Robotica (2009) volume 27, pp. 103–117. © 2008 Cambridge University Press
doi:10.1017/S0263574708004554 Printed in the United Kingdom

Uncoupled actuation of overconstrained 3T-1R hybrid parallel manipulators

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(Received in Final Form: March 19, 2008. First published online: April 24, 2008)

SUMMARY

Based on the Lie-group-algebraic properties of the displacement set and intrinsic coordinate-free geometry, several novel 4-dof overconstrained hybrid parallel manipulators (HPMs) with uncoupled actuation of three spatial translations and one rotation (3T-1R) are proposed. In these HPMs, three limbs are those of Cartesian translational parallel mechanisms (CTPMs) and the fourth limb includes an Oldham-type constant velocity shaft coupling (CVSC). The Lie subgroup of Schoenflies (X) displacements of the displacement Lie group and its mechanical generators with nine categories of their general architectures are recalled. A comprehensive enumeration of all possible Oldham-type CVSC limbs is derived from X-motion generators. Their constant velocity (CV) transmissions are verified by group-algebraic approach. Then, combining one CTPM and one CVSC, we synthesize a lot of uncoupled 3T-1R overconstrained HPMs, which are classified into nine distinct classes of general architectures. In addition, all possible architectures with at least one hinged parallelogram or with one cylindrical pair are disclosed too. At last, related non-overconstrained HPMs are attained by the addition of one idle pair in each limb of the previous HPMs.

KEYWORDS: Lie group; Uncoupled actuation; Overconstrained; Hybrid manipulator; Oldham coupling; Constant velocity shaft coupling; Schoenflies motion; Schönflies

Nomenclature

M, M', O, A, B, Q, ...:	The italic capital letters designate geometric points that are elements of the Euclidean affine three-dimensional (3D) space.
T, **R** :	The bold-faced letters are employed to designate geometric vectors belonging to the Euclidean vector 3D space which is associated to the affine 3D space.
u, **v**, **w**, **i**, **j**, **k**, **s**:	The bold-faced small letters are unit vectors.
$(O, \mathbf{u}, \mathbf{v}, \mathbf{w})$:	An orthonormal frame of reference of the Euclidean affine 3D space.
$(A, \mathbf{k})$, $(M, \mathbf{u})$, $(Q, \mathbf{u})$:	Frames of reference of affine 1D subspaces of the Euclidean affine 3D space; these frames characterize axes of rotation.
$(\mathbf{OM})$:	Position vector of M. It represents the translation providing M from O and $M = O +$ $(\mathbf{OM})$ or $M - O = (\mathbf{OM})$.
$\mathbf{u}\times$:	Skew-symmetric operator of the vector product by **u**.
a, b, c:	Real parameters.
θ, α, β:	Greek letters designate angles of rotation.
$\exp(\theta\mathbf{u}\times)$:	The exponential series of the linear operator $(\theta\mathbf{u}\times)$; $(\mathbf{OM}') = \exp(\theta\mathbf{u}\times)(\mathbf{OM})$ $= (\mathbf{OM}) + \theta\mathbf{u}\times(\mathbf{OM}) +$ $(\theta^2/2)(\mathbf{u}\times)^2(\mathbf{OM}) + \cdots +$ $(\theta^n/n!)(\mathbf{u}\times)^n(\mathbf{OM}) + \cdots$ $= (\mathbf{OM}) + \sin\theta\mathbf{u}\times(\mathbf{OM}) +$ $(1 - \cos\theta)(\mathbf{u}\times)^2(\mathbf{OM})$ which is the modern form of the Rodrigues formula expressing the rotation of the vector $(\mathbf{OM})$ as well as the rotation of point M around the axis $(O, \mathbf{u})$. Any point M becomes another point denoted M' under a point transformation, which may be the foregoing rotation.[8]
$\{\boldsymbol{R}(A, \mathbf{k})\}$:	The set of rotations around a given axis $(A, \mathbf{k})$; this set is a 1D Lie subgroup of the displacement Lie group $\{\boldsymbol{D}\}$. A rotation of $\{\boldsymbol{R}(A,\boldsymbol{k})\}$ is expressed by $(\mathbf{AM}') = \exp(\alpha\mathbf{k}\times)(\mathbf{AM})$.

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https://doi.org/10.1017/S0263574708004554 Published online by Cambridge University Press

$\{\boldsymbol{D}\}$: The 6D displacement Lie group; in any frame of reference, the displacement group is represented by a matrix group usually denoted $SE(3)$.[37,38]

$M'=O+\mathbf{T}+\exp(\mathbf{R}\times)(\mathbf{OM})$: Coordinate-free expression of a general displacement $\boldsymbol{D}$. It is a point transformation: M becomes M' by $\boldsymbol{D}$ or $M \xrightarrow{D} M'$. One can consider $\boldsymbol{D}$ to be an operator acting on points: $M' = \boldsymbol{D}M$. The product of two displacements is defined by the succession of two point transforms: $M \xrightarrow{D1} M'$ is followed by $M' \xrightarrow{D2} M"$. The transform $\boldsymbol{P}$ of $M \xrightarrow{P} M"$ is the product $\boldsymbol{P} = \boldsymbol{D}_2\boldsymbol{D}_1$. When $\mathbf{R}$ is restricted to be the zero vector, the displacements are the point transformations: $M' = O + \mathbf{T} + (\mathbf{OM}) = M + \mathbf{T}$, which are called translations. When $\mathbf{T}$ can be any geometric vector, the set of translations is the 3D Lie subgroup $\{\boldsymbol{T}\}$ of $\{\boldsymbol{D}\}$. The comprehensive list of the Lie subgroups of the Lie group of displacements has been published several times[7–12,38] using the same notation, for instance, in Selig's book.[38] Using a frame of reference, any subgroup of displacements is represented by a subgroup of matrices acting on coordinate arrays. Selig classified the conjugacy classes of matrix subgroups of $SE(3)$.

$\{\boldsymbol{T}\}$: The set of the spatial translations; it is the spatial translation Lie group.

$\{\boldsymbol{X}(\mathbf{u})\}$: The set of spatial translations and rotations around any axis parallel to a given unit vector $\mathbf{u}$; this set is a 4D Lie subgroup of the displacement Lie group $\{\boldsymbol{D}\}$. In the paper, the meaning of $\{\boldsymbol{X}(\mathbf{u})\}$ is emphasized.

$\{\boldsymbol{H}(N, \mathbf{u}, p)\}$: The set of helical motions around a given axis $(N, \mathbf{u})$ with a given pitch p; this set is a 1D Lie subgroup of the displacement Lie group $\{\boldsymbol{D}\}$. A helical motion belonging to $\{\boldsymbol{H}(N, \mathbf{u}, p)\}$ is expressed by $(\mathbf{NM}') = p(\beta/2\pi)\mathbf{u} + \exp(\beta\mathbf{u}\times)(\mathbf{NM})$.

$\{\boldsymbol{T}(\mathbf{i})\}$, $\{\boldsymbol{T}(\mathbf{j})\}$: The 1D Lie groups of rectilinear translations parallel to $\mathbf{i}$ and $\mathbf{j}$, respectively.

$\{\boldsymbol{T}(\perp\mathbf{k})\}$: The 2D Lie group of planar translations parallel to a plane perpendicular to $\mathbf{k}$.

$\{\boldsymbol{G}(\mathbf{k})\}$: The 3D Lie group of planar gliding motions parallel to a plane perpendicular to $\mathbf{k}$.

$\{\boldsymbol{L}(1, 2)\}$: The set of feasible displacements of a rigid body (2) with respect to another body (1) in a kinematic chain and it is called *kinematic bond* between the bodies. All subsets of displacements are conventionally indicated with curly brackets in order to facilitate the understanding by a special notation characterizing a type of mathematical entity.

$\{\boldsymbol{G}_1\}$, $\{\boldsymbol{G}_2\}$: The displacement subsets produced by two serial chains between particular links of a single loop chain.

$\{\boldsymbol{E}\}$: The improper subgroup of $\{\boldsymbol{D}\}$ containing only the identity transformation $\boldsymbol{E}$.

1. Introduction

The uncoupled 4-dof (degree of freedom) hybrid parallel manipulators (HPMs) proposed in the paper produces end-effector motion having 3 dofs of translation and 1 dof of rotation, which is often called a (3T-1R) motion. The 3T-1R motion of the end effector is obtained by directly controlling the four fixed actuators in a fully uncoupled manner. These manipulators work like a classical serial machine with the (X, Y, Z, θ) motion. The 3-dof translational motion is achieved by Cartesian translational parallel mechanisms (CTPMs),[1–6] which are actuated in a fully uncoupled manner by three orthogonal fixed prismatic pairs. The independent control of the end-effector rotation around a movable axis keeping parallel to a fixed direction is obtained by means of a fourth limb, which includes a constant velocity shaft coupling (CVSC) implemented between two parallel revolute joints. Primitive generators of Schoenflies (X) motion[7–12] realize the constant velocity transmission. This kind of CVSC is based on the transmission principle of Oldham-style coupling.

It is worth noting that a 3T-1R motion is a Schoenflies motion. *Schoenflies* is also spelt *Schönflies*.[13–15] Hence, in other words, the paper deals with a special category of Schoenflies motion generators, which are kinematic chains having a special hybrid topology. One may notice that Schoenflies motion generators, which are realized by

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a fully parallel arrangement of limbs, are described in recent publications, for instance by Gogu[16] and Kong and Gosselin.[17]

The Schoenflies motion generators that are serial open chains without internal passive mobility can be regarded as the primitive generators of X motion; they have many applications. For instance, each joint of a primitive generator of X motion can be actuated thus making up a serial 4-dof manipulator often called a SCARA robot.[18] The use of this kind of generators to synthesize pure translational parallel manipulators is disclosed.[4–6,18,19] The special case of decoupled 3-dof translational parallel manipulators is also published.[1–3] A new application of the primitive generators of X motion is the synthesis of a family of CVSCs for parallel axes and their implementation in special hybrid 4-dof manipulators. Till now, several types of 4-dof parallel manipulators have been proposed and studied.[20–35] In our paper, we focus on the type synthesis of overconstrained 3T-1R hybrid manipulators with fully uncoupled actuation of the 4-dof motion of the end effector. Our systematic method is based on Lie-group-algebraic properties of the displacement set and intrinsic coordinate-free geometry.[36]

In what follows, we begin with the Schoenflies displacement Lie group and its primitive mechanical generators. In a second step, we focus on the synthesis of Oldham-style CVSC chains that are essential subchains of the fourth limbs. Algebraic method is employed to verify the constant velocity (CV) transmission between two parallel R axes, and the chains with a CVSC subchain are comprehensively enumerated. At last, a category of manipulators with a special hybrid topology is derived by gathering a CTPM with a limb including an Oldham-style CVSC chain. That way, we systematically synthesize several novel uncoupled 4-dof 3T-1R overconstrained hybrid manipulators.

