH) \leqslant k\text{-dim}_{RG}(RG/K).$$

If this inequality is strict, we obtain a non-zero RG-submodule of RG/K of dimension smaller than $k\text{-dim}(RG/K)$, a contradiction.

(ii) Since $K \cap RT$ is a semiprime ideal of RT, so is $I = \cap_g (K \cap RT)^g$, and RT/I is K-homogeneous of dimension α, say, by (i). Clearly, $I \subseteq K^G \cap RT$, while

$$K^G \cap RT \subseteq KRG \cap RT = K \cap RT.$$

Since $K^G \cap RT$ is G-invariant, $K^G \cap RT \subseteq I$, and so $I = K^G \cap RT$.

Since $K^G = \cap_g K^g RG$, RG/K^G is K-homogeneous, first as RT-module, then as RG-module, using ((22), lemma 8). Let $c \in \mathscr{C}_{RT}(RT/I)$. Then

$$k\text{-dim}_{RT}(RT/I + cRT) < \alpha, \quad \text{so} \quad k\text{-dim}_{RG}(RG/IG + cRG) < \alpha,$$

by ((22), lemma 8). Hence,

$$k\text{-dim}_{RG}(RG/K^G + cRG) < \alpha = k\text{-dim}_{RG}(RG/K^G),$$

by (i). Thus, $c + K^G$ is left regular in RG/K^G; a similar argument using left ideals and K-symmetry proves that $c + K^G$ is right regular. That is, $\mathscr{C}_{RT}(RT/I) \subseteq \mathscr{C}_{RG}(RG/K^G)$.

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Therefore, by ((8), (2·14) and (2·7)), if L is a prime of RG minimal over K^G,

$$L \cap RT \cap \mathscr{C}_{RT}(I) = \varnothing.$$

Hence, since $L \cap RT$ is G-prime, it is an intersection of primes minimal over I, ((8), (2·14)). Thus,

$$\begin{aligned} k\text{-dim}_{RG}(RG/L) &= k\text{-dim}_{RT}(RT/L \cap RT), \quad \text{by (i)}, \\ &= k\text{-dim}_{RT}(RT/I), \\ &= k\text{-dim}_{RT}(RT/K \cap RT), \\ &= k\text{-dim}_{RH}(RH/K), \quad \text{by (i)}. \end{aligned}$$

The Hirsch number of a polycyclic group G is denoted by $h(G)$.

LEMMA 4·3. *If H is an orbital subgroup of G, with $N = N_G(H)$, and L is a G-orbital ideal of RH, then*

$$k\text{-dim}_{RG}(RG/L^G) = k\text{-dim}_{RH}(RH/L) + h(N/H).$$

Proof. Fix a subgroup A of finite index in N such that A/H is poly-(infinite cyclic), and let $M = \cap_g A^g$. By ((22), lemma 8),

$$k\text{-dim}_{RG}(RG/LG) = k\text{-dim}_{RM}(RG/LG) = k\text{-dim}_{RA}(RG/LG).$$

Since $H \subseteq A$, $k\text{-dim}_{RA}(RG/LG) = k\text{-dim}_{RA}(RA/LA)$. The result now follows from ((4), lemma 3·8).

Let I be an ideal of RG. Recall that there exists a unique normal subgroup H of G such that $I = (I \cap RK)RG$, for a normal subgroup K of G, if and only if $H \subseteq K$, ((17), page 304). We shall often denote this subgroup H, *the controller of* I, by $\text{cont}_G(I)$.

LEMMA 4·4. *Let E be a normal subgroup of finite index in the group F, let C be a normal subgroup of F with F/C torsion-free abelian, and let $D = C \cap E$. Let P be a prime ideal of RF, and let J be a prime of RE minimal over $P \cap RE$.*

(i) *If C controls the semiprime ideal N of RF, then C controls each of the primes minimal over N.*

(ii) *If P is controlled by C, then J is controlled by D.*

(iii) *If J is controlled by D, then P is controlled by C.*

Proof. (i) Let Q be a prime minimal over N, and let $X_1, \ldots, X_n$ be the primes of RC minimal over $N \cap RC$. Then X_i^F is prime, $1 \leqslant i \leqslant n$, by ((19), lemma 9), and

$$\prod_{i=1}^{n} (X_i^F) \subseteq (N \cap RC)RF = N \subseteq Q,$$

so that $X_1^F \subseteq Q$, say. Since $N \subseteq X_1^F$, $X_1^F = Q$ as required.

(ii) Let $\pi_X : RG \to RX$, for a subgroup X of G, denote the projection map. Then

$$\pi_D(P \cap RE) \subseteq \pi_C(P) \cap RE \subseteq P \cap RC \cap RE = P \cap RD,$$

since $\pi_C(P) = P \cap RC$. Thus $P \cap RE = (P \cap RD)RE$, by ((17), lemma 1·1·5(ii)). The result now follows from (i).

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(iii) Let $X_1, \ldots, X_n$ be the primes of RC minimal over $(J \cap RD)RC$. Then X_i^F is prime, $1 \leqslant i \leqslant n$, by ((19), lemma 9). Some product $\Pi_j X_{i_j}^F$ lies in $(J \cap RD)RF$, and is F-invariant; it is therefore contained in $J^F = (\cap_{f \in F} J^f)RF \subseteq P$. Hence, there exists i, $1 \leqslant i \leqslant n$, such that $X_i^F \subseteq P$, since P is prime. However,

$$X_i^F \cap RE \supseteq \cap_{f \in F}\{X_i \cap RD)RE\}^f \supseteq \cap_{f \in F}\{(J \cap RD)RE\}^f = P \cap RE.$$

Therefore, by ((14), lemma 1·3(ii)), $X_i^F = P$, as claimed.

The assumption in lemma 4·4 that F/C is torsion-free abelian can be greatly weakened; use Theorem 4·6 below in place of ((19), lemma 9) in the above proofs.

For the definition of a *source* of a prime of RG, see (14). We write $\mathrm{nio}(G)$ for the intersection of the normalizers of the isolated orbital subgroups of G, so $|G : \mathrm{nio}(G)| < \infty$, ((19), theorem C 2).

LEMMA 4·5. *Let P be a prime of RG, with source L, an ideal of RH. Let $W = \mathrm{cont}_H(L)$ (so W is G-orbital). Let $T = \mathrm{nio}(G)$. Then there exists a prime Q of RT, minimal over $P \cap RT$, such that*

$$i_G(\mathrm{cont}_T(Q)) = i_G(W).$$

Proof. Let $N = i_G(L^+)$, $A = N_G(N)$, so $T \subseteq A$ and L^A is a prime of RA, ((14), theorem 2·4). Now $L^+ \lhd H$ and H/L^+ is finite-by-abelian. Hence, putting $i_H(W) = \overline{W}$, then $N \subseteq \overline{W} \subseteq H$, and $H/\overline{W}$ is torsion-free abelian. Let $E = H \cap T$, and let J be a prime of RE minimal over the H-prime ideal $L \cap RE$.

By ((14), theorem 2·4(iii)), there exists a prime Q of RT minimal over $P \cap RT$ with $J^T = Q$. By Lemma 4·4(ii), $J = [J \cap R(T \cap \overline{W})]RE$. Set $I = J \cap R(T \cap \overline{W})$. Then $Q = J^T = (I^E)^T = I^T$. Since T normalizes N, it normalizes $H = \delta_G(N)$, and so $E \lhd T$. Thus $\mathrm{cont}_T(Q) \subseteq E \subseteq H$. Also, since $Q = I^T$, $Y = \mathrm{cont}_T(Q) \subseteq i_T(T \cap \overline{W})$, since the latter subgroup is also normal in T. Therefore, $|\langle Y, \overline{W}\rangle : \overline{W}| < \infty$, and since $H/\overline{W}$ is torsion-free, $Y \subseteq \overline{W}$.

Let $\overline{Y} = i_H(\mathrm{cont}_T(Q))$, so that $\overline{Y} \subseteq \overline{W}$. By ((14), theorem 2·4(i), (iii)),

$$N = i_G(Q^+) = i_H(Q^+),$$

so that $N \subseteq \overline{Y}$. Also, by Lemma 4·4(i), $\overline{Y} \cap E$ controls each of the minimal primes over $Q \cap RE$, since $Y \subseteq \overline{Y} \cap E$; in particular $\overline{Y} \cap E$ controls J. By Lemma 4·4(iii), $\overline{Y}$ controls L. Thus $\overline{W} \subseteq \overline{Y}$, and so $\overline{Y} = \overline{W}$. Applying i_G now yields the lemma.

The next result is a consequence of the extension by Passman and Lorenz (14) of the main result of (19). It answers a question of Roseblade ((19), page 418).

THEOREM 4·6. *Let H be an isolated orbital subgroup of G, and let L be an orbital prime ideal of RH. Then L^G is prime.*

Proof. By ((14), theorem 2·3), there exists an H-orbital subgroup W of H, and an H-orbital prime ideal K of RW, such that $L = K^H$ and $W = \delta_H(K^+)$. Let $N = i_H(K^+)$, so that $N = i_G(N)$, since H is isolated, and $N \lhd W$, with W/N abelian, by ((19), §7·1). Let $A = N_G(N)$, so $W \lhd A$. Since L is G-orbital, so are K, K^+, and $i_G(K^+) = N$, so that $|G : A| < \infty$. Fix a subgroup M of finite index in K^+, with M characteristic in N, so that $M \lhd A$, and let an overbar denote images in A/M and $R(A/M)$.

