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Subsequence Sums

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Let G be a cyclic group of order n and let $\mu = \{x_1, x_2, \ldots, x_m\}$ be a sequence of elements of G. Let k be the number of distinct values taken by the sequence μ. Let $n \wedge \mu$ be the set of the n-subsequence sums.

We show that one of the following conditions holds:

- μ has a value repeated $n-k+3$ times,
- $n \wedge \mu$ contains a non-null subgroup,
- $|n \wedge \mu| \geqslant m-n+k-2$.

We conjecture that the last condition could be improved to $|n \wedge \mu| \geqslant m-n+k-1$. This conjecture generalizes several known results. We also obtain a generalization of a recent result due to Bollobás and Leader.

1. Introduction

Let G be an abelian group of order $n \geqslant 2$ and let μ be a sequence of elements of G. The length (resp. the number of distinct values) of μ will be denoted by $|\mu|$ (resp. $\#\mu$). The set of sums of the n-subsequence of μ will be denoted by $n \wedge \mu$.

The Erdős–Ginzburg–Ziv theorem [7] states that $0 \in n \wedge \mu$, if $|\mu| \geqslant 2n-1$. Olson proved in [18] that $n \wedge \mu$ contains a non-null subgroup if no value is repeated $n+1$ times.

It was observed by Brakemeier [3, 4] that the condition $|\mu| \geqslant 2n-1$ is needed only if $\#\mu = 2$. He proved [3, 4] that $0 \in n \wedge \mu$, if G is cyclic, $\#\mu \geqslant (n+3)/2$ and $|\mu| = n+2$. Another proof of this result was obtained independently by Gallardo, Grekos and Pinko in [10]. Assuming n to be a prime, Brakemeier showed that $0 \in n \wedge \mu$ if $|\mu| \geqslant 2n - \#\mu + 1$. Yet another proof was independently obtained by Bialostocki and Lotspeich [1]. They conjectured the validity of this result if n is composite. Hamidoune, Ordaz and Ortunio [14] obtained a proof of this conjecture.

An addition theorem generalizing the Erdős–Ginzburg–Ziv theorem was obtained recently by Bollobás and Leader [2]. This result states that either $0 \in n \wedge \mu$, or $|n \wedge \mu| \geqslant$

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$|\mu| - n + 1$. Clearly the second condition implies $0 \in n \wedge \mu$ if $|\mu| = 2n - 1$. On the other hand, the Bollobás–Leader theorem applies to sequences with a small length.

The above results suggest the following conjectures.

Conjecture 1.1. *Let G be a cyclic group of order n and let μ be a sequence of elements of G. Then one of the following conditions holds:*

- *μ has a value repeated $n - \#\mu + 3$ times,*
- *$n \wedge \mu$ contains a non-null subgroup,*
- *$|n \wedge \mu| \geqslant |\mu| - n + \#\mu - 1$.*

Conjecture 1.2. *Let G be a cyclic group of order n and let μ be a sequence of elements of G. Then one of the following conditions holds:*

- *$0 \in n \wedge \mu$,*
- *$|n \wedge \mu| \geqslant |\mu| - n + \#\mu - 1$.*

Assume that G is a cyclic group and that μ contains no element repeated $n - \#\mu + 3$ times. We show that either $n \wedge \mu$ contains a non-null subgroup H, or $|n \wedge \mu| \geqslant |\mu| - n + \#\mu - 2$. This result is slightly weaker than Conjecture 1.1. In order to limit the size of the present paper, our results related to Conjecture 1.2 will be presented in a forthcoming paper.

Our results will be obtained by combining addition theorems and strings (a notion introduced by the author in [12]). In order to make the present paper self-contained, we shall define strings later and prove their basic properties.

The organization of the paper is as follows. Section 2 contains some known results needed in our proofs. Section 3 contains the basic properties of strings which allow us to adapt the proof of Erdős, Ginzburg and Ziv in the prime case to finite abelian groups. Recall that in such a proof the sequence μ is partitioned into sets of cardinality 2, assuming that no element is repeated $(n + 1)$ times. Put $\mu = \{x_i;\ 1 \leqslant i \leqslant m\}$. Let q be a positive integer such that $q \leqslant m \leqslant 2q - 1$. We prove in Section 3 that either $|q \wedge \mu| \geqslant m - q + 1$, or $qx_i \in q \wedge \mu$, for some $1 \leqslant i \leqslant m$. For $q = n$, this result is the Bollobás–Leader theorem.