2. Schoenflies Displacement Group and Its Primitive Generators

A 3T-1R motion is a noteworthy type of motion, which is also called Schoenflies motion[11–15] and often termed X motion for brevity. A specified set of X motions is endowed with the algebraic structure of a Lie group. Such a Lie group is a 4D Lie subgroup of the 6D Lie group of displacements. The word *displacement* designates an orientation-preserving isometry in the Euclidean 3D space of classical geometry. A displacement is the mathematical model of a rigid-body motion. When a frame of reference of the Euclidean space is given, a displacement is represented by a matrix acting on coordinate arrays. Such a matrix belongs to the matrix Lie group classically denoted $SE(3)$.[37,38] A specified group of X displacements (or motions) is denoted $\{\boldsymbol{X}(\mathbf{w})\}$ and contains any spatial translation and any rotation provided that the axis of rotation is parallel to a given unit vector $\mathbf{w}$. All the primitive mechanical generators of a Schoenflies group are obtained by serial concatenation of 1-dof Reuleaux pairs or hinged parallelograms and were enumerated by Hervé and Sparacino.[9] Later, Hervé pointed out some relation between the instantaneous motion type described by Schoenflies and the 4D displacement group denoted by X in his previous publications.[7,10,11] Then, Hervé decided to honor the great mathematician Arthur Schoenflies and he used his name to designate an X displacement group. In recent years, several authors adopted this eponym to designate this 4-dof motion type, which was also called SCARA motion in less recent literature on robotics. For instance, Gogu[16] proposed fully parallel mechanisms generating a Schoenflies motion of the end effector through a synthesis method based on transformations, which are linear with respect to the parameters of the relative motion in the joints when the motion has only infinitesimal amplitude. However, one should notice that a rotation is expressed with trigonometric functions of the angle of rotation and a trigonometric function is not a linear function. In our paper, what is called X-motion generators are the primitive generators obtained by serial concatenation of 1-dof pairs and hinged parallelograms. Nevertheless, the X-motion generators with internal mobility are not considered. The enumeration of all Schoenflies motion generators with a fully parallel topology is a complex problem beyond the scope of the current paper.

In the notation $\{\boldsymbol{X}(\mathbf{w})\}$, $\boldsymbol{X}$ stands for Schoenflies (or Schönflies) motion type, and $\mathbf{w}$ is a given unit vector characterizing the axes of the feasible rotations; the curly bracket is a conventional way to designate a set. Assume that $(O, \mathbf{u}, \mathbf{v}, \mathbf{w})$ is an orthonormal frame of reference and M is any point of the 3D Euclidean affine space. A displacement $\boldsymbol{X}(\mathbf{w}; a, b, c, \theta)$ belonging to $\{\boldsymbol{X}(\mathbf{w})\}$ transforms M into the point M' according to the formula $M \to M' = O + a\mathbf{u} + b\mathbf{v} + c\mathbf{w} + \exp(\theta\mathbf{w}\times)(\mathbf{OM})$, where a, b, c, and θ are four canonical parameters of the 4D Lie group, $\mathbf{OM} = M - O$ is the vector obtained from the ordered couple of points O and M, and $\mathbf{w}\times$ is the skew-symmetric linear operator of the vector product by $\mathbf{w}$. The operator $\exp(\theta\mathbf{w}\times)$ is the exponential series of the operator $(\theta\mathbf{w}\times)$. This exponential operator is not linear with respect to the angle θ but with respect to the position vector $(\mathbf{OM})$ of M. If the four canonical parameters of $\{\boldsymbol{X}(\mathbf{w})\}$ have infinitesimal amplitudes, namely if a, b, c, and θ become $\mathrm{d}a$, $\mathrm{d}b$, $\mathrm{d}c$, and $\mathrm{d}\theta$, respectively, then we obtain an infinitesimal displacement in $\{\boldsymbol{X}(\mathbf{w})\}$. Any point M becomes $M' = M + \mathbf{dM}$ and the infinitesimal vector $\mathbf{dM}$ can readily be calculated. For the infinitesimal motion, we have $\exp(\mathrm{d}\theta\,\mathbf{w}\times)(\mathbf{OM}) = \mathbf{OM} + \mathrm{d}\theta\mathbf{w}\times(\mathbf{OM})$ and

$$(\mathbf{OM}') = (\mathbf{OM}) + \mathbf{dM} = \mathrm{d}a\,\mathbf{u} + \mathrm{d}b\,\mathbf{v} + \mathrm{d}c\,\mathbf{w} + \exp(\mathrm{d}\theta\mathbf{w}\times)$$
$$(\mathbf{OM}) = \mathrm{d}a\,\mathbf{u} + \mathrm{d}b\,\mathbf{v} + \mathrm{d}c\,\mathbf{w} + (\mathbf{OM}) + \mathrm{d}\theta\,\mathbf{w}\times(\mathbf{OM}). \quad (1)$$

Hence,

$$\mathbf{dM} = \mathrm{d}a\,\mathbf{u} + \mathrm{d}b\,\mathbf{v} + \mathrm{d}c\,\mathbf{w} + \mathrm{d}\theta\,\mathbf{w}\times(\mathbf{OM}) \quad (2)$$

or

$$\begin{pmatrix}\mathbf{dM}\\ 0\end{pmatrix} = \$\,(\mathbf{w}; \mathrm{d}\theta;\ \mathrm{d}a,\ \mathrm{d}b,\ \mathrm{d}c)\begin{pmatrix}\mathbf{OM}\\ 1\end{pmatrix}. \quad (3)$$

The operator $\$(\mathbf{w}; \mathrm{d}\theta; \mathrm{d}a, \mathrm{d}b, \mathrm{d}c)$ is the twist of an $\{\boldsymbol{X}(\mathbf{w})\}$ motion. This operator is linear with respect to the position vector $(\mathbf{OM})$ and also with respect to the infinitesimal motion parameters $\mathrm{d}\theta$, $\mathrm{d}a$, $\mathrm{d}b$, and $\mathrm{d}c$. The set of these twists is a 4D vector space called a four-system of screws

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in literature. The canonical vector base of the 6D vector space of all the screws (twists and wrenches) is derived from the Cartesian frame of reference (O, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$) and is $\begin{pmatrix}\mathbf{u}\times & \mathbf{0}\\ 0 & 0\end{pmatrix}$, $\begin{pmatrix}\mathbf{v}\times & \mathbf{0}\\ 0 & 0\end{pmatrix}$, $\begin{pmatrix}\mathbf{w}\times & \mathbf{0}\\ 0 & 0\end{pmatrix}$, $\begin{pmatrix}\mathbf{0} & \mathbf{u}\\ 0 & 0\end{pmatrix}$, $\begin{pmatrix}\mathbf{0} & \mathbf{v}\\ 0 & 0\end{pmatrix}$, and $\begin{pmatrix}\mathbf{0} & \mathbf{w}\\ 0 & 0\end{pmatrix}$. The components in that canonical vector base of a twist representing an infinitesimal $\{\boldsymbol{X}(\mathbf{w})\}$ motion are (0, 0, dθ; da, db, dc). The set of these twists can be denoted by $\{\boldsymbol{x}(\mathbf{w})\}$ with a small $\boldsymbol{x}$ rather than a capital $\boldsymbol{X}$. From Lie's theory of continuous groups of transformations, there is a one-to-one correspondence between the 4D Lie group $\{\boldsymbol{X}(\mathbf{w})\}$ and its 4D Lie algebra $\{\boldsymbol{x}(\mathbf{w})\}$. In the original work of the very great mathematician Sophus Lie, what is now called *Lie algebra* was called *infinitesimal group*.

In a given kinematic chain, the set of the feasible relative displacements between two rigid bodies is a subset of the displacement Lie group, which is called the *kinematic bond* between the bodies. The serial concatenation of two kinematic pairs produces a kinematic bond between the distal bodies of the chain and this bond is the product of the pair bonds. In classical general theory of groups, the product of sets is defined as being the set of products. Generally, the product of two subgroups is not a subgroup. But the mainly useful fact is the closure of the product of two elements in any subgroup. By definition, a Lie subgroup is a subset of a Lie group, which still is an algebraic Lie group for the same product. From group theory, the intersection of two subgroups is always a subgroup that may be the improper displacement subgroup $\{\boldsymbol{E}\}$ of dimension 0, which only contains the identity transform $\boldsymbol{E}$. From these elementary theorems of group theory, a group $\{\boldsymbol{X}(\mathbf{u})\}$ can be decomposed into products of its 1D subgroups. It is straightforward to verify that, $\mathbf{u}$ being given, a 1D Lie group of helical displacements $\{\boldsymbol{H}(N, \mathbf{u}, p)\}$ with any given axis (N, $\mathbf{u}$) and any given pitch p is a Lie subgroup of $\{\boldsymbol{X}(\mathbf{u})\}$. Consequently, a generic expression of the decomposition of the 4D subgroup $\{\boldsymbol{X}(\mathbf{u})\}$ into a product of four of its 1D subgroups, which are generated by the 1-dof Reuleaux lower pairs, is $\{\boldsymbol{X}(\mathbf{u})\} = \{\boldsymbol{H}(N_1, \mathbf{u}, p_1)\}\{\boldsymbol{H}(N_2, \mathbf{u}, p_2)\}\{\boldsymbol{H}(N_3, \mathbf{u}, p_3)\}\{\boldsymbol{H}(N_4, \mathbf{u}, p_4)\}$ in which one, two, or three of the factors $\{\boldsymbol{H}(N_i, \mathbf{u}, p_i)\}$ may be replaced by a 1D subgroup $\{\boldsymbol{T}(\mathbf{s}_i)\}$ of translation parallel to the unit vectors $\mathbf{s}_i$, provided that the vectors $\mathbf{s}_i$ are linearly independent. One, two, or three pitches may also be equal to zero. The relation $\{\boldsymbol{X}(\mathbf{u})\} \supseteq \{\boldsymbol{H}(N_1, \mathbf{u}, p_1)\}\{\boldsymbol{H}(N_2, \mathbf{u}, p_2)\}\{\boldsymbol{H}(N_3, \mathbf{u}, p_3)\}\{\boldsymbol{H}(N_4, \mathbf{u}, p_4)\}$ is the direct consequence of the closure of the product in the group $\{\boldsymbol{X}(\mathbf{u})\}$. Obviously, the equality is valid in a neighborhood of the identity $\boldsymbol{E}$ if and only if (iff) the product in its right side is a 4D manifold, or else the product as well as the corresponding chain is singular. The equality, generally, is not valid for unlimited variation of the parameter in the factors $\{\boldsymbol{H}(N_i, \mathbf{u}, p_i)\}$. The singularity detection can be done by studying the linear dependency of the twists, which represent infinitesimal motion in the joints. Furthermore, a phenomenon of discontinuous mobility[39] may occur in special X-motion generators.