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Now $\bar{K}$ is an almost faithful $\bar{A}$-orbital prime of $R\overline{W}$. Since $\overline{W} \lhd \bar{A}$ and $\overline{W}$ is finite-by-abelian, ((19), theorem D) asserts that

$$\bar{K} = (\bar{K} \cap R\bar{O})\, R\overline{W},$$

where $\bar{O} = \Delta(\bar{A}) \cap \overline{W}$. Since $\bar{K} \cap R\bar{O}$ is $\overline{W}$-prime, this implies that

$$K = (K \cap RO)\, RW = IW,$$

for all primes I of RO minimal over $K \cap RO$.

Let $U = \delta_G(M) = \delta_G(N)$. Since $O = \delta_A(M) \cap W$, we have $O \subseteq U \cap H$. On the other hand, $U \cap H \subseteq W$, and $N \lhd \delta_G(N)$, so $\delta_G(N) = \delta_A(N)$. Thus,

$$U \cap H = \delta_G(N) \cap H = \delta_A(M) \cap H \subseteq \delta_A(M) \cap W = 0.$$

That is, $O = U \cap H$. Now $N \subseteq U \cap H$, so that, since N is isolated, $U \cap H \lhd U$, and $U/U \cap H$ is abelian. Let $T/U \cap H$ be the torsion subgroup of $U/U \cap H$, so that T is orbital. By ((19), lemma 1), $|\langle T, H\rangle : H| < \infty$, and $\langle T, H\rangle$ is orbital, so that $T \subseteq H$. Hence, U/O is torsion-free abelian.

Let I be any prime of RO minimal over $K \cap RO$. Then I^U is prime, by ((19), lemma 9). Note that $K^+ \subseteq I^+ \subseteq N$, the second inclusion holding since $\cap_{w \in W} (I^+)^w \subseteq K^+$, and W/N is abelian. Therefore, since $N \lhd U$ and $|N : K^+| < \infty$, $|N : (I^U)^+| < \infty$. Thus, by ((14), theorem 2·4), $(I^U)^G$ is a prime ideal of RG. But

$$(I^U)^G = I^G = (I^W)^G = K^G = (K^H)^G = L^G,$$

and so the theorem is proved.

We can now prove the main result of this section.

THEOREM 4·7. *Let P be a prime ideal of RG, and let $T = \mathrm{nio}(G)$.*

(i) *The following are equivalent*:

(*a*) $H \in \mathscr{S}_P$;

(*b*) $H = i_G(W)$, *where W is the controller of a source I of P*;

(*c*) $H = i_G(Y)$, *where Y is the controller of a prime Q of RT, minimal over $P \cap RT$.*

(ii) $\mathscr{S}_P$ *is non-empty; it is a single conjugacy class of isolated orbital subgroups of G.*

(iii) $\mathscr{T}_P = \{L^g : g \in G\}$, *where L is a G-orbital prime of RH, $H \in \mathscr{S}_P$, and $L^G = P$.*

(iv) *If $G = T$ is orbitally sound, P has a unique inductor H, and $\mathscr{T}_P$ consists of those prime ideals of RH minimal over $P \cap RH$.*

Proof. (ii) Since the primes Q in (i)(*c*) form a conjugacy class, (ii) will follow from (i).

(i) We begin by showing that, if H satisfies (*b*), then H has properties (i), (ii) and (iii) of an inductor. Let I be a prime ideal of RV, say, so that V is orbital and $W \lhd V$. Let $L_1, \ldots, L_n$ be the (necessarily orbital) primes of RH minimal over $(I \cap RW)^H$. Then

$$P = I^G = (I \cap RW)^G = ((I \cap RW)^H)^G \subseteq L_i^G,$$

for $1 \leqslant i \leqslant n$. By Lemma 4·2(ii) and Lemma 4·3, $k\text{-dim}\,(RG/P) = k\text{-dim}\,(RG/L_i^G)$, for all i. Hence, by ((9), proposition 6·1), $P = L_i^G$, for $1 \leqslant i \leqslant n$. Now $I^+ = (I \cap RW)^+$, so

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$W \subseteq \delta_G(I^+)$. Let W_0 be the intersection of the H-conjugates of W. Then $|H:W_0| < \infty$, and $W_0 \subseteq \delta_G((I^+ \cap W_0)^h)$, for all $h \in H$. Hence $W_0 \subseteq \delta_G(\cap_{h\in H}(I^+ \cap W_0)^h)$. Since

$$\bigcap_{h\in H}(I^+ \cap W_0)^h \subseteq L_i^+, \quad H \subseteq i_G\delta_G(L_i^+),$$

for $1 \leqslant i \leqslant n$. Thus the pair L_1, H satisfies hypotheses (i), (ii) and (iii) for an inductor.

Next, we show that (*) if $H \in \mathscr{S}_P$, there exists Y as in (c) with $i_G(Y) \subseteq H$. This will complete the proof of (i). For let $H \in \mathscr{S}_P$, and let Y be given by (*). By Lemma 4·5, $i_G(Y) = i_G(W)$ for some W satisfying (b). Thus, by the minimality of H and the previous paragraph, $H = i_G(Y) = i_G(W)$. Therefore, H satisfies (a) implies H satisfies (b), and H satisfies (b) implies H satisfies (c), by Lemma 4·5. If H satisfies (c), then H contains $X \in \mathscr{S}_P$, by Lemma 4·5 and the paragraph above. We have just observed that X must satisfy (c), so X and H are conjugate, as seen above. This can only happen if $X = H$, so $H \in \mathscr{S}_P$, as required.

It remains to prove (*). Let L be a prime of RH, where $H \in \mathscr{S}_P$, such that L is a subsource for P. Let $B = H \cap T$, so that $B = i_T(B)$, and B is normal in T, since T is orbitally sound. Let $J_1, \ldots, J_n$ be the prime ideals of RB minimal over $L \cap RB$. By Theorem 4·6 (or ((19), corollary 20)), J_i^T is prime for $1 \leqslant i \leqslant n$. Some finite product $\Pi_j J_{i_j}$ of the J_i's lies in $L \cap RB$, and so, by Lemma 4·1,

$$\prod_j J_{i_j}^T \subseteq (\prod_j J_{i_j})^T \subseteq (L \cap RB)^T. \tag{1}$$

Now $(L \cap RB)^T + (L^G \cap RT) \subseteq LRG \cap RT$, and so

$$\begin{aligned} k\text{-dim}_{RT}(RT/(L \cap RB)^T + (L^G \cap RT)) &\geqslant k\text{-dim}_{RT}(RT/LRG \cap RT) \\ &= k\text{-dim}_{RG}(RG/LRG) \\ &= k\text{-dim}_{RG}(RG/L^G) \\ &= k\text{-dim}_{RT}(RT/L^G \cap RT), \end{aligned}$$

by Lemma 4·2(i). Since $L^G \cap RT$ is semiprime, there exists a prime Q of RT, minimal over $L^G \cap RT$, such that $(L \cap RB)^T \subseteq Q$, by Goldie's theorem and ((9), proposition 6·1). Therefore (1) implies that J_1^T, say, lies in Q. It follows easily from Lemmas 4·2 and 4·3 that $k\text{-dim}_{RT}(RT/J_1^T) = k\text{-dim}_{RT}(RT/Q)$, so $J_1^T = Q$, by ((9), proposition 6·1). Thus $\text{cont}_T(Q) \subseteq B \subseteq i_G(B) = i_G(H)$, and so (*) is proved.

(iii) If L is a subsource of P, so, clearly, is any G-conjugate of L. For the converse, let L_1 and L_2 be subsources of P. By (ii), it may be assumed that L_1 and L_2 are ideals of RH, where $H \in \mathscr{S}_P$. Let $A = N_G(H)$, and let $P \cap RT = \cap_{g\in G} Q^g$, where Q is chosen so that $H = i_G(Y)$ where $Y = \text{cont}_T(Q)$, by (i). Let $C = N_G(Q)$. Then C normalizes Y, so $C \subseteq A$. Put $D = H \cap T$.

We claim that $L_1^A \cap RT = \cap_{a\in A} Q^a$. The proof of (*) in (i) shows that there exists a prime ideal J of RD, minimal over $L_1 \cap RD$, with $J^T = Q$. Now $L_1 \cap RD$ is H-prime and J has finitely many H-conjugates, so

$$\Pi_{h\in H}(J^h)^T \subseteq (\Pi_{h\in H} J^h)^T \subseteq (L_1 \cap RD)^T, \tag{2}$$

by Lemma 4·1. Now $D = i_T(D)$ is normal in T, so, for $h \in H$,

$$(J^h)^T = (\bigcap_{t\in T} J^{ht})RT = [(\bigcap_{t\in T} J^t)RT]^h = Q^h. \tag{3}$$

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Also,

$$\bigcap_{a\in A}[(L_1\cap RD)^T]^a \subseteq L_1^A\cap RT. \tag{4}$$

It follows from (2), (3) and (4) that some product of A-conjugates of Q lies in $L_1^A\cap RT$. If X is a minimal prime over $L_1^A\cap RT$, then $Q^a\subseteq X$ for some $a\in A$. By Lemmas 4·2 and 4·3,

$$k\text{-dim}\,(RT/Q^a) = k\text{-dim}\,(RT/X), \quad \text{and so} \quad X = Q^a.$$

By Theorem 4·6, L_1^A is prime, so $L_1^A\cap RT$ is A-prime, and thus

$$L_1^A\cap RT = \bigcap_{a\in A} X^a = \bigcap_{a\in A} Q^a,$$

as claimed. Similarly, $L_2^A\cap RT = \bigcap_{a\in A} Q^a$.