In Section 4, we prove an addition theorem, which will be combined with the properties of a string in order to get the main result.

2. Tools

Let G be an abelian group and let H be a subgroup. The subgroup generated by a subset X will be denoted by $\langle X \rangle$. Recall that a subset X is said to be *H-periodic* if $X + H = X$. The *stabilizer* of A is $\mathrm{St}(A) = \{x | A + x = A\}$. We shall use the following lemma.

Lemma 2.1. *Let L and M be non-empty subsets of a finite abelian group G and let $x, y \in G$. Then*

$$L + \mathrm{St}(L) = L; \tag{2.1}$$

$$\mathrm{St}(L) \subset \mathrm{St}(L + M); \tag{2.2}$$

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$$\text{if } |L+M| = |L| \text{ then } \mathrm{St}(L+M) = \mathrm{St}(L); \tag{2.3}$$

$$\text{if } |L+M| > |L| \text{ then } |L+M| \geqslant |L| + |\mathrm{St}(L)|; \tag{2.4}$$

$$|L + \{x, y\}| = |L| \text{ if and only if } y - x \in \mathrm{St}(L). \tag{2.5}$$

This lemma is proved in [12]. Alternatively, the reader may treat the proof of Lemma 2.1 as an exercise.

Recall the following easy and well-known lemma.

Lemma 2.2. ([15]) *Let A, B be subsets of a finite group such that $|A| + |B| \geqslant |G| + 1$. Then $A + B = G$.*

Let A, B be subsets of an abelian group G. Let k be a nonnegative integer. We shall write

$$k \wedge A = \left\{ \sum_{x \in B} x \mid B \subset A \text{ and } |B| = k \right\},$$

and

$$A \wedge B = \{x + y \mid x \in A, \quad y \in B \text{ and } x \neq y\}.$$

We have $2 \wedge A = A \wedge A$.

Note that $|2 \wedge A| \geqslant |A| - 1$. It follows that $2 \wedge A = \emptyset$ if and only if $|A| \leqslant 1$.

Observe that $|2 \wedge A| \geqslant 3$, if $|A| \geqslant 3$. Indeed, there are distinct elements $a, b, c \in A$. Clearly $a + b, a + c, b + c$ are distinct elements of $2 \wedge A$. We shall use these observations without reference.

For an abelian group G, put $\epsilon(G) = |\{x \mid 2x = 0\}|$. Note that the condition $\epsilon(G) \leqslant 2$ holds if either G is cyclic or $|G|$ is odd. We also recall a lemma which is the restricted sum analogue of Lemma 2.2.

Lemma 2.3. ([13]) *Let G be a finite abelian group such that $\epsilon(G) \leqslant 2$. Let A, B be subsets of G such that $|A| + |B| \geqslant m + 1 + \epsilon(G)$. Then $A \wedge B = G$. In particular, $2 \wedge A = G$, if $|A| \geqslant (|G| + 3)/2$.*

Proof. Suppose the contrary and take $a \in G \setminus (A \wedge B)$. Let $X = \{x : a = 2x\}$. Clearly $|X| \leqslant \epsilon(G)$. Take $X = X_1 \cup X_2$, where $0 \leqslant |X_i| \leqslant 1$, for all i. By Lemma 2.2, $x \in (A \setminus X_1) + (B \setminus X_2)$. It follows that $x \in A \wedge B$, a contradiction. □

We shall use the following lemma.

Lemma 2.4. *Let H be a subgroup of $\mathbb{Z}_m$. Let A, B and C be subsets of $\mathbb{Z}_m$ such that $|A + H| = |B + H| = |C + H| = |H|$. Assume $|A| + |B| > |H|$ and $|C| \geqslant (|H| + 3)/2$. Then $A + B = A + B + H$ and $2 \wedge C = 2C = 2C + H$.*

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Proof. The hypothesis implies that A, B, C are non-empty. Choose $a \in A$, $b \in B$ and $c \in C$. By Lemma 2.2, $(A - a) + (B - b) = H$. By Lemma 2.3, $2 \wedge (C - c) = (2 \wedge C) - 2c = H$. The lemma follows since $|2 \wedge (C - c)| = |2 \wedge C|$. □

We shall also use Kneser's theorem.