The $\{\boldsymbol{X}(\mathbf{u})\}$ primitive mechanical generators are obtained by serial concatenation of kinematic pairs embodying the subsets of $\{\boldsymbol{X}(\mathbf{u})\}$ that are the factors $\{\boldsymbol{H}(N_i, \mathbf{u}, p_i)\}$ of the foregoing product. In other words, any serial concatenation of four 1-dof kinematic pairs, each of which generates 1-dof motion included in $\{\boldsymbol{X}(\mathbf{u})\}$, generally makes up a mechanical generator of the group $\{\boldsymbol{X}(\mathbf{u})\}$. Defective generators have a connectivity lower than four and then an intermediate link has a passive internal mobility. A comprehensive list of all possible combinations of 1-dof kinematic pairs generating Schoenflies motion was published in a succinct paper in 1991.[9] A much more detailed discussion on the primitive generators of X motion is the topic of another paper that is not yet published. Here, the primitive X-motion generators are sorted into nine categories of general architectures as shown in Table I. Including their kinematic inversions, there are 79 general architectures of X-motion generators. One of the generators of the $\{\boldsymbol{X}(\mathbf{u})\}$ group is HHHH where the screws H have axes parallel to $\mathbf{u}$ and the four pitches must not be equal. A P pair is the limit case of such an H pair with a pitch becoming infinite (P parallel to $\mathbf{u}$) or the limit case of an H with an axis going at the infinity while keeping parallel to $\mathbf{u}$ (P perpendicular to $\mathbf{u}$) or the combination of both previous cases (oblique P). An R pair is an H with a zero pitch. Nine kinematic chains that generate X motion are graphically displayed in Fig. 1.

In any kinematic chain, if the set of the feasible displacements of a body (2) with respect to another body (1) is $\{\boldsymbol{L}(1, 2)\}$, then the set of the feasible displacements of the body (1) with respect to the body (2) is the set $\{\boldsymbol{L}(2, 1)\}$ of the inverses of the displacements belonging

Table I. Nine categories of general generators of Schoenflies motion.

Category	Type	Amount	All possible X-motion generators (including the inversions)
I	HHHH	15	RRRH RRHR RRHH RHRH RHHR HRRH HHHR HHRH HHHH HRRR RHRR HHRR HRHR RHHH HRHH
II	PHHH	16	PRRR PRRH PRHR PHRR PRHH PHRH PHHR PHHH RRRP HRRP RHRP RRHP HHRP HRHP RHHP HHHP
III	HPHH	16	RPRR HPRR RPHR RPRH HPHR HPRH RPHH HPHH RRPR RRPH RHPR HRPR RHPH HRPH HHPR HHPH
IV	PPHH	8	PPRH PPRR PPHR PPHH HRPP RRPP RHPP HHPP
V	PHPH	8	PRPR PRPH PHPR PHPH RPRP HPRP RPHP HPHP
VI	PHHP	4	PRRP PRHP PHHP PHRP
VII	HPPH	4	HPPR RPPR HPPH RPPH
VIII	PPPH	4	PPPR PPPH RPPP HPPP
IX	PPHP	4	PPHP PPRP PHPP PRPP

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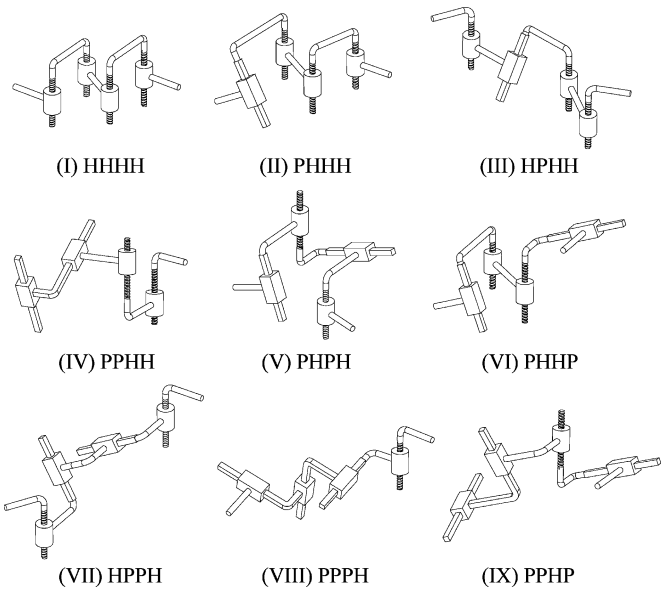


Fig. 1. General architectures of nine X-motion primitive generators.

to $\{\boldsymbol{L}(1, 2)\}$, which is expressed by the notation $\{\boldsymbol{L}(2, 1)\} = \{\boldsymbol{L}^{-1}(1, 2)\}$. If $\{\boldsymbol{L}(1, 2)\}$ is a group, then $\{\boldsymbol{L}(2, 1)\} = \{\boldsymbol{L}^{-1}(1, 2)\} = \{\boldsymbol{L}(1, 2)\}$. Consequently, reversing the order of joints in a chain generating the $\{\boldsymbol{X}(\mathbf{u})\}$ group also yields a mechanical generator of $\{\boldsymbol{X}(\mathbf{u})\}$. This operation is usually called *kinematic inversion* of the chain. All combinations are enumerated in the table for completeness.

The relative motion between the opposite bars of a 1-dof hinged parallelogram is 1-dof circular translation that is a 1D submanifold of the 2D Lie group of planar translations; the plane is that of the parallelogram. Consequently, for a small motion, a hinged parallelogram is equivalent to a prismatic pair. Replacing all P pairs with hinged parallelograms, we obtain all possible X-motion generators with hinged parallelograms. The general architectures with at least one hinged parallelogram corresponding to Table I are enumerated in Table II in which architectures that are equivalent by kinematic inversion are included. Hence, H and P pairs can be replaced by R and Pa, respectively, and the corresponding architectures are readily obtained. It is well known that flattened parallelograms are singular and they must be avoided. Detailed exploration on the singular postures of X-motion primitive generators is the topic of another paper in our further work. As shown in Table II, we have a total of 160 X-motion primitive generators with at least one hinged parallelogram.

3. Oldham-Style Constant Velocity Shaft Couplings

The Oldham coupling is a three-piece shaft coupling; its centerpiece is loose, which means that the coupling can be readily disassembled. Two sliding joints that are called prismatic P pairs in theoretical kinematics connect the three pieces. Hence, an Oldham coupling is a technical application of a PP subchain. The two P pairs must be nonparallel and, generally, they are perpendicular. This coupling transmits rotation with the constant velocity ratio of one when the drive and driven shafts are parallel and can be used to connect two rotating shafts with parallel axes separated by a distance that may be constant or variable. Figure 2(a) shows a sketch of the basic Oldham chain. In the best configuration, the two parallel shafts are coaxial and then there is no relative motion in the two P pairs, which provides the best avoidance of power loss by friction. This mechanism is an inversion of the Scotch yoke. In the closed loop chain with an Oldham coupling of

Table II. Schoenflies motion generators with at least one hinged parallelogram.

Category	Type	Amount	All X-motion generators with at least one hinged parallelogram (including the inversions)
I	P_aHHH	16	P_aRRR P_aRRH P_aRHR P_aHRR P_aRHH P_aHRH P_aHHR P_aHHH $RRRP_a$ $HRRP_a$ $RHRP_a$ $RRHP_a$ $HHRP_a$ $HRHP_a$ $RHHP_a$ $HHHP_a$
II	HP_aHH	16	RP_aRR HP_aRR RP_aHR RP_aRH HP_aHR HP_aRH RP_aHH HP_aHH RRP_aR RRP_aH RHP_aR HRP_aR RHP_aH HRP_aH HHP_aR HHP_aH
III	$(P_a)(P_a)HH$	24	P_aPRH P_aPRR P_aPHR P_aPHH $HRPP_a$ $RRPP_a$ $RHPP_a$ $HHPP_a$ PP_aRH PP_aRR PP_aHR PP_aHH HRP_aP RRP_aP RHP_aP HHP_aP P_aP_aRH P_aP_aRR P_aP_aHR P_aP_aHH $HRP_aP_aRRP_aP_aRHP_aP_aHHP_aP_a$
1V	$(P_a)H(P_a)H$	24	P_aRPR P_aRPH P_aHPR P_aHPH $RPRP_a$ $HPRP_a$ $RPHP_a$ $HPHP_a$ PRP_aR PRP_aH PHP_aR PHP_aH RP_aRP HP_aRP RP_aHP HP_aHP P_aRP_aR P_aRP_aH P_aHP_aR P_aHP_aH RP_aRP_a HP_aRP_a RP_aHP_a HP_aHP_a
V	$(P_a)HH(P_a)$	12	P_aRRP P_aRHP P_aHRP P_aHHP $PRRP_a$ $PRHP_a$ $PHRP_a$ $PHHP_a$ $P_aRRP_aP_aRHP_aP_aHRP_aP_aHHP_a$
VI	$H(P_a)(P_a)H$	12	HP_aPR RP_aPR RP_aPH HP_aPH HPP_aR RPP_aR RPP_aH HPP_aH RP_aP_aR RP_aP_aH HP_aP_aR HP_aP_aH
VII	$(P_a)(P_a)(P_a)H$	28	P_aPPR P_aPPH $RPPP_a$ $HPPP_aPP_aPR$ PP_aPH RPP_aP HPP_aP PPP_aR PPP_aH RP_aPP HP_aPP P_aP_aPR P_aP_aPH RPP_aP_a $HPP_aP_aP_aPP_aR$ P_aPP_aH $RP_aPP_aHP_aPP_aPP_aP_aR$ PP_aP_aH RP_aP_aP HP_aP_aP $P_aP_aP_aR$ $P_aP_aP_aH$ $RP_aP_aP_a$ $HP_aP_aP_a$
VIII	$(P_a)(P_a)H(P_a)$	28	P_aPRP P_aPHP $PRPP_a$ $PHPP_aPP_aRP$ PP_aHP PRP_aP PHP_aP $PPRP_a$ $PPHP_a$ P_aRPP P_aHPP P_aP_aRP P_aP_aHP PRP_aP_a $PHP_aP_aP_aPRP_a$ $P_aPHP_aP_aRPP_aP_aHPP_aPP_aRP_aPP_aHP_aP_aRP_aP$ P_aHP_aP $P_aP_aRP_a$ $P_aP_aHP_a$ $P_aRP_aP_a$ $P_aHP_aP_a$

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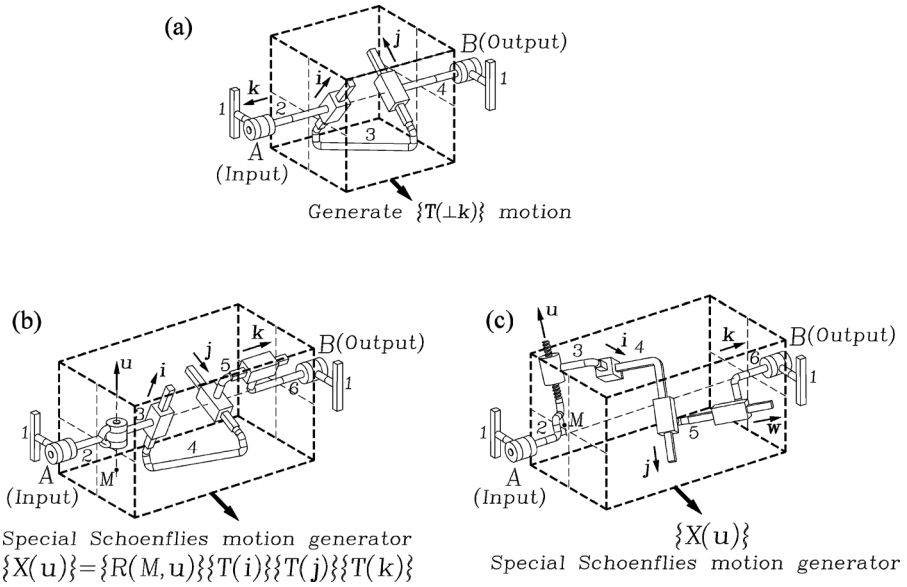


Fig. 2. Derivation of Oldham-style CV shaft couplings. (a) An Oldham coupling (basic) (b) A R-RPPP-R (special) (c) A R-HPPP-R (general).