It now follows from ((14), theorem 1·7), with RT playing the role of R, that $L_1^A = L_2^A$. Since $L_i^A = (\bigcap_{a\in A} L_i^a)RA$, $i = 1, 2$, we have

$$\bigcap_{a\in A} L_1^a = \bigcap_{a\in A} L_2^a.$$

But these are finite intersections of prime ideals, and so $L_1^a = L_2$, for some $a\in A$.

(iv) In this case, isolated orbital subgroups of G are normal, so the result follows from (i), (ii) and (iii).

Note that if L is a G-orbital prime ideal of RH, where H is isolated and orbital in G, $H\subseteq i_G\delta_G(L^+)$, and $i_H(\text{cont}_H(L)) = H$, then L^G is prime; this is a special case of Theorem 4·6. In this case, $H\in\mathscr{S}_P$ and $L\in\mathscr{T}_P$.

Remark 4·8. Much of this section, in particular Theorems 4·6 and 4·7, is valid with an arbitrary commutative Noetherian ring as coefficient ring. This is clear in the case of Theorem 4·6, as its proof does not depend on Lemma 4·3, the only place where the assumption that R is absolutely Hilbert is used. To prove Theorem 4·7 for an arbitrary coefficient ring R, we may assume without loss that R is a domain and $P\cap R = 0$, then pass to a suitable finitely generated subfield K of the quotient field of R, and finally to an absolutely Hilbert order in K. No doubt, this cumbersome method of proof could be avoided by devising, with a little extra care, a proof of 4·7 which did not involve the use of Krull dimension.

5. *The existence of links*

Our objective in this section is Theorem 5·13. For convenience, the proof is split into two parts.

(*a*) *Sufficient conditions for a link*

The problem here is to show that links in inductors 'lift' to links in RG. Two situations must be considered separately: firstly, the case of a normal inductor and, secondly, a step from RA to RG, where $|G:A| < \infty$.

The first lemma is in essence contained in (5).

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LEMMA 5·1. *Let H be subnormal in G, and let X be an essential submodule of the RH-module U. Then, as RG-modules, $X \otimes RG$ is essential in $U \otimes RG$.*

Proof. Clearly it may be assumed that $H \lhd G$. Then, as RH-modules,

$$U \otimes RG \cong \sum_i{}^{\oplus} U \otimes g_i, \qquad X \otimes RG \cong \sum_i{}^{\oplus} X \otimes g_i,$$

where $\{g_i\}$ is a transversal to H in G. For all i, $X \otimes g_i$ is RH-essential in $U \otimes g_i$, so $X \otimes RG$ is RH-essential in $U \otimes RG$. *A fortiori*, it is RG-essential.

LEMMA 5·2. *Let P and Q be prime ideals of RG, such that there exists $H \lhd G$, and G-orbital prime ideals K and L of RH, with $K^G = P$, $L^G = Q$. If $K \rightsquigarrow L$, then $P \rightsquigarrow Q$.*

Proof. Let $X \subseteq U$ be a pair of RH-modules realizing the link $K \rightsquigarrow L$. Then $\overline{X} = X \otimes RG$ is essential in $\overline{U} = U \otimes RG$, by Lemma 5·1. By ((14), lemma 3·3), $r_{RG}(\overline{X}) = P$ and $r_{RG}(\overline{U}/\overline{X}) = Q$. Also, $\overline{X}$ embeds in RG/KG, and $\overline{U}/\overline{X}$ in RG/LG. Since $k\text{-dim}\,(RG/P) = k\text{-dim}\,(RG/KG)$ and $k\text{-dim}\,(RG/Q) = k\text{-dim}\,(RG/LG)$, $\overline{U}$ contains a pair of submodules realizing the link $P \rightsquigarrow Q$, as claimed.

Before dealing with finite extensions, we give two subsidiary lemmas. The first is a consequence of Theorem 2·2, (i) $\Leftrightarrow$ (iii). For $n \geqslant 1$, $M_n(S)$ denotes the ring of $n \times n$ matrices over the ring S.

LEMMA 5·3. *Let $n \geqslant 1$, let S be a weakly K-symmetric smooth w.i.i. Noetherian ring, and let K, L be prime ideals of S, with $K \rightsquigarrow L$. Then $M_n(K) \rightsquigarrow M_n(L)$ in $M_n(S)$.*

The next lemma slightly generalizes ((14), lemma 1·5(iii)). Of course, the proof is well-nigh identical, so we omit it.

LEMMA 5·4. *Let $S * G$ be a crossed product, where G is a finite group and S is a G-simple semisimple Artinian ring with orthogonal centrally primitive idempotents $e_1, \ldots, e_n$, with $1 = e_1 + \ldots + e_n$. Let $H = C_G(e_1)$, and let $H \subseteq A$ be a subgroup of G. Then $S * G \cong M_t(W)$ where $t = |G:A|$, and where $W \cong eS * Ge = eS * A$, for an idempotent e which is a sum of some of the e_i.*

LEMMA 5·5. *Let A be a subgroup of finite index in G, and K and L be prime ideals of RA, with $K^G = P$ and $L^G = Q$ both prime. Let $T = \text{nio}\,(G)$, and suppose that A contains the normalizer of the primes of RT minimal over $P \cap RT$ and over $Q \cap RT$. If $K \rightsquigarrow L$, then $P \rightsquigarrow Q$.*

Proof. Let $X \subseteq U$ be a pair of RA-modules realizing $K \rightsquigarrow L$. Then

$$XG = X \otimes RG \subseteq U \otimes RG = UG,$$

and UG is an $RG/(LK)^G$-module. Let

$$Y = \{\alpha \in UG : \alpha P = 0\}, \quad \text{so} \quad XG \subseteq Y.$$

Suppose first that Y is essential in UG. Now $r_{RG}(UG/Y) \supseteq L^G = Q$. If $r(UG/Y) = Q$ and $k\text{-dim}\,(UG/Y) = k\text{-dim}\,(RG/Q)$, then UG contains a submodule realizing a link $P \rightsquigarrow Q$. Otherwise, $k\text{-dim}\,(UG/Y) < k\text{-dim}\,(RG/Q) = k\text{-dim}\,(RG/P) = \alpha$, say, by Theorem A and Lemma 4·3. Now UG is K-homogeneous of dimension α, but, since P is w.i.i. ((4), theorem 3·13),

$$k\text{-dim}\,((UG)P) = k\text{-dim}\,((UG)P/YP) < \alpha,$$

so $(UG)P = 0$. Thus $P \cap RT \subseteq r_{RT}(U)$.

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Suppose now that Y is not essential in UG. Let $\{1 = g_1, g_2, \ldots, g_n\}$ be a right transversal to A in G, and let $0 \neq u = \Sigma_{i=1}^{n} u_i g_i \in UG$, with $uRG \cap Y = 0$. There exists j, $1 \leqslant j \leqslant n$, such that $u_j \notin X$; clearly we may assume that $j = 1$. Thus $r_{RA}(u_1 RA) = I$, say, where I is a K-primary ideal of RA, and $LK \subseteq I$. Note that $UI = 0$, by weak ideal invariance of I. Now uRG embeds in a factor of UG/XG, so $r_{RG}(uRG) \supseteq Q$. Since UG is K-homogeneous, it follows that $r_{RG}(uRG) = Q$. But $uRA \subseteq u_1 RA \oplus \Sigma_{i=2}^{n} u_i g_i RA$ as RA-modules, so $Q \cap RA \subseteq I = r_{RA}(u_1 RA)$.

Now $T = \mathrm{nio}(G) \subseteq A$. The previous two paragraphs show that if UG does not contain a submodule realizing a link $P \rightsquigarrow Q$, then either $r_{RT}(U) \supseteq P \cap RT$, or $r_{RT}(U) \supseteq Q \cap RT$. Thus, there is a link $K/JA \rightsquigarrow L/JA$ in RA/JA, where J is a G-prime ideal of RT. By ((14), lemma 1·5(i)), if we set $\bar{R} = RT/J$, so that

$$RG/JG \cong \bar{R} * (G/T),$$

then $\bar{R} * (A/T)$ and $\bar{R} * (G/T)$ have Artinian quotient rings Q_1 and Q_2, obtained by inverting the regular elements of $\bar{R}$. Let $\bar{K}, \bar{L}$ be the images of K and L in $\bar{R} * (A/T)$, so $\bar{K} \rightsquigarrow \bar{L}$, and hence $\bar{K}Q_1 \rightsquigarrow \bar{L}Q_1$ in Q_1, as in ((15), proposition 2).

Note that if W is a minimal prime of $\bar{R}$, WQ_1 will be generated by $(1-f)$ for a (unique) centrally primitive idempotent f of the quotient ring $Q(\bar{R})$ of $\bar{R}$. Since $N_G(W) \subseteq A$, $C_G(f) \subseteq A$. By Lemma 5·4,

$$Q_2 \cong M_q(S), \tag{5}$$

where $q = |G:A|$, and there is an idempotent e of $Q(\bar{R})$ with $A \subseteq C_G(e)$ such that

$$S \cong eQ_2 e = eQ(\bar{R}) * (A/T) = eQ_1. \tag{6}$$

Moreover, since $\bar{R}$ is G-prime, we can if necessary conjugate by an element of G to ensure that $(1-e) \in \bar{K}Q_1$. Since $(1-e)Q_1$ has the AR property, by ((17), theorem 11·2·2), $(1-e) \in \bar{L}Q_1$, by Lemma 2·3. Thus $\bar{K}Q_1 = e\bar{K}Q_1 + (1-e)Q_1$ and

$$\bar{L}Q_1 = e\bar{L}Q_1 + (1-e)Q_1.$$

Let $E \subseteq V$ be a realization of the link $\bar{K}Q_1 \rightsquigarrow \bar{L}Q_1$. Then $V(1-e) = V(1-e)^2 = 0$, and so $e\bar{K}Q_1 \rightsquigarrow e\bar{L}Q_1$ in the ring eQ_1. It follows from Lemma 5·3 and the isomorphisms (5) and (6) that there is a link between the corresponding ideals of Q_2. But the proof of ((14), theorem 1·7, step 1) shows that these are $(\bar{K}Q_1)^{G/T}$ and $(\bar{L}Q_1)^{G/T}$ respectively. Passing back from Q_2 to RG/JG, as in ((15), proposition 2), we obtain a link $P \rightsquigarrow Q$, as desired.