Theorem 2.5. ([16, 17]) *Let G be an abelian group, and let A, B be non-empty subsets of G. Put $H = \mathrm{St}(A + B)$. Then*

$$|A + B| \geqslant |A + H| + |B + H| - |H|.$$

3. Multisets

Additive properties are independent of sequence reordering. It is quite natural to use unordered sequences. An unordered sequence of elements of a set A may be defined as an element of the free abelian monoid with basis A. Such an approach is the usual one in factorization theory and the related zero sum theory [8]. Here, we shall use an equivalent approach via multisets.

Let A be a set and let $B \subset A$. By a multisubset of A, we mean a map F from A into $\mathbb{N}$. The number of repetitions of $x \in A$ is $F(x)$. The length of F is $|F| = \sum_{x \in A} F(x)$. The support of F is $\|F\| = \{x \mid F(x) \neq 0\}$. The cardinality of the support of F will be denoted by $\#F$. Obviously $\#F = \sum_{x \in A} \min(1, F(x))$. A multiset F such that $F(x) \leqslant 1$, for all x, will be identified with the set $\|F\|$. We shall also identify an element $x \in A$ with the indicator of the set $\{x\}$. More generally, a set $\{x_1, \ldots, x_k\}$ could be identified with the multisubset $x_1 \cdots x_k$.

To any finite family $\mu = \{x_i; i \in I\}$ of elements of A, we associate a multisubset F, where $F(x) = |\{i \mid x = x_i\}|$. Let $\mu = \{x_i; i \in I\}$ be a finite family of elements of an abelian group G and let $0 \leqslant k \leqslant |I|$. The set of k-subfamily sums is

$$k \wedge \mu = \left\{ \sum_{i \in J} x_i \mid J \subset I \text{ and } |J| = k \right\}.$$

Clearly

$$k \wedge \mu = \sum_{i \in I} x_i - (|I| - k) \wedge F.$$

Let F be a multiset. Let μ_1, μ_2 be two sequences represented by F. Clearly $k \wedge \mu_1 = k \wedge \mu_2$. So we may write $k \wedge F = k \wedge \mu_1$.

Let F, E be multisubsets of A. We shall write $F \leqslant E$ if, for all x, $F(x) \leqslant E(x)$. Assume $F \leqslant E$. We let EF^{-1} denote the unique multisubset D of A verifying the relation $E = DF$. We shall write $F \cap E$ for the largest multisubset D, such that $D \leqslant F$ and $D \leqslant E$. Let $B \subset A$. We let F_B denote the restriction of F to B. Note that $F_B(x) = 0$, if $x \notin B$. Otherwise $F_B(x) = F(x)$.

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4. Strings

The notion of a string was introduced in [12]. We use in this paper a slightly modified notion of strings. In the present section, we adapt the properties of strings obtained in [12] to this new situation.

Let G be a finite abelian group. Let q be an integer. Let F be a multisubset of E.

An *F-chain* of *length* $r+1$ is a sequence $\mathcal{A}=\{A_1,\dots,A_{r+1}\}$ of multisubsets of F satisfying the following conditions:

(i) $A_{r+1}\neq\emptyset$,

(ii) $F=A_1,\dots,A_{r+1}$,

(iii) for all $1\leqslant i\leqslant r$, $|A_i|=\#(A_i)=2$,

(iv) for all $1\leqslant j\leqslant r-1$, $|\sum_{1\leqslant i\leqslant j}A_i|<|\sum_{1\leqslant i\leqslant j+1}A_i|$.

It follows that, for all $1\leqslant j\leqslant r-1$,

$$\left|\sum_{1\leqslant i\leqslant j}A_i\right|\geqslant j+1. \tag{4.1}$$

According to our definition the empty sequence is an F-chain with length $=0$. In particular, chains always exist.

An F-chain with a maximal length will be called an *F-string*. Let $\mathcal{A}=\{A_1,\dots,A_{r+1}\}$ and $\mathcal{B}=\{B_1,\dots,B_{r+1}\}$ be F-strings. We shall write $\mathcal{A}<\mathcal{B}$ if and only if one of the following conditions holds:

(i) $\#A_{r+1}<\#B_{r+1}$,

(ii) $\#A_{r+1}=\#B_{r+1}$ and $|A_1+\cdots+A_r|<|B_1+\cdots+B_r|$.