Fig. 2(a), the subchain PP generates the 2D group of planar translations $\{\boldsymbol{T}(\perp\mathbf{k})\}$; the plane is perpendicular to the unit vector $\mathbf{k}$. The displacement subsets $\{\boldsymbol{G}_1\}$ and $\{\boldsymbol{G}_2\}$ produced by two serial chains connecting links 1 and 4 are expressed respectively as:

$$\{\boldsymbol{G}_1\} = \{\boldsymbol{R}(A,\mathbf{k})\}\{\boldsymbol{T}(\mathbf{i})\}\{\boldsymbol{T}(\mathbf{j})\} = \{\boldsymbol{G}(\mathbf{k})\} \textit{ and } \{\boldsymbol{G}_2\} = \{\boldsymbol{R}(B,\mathbf{k})\} \tag{4}$$

where $\{\boldsymbol{R}(A,\mathbf{k})\}$ denotes the 1D group of rotations around the axis $(A,\ \mathbf{k})$; $\{\boldsymbol{T}(\mathbf{i})\}$ is the 1D group of rectilinear translations parallel to the unit vector $\mathbf{i}$; $\{\boldsymbol{G}(\mathbf{k})\}$ is the 3D group of planar displacements (planar gliding) parallel to a plane perpendicular to $\mathbf{k}$. It may be helpful to recall that the equality $\{\boldsymbol{R}(A,\mathbf{k})\}\{\boldsymbol{T}(\mathbf{i})\}\{\boldsymbol{T}(\mathbf{j})\} = \{\boldsymbol{G}(\mathbf{k})\}$ expresses the decomposition of the 3D group $\{\boldsymbol{G}(\mathbf{k})\}$ into a product of three of its subgroups. The equality is valid in a neighborhood of

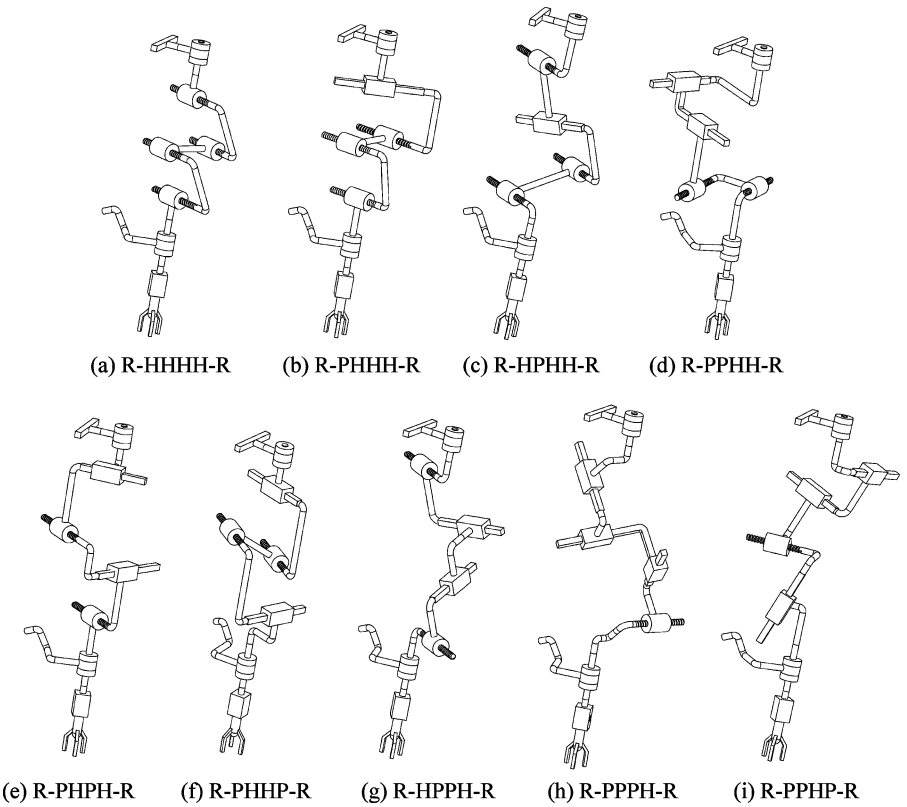


Fig. 3. Oldham-style general CVSC limb chains.

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the identity transform provided that the product is 3D. Then, the kinematic bond $\{\boldsymbol{L}(1,4)\}$ of link 4 relative to link 1 is:

$$\{\boldsymbol{L}(1,4)\} = \{\boldsymbol{G}_1\} \cap \{\boldsymbol{G}_2\} = \{\boldsymbol{G}(\mathbf{k})\} \cap \{\boldsymbol{R}(B,\mathbf{k})\} = \{\boldsymbol{R}(B,\mathbf{k})\} \tag{5}$$

because of $\{\boldsymbol{R}(B,\mathbf{k})\} \subset \{\boldsymbol{G}(\mathbf{k})\}, \forall B$. Hence, this Oldham chain is movable with 1 dof of finite mobility.

Let us assume that one revolute pair rotating around $(M,\mathbf{u})$ and one prismatic pair parallel to $\mathbf{k}$ are added on both sides of the PP subchain, as shown in Fig. 2(b). The obtained subchain RPPP is one among the primitive Schoenflies motion generators. This subchain produces the displacement subgroup $\{\boldsymbol{X}(\mathbf{u})\} = \{\boldsymbol{R}(M,\mathbf{u})\}\{\boldsymbol{T}(\mathbf{i})\}\{\boldsymbol{T}(\mathbf{j})\}\{\boldsymbol{T}(\mathbf{k})\} = \{\boldsymbol{R}(M,\mathbf{u})\}\{\boldsymbol{T}\}$ (also equal to $\{\boldsymbol{T}\}\{\boldsymbol{R}(M,\mathbf{u})\}$), $\{\boldsymbol{T}\}$ being the notation of the 3D group of spatial translations. In the closed loop chain of Fig. 2(b), the kinematic bond $\{\boldsymbol{L}(1,6)\}$ between link 1 and link 6 is the intersection of two displacement subsets $\{\boldsymbol{G}_1\}$ and $\{\boldsymbol{G}_2\}$:

$$\begin{aligned}\{\boldsymbol{G}_1\} &= \{\boldsymbol{R}(A,\mathbf{k})\}\{\boldsymbol{R}(M,\boldsymbol{u})\}\{\boldsymbol{T}(\mathbf{i})\}\{\boldsymbol{T}(\mathbf{j})\}\{\boldsymbol{T}(\mathbf{k})\}\\ &= \{\boldsymbol{R}(A,\mathbf{k})\}\{\boldsymbol{X}(\mathbf{u})\}\\ &= \{\boldsymbol{R}(A,\mathbf{k})\}\{\boldsymbol{T}\}\{\boldsymbol{R}(Q,\mathbf{u})\} \quad (\forall Q) \end{aligned} \tag{6}$$

$$\{\boldsymbol{G}_2\} = \{\boldsymbol{R}(B,\mathbf{k})\}. \tag{7}$$

Therefore, the motion of link 6 relative to link 1 is expressed by:

$$\begin{aligned}\{\boldsymbol{L}(1,6)\} &= \{\boldsymbol{G}_1\} \cap \{\boldsymbol{G}_2\}\\ &= \{\boldsymbol{R}(A,\mathbf{k})\}\{\boldsymbol{T}\}\{\boldsymbol{R}(Q,\mathbf{u})\} \cap \{\boldsymbol{R}(B,\mathbf{k})\}\\ &= \{\boldsymbol{R}(A,\mathbf{k})\}\{\boldsymbol{T}(\perp\mathbf{k})\{\boldsymbol{T}(\mathbf{k})\}\{\boldsymbol{R}(Q,\mathbf{u})\} \cap \{\boldsymbol{R}(B,\mathbf{k})\}\\ &= \{\boldsymbol{G}(\mathbf{k})\}\{\boldsymbol{T}(\mathbf{k})\}\{\boldsymbol{R}(Q,\mathbf{u})\} \cap \{\boldsymbol{R}(B,\mathbf{k})\}\\ &= \{\boldsymbol{R}(B,\mathbf{k})\}. \end{aligned} \tag{8}$$

$\{\boldsymbol{L}(1,6)\} = \{\boldsymbol{R}(B,\mathbf{k})\}$ results from the set relations $\{\boldsymbol{R}(B,\mathbf{k})\} \subset \{\boldsymbol{G}(\mathbf{k})\} \subset \boldsymbol{G}(\mathbf{k})\}\{\boldsymbol{T}(\mathbf{k})\}\{\boldsymbol{R}(Q,\mathbf{u})\}$. This confirms that the motion between links 1 and 6 has 1 dof of finite mobility. It is noteworthy that, in Fig. 2(b), the R pair that is added to the chain of Fig. 2(a) is useless and, therefore, is idle. However, the added P pair permits an axial displacement, which does not affect the rotation in the coupled shafts. Hence, the chain of Fig. 2(b) achieves the CVSC as well as the chain of Fig. 2(a). In Fig. 2(a), the Oldham CVSC is a PP generator of the 2D group $\{\boldsymbol{T}(\perp\mathbf{k})\}$ while, in Fig. 2(b), the new CVSC is a RPPP generator of the 4D group $\{\boldsymbol{X}(\mathbf{u})\}$. A more general form of such a new CVSC is shown in Fig. 2(c). It is clear that the RPPP generator of X motion can be replaced by any one of its kinematic equivalencies chosen among the X-motion generators listed in Table I. Nine general architectures of CVSC chains are displayed in Fig. 3. Apparently, their kinematic inversions are also adequate CVSC chains. In addition, replacing prismatic pairs by hinged parallelograms also yields valid CVSC chains, which are depicted in Fig. 4.

Revolute R pairs that are H pairs with zero pitch can be used in replacement of H pairs but four pitches must not be equal and the derived chains are adequate CVSC chains. Undoubtedly, the kinematic inversions of these chains are also valid chains with the CV transmission. Moreover, replacing three P pairs by hinged parallelograms also leads to more CV chains. For brevity, we omit the demonstrations of these possible CV architectures here.