THEOREM 5·6. *Let P and Q be primes of RG, such that*

(i) $\mathscr{S}_P = \mathscr{S}_Q$;

(ii) *there exist $H \in \mathscr{S}_P$ and ideals K, L of RH, with $K \in \mathscr{T}_P$, $L \in \mathscr{T}_Q$, and $K \rightsquigarrow L$.*

Then $P \rightsquigarrow Q$.

Proof. Let $A = N_G(H)$. Then K^A and L^A are prime by Theorem 4·6. Thus $K^A \rightsquigarrow L^A$ by Lemma 5·2. Let I be a minimal prime over $P \cap RT$, where $T = \mathrm{nio}(G)$. Then $H = i_G(\mathrm{cont}(I))$, so $N_G(\mathrm{cont}(I)) \subseteq A$. Since $N_G(I) \subseteq N_G(\mathrm{cont}((I))$, $N_G(I) \subseteq A$. Similarly, $N_G(J) \subseteq A$, where J is a prime of RT minimal over $Q \cap RT$. Therefore Lemma 5·5 applies, yielding $P \rightsquigarrow Q$.

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(*b*) *Necessary conditions for a link*

We approach the result via several lemmas.

LEMMA 5·7 ((5), § 2). *Let H be a subgroup of finite index in a group T, and let S be a ring. If E is an injective SH-module, then $E \otimes ST$ is an injective ST-module.*

LEMMA 5·8. *Let H be normal in G, with $|G:H| < \infty$, and let P and Q be primes of RG with $P \rightsquigarrow Q$. Then either (a) $P \cap RH = Q \cap RH$, or (b) there exist prime ideals I and J of RH, minimal over $P \cap RH$ and $Q \cap RH$ respectively, with $I \rightsquigarrow J$.*

Proof. Let $E = E_{RH}(RH/P \cap RH) \otimes RG$, so E is RG-injective by Lemma 5·7. Now $RH/P \cap RH$ is semiprime, so the classical quotient ring S of $RH/P \cap RH$ exists and is semisimple Artinian, by Goldie's theorem. Hence $S \otimes RG$ is the classical quotient ring of $RG/(P \cap RH)RG$, by ((23), lemma 2·6), and $S \otimes RG$ is quasi-Frobenius, by ((18), lemma 4). This implies that $P/(P \cap RH)RG$ is a right annihilator prime of $RG/(P \cap RH)RG$, ((24), page 276). Thus $E_{RG}(U) \subseteq E$, where U is a uniform right ideal of RG/P.

Put $X = \{x \in E(RH/P \cap RH) : x(P \cap RH)\} = 0$, and let Y/X be the sum of all the critical submodules of $E(RH/P \cap RH)/X$, so that X is essential in $E(RH/P \cap RH)$, and Y/X is essential in $E(RH/P \cap RH)/X$. By Lemma 5·1, YG/XG is essential in E/XG. Let $P_1, P_2, \ldots$ be the prime ideals of RH such that $I \rightsquigarrow P_i$, as I ranges through the primes minimal over $P \cap RH$. Then Y/X is a direct sum of RH/P_i-modules, for $i = 1, 2, \ldots$, and so YG/XG is a direct sum of RG/P_i^G-modules, for various i.

Let $A \subseteq B$ be a pair of RG-modules realizing the link $P \rightsquigarrow Q$. Then A embeds in U, so A, and hence B, embed in the injective module E. If $B \subseteq XG$, then $B(P \cap RH)RG = 0$, so that $P \cap RH \subseteq Q \cap RH$. Since $k\text{-dim}\,(RG/P) = k\text{-dim}\,(RG/Q)$, this implies that (*a*) holds. For if $P \cap RH \neq Q \cap RH$, then since both these ideals are G-prime, $Q \cap RH \cap \mathscr{C}_{RH}(P \cap RH) \neq \varnothing$, contradicting Lemma 4·2(i). Otherwise, $B \not\subseteq XG$. Now $XG = \{e \in E : e(P \cap RH) = 0\}$. For, if $e \in E$ and $e(P \cap RH)RG = 0$, then $e = e_1 + \ldots + e_n$, where $e_i \in E(RH/P \cap RH) \otimes g_i$ and $g_1, \ldots, g_n$ is a transversal to H in G. Then $e_i(P \cap RH) = 0$, since H is normal, so that $e_i \in X \otimes g_i$. Thus, $A \subseteq XG$, and so $(B + XG/XG)Q = 0$. If $r(B + XG/XG) \supsetneq Q$, then weak ideal invariance of $(P \cap RH)RG$ would yield $B(P \cap RH)RG = 0$, a contradiction. Thus, since YG/XG is essential, it follows that $P_i^G \subseteq Q$ for some i. Now $P_i^G = (\cap_{g \in G} P_i^g)RG$. Also,

$$k\text{-dim}\,(RG/Q) = k\text{-dim}\,(RG/P_i^G), \quad \text{so} \quad Q \cap RH = \cap_{g \in G} P_i^g;$$

that is, (*b*) is true.

The first consequence of Lemma 5·8 is

COROLLARY 5·9. *Let P and Q be primes of RG such that $P \rightsquigarrow Q$. Then $\mathscr{S}_P = \mathscr{S}_Q$.*

Proof. Apply Lemma 5·8 with $H = \text{nio}\,(G)$, so that either (*a*) or (*b*) holds. If (*a*), then $\mathscr{S}_P = \mathscr{S}_Q$ by Theorem 4·7(i), $(a) \Leftrightarrow (c)$. Suppose that (*b*) holds, so there exist primes I and J or RH minimal over $P \cap RH$ and $Q \cap RH$ respectively, with $I \rightsquigarrow J$. Let

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$T = i_H(\mathrm{cont}_H(I))$, so that $T \lhd H$. By Theorem 3·4, $J = (J \cap RT)RH$, so that $i_H(\mathrm{cont}_H(J)) \subseteq T$. By Theorem 2·2, $J \underset{l}{\rightsquigarrow} I$, so we deduce as above that

$$T \subseteq i_H(\mathrm{cont}_H(J)).$$

Hence, equality holds, and the result now follows from Theorem 4·7 (i).

The next part of the proof is conveniently expressed in terms of the socle series of a module, which will also prove useful in §7. Let M be an RG-module with an essential submodule U which is a sum of type I critical modules of dimension α. Define the *α-socle series of M* as in ((11), §4), by first setting $\mathrm{soc}_n^\alpha(M) = 0$. For $n \geqslant 1$, let

$$Y/\mathrm{soc}_{n-1}^\alpha(M) \subseteq \mathrm{soc}_n^\alpha(M)/\mathrm{soc}_{n-1}^\alpha(M)$$

be the sum of all the finitely generated α-critical submodules of $M/\mathrm{soc}_{n-1}^\alpha(M)$. Then $\mathrm{soc}_n^\alpha(M) = \{m \in M : k\text{-dim}\,(mRG + Y/Y) < \alpha\}$. As noted in ((1), §5), the arguments of ((11), §4) carry over to show that $M = \cup_n \mathrm{soc}_n^\alpha(M)$, and the critical submodules of the factors $\mathrm{soc}_n^\alpha(M)/\mathrm{soc}_{n-1}^\alpha(M)$ are all of type I. For $n \geqslant 1$, the various prime annihilators $P_1^n, P_2^n, \ldots$ of these critical factors satisfy the following condition: If P_j^n is one such ideal (and $n > 1$), there exists an annihilator P_i^{n-1} such that $P_i^{n-1} \rightsquigarrow P_j^n$, as is easily deduced from ((11), proposition 4·2).

Assume now that A is a finitely generated submodule of M. Then, for each n, the set $\{P_1^n, P_2^n, \ldots, P_{t(n)}^n\}$ of annihilators of critical submodules of $\mathrm{soc}_n^\alpha(A)/\mathrm{soc}_{n-1}^\alpha(A)$ is finite. Set $S_n = \cap_i P_i^n$, so RG/S_n is a semiprime K-homogeneous ring of dimension α, for all $n \geqslant 1$. Note that $\mathrm{soc}_n^\alpha(A)[S_n S_{n-1} \ldots S_1] = 0$. For, if $Y/\mathrm{soc}_{n-1}^\alpha(A)$ is the sum of the critical submodules of $\mathrm{soc}_n^\alpha(A)/\mathrm{soc}_{n-1}^\alpha(A)$, then $YS_n \subseteq \mathrm{soc}_{n-1}^\alpha(A)$. Since $\mathrm{soc}_n^\alpha(A)/Y$ has dimension less than α, weak ideal invariance shows that $\mathrm{soc}_n^\alpha(A)S_n \subseteq \mathrm{soc}_{n-1}^\alpha(A)$. It is now possible to give a more natural definition of the α-socle series of M:

LEMMA 5·10. *In the above notation,* $\mathrm{soc}_n^\alpha(M) = \{m \in M : r(mRG)$ *contains the product of n semiprime ideals with K-homogeneous factor rings of dimension α*$\}$.

Proof. By ((11), proposition 4·2), $\mathrm{soc}_n^\alpha(A) = A \cap \mathrm{soc}_n^\alpha(M)$ for all $n \geqslant 1$ and all submodules A of M. It is thus enough to prove the lemma for M finitely generated. We have already shown that $\mathrm{soc}_n^\alpha(M)$ is contained in the right-hand module. To prove the reverse inclusion, argue by induction on n.

LEMMA 5·11. *Let P and Q be primes of RG, with $P \rightsquigarrow Q$. Let A be a subgroup of G which contains the normalizer of an inductor H of P. Then there exist primes K and L of RA, with $K^G = P$, $L^G = Q$, and $K \rightsquigarrow L$.*

Proof. Let I be a subsource of P in RH, so $I^A = F$ is prime by Theorem 4·6. Let $E = E_{RA}(RA/F)$, so $E \otimes RG$ is RG-injective, by Lemma 5·7. Let $X \subseteq U$ be a realization of $P \rightsquigarrow Q$. Since $(RA/F) \otimes RG = RG/FG$ has annihilator $F^G = P$, and

$$k\text{-dim}\,(RG/FG) = k\text{-dim}\,(RG/P) = \alpha,$$

by Lemma 4·3, it follows that $E \otimes RG$ contains a copy of $E_{RG}(X)$. Hence, U embeds in $E \otimes RG$. Let $M \subseteq E$ be a finitely generated RA-module with $U \subseteq M \otimes RG$.

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Now $k\text{-dim}_{RA}(M) = k\text{-dim}_{RG}(M \otimes RG) = \alpha$, and M is K-homogeneous. Let

$$0 = Y_0 \subseteq Y_1 \subseteq \ldots \subseteq Y_n = M$$

be the α-socle series of M. By definition, M/Y_i is K-homogeneous and has dimension α, $0 \leqslant i < n$. By ((22), lemma 8),

$$M \otimes RG/Y_i \otimes RG \quad \text{is also } K\text{-homogeneous of dimension } \alpha. \tag{7}$$

If $X \nsubseteq Y_1 \otimes RG$, then $X \cap (Y_1 \otimes RG) = 0$; for, otherwise, $X/X \cap (Y_1 \otimes RG)$ is non-zero and has dimension less than α, since X is critical, contradicting (7). If

$$X \cap (Y_1 \otimes RG) = 0,$$

then $U \cap (Y_1 \otimes RG) = 0$, and we replace M by M/Y_1. Thus, we may assume now that $X \subseteq Y_1 \otimes RG$ (although of course we no longer know that $r(Y_1 \otimes RG) = F^G = P$).

Suppose $X \subsetneqq U \cap (Y_1 \otimes RG)$; we shall obtain a contradiction. Now U/X is α-critical, so $k\text{-dim}\,(U + (Y_1 \otimes RG)/Y_1 \otimes RG) < \alpha$. By (7), $U \subseteq Y_1 \otimes RG$. Let

$$S = r(Y_1 \otimes RG);$$

we claim that S is semiprime and that $k\text{-dim}\,(RG/S) = \alpha$. By Lemma 5·10,

$$r_{RA}(Y_1) = P_1 \cap \ldots \cap P_t = W,$$

say, where P_i is a prime ideal of RA, $1 \leqslant i \leqslant t$. By construction, exist links $F \rightsquigarrow \rightsquigarrow P_i$ for all $i = 1, \ldots, t$, and so H is an inductor for each P_i, by Corollary 5·9. Hence, by Theorem 4·6, P_i^G is prime, $1 \leqslant i \leqslant t$, since H is isolated in G. By ((14), lemma 3·3), $S = W^G = P_1^G \cap \ldots \cap P_t^G$, so the claim is proved, noting that

$$k\text{-dim}\,(RG/P_i^G) = k\text{-dim}\,(RA/P_i) = k\text{-dim}\,(RA/F) = \alpha,$$

for all i, by Theorem A. Hence, $S \subseteq r(U)$. This contradicts the fact that $X \subseteq U$ realizes the link $P \rightsquigarrow Q$. Therefore, $X = U \cap (Y_1 \otimes RG)$.

Let $1 = g_1, g_2, \ldots, g_r$ be a right transversal to A in G, and let

$$\pi: M \otimes RG \to M: \sum_{i=1}^{r} m_i \otimes g_i \mapsto m_1,$$

so π is an RA-homomorphism. Let $u \in U \setminus (Y_1 \otimes RG)$, say $u = \Sigma_i v_i \otimes g_i$, where $v_i \in M$. Since $\pi(X) \subseteq Y_1$, at least one of the v_i is not in $\pi(X)$; so, replacing u by ug ($g \in G$) if necessary, we may assume $v_1 \notin \pi(X)$. Indeed, multiplying u successively by $g\gamma$, for various $g \in G$ and $\gamma \in RA$, we may arrange matters so that there is a prime ideal L of RA such that $v_i L \subseteq \pi(X)$, $1 \leqslant i \leqslant r$, and

$$r_{RA}(v_1 RA + \pi(X)/\pi(X)) = L. \tag{8}$$

Then,

$$\begin{aligned}(uRG)L^G = (\sum_i v_i \otimes g_i)L^G \subseteq (\sum_i v_i \otimes g_i L^{g_i G}) \cap U \\ \subseteq U \cap (\pi(X) \otimes RG) \subseteq (Y_1 \otimes RG) \cap U = X.\end{aligned} \tag{9}$$

Hence, $k\text{-dim}\,(RG/L^G) = k\text{-dim}\,(RA/L) \geqslant \alpha$. But, from (8), $k\text{-dim}\,(RA/L) \leqslant \alpha$. Thus, $k\text{-dim}\,(RA/L) = \alpha$, and L is the annihilator of a critical composition factor of M.

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Therefore, $F \rightsquigarrow \rightsquigarrow L$, so L^G is prime, by Theorem 4·6. It follows from (9) that $L^G = Q$.

Since $U \cap (Y_1 \otimes RG) = X$, there exists i such that $v_i \notin Y_1$, and we can replace u by ug_i^{-1} to ensure that $v_1 \notin Y_1$. (This does not affect (8); for we still have $v_1 L \subseteq \pi(X)$, and if $L \subseteq T \lhd RA$ with $v_1 T \subseteq \pi(X)$, then $k\text{-dim}\,(RA/T) < \alpha$, since L is prime, and $v_1 T \subseteq Y_1$, contradicting the fact that M/Y_1 is α-homogeneous.)

Consider now the submodule $v_1 RA$ of M. It has submodules

$$v_1 RA \cap \pi(X) = D \subseteq W = v_1 RA \cap Y_1,$$

and we have just seen that

$$r(v_1 RA/D) = r(v_1 RA/W) = L.$$

Now $D \subseteq Y_1$, so $r(D) = K_1 \cap \ldots \cap K_s$, say, where the K_j are prime ideals with inductor H. Thus, RG/K_j^G is a prime ring of dimension α, $1 \leqslant j \leqslant s$. Since $r(X) = P$, it follows easily that K_j^G must equal P, $1 \leqslant j \leqslant s$. Now,

$$(v_1 RA)\, L(K_1 \cap \ldots \cap K_s) = 0.$$

Since Y_1 is essential in M, W is essential in $(v_1 RA)$. Thus, for some j, $(v_1 RA)$ has a factor realizing a link $K_j \rightsquigarrow L$. Setting $K_j = K$ gives the required prime ideals.

We can now prove the main result of part (b).

THEOREM 5·12. *Let P and Q be primes of RG, with $P \rightsquigarrow Q$, and let $H \in \mathscr{S}_P = \mathscr{S}_Q$. There exist primes K and L of RH, with $K \in \mathscr{T}_P$, $L \in \mathscr{T}_Q$, such that $K \rightsquigarrow L$.*

Proof. Let $A = N_G(H)$. By the previous lemma, there exist primes I and J of RA with $I^G = P$, $J^G = Q$, and $I \rightsquigarrow J$. It's easy to see that $\{H\} = \mathscr{S}_I$, so $H = \mathscr{S}_J$ by Corollary 5·9. Thus $I = (I \cap RH)\,RA$ and $J = (J \cap RH)\,RA$. Let $k\text{-dim}\,(RA/I) = \alpha$, and $h(A/H) = n$. Then $k\text{-dim}\,(I \cap J/JI) = \alpha$, by Theorem 2·2. By ((22), lemma 10),

$$k\text{-dim}_{RH}((I \cap RH) \cap (J \cap RH)/(J \cap RH)\,(I \cap RH)) = \alpha - n = k\text{-dim}(RH/I \cap RH).$$

The proof of Theorem 2·2(iii) $\Rightarrow$ (i) can easily be adapted to show that there exist primes K and L of RH, minimal over $I \cap RH$ and $J \cap RH$ respectively, such that $K \rightsquigarrow L$. Since $K \in \mathscr{T}_I$, and $K^G = (K^A)^G = I^G = P$, $K \in \mathscr{T}_P$. Similarly, $L \in \mathscr{T}_Q$.