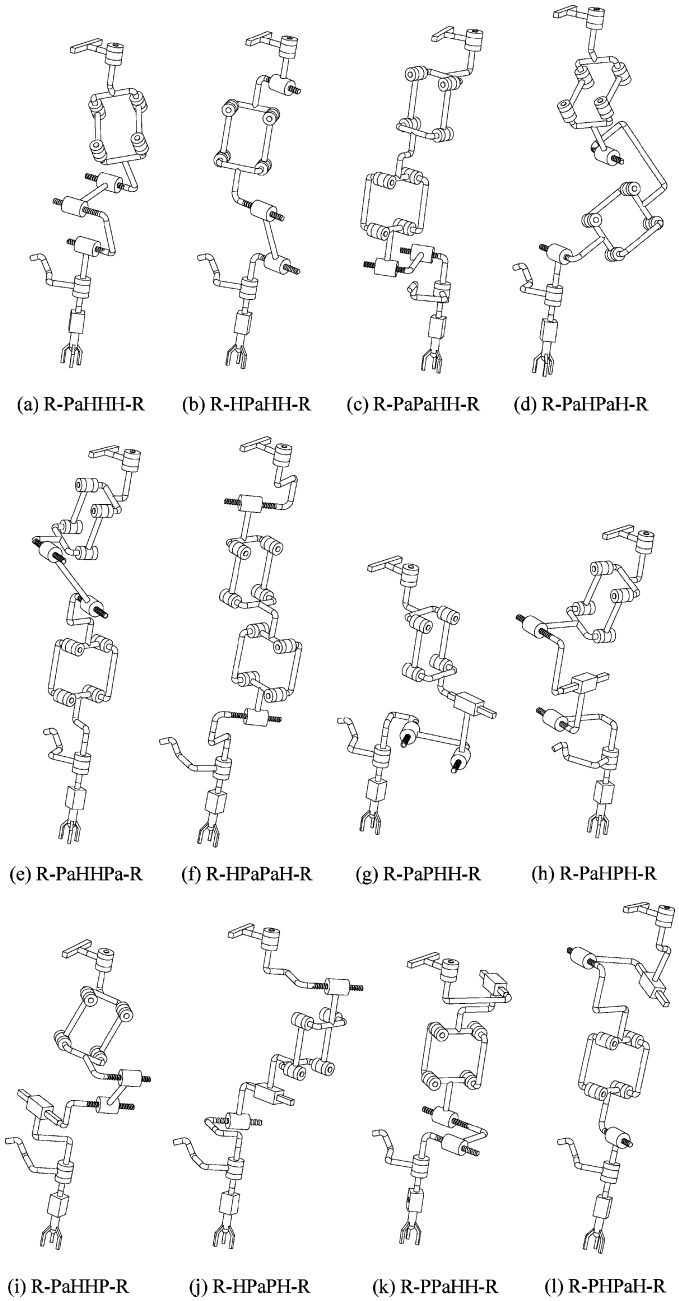


Fig. 4. General CVSC limb chains with hinged parallelograms.

In the foregoing CVSC chains, special geometric layout of contiguous pairs can be adopted. For instance, a subchain HH with coinciding axes and distinct pitches or a subchain HP and PH with P parallel to H-axis is equivalent to a cylindrical C pair. These chains still achieve a CV transmission between two parallel revolute pairs. Figure 5 shows this kind of CVSC limbs without H pair. In the figure, the notation $\underline{\text{C}}$ indicates a C pair that does not rotate in the HPMs. In the following section on the type synthesis of 3T-1R HPMs, which is the essential topic of the paper, the subchain $\underline{\text{B}}$ of the chain of

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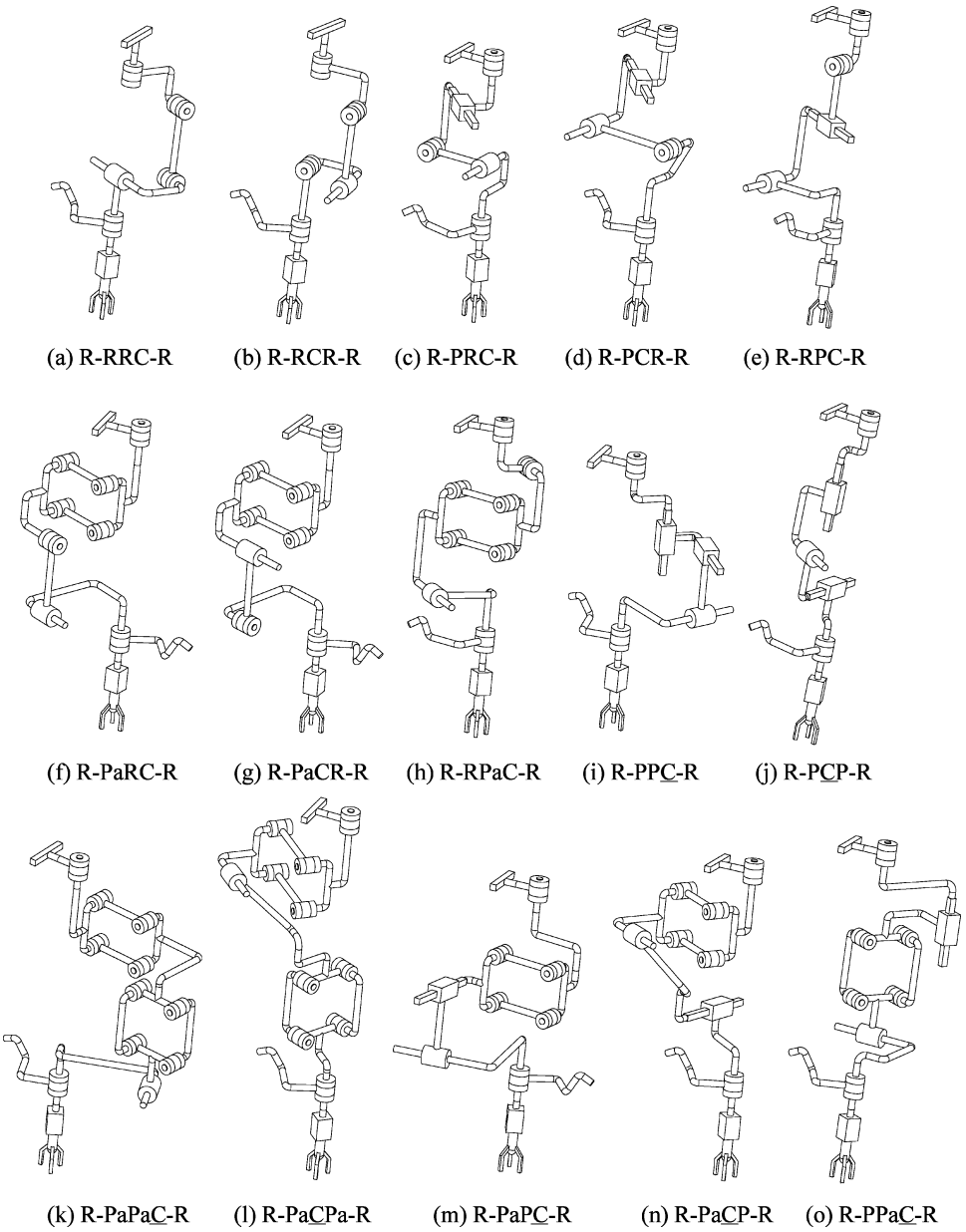


Fig. 5. CVSC limb chains with a C pair.

the fourth limb denotes a CVSC, which can be any one of the generators of Schoenflies motion provided that their postures are not singular.

4. Uncoupled Actuation of Overconstrained 3T-1R Hybrid Manipulators

In a 3T-1R HPM, the end-effector motion is represented by a product $\{\boldsymbol{T}\}\{\boldsymbol{R}(N, \mathbf{i})\}$. Four fixed actuators can directly control this kind of 4-dof hybrid mechanism in an uncoupled (or decoupled) manner. The manipulator works like a classical (X, Y, Z, θ) serial machine. Parallel mechanisms achieving the Cartesian control of the 3-dof translation of the end effector were first introduced by Kong and Gosselin[3] and by Kim and Tsai.[2] A new type of CTPM implementing screw pairs was proposed by Hervé,[4] and a detailed study of all Cartesian parallel mechanism was done recently by Lee and Hervé.[6] Referring to Fig. 6, a Cartesian translational parallel manipulator (CTPM) with three limbs including three orthogonal fixed prismatic pairs produces the 3-dof translation $\{\boldsymbol{T}\}$ of an intermediate platform. One body of a revolute pair that is a journal is rigidly connected to that platform. The second body of the R pair that is a shaft is connected to the fixed frame by a fourth articulated limb. The independent control of the rotation $\{\mathbf{R}(N, \mathbf{i})\}$ in the R pair jointed on the translating platform is obtained by the fourth limb, which is made up of two R pairs connected by a CVSC for parallel axes. There are several types of kinematic chains achieving the CV transmission between two revolute joints with parallel axes. The PP subchain generating a planar translation $\{\boldsymbol{T}(\perp\mathbf{i})\}$ is called Oldham joint that is a well-known CVSC. However, the closed chain implementing an Oldham joint is a planar chain and, consequently, the 3-dof translation of one of its body with respect to another body

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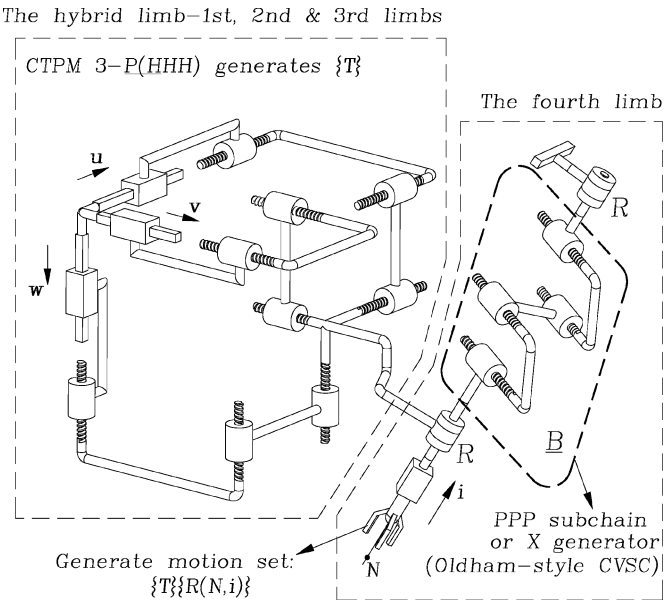


Fig. 6. Assembly principle of uncoupled 3T-1R overconstrained HPMs.

is forbidden. In an uncoupled 3T-1R HPM, the intermediate platform has to move in a pure 3-dof translation with respect to the fixed base. The fourth limb has the structural type RBR where the R pairs have parallel axes and B denotes an unknown kinematic chain of a CVSC for parallel shaft axes. Necessarily, the subchain B generates a displacement subset $\{\boldsymbol{B}\}$ that must contain $\{\boldsymbol{T}\}$. As a matter of fact, the motion set generated by the chain RBR has to contain the motion between the fixed base and the journal hold by the intermediate platform, whose motion is 3-dof translation. The condition $\{\boldsymbol{B}\} \supseteq \{\boldsymbol{T}\}$ implies $\{\boldsymbol{B}\} \cap \{\boldsymbol{T}\} = \{\boldsymbol{T}\}$. The closed chain RBR has to be movable with 1 dof and, therefore, the dimension of the subset $\{\boldsymbol{B}\}$ must not be greater than 5.