Combining the results of parts (a) and (b), we get

THEOREM 5·13. *Let P and Q be prime ideals of RG. The following are equivalent:*

(i) $P \rightsquigarrow Q$;

(ii) (a) $\mathscr{S}_P = \mathscr{S}_Q$;

(b) *there exist $H \in \mathscr{S}_P$ and ideals K and L of RH, with $K \in \mathscr{T}_P$, $L \in \mathscr{T}_Q$, and $K \rightsquigarrow L$;*

(iii) (a) $\mathscr{S}_P = \mathscr{S}_Q$;

(b) *for all $H \in \mathscr{S}_P$ and all ideals K of RH with $K \in \mathscr{T}_P$, there exists $L \in \mathscr{T}_Q$, L an ideal of RH, with $K \rightsquigarrow L$.*

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Proof. (i) $\Rightarrow$ (ii) (*a*) Corollary 5·9.

(*b*) Theorem 5·12.

(ii) $\Rightarrow$ (iii) Suppose (ii) holds. If $g \in G$, then from Theorem 2·2 it is clear that $K^g \rightsquigarrow L^g$ in RH^g. Thus (iii) follows from Theorem 4·7.

(iii) $\Rightarrow$ (i) Theorem 5·6.

A *vertex* of a prime P of RG is the isolator in G of L^+, where L is a source of P, (14). Although primes in the same connected component of the graph of links have the same set of inductors, their vertices need not be the same. This is illustrated by the following example. (See, however, part (v) of the result which follows it.)

EXAMPLE 5·14. Let G have a normal plinth A of Hirsch number greater than one, with G/A infinite cyclic. Let $P = \mathfrak{g}$, the augmentation ideal of $\mathbb{Z}G$. By (7), there exist infinitely many distinct primes Q for which there is a path $P \rightsquigarrow P_1 \rightsquigarrow \ldots \rightsquigarrow P_n = Q$. As noted at the end of §2, $\mathbb{Z}G/Q$ is a finitely generated abelian group for any such prime Q. Now $Q = P$ is one such prime. Suppose that $P \rightsquigarrow Q$, and $Q \neq P$. Since the augmentation ideal $\mathfrak{a}G$ has the AR property, ((17), theorem 11·2·9), $\mathfrak{a}G \subseteq Q$, by Lemma 2·3, so that $A \subseteq Q^+$. Suppose that $A \neq Q^+ = H$, so $|G:H| < \infty$. Let $X \subseteq U$ be a realization of $P \rightsquigarrow Q$. Let $\mathfrak{h}$ denote the augmentation ideal of $\mathbb{Z}H$. Then $\mathfrak{h}G \subseteq Q \cap P$, so $(\mathfrak{h}G)^2 \subseteq r(U)$. Let $T = H'$, so T has finite index in A, since A is a plinth and $H \lhd G$. Then $\mathfrak{t}G \subseteq (\mathfrak{h}G)^2 \subseteq r(U)$. Hence, $U \otimes \mathbb{Q}$ is a uniform $\mathbb{Q}G/\mathfrak{t}\mathbb{Q}G$-module, with a non-zero submodule killed by $P\mathbb{Q}/\mathfrak{t}\mathbb{Q}G$. But $\mathbb{Q}G/\mathfrak{t}\mathbb{Q}G \cong \mathbb{Q}(G/T)$, so $P\mathbb{Q}/\mathfrak{t}\mathbb{Q}G$ has the AR property, by (17), theorem 11·2·4). Hence $P^n \subseteq r(U \otimes \mathbb{Q})$, for some n and, since $U \subseteq U \otimes \mathbb{Q}$, it follows that $P \subseteq Q$. By theorem A, $P = Q$, a contradiction. Hence $Q^+ = A$ for all primes $Q \neq P$ such that $P \rightsquigarrow Q$. In this case, P has vertex G, while Q has vertex A.

The last result of this section shows that linked primes have many properties in common.

THEOREM 5·15. *Let P and Q be primes of RG, let $H \in \mathscr{S}_P$, and let K and L be subsources of P and Q respectively, in RH. Suppose $P \rightsquigarrow Q$, and that L is chosen so that $K \rightsquigarrow L$.*

(i) *Let $R/P \cap R$ have characteristic* (*a*) 0 *or* (*b*) $p > 0$, *and let T be a normal subgroup of H which is* (*a*) *finite-by-nilpotent or* (*b*) *p-nilpotent. Then $K \cap RT = L \cap RT$.*

(ii) *k-dim $(RH/K) = k$-dim (RH/L), and this number is the maximal length of a chain of primes in RH/K (provided k-dim$_R(R/P \cap R) < \infty$).*

(iii) *If $R = \mathbb{Z}$ or R is a field, then* ht(P) = ht(Q), *and* ht(K) = ht(L), *where* ht(I) *denotes the height of the prime I.*

(iv) *K^+ and L^+ have the same Fitting subgroup.*

(v) *If $R/P \cap R$ has positive characteristic, then $K^+ \cap L^+$ has finite index in both K^+ and L^+.*

Proof. (i) Let S be the localization of R at $P \cap R$. If $X \subseteq U$ is a realization of $K \rightsquigarrow L$, then U is $\mathscr{C}_R(P \cap R)$-torsion free, so $X \otimes S \subseteq U \otimes S$ realizes the link $KS \rightsquigarrow LS$ in SH. By (*a*) ((17), corollary 11·3·12) or (*b*) (20), $K \cap ST$ has the AR property. In case (*b*)

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we may assume that $|H:T| < \infty$, by ((17), lemma 11·2·16). Hence, $(K \cap ST)SH$ has the AR property, by (*a*) ((17), theorem 11·2·9) or (*b*) ((14), lemma 4·1). By Lemma 2·3, $(K \cap ST)SH \subseteq LS$ and $K \cap R = L \cap R$, so $LS \cap RH = L$ and thus $(K \cap RT)RH \subseteq L$. Therefore $K \cap RT \subseteq L \cap RT$. Since $L \underset{i}{\rightsquigarrow} K$ by Theorem 2·2, the reverse inclusion follows similarly.

(ii) The first part is given by Theorem A. Now H/K^+ and H/L^+ are abelian-by-finite groups, by property (iii) of inductors and ((17), lemma 4·1·5). Thus RH/K and RH/L are P.I. rings by ((17), lemma 5·1·11). For such rings, k-dim and the classical Krull dimension are equal, ((9), theorem 8·12).

(iii) The proof of ((14), theorem IV (ii)) may be used to show that

$$\mathrm{ht}(P) = \epsilon + p_A(H) - k\text{-dim}(RH/K),$$

where $A = N_G(H)$, $p_A(H)$ is the A-plinth length of H, (defined in [(19), §2·3]), and $\epsilon = 0$ if R is a field, $\epsilon = 1$ if $R = \mathbb{Z}$. Thus, (iii) follows from (ii).

(iv) This follows from (i).

(v) This is proved in the same way as (i), using ((17), theorem 11·2·15 and lemma 11·2·16).

It seems plausible that in (i) above T need only be an orbital subgroup of H, if the conclusion is restated as $(K \cap RT)^H = (L \cap RT)^H$. In (iii), it is likely that it is only necessary to assume that R is absolutely Hilbert and catenary – but it seems pointless to get bogged down in the details of arcane coefficient rings.

6. *The structure of the graph of links*

What are the possibilities for a connected component of the graph of links of RG? First, consider the case of a prime P for which there are *no* links $P \rightsquigarrow Q$. This amounts to saying that $E_{RG}(RG/P)$ is simply the quotient ring $Q(RG/P)$. In this case ((17), lemma 3·2·5) shows that $P \cap \mathscr{C}_{RG}(0) = \varnothing$. Since RG has an Artinian quotient ring, (23), P must be a minimal prime of RG, ((8), 2·7 and 2·17). Hence, P has inductor $H = i_G(1)$, ((19), corollary 22) and, if K is a subsource of P, then $(*)K \not\rightsquigarrow L$ for all primes L of RH. Let k be the quotient field of R, and note that $P \cap R = 0$. Then $(*)$ holds if and only if kH/Kk is an injective kH-module; that is, if and only if Kk is idempotently generated.

Somewhat more generally, what if there is possibly a link $P \rightsquigarrow P$, but no other? By a result of Jategaonkar (11), this happens if and only if $\mathscr{C}_{RG}(P) = \mathscr{C}$ satisfies the right Ore condition, and $P_{\mathscr{C}}$ has the right AR property in $RG_{\mathscr{C}}$. In particular, we have the following sufficient criteria:

THEOREM 6·1. *Let P be a prime of RG, with $H \in \mathscr{S}_P$. Then P is a disconnected point of the graph of links of* Spec (RG) *provided either*

(i) *$R/P \cap R$ has characteristic zero and H is finite-by-nilpotent, or*

(ii) *$R/P \cap R$ has characteristic $p > 0$ and H is p-nilpotent.*

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Proof. (i) This follows from Theorem 5·15(i).

(ii) Adapt (20), along the lines of the proof of Theorem 5·15(v), to show that all the ideals of RH have the AR property. Thus, $K \rightsquigarrow L$ in RH only if $K = L$.

It seems probable that, in (ii) above, P actually has the AR property, and this is doubtless not hard to confirm. (It is certainly so if H is subnormal in G.) Modular group algebras of finite groups can easily be provided to show that Theorem 6·1(ii) does not afford a necessary condition for P to be a disconnected point. Perhaps, however, when $R/P \cap R$ has characteristic 0, (i) does give a necessary and sufficient condition?

If $P \in \mathrm{Spec}\,(RG)$ is *not* a disconnected point, the next possibility is that the connected component containing P is finite. This amounts to saying that P belongs to a *clan* (15), as can be proved by the method of ((15), theorem 5).

THEOREM 6·2. *A prime P of RG belongs to a clan provided either* (i) *$R/P \cap R$ has positive characteristic p; or* (ii) *P has a nilpotent-by-finite inductor H.*

Proof. (i) There is a p-nilpotent normal subgroup A of finite index in G, ((17), lemma 11·2·16). Then $(P \cap RA)RG$ has the AR property, by the proof of Theorem 6·1(ii) and ((4), lemma 4·1). By Lemma 2·3, every prime Q in the same component as P contains $(P \cap RA)RG$. If Q is such a prime, $k\text{-dim}\,(RG/P) = k\text{-dim}\,(RG/Q)$, by Theorem A, so Q is minimal over $(P \cap RA)RG$, by Lemma 4·2.

(ii) Let K be a subsource of P in RH, and let N be a nilpotent normal subgroup of finite index in H. As above, $(K \cap RN)RH$ has the AR property and hence K lies in a finite component of the graph of links of RH. Thus Theorem 5·13 gives the result.

Remark 6·3. The sufficient conditions given in Theorem 6·2 may actually be necessary. This is certainly so if P has a subsource K in RH such that

$$\dim_k ((RH/K) \otimes k) < \infty,$$

where k is the quotient field of $R/K \cap R$. For in (7) it is shown that, if V is a finite-dimensional kH-module, where k is a field of characteristic zero and H is polycyclic-by-finite but not nilpotent-by-finite, then $E_{kH}(V)$ has infinitely many distinct isomorphism types of simple kH-module appearing in its socle series. Set

$$V = (RH/K) \otimes k,$$

and note that

$$E_{kH}(V) = E_{(R/K \cap R)H}(RH/K) \subseteq E_{RH}(RH/K),$$

to confirm the claim.

Consider now a prime P in an infinite connected component of the graph of links. Here, a degree of finiteness still remains:

THEOREM 6·4. *If P is a prime of RG, there are only finitely many primes Q such that $P \rightsquigarrow Q$, and only finitely many primes I such that $I \rightsquigarrow P$.*

Proof. By Theorem 2·2, we need only prove the first statement. By Theorem 5·13, it may be assumed that P is its own subsource. Suppose first that P^+ is finite. Then $G = i_G\delta_G(\{1\})$, so G is abelian-by-finite ((17), lemma 4·1·5). Therefore P is in a finite

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component, by Theorem 6·2, Now suppose that P^+ is infinite, so it contains an infinite abelian normal subgroup A. By ((17), theorem 11·2·9), the augmentation ideal $\mathfrak{a}G$ has the AR property. Let Q be a prime with $P \rightsquigarrow Q$, so, by Lemma 2·3, $\mathfrak{a}G \subseteq Q$. Let overbar denote images in $RG/\mathfrak{a}G = \overline{RG}$. By induction on the Hirsch number, we can assume that the theorem is true in $\overline{RG}$, and so it may be assumed that $\bar{P} \not\rightsquigarrow \bar{Q}$. By Theorem 2·2, $Q \underset{l}{\rightsquigarrow} P$, but $\bar{Q} \underset{l}{\not\rightsquigarrow} \bar{P}$.

Let $E'_S(M)$ denote the injective hull of a *left* S-module M. Let

$$U' = \{e \in E'_{RG}(RG/Q) : (\mathfrak{a}G)e = 0\}.$$

Then

$$U' = E'_{\overline{RG}}(RG/Q) \cong \mathrm{Hom}_{RG}(\overline{RG}, E').$$

The exact sequence of left modules $0 \to \mathfrak{a}G \to RG \to \overline{RG} \to 0$ yields, on applying $\mathrm{Hom}_{RG}(\ \ , E')$, a sequence

$$0 \to U' \to E' \to \mathrm{Hom}_{RG}(\mathfrak{a}G, E') \to 0,$$

which is exact since E' is injective.

Let $X \subseteq Y$ be a realization of the link $Q \underset{l}{\rightsquigarrow} P$, so that $Y \subseteq E'$. Since $\bar{Q} \underset{l}{\not\rightsquigarrow} \bar{P}$, we cannot have $Y \subseteq U'$. Thus, since Y/X embeds in each of its non-zero submodules ((1), lemma 5·1), $Y \cap U' = X$. Therefore Y/X embeds in $E'/U' = \mathrm{Hom}_{RG}(\mathfrak{a}G, E')$, and so there exists $0 \neq f \in \mathrm{Hom}_{RG}(\mathfrak{a}G, E')$ with $Pf = 0$. That is, $(\mathfrak{a}GP)f = 0$, so there is a non-zero homomorphism from $T = {}_{RG}|(\mathfrak{a}G/\mathfrak{a}GP)$ to E'. Hence, there is a subfactor of T isomorphic to a uniform left ideal of RG/Q. By Lemma 2·1 and Theorem A,

$$\begin{aligned} l\text{-dim}\,(\mathfrak{a}G/\mathfrak{a}GP) = k\text{-dim}\,(\mathfrak{a}G/\mathfrak{a}GP) &\leqslant k\text{-dim}\,(RG/P) \\ &= k\text{-dim}\,(RG/Q) = l\text{-dim}\,(RG/Q). \end{aligned}$$

If this inequality is strict, then no such Q exists, while if equality holds then there are finitely many possibilities for Q, namely the annihilators of the critical composition factors of ${}_{RG}|(\mathfrak{a}G/\mathfrak{a}GP)$. This completes the proof.

Theorem 6·3 is not really a result about group rings; in fact, essentially the same proof works for any Noetherian poly-AR ring satisfying the hypotheses of Theorem 2·2.

7. *The structure of injective RG-modules*

Since injective RG-modules are direct sums of uniform injectives ((24), proposition V·4·5), in describing injective hulls of type I RG-modules it is enough to treat the case of $E = E_{RG}(U)$, where U is a uniform right ideal of a prime factor ring of RG, of dimension α, say. We can interpret Theorem 6·4 in terms of the α-socle series of E, defined before Lemma 5·10. Theorem 6·4 implies that *the set of annihilator primes of* $\mathrm{soc}^\alpha_n(E)/\mathrm{soc}^\alpha_{n-1}(E)$ *is finite, for all* $n \geqslant 1$. List them as $\{P^n_1, \ldots, P^n_{t(n)}\}$, (so $t(1) = 1$ and $P^1_1 = r(U)$), and put $S_n = \cap_i P^n_i$. Lemma 5·10 shows that

$$\mathrm{soc}^\alpha_n(E) = \{e \in E : eS_nS_{n-1}\ldots S_1 = 0\}, \quad \text{for all } n \geqslant 1.$$

In discussing further the global structure of E, it is easiest to consider first a special case which always pertains when G is orbitally sound. Recall that a ring S is *P-primary* if there is a unique prime ideal P of S such that $l(P) \neq 0$.

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THEOREM 7·1. *Let P be a prime ideal of RG with a unique inductor H, and let U be a uniform right ideal of RG/P, with $E = E_{RG}(U)$. Let K be a subsource of P, and let U_0 be a uniform right ideal of RH/K. Let M be a finitely generated submodule of E, with α-socle series*

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M,$$

where $r(M_i/M_{i-1}) = S_i = \cap_j P_{ij}$, $1 \leqslant i \leqslant n$ (so $P = S_1 = P_{11}$), as above. Let

$$X = \{e \in E : eS_nS_{n-1}\ldots S_1 = 0\}, \quad \text{and let} \quad r(X) = I.$$

(i) *$I = (I \cap RH)RG$, and RG/I has an Artinian quotient ring Q.*

(ii) *Let $X_0 = E_{RH/I\cap RH}(U_0)$; then $X_0 \otimes_{RH} Q$ is a finite direct sum of copies of X, (as RG-modules and as Q-modules).*

(iii) *X is a finitely generated Q-module.*

(iv) *If G/H is poly-(infinite cyclic), then $X = X_0 \otimes_{RH} Q$.*

Proof. By Theorem 5·13, H is the inductor of P_{ij}, for all i, j, and so H controls each of $S_1, \ldots, S_n$. Let $K_i = S_i \cap RH$, $1 \leqslant i \leqslant n$, so that $K_1 = \bigcap_{g\in G} K^g$. Let A be a normal subgroup of finite index in G such that A normalizes K, $H \subseteq A$, and A/H is poly-(infinite cyclic). Let $k\text{-dim}\,(RG/I) = k\text{-dim}\,(M) = \alpha$, say. Now RG/I is K-homogeneous, since finitely generated submodules of X are, and is P-primary, since P is the only annihilator prime of a submodule of X. Therefore, RG/I has an Artinian quotient ring Q ((13), theorem 8) and ((4), theorem 3·13).

Let $h(A/H) = t$, so that $k\text{-dim}\,(RH/K_i) = \alpha - t = \beta$, say, for $1 \leqslant i \leqslant n$, by Lemma 4·3. Since $K_nK_{n-1}\ldots K_1 \subseteq I \cap RH \subseteq \cap_i K_i$, $k\text{-dim}\,(RH/I \cap RH) \leqslant \beta$. Indeed, if $Y/I \cap RH$ is a non-zero right ideal of $RH/I \cap RH$, then $k\text{-dim}\,(Y/I \cap RH) = \beta$. For otherwise, $k\text{-dim}_{RG}(YG + I/I) < \alpha$, by Lemma 4·3, a contradiction. That is, $RH/I \cap RH$ is K-homogeneous, and so it has an Artinian quotient ring Q_0, by (13) and (4) again.

Now X_0 is an injective Q_0-module ((10), theorem 2·3). Let $\bar{X} = X_0 \otimes RA$, so $\bar{X}$ is an essential extension of U_0A, by Lemma 5·1. Also, K^A is prime, by Theorem 4·6, and U_0A is uniform, as in ((3), lemma 2·5), and isomorphic to a right ideal of RA/K^A. Let $RA/(I \cap RH)RA = \overline{RA}$. Thus $\bar{X} \subseteq E_{\overline{RA}}(U_0A) = D$.