In summary, we can classify the uncoupled 3T-1R HPMs into three families: (1) $\{\boldsymbol{B}\} = \{\boldsymbol{T}\}$ in the chain RBR; (2) $\{\boldsymbol{B}\}$ is a 4D displacement set containing $\{\boldsymbol{T}\}$ and the groups $\{\boldsymbol{X}(\mathbf{s})\}$ designate the only displacement sets having this property, $\mathbf{s} \neq \mathbf{i}$ as it will be explained in the next section; (3) $\{\boldsymbol{B}\}$ is a 5D displacement set containing $\{\boldsymbol{T}\}$. The mechanisms of the first two families of HPMs are overconstrained and they constitute the essential subject of the paper. The description of mechanisms belonging to the third family of nonoverconstrained HPMs is intricate and lies outside the scope of the current paper but will be presented in future work.

4.1. The first family of uncoupled 3T-1R HPMs

The subchain B of this family is obviously achieved by a PPP chain provided that the Ps are not parallel to a plane. Moreover, the P pairs can be replaced by hinged parallelograms (Pa). These manipulators are enumerated in Table III, which includes all possible replacement of one or several P pairs by hinged parallelograms in the PPP chain. Typical examples are graphically displayed in Fig. 7. The other ones are obtained by replacements of H with R and of P with Pa from the previous figures.

4.2. The second family of uncoupled 3T-1R HPMs

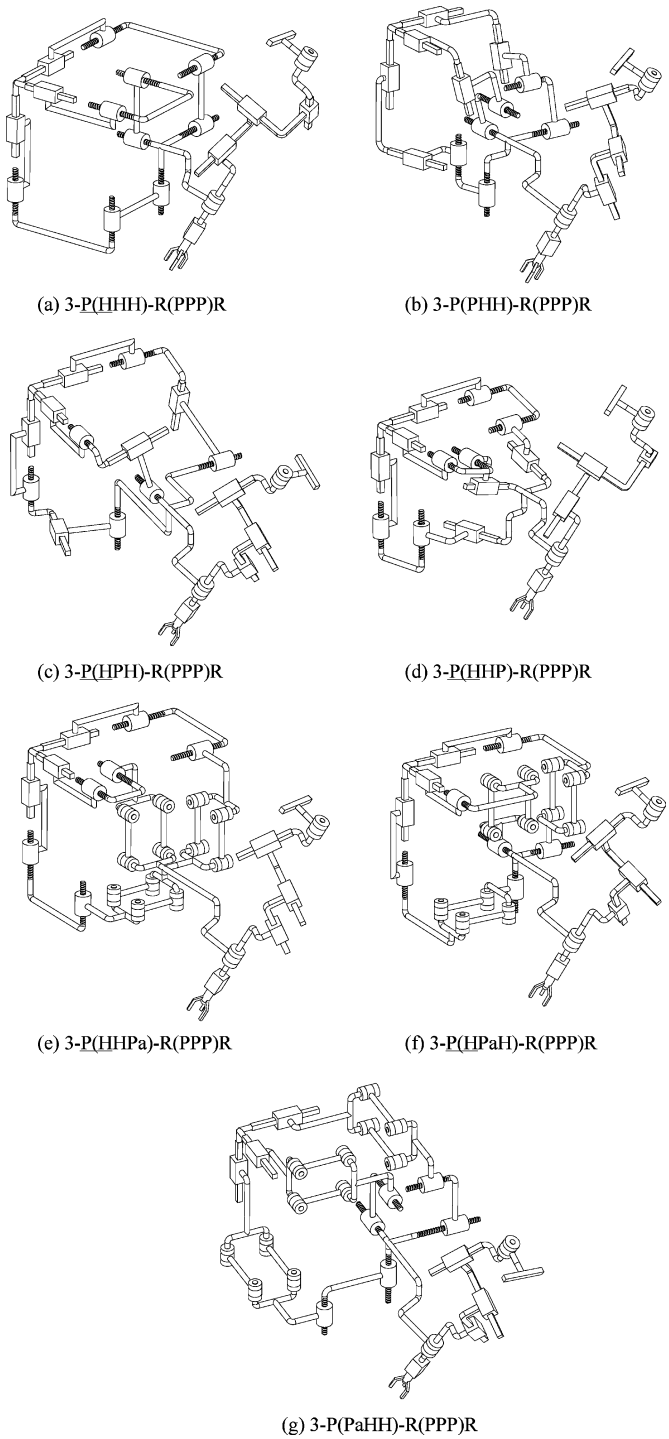
In the second family of 3T-1R HPMs, the B chain of the CVSC in the fourth limb can be any one chosen from the list of the generators of Schoenflies motion, which are enumerated in previous section. These chains that achieve the CV transmission of the rotation from a fixed R pair to a translating R pair are related to the well-known Oldham coupling and they are called the Oldham-style CVSC. The synthesized 3T-1R HPMs have the structural type T-RBR where T is one among the CTPMs and B is one of the generators of X motion. In the chain RBR of the fourth limb, the two R axes are parallel (or collinear) to $\mathbf{i}$ at the home configuration of the manipulator and remain parallel because the translational motion of the intermediate platform with respect to the fixed base does not modify the initial parallelism. The B chain generates a specified X motion, for instance $\{\boldsymbol{X}(\mathbf{s})\}$ at the home configuration. The vector $\mathbf{s}$ must not be parallel to $\mathbf{i}$. As a matter of fact, if $\mathbf{s} = \mathbf{i}$, then the RBR is a chain associated to the 4D group $\{\boldsymbol{X}(\mathbf{i})\}$. The rigid connection of the distal bodies of this defective RBR chain is a closed chain that is movable with 2 dofs. This fact can be verified using the Chebyshev–Grübler–Kuztbach formula generalized for a rigid-body motion space (actually a Lie group) of dimension 4.

The number of combinations is simply obtained by multiplying the number of CTPMs by the number of X-motion generators. It is clear that in the X-motion generators including three Ps or Pas, the H (or R) pair is useless in the

Table III. The first family of uncoupled 3T-1R overconstrained HPMs.

Type	Amount	The first family of uncoupled 3T-1R HPMs
No Pa in CTPM	8	3(*Py*)-R(PPP)R 3(*Pr*)-R(PPP)R
With Pa in CTPM	6	3(*Pya*)-R(PPP)R 3(*Pra*)-R(PPP)R
Only Pa in RBR chain	56	3(*Py*)-R(*Ppa*)R 3(*Pr*)-R(*Ppa*)R 3(*Py*)-R(*Ppb*)R 3(*Pr*)-R(*Ppb*)R 3(*Py*)-R(*Ppc*)R 3(*Pr*)-R(*Ppc*)R
With Pa in any limb chain	42	3(*Pya*)-R(*Ppa*)R 3(*Pra*)-R(*Ppa*)R 3(*Pya*)-R(*Ppb*)R 3(*Pra*))-R(*Ppb*)R 3(*Pya*)-R(*Ppc*)R 3(*Pra*)-R(*Ppc*)R
Notations:		
(*Py*): P(HHH), P(PHH), P(HPH) or P(HHP)		(*Pr*): P(RRR), P(PRR), P(RPR) or P(RRP)
(*Pya*): P(PaHH), P(HPaH) or P(HHPa)		(*Pra*): P(PaRR), P(RPaR) or P(RRPa)
(*Ppa*): (PaPP), (PPaP) or (PPPa)		(*Ppb*): (PaPaP), (PaPPa) or (PPaPa)
(*Ppc*): (PaPaPa)		

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(a) 3-P(HHH)-R(PPP)R (b) 3-P(PHH)-R(PPP)R

(c) 3-P(HPH)-R(PPP)R (d) 3-P(HHP)-R(PPP)R

(e) 3-P(HHPa)-R(PPP)R (f) 3-P(HPaH)-R(PPP)R

(g) 3-P(PaHH)-R(PPP)R

Fig. 7. The first family of uncoupled 3T-1R HPMs with general architectures.

CV transmission and, therefore is idle (or inactive). Table IV shows nine distinct main categories. The architectures of categories VIII & IX in Table IV will degenerate into the first family. Except the last two categories, every representative HPM with idle pair for each one is revealed in Fig. 8.

Moreover, replacing any arbitrary prismatic pair (P) with a hinged parallelogram (Pa) or any arbitrary screw pair (H) with revolute pair, more uncoupled 3T-1R HPMs are derived and enumerated in Table V, in which we have omitted the manipulators with idle pair. For clearness and brevity,

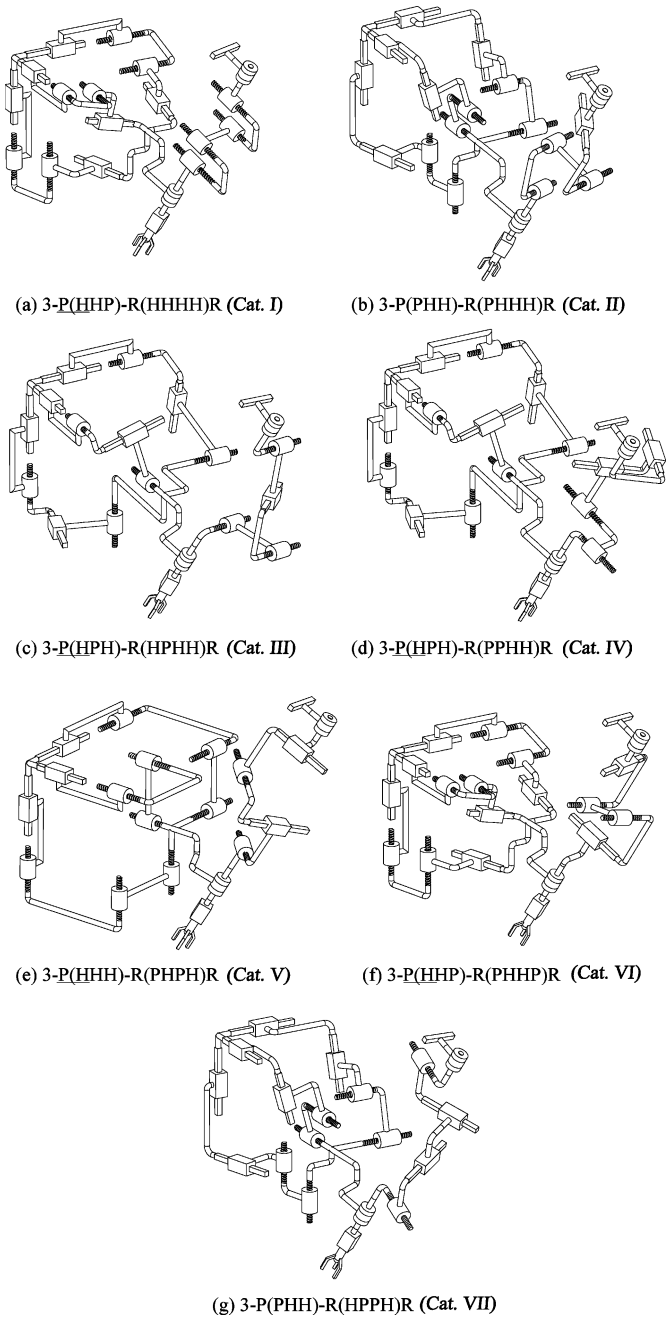
Table IV. The second family of uncoupled 3T-1R overconstrained HPMs.