Now $\overline{RA}$ has a partial quotient ring $Q_0 \otimes RA$, and so has an Artinian quotient ring S (12). Note that D is an S-injective module (10). We claim that $\bar{X} \otimes S = D$. To prove this amounts to showing that $D/\bar{X}$ is $\mathscr{C}_{\overline{RA}}(0)$-torsion. Let $\{t_i : i \in I\}$ be a transversal to H in A. Since A normalizes K, $X_0 \otimes t_i$ is the $(RH/I \cap RH)$-injective hull of U_0, and the Q_0-injective hull of U_0Q_0, for all $i \in I$. By ((24), proposition V·4·3), $\bar{X} = \Sigma_i^{\oplus} X_0 \otimes t_i$ is injective as a Q_0-module. If $D/\bar{X}$ is not $\mathscr{C}_{\overline{RA}}(0)$-torsion, it has a uniform submodule $Y/\bar{X}$ whose annihilator is a minimal prime of $\overline{RA}$, that is, a minimal prime $\bar{C}^A$ over one of $\overline{K_1A}, \ldots, \overline{K_nA}$, where C is a prime of RH minimal over K_j, say. Since all the critical composition factors of $E_{RA}(U_0A)$ are type I, by Theorem A, $Y/\bar{X}$ must be isomorphic to a uniform right ideal of RA/C^A. If W/C is a uniform right ideal of RH/C, then WA/CA is uniform, as in ((3), lemma 2·5), and so subisomorphic to $Y/\bar{X}$ ((1),

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lemma 5·1). We may thus assume that $Y/\overline{X} = W \otimes RA$. Now $Q_0 \otimes RA$ is a free right and left Q_0-module, so, by ((6), proposition 4·1·3),

$$\mathrm{Ext}^1_{Q_0 \otimes RA}(WQ_0 \otimes RA,\ X_0 \otimes RA) = \mathrm{Ext}^1_{Q_0}(WQ,\ X_0 \otimes RA). \tag{1}$$

Since $\overline{X} = X_0 \otimes RA$ is Q_0-injective, the right-hand side of (1) is zero. Thus so is the left-hand side, and the claim is proved. So far, we have shown that

$$E_{\overline{RA}}(U_0 A) = E_{RH/I \cap RH}(U_0) \otimes S.$$

Hence,

$$E_{\overline{RA}}(U_0 A) \otimes RG = E_{RH/I \cap RH}(U_0) \otimes S \otimes RG. \tag{2}$$

By Lemma 5·7, the left-hand side of (2) is an injective $\overline{RG}$-module, where

$$\overline{RG} = RG/(I \cap RH)RG.$$

By Lemma 5·1, it contains the essential submodule $U_0 G$, whose annihilator is $K^G = P$. Since $k\text{-dim}(U_0 G) = k\text{-dim}(U)$, $U_0 G$ contains an essential direct sum of copies of U, and so

$$T = E_{\overline{RA}}(U_0 A) \otimes RG \cong \underbrace{E_{\overline{RG}}(U) \oplus \ldots \oplus E_{\overline{RG}}(U)}_{m}, \tag{3}$$

where $m \leqslant |G:A|$.

Let $L = r_{RH}(E_{RH/I \cap RH}(U_0))$, so $I \cap RH \subseteq L$. Thus $r_{RG}(T) = L^G$, by (2) and ((14), lemma 3·3). Note that $(I \cap RH)RG \subseteq L^G$. Now T, being $\overline{RG}$-injective, contains a copy of the RG/I-injective hull of U, namely X. Since $r(X) = I$, $L^G \subseteq I$. However, since $X = \{e \in E: eS_n \ldots S_1 = 0\}$, and $S_n \ldots S_1 \subseteq (I \cap RH)RG$, $T \subseteq \underbrace{X \oplus \ldots \oplus X}_{m}$. Hence $I \subseteq L^G$. It follows that $L^G = I$. Since $L^G = (\cap_g L^g)RG$, I is controlled by RH, so $I = (I \cap RH)RG$. This proves (i), noting that $Q = S \otimes RG$. Now (ii) follows from (2) and (3), since $\overline{RG} = RG/I$.

For (iii), observe that the simple images of Q_0 are the simple factors of the quotient rings of RH/K_i, $1 \leqslant i \leqslant n$. Since H is an inductor of all the P_{ij}, by Theorem 5·13, H/K_i^+ is abelian-by-finite, so that each simple image of $Q(RH/K_i)$ is finite-dimensional over a (commutative) subfield, for $1 \leqslant i \leqslant n$. That is, each such simple image is an Artinian P.I. ring, and so is finitely generated over its centre, ((17), theorem 5·3·4). By ((21), theorem 3), therefore, X_0 has finite length as a Q_0-module, so that $X_0 \otimes Q$ has finite length as a Q-module. Hence, (iii) follows from (ii).

Finally, the fact that $U_0 G$ is a uniform RG-module under hypothesis (iv), as noted in ((3), lemma 2·5), implies that in this case $m = 1$ in (3).

Theorem 7·1 gives a recipe for constructing injective hulls of type I RG-modules. Given a type I critical module U, let $P = r(U)$, and let $K \lhd RH$ be a subsource of P. Suppose first that H is normal. Let U_0 be a uniform right ideal of RH/K, and view $E_{RH}(U_0)$ as a direct limit of the modules $U_I = E_{RH/I}(U_0)$, as I ranges through the G-invariant K-primary ideals of RH. Then the direct limit of the RG-modules $U_I \otimes_{RH} Q_I$, where Q_I is the Artinian quotient ring of RG/IG, is just a direct sum of finitely many copies of $E_{RG}(U)$. It is obvious from Theorem 7·1 how to define the connecting monomorphisms in the direct limit.

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Now drop the assumption that H is normal, and set $A = N_G(H)$. Let U_1 be a uniform right ideal of the prime ring RA/K^A, and construct $E_{RA}(U_1)$ as above. Then

$$E' = E_{RA}(U_1) \otimes RG$$

is RG-injective, by Lemma 5·1, and contains $U_1 \otimes RG$, whose annihilator is P ((14), lemma 3·3). Thus $E_{RG}(U)$ is a direct summand of E'. (In fact, we conjecture that E' is a (finite) direct sum of copies of $E_{RG}(U)$. This is certainly so when A is subnormal in G.)

It follows that, 'locally', injective hulls of type I RG-modules resemble modules over P.I. rings, since the K-primary rings RH/I occurring above are P.I. One consequence of this is Theorem 7·1 (iii). Viewed globally, this yields the following result, whose validity might have been guessed from the fact that injective hulls of simple RG-modules are Artinian (16).

COROLLARY 7·2. *Let W be a finitely generated type I RG-module, with $k\text{-dim}(W) = \alpha$, and set $E = E_{RG}(W)$. For all $n \geqslant 1$, $\text{soc}_n^\alpha(E)/\text{soc}_{n-1}^\alpha(E)$ has finite uniform dimension, and $\text{soc}_n^\beta(E) = 0$ if $\beta > \alpha$.*

Proof. First, $\text{soc}_n^\beta(E) = 0$ if $\beta > \alpha$ by Theorem A. It is clearly enough to treat the case where W is α-critical. Observe that the corollary amounts to saying that there exists an Artinian ring Q_n, the quotient ring of a factor ring of RG, over which $\text{soc}_n^\alpha(E)$ is finitely generated. Thus, in the light of the discussion preceding Theorem 7·1, the case when G is orbitally sound follows from Theorem 7·1 (iii), on choosing an appropriate submodule M of $\text{soc}_n^\alpha(E)$.

In general, assume W is critical, $r(W) = P$, and let $T = \text{nio}(G)$. By ((18), lemma 4), $RG/(P \cap RT)RG$ is a quasi-Frobenius ring, so $P/(P \cap RT)RG$ is an annihilator prime of $RG/(P \cap RT)RG$ ((24), page 276). Hence, E is a summand of

$$\bar{E} = E_{RT}(RT/P \cap RT) \otimes RG = E_0 \otimes RG.$$

By ((11), proposition 4·2), $\text{soc}_n^\alpha(E) = \text{soc}_n^\alpha(\bar{E}) \cap E$, $n \geqslant 1$. By Lemma 5·10

$$\text{soc}_n^\alpha(\bar{E}) \subseteq \text{soc}_n^\alpha(E_0) \otimes RG, \quad n \geqslant 1.$$

From the previous paragraph, $\text{soc}_n^\alpha(E_0)$ is a finitely generated Q_n-module, where $Q_n = Q(RT/I_n)$, say. Now I_n is K-homogeneous and G-orbital, so $Q_n' = Q(RT/\cap_g I_n^g)$ exists and is Artinian ((13) and (4), theorem 3·13). Since $\text{soc}_n^\alpha(E_0) \otimes RG$ is a finitely generated module over the Artinian ring $Q_n' \otimes RG$ (the quotient ring of RG/I_n^G), so are $\text{soc}_n^\alpha(\bar{E})$ and $\text{soc}_n^\alpha(E)$, as required.

It should not be thought that the local resemblance between injective type I RG-modules and injective modules over Noetherian P.I. rings leads in all respects to a global resemblance, as in the above result. For example, Bruno Müller has shown that if M is an α-critical module over a Noetherian P.I. ring, then only finitely many prime ideals appear as annihilators of critical composition factors of $E(M)$ (despite the fact that $r(M)$ may belong to an infinite component of the graph of links). Contrast this with Remark 6·3.

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