Category I	Category II	Category III
Category IV	Category V	Category VI
3(*Py*)-R(PPHH)R	3(*Py*)-R(PHPH)R	3(*Py*)-R(PHHP)R
Category VII	Category VIII	Category IX
3(*Py*)-R(HPPH)R	3(*Py*)-R(PPPH*)R	3(*Py*)-R (PPH*P)R

Notations: (*Py*): P(HHH), P(HHP), P(HPH) or P(PHH)

* Indicates the idle pair.

only six illustrative architectures are sketched in Fig. 9. Obviously, the kinematic inversion of the limb chain also yields uncoupled 3T-1R HPMs.



(a) 3-P(HHP)-R(HHHH)R *(Cat. I)* (b) 3-P(PHH)-R(PHHH)R *(Cat. II)*

(c) 3-P(HPH)-R(HPHH)R *(Cat. III)* (d) 3-P(HPH)-R(PPHH)R *(Cat. IV)*

(e) 3-P(HHH)-R(PHPH)R *(Cat. V)* (f) 3-P(HHP)-R(PHHP)R *(Cat. VI)*

(g) 3-P(PHH)-R(HPPH)R *(Cat. VII)*

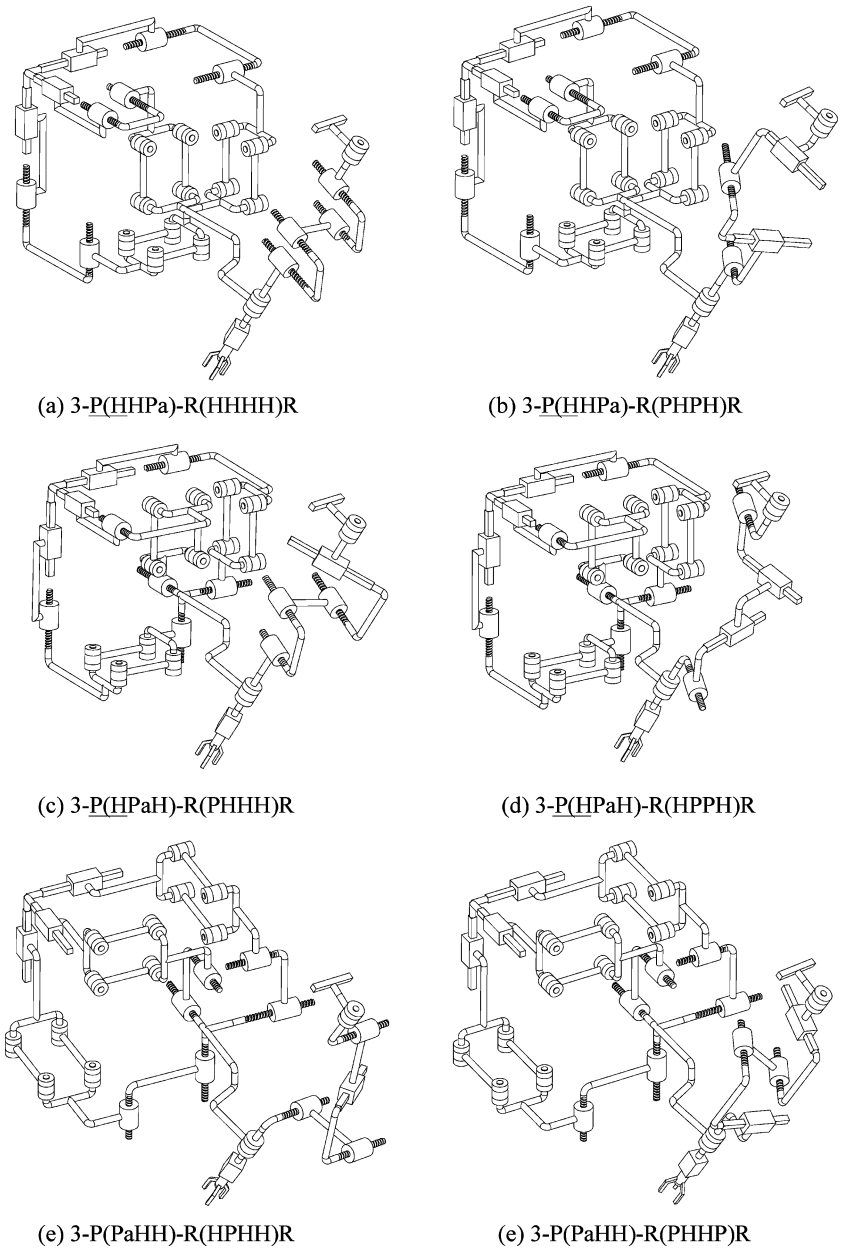
Fig. 8. The second family of uncoupled 3T-1R HPMs with general architectures.

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Table V. Uncoupled 3T-1R overconstrained HPMs with at least one Pa.

Category	Amount	All possible HPM architectures with at least one hinged parallelogram
I	3	3*(Pya)*-R(HHHH)R
II	10	3*(Py)*-R(PaHHH)R 3*(Pya)*-R(PHHH)R 3*(Pya)*-R(PaHHH)R
III	10	3*(Py)*-R(HPaHH)R 3*(Pya)*-R(HPHH)R 3*(Pya)*-R(HPaHH)R
IV	24	3*(Py)*-R(PaPHH)R 3*(Py)*-R(PPaHH)R 3*(Py)*-R(PaPaHH)R 3*(Pya)*-R(PPHH)R 3*(Pya)*-R(PaPHH)R 3*(Pya)*-R(PPaHH)R 3*(Pya)*-R(PaPaHH)R
V	24	3*(Py)*-R(PaHPH)R 3*(Py)*-R(PHPaH)R 3*(Py)*-R(PaHPaH)R 3*(Pya)*-R(PHPH)R 3*(Pya)*-R(PaHPH)R 3*(Pya)*-R(PHPaH)R 3*(Pya)*-R(PaHPaH)R
VI	24	3*(Py)*-R(PaHHP)R 3*(Py)*-R(PHHPa)R 3*(Py)*-R(PaHHPa)R 3*(Pya)*-R(PHHP)R 3*(Pya)*-R(PaHHP)R 3*(Pya)*-R(PHHPa)R 3*(Pya)*-R(PaHHPa)R
VII	24	3*(Py)*-R(HPaPH)R 3*(Py)*-R(HPPaH)R 3*(Py)*-R(HPaPaH)R 3*(Pya)*-R(HPPH)R 3*(Pya)*-R(HPaPH)R 3*(Pya)*-R(HPPaH)R 3*(Pya)*-R(HPaPaH)R

Notations: (*Py)*: P(HHH), P(HHP), P(HPH) or P(PHH) *(Pya)*: P(PaHH), P(HPaH) or P(HHPa)



(a) 3-P(HHPa)-R(HHHH)R (b) 3-P(HHPa)-R(PHPH)R

(c) 3-P(HPaH)-R(PHHH)R (d) 3-P(HPaH)-R(HPPH)R

(e) 3-P(PaHH)-R(HPHH)R (e) 3-P(PaHH)-R(PHHP)R

Fig. 9. Uncoupled 3T-1R overconstrained HPMs with hinged parallelograms.

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Table VI. Uncoupled 3T-1R overconstrained HPMs without H pairs.

Type I	Type II	Type III	Type IV
3(*Pr*)-R(PRRR)R	3(*Pr*)-R(RPRR)R	3(*Pr*)-R(PPRR)R	3(*Pr*)-R(PRPR)R
Type V	Type VI	Type VII	Type VIII
3(*Pr*)-R(PRRP)R	3(*Pr*)-R(RPPR)R	3(*Pr*)-R(PPPR*)R	3(*Pr*)-R(PPR*P)R

Notations: (*Pr*): P(RRR), P(PRR), P(RPR) or P(RRP)

* Indicates the idle pair.

Table VII. Uncoupled 3T-1R overconstrained HPMs with one cylindrical pair.

Type	Amount	All possible HPMs with one cylindric pair (including the inversions)
I	12	*3(Pr)*-R(RRC)R *3(Pr)*-R(CRR)R *3(Pr)*-R(RCR)R
II	16	*3(Pr)*-R(PRC)R *3(Pr)*)-R(CRP)R *3(Pr)*-R(PCR)R *3(Pr)*-R(RCP)R
III	8	*3(Pr)*-R(RPC)R *3(Pr)*-R(CPR)R
IV	8	*3(Pr)*-R(PPC)R *3(Pr)*-R(CPP)R
V	4	*3(Pr)*-R(PCP)R

Notations: (*Pr*) means P(RRR), P(PRR), P(RPR) or P(RRP)

In special realizations, instead of H pairs, R pairs that are H with a zero pitch can be used. That way, a lot of special HPMs can be easily derived and comprehensively enumerated. The inversions of CVSC chain in the derived manipulators also produce uncoupled HPMs. For brevity, we omit them here. Nevertheless, for easing applications, Table VI shows the manipulators derived by replacement of all H pairs with R pairs from Table IV. Certainly, the inversion of CVSC limb also yields an adequate 3T-1R HPM.

The combinations with H pairs of CTPMs, which generalize the CTPMs with no H pair introduced by Kong and Gosselin[3] and also by Kim and Tsai,[2] are disclosed by authors[6] recently. Then, implementing CVSCs with one cylindrical pair in Figs. 5(a)–(h) and CTPMs with screw pairs, we synthesize several new architectures of 3T-1R HPMs. For the use of more applications, all possible HPMs without H pairs are tabulated in Table VII and some examples are shown in Fig. 10. For completeness, it is worth mentioning that all possible uncoupled 3T-1R 4-dof HPMs with hinged parallelograms in CTPM limbs but without H pair in the limbs can be derived from Table V. In particular, from Tables V and VI, and Type *III* of Table VII, we deduce four very simple architectures, which are shown in Fig. 11.

4.3. Related non-overconstrained uncoupled 3T-1R HPMs

Any overconstrained mechanism cannot work in the presence of manufacturing errors. On the other hand, one can always derive a nonoverconstrained mechanism from an overconstrained one by the addition of mobility. Some rigid links are replaced by kinematics pairs, which, generally, keep inactive or idle when the manufacturing errors are negligible. Clearly, this can be done in the foregoing overconstrained HPMs. In each limb, the added feasible motion has to be independent of the motion set generated by the limb, or else the limb connectivity is not modified and the limb acquires an internal passive mobility and the whole mechanism remains overconstrained. In the HPMs proposed in this paper, each limb includes a subchain generating Schoenflies motion. Hence, the addition of a P pair in an X-motion generator leads to another X-motion generator having internal mobility and, therefore, it is not an effective way to eliminate the overconstraints. For the same reason, the addition of an H

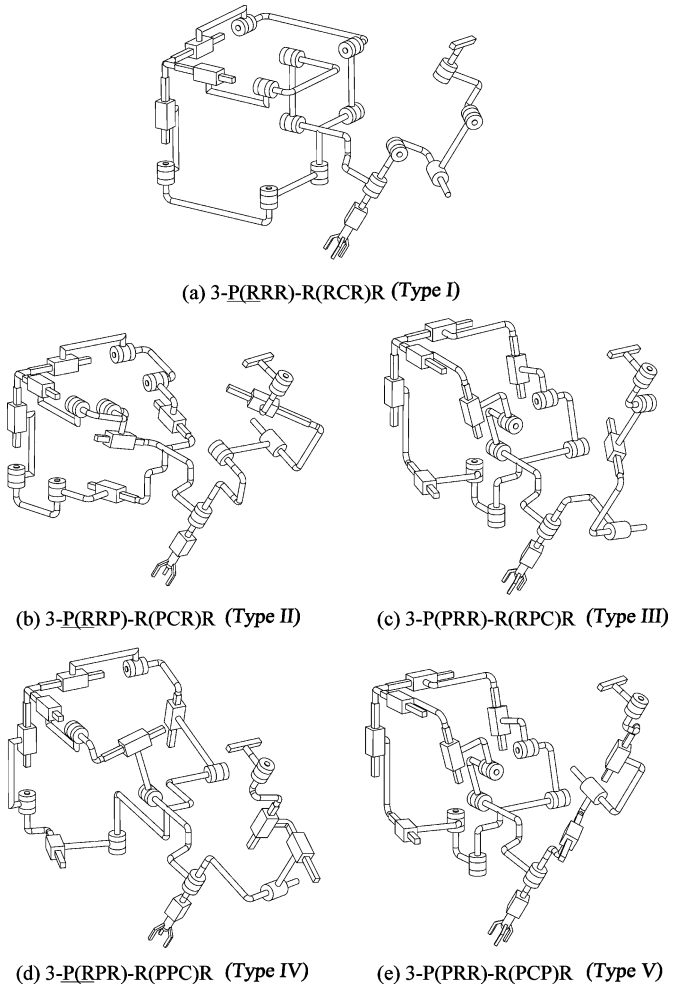
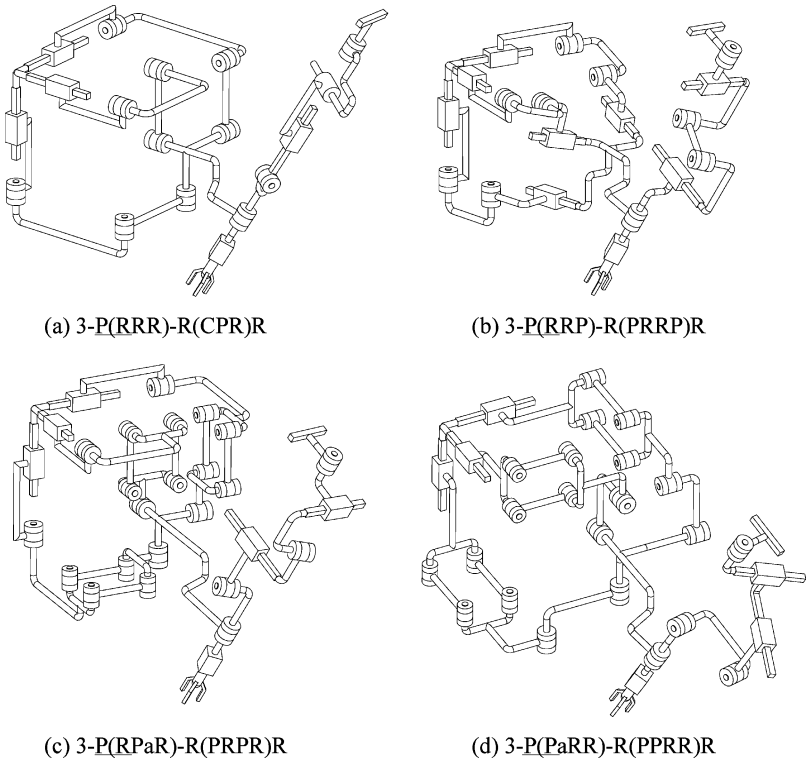


Fig. 10. Uncoupled 3T-1R overconstrained HPMs with a cylindrical pair.

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(a) 3-P(RRR)-R(CPR)R (b) 3-P(RRP)-R(PRRP)R

(c) 3-P(RPaR)-R(PRPR)R (d) 3-P(PaRR)-R(PPRR)R

Fig. 11. Simple examples for easing applications.

(or an R) pair having its axis parallel to the axes of rotation of the X motion has to be avoided.

In a nonoverconstrained CTPM, each of the three limbs generates 5-dof motion, according to the Chebyshev–Grübler–Kutzbach formula for fully spatial mechanisms. Hence, while avoiding the aforesaid parallelism, the arbitrary addition of one H (or R) pair in each of the three 4-dof limbs leads to a nonoverconstrained CTPM with three idle H (or R).

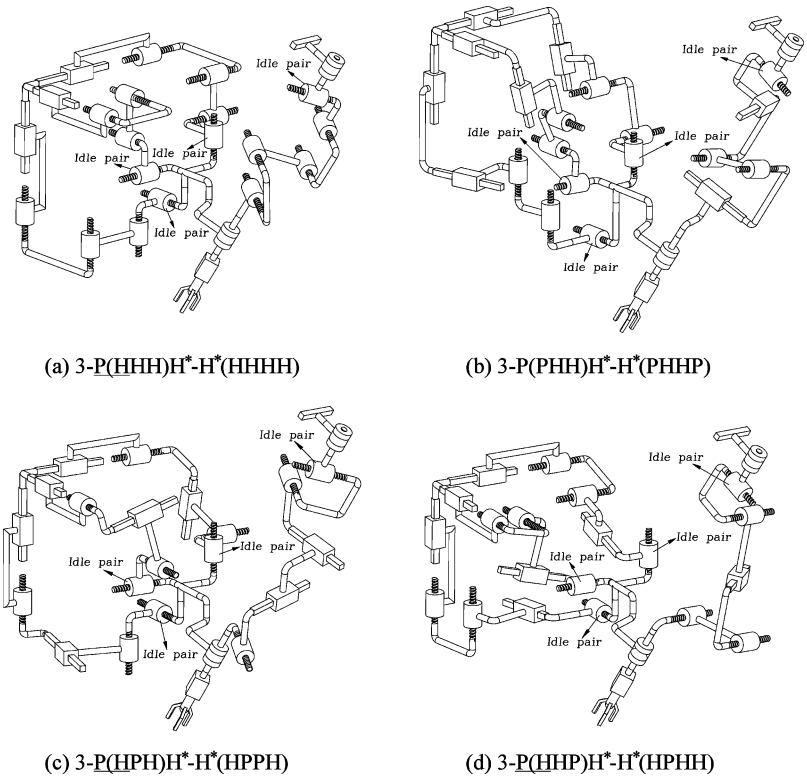
In the overconstrained HPMs that are disclosed in this paper, the fourth limb connecting the base to the end effector produces 6-dof motion and it has the structural type R(X)R where (X) is any one of the primitive generators of X motion. The elimination of overconstraints requires a 7-dof fourth limb. Hence, the addition of an H (or R) in the fourth limbs can be done provided that the H-axis is not parallel to the (X) axes of rotation. Moreover, the addition of an H (or R) with an axis parallel to the fixed R is not effective in the elimination of overconstraints. As a matter of fact, in that case, the vector space of the twists in the limb joints is 5D instead of being 6D; the fourth limb has an internal mobility that, generally, only has infinitesimal amplitude. The discussion of this singular phenomenon is beyond the scope of this paper. The addition of an H (or R) with an axis that is parallel neither to the (X) axes nor to the fixed R-axis leads to a spatial chain without overconstraint.

Such nonoverconstrained systems directly originate from the overconstrained ones. Drawings in Fig. 12 illustrate some examples of the possible incorporation of idle pairs in the HPMs, which are disclosed in the paper. Clearly, the number of mechanism architectures is very great. However, one can consider that these nonoverconstrained HPMs with idle pairs share the same structural type with the overconstrained ones. Nevertheless, nonoverconstrained HPMs that are not directly associated to an overconstrained mechanism can also be devised by using kinematic equivalencies that result from the Lie-group-algebraic structure of the Euclidean displacement set. The type synthesis of these manipulators lies outside the scope of this paper.

5. Conclusions

The Oldham-type CVSCs are realized by Schoenflies motion generators, which are systematically enumerated. In this way, a new application of the Schoenflies motion generators is revealed. Two families of novel uncoupled 3T-1R overconstrained HPMs are systematically introduced via Lie-group-algebraic properties of the displacement set and intrinsic coordinate-free geometry. Each HPM is obtained through a gathering of a three-limb CTPM with a fourth limb including a CVSC chain. In the first family of the fourth limbs, a PPP chain is the CVSC and in the second one an X-motion generator is used as a CVSC. In these 4-dof HPMs, the spatial translation and the rotation of the end effector are controlled from the fixed base in a fully uncoupled manner. In a CTPM, the 3-dof spatial translation of the moving platform is achieved through the uncoupled actuation of three fixed orthogonal prismatic pairs. The CVSC in the fourth limb transmits rotation between two parallel revolute pairs. One R is fixed and actuated; the other R is jointed with the CTPM translating platform. The non-overconstrained HPMs, which are directly derived from the overconstrained ones by the addition of one idle H (or R) pair in each of four limbs are also presented and some examples are illustrated by figures. Beyond the aforementioned findings, the group-theory

https://doi.org/10.1017/S0263574708004554 Published online by Cambridge University Press



(a) 3-P(HHH)H*-H*(HHHH) (b) 3-P(PHH)H*-H*(PHHP)

(c) 3-P(HPH)H*-H*(HPPH) (d) 3-P(HHP)H*-H*(HPHH)

Fig. 12. Uncoupled 3T-1R non-overconstrained HPMs with idle pairs.

synthesis method used in the paper will be applied to the third family of non-overconstrained 4-dof HPMs in future works.

Acknowledgment

The authors are very thankful to the National Science Council for supporting this research under grants NSC 93-2212-E-151-012 & NSC 95-2221-E-151-011.

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