A *maximal* string is a string that is maximal with respect to the order relation $\mathcal{A}=\mathcal{B}$ or $\mathcal{A}<\mathcal{B}$.

Lemma 4.1. ([12]) *Let $\mathcal{A}=\{A_1,\dots,A_{r+1}\}$ be an F-chain. Then $\mathcal{B}=\{A_1,\dots,A_{j-1},A_{j+1},\dots,A_r,A_j,A_{r+1}\}$ is an F-chain, for any $1\leqslant j\leqslant r-1$.*

Proof. Put $A_i=\{a_i,b_i\}$, $1\leqslant i\leqslant r$. Suppose that $\mathcal{B}$ is not a chain. Then there exists $m\in\{1,\dots,j-1,j+1,\dots,r-1\}$ such that

$$\left|\sum_{i\in\{1,\dots,j-1,j+1,\dots,m\}}A_i\right|=\left|\sum_{i\in\{1,\dots,j-1,j+1,\dots,m+1\}}A_i\right|.$$

Clearly $m\geqslant j-1$. By (2.5), $a_{m+1}-b_{m+1}\in\mathrm{St}\big(\sum_{i\in\{1,\dots,j-1,j+1,\dots,m\}}A_i\big)$. By (2.2), $a_{m+1}-b_{m+1}\in\mathrm{St}\big(\sum_{1\leqslant i\leqslant m}A_i\big)$. Hence $m\geqslant r$, a contradiction. □

Lemma 4.2. *Let $\mathcal{A}=\{A_1,\dots,A_{r+1}\}$ be an F-string such that $|A_{r+1}|\geqslant 3$. Let $w\in\|\mathcal{A}_{r+1}\|$. Put $Q=\mathrm{St}\big(\sum_{i\leqslant r}A_i\big)$. Then*

$$\|\mathcal{A}_{r+1}\|-w\subset Q. \tag{4.2}$$

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Proof. Suppose the contrary, and choose a in $\|\mathcal{A}_{r+1}\|$ such that $a - w \notin Q$. It follows that $\{A_1, \ldots, A_{r+1}, aw\}$ is an F-chain, contradicting our assumption that $\mathcal{A}$ is a string. □

Proposition 4.3. *Let $\mathcal{A} = \{A_1, \ldots, A_{r+1}\}$ be a maximal F-string such that $|\mathcal{A}_{r+1}| \geqslant 3$. Let $w \in \|\mathcal{A}_{r+1}\|$ and put $Q = \mathrm{St}\left(\sum_{i \leqslant r} A_i\right)$. Also assume that $|\sum_{1 \leqslant i \leqslant r} A_i| \leqslant |F| - r - 1$.*

Then, for all $1 \leqslant i \leqslant r$,

$$A_i \cap (Q + w) \neq \emptyset. \tag{4.3}$$

Proof. Take $a_i, b_i \in G$ such that $A_i = a_i b_i$, $1 \leqslant i \leqslant r$. Put $Q' = \mathrm{St}\left(\sum_{i \leqslant r-1} A_i\right)$. Let us prove the relation

$$A_{r+1} \cap (Q + w) \neq \emptyset. \tag{4.4}$$

Suppose the contrary. By (4.2),

$$A_r \cap \mathcal{A}_{r+1} = \emptyset.$$

By (2.4),

$$\left|\sum_{1 \leqslant i \leqslant r} A_i\right| \geqslant \left|\sum_{1 \leqslant i \leqslant r-1} A_i\right| + |Q'| \geqslant r + |Q'|.$$

By (4.2), $\#A_{r+1} \leqslant |Q'|$. Therefore we have

$$\begin{aligned} |\mathcal{A}_{r+1}| &= |F| - 2r \\ &\geqslant \left|\sum_{1 \leqslant i \leqslant r} A_i\right| - r + 1 \\ &\geqslant \left|\sum_{1 \leqslant i \leqslant r-1} A_i\right| + |Q'| - r + 1 \\ &\geqslant |Q'| + 1 \\ &\geqslant \#A_{r+1} + 1. \end{aligned}$$

In particular, there is a $u \in G$ such that

$$u^2 \leqslant A_{r+1}.$$

Since $a_r - b_r \notin Q'$, either $a_r - u \notin Q'$ or $b_r - u \notin Q'$. Without loss of generality, we may assume $a_r - u \notin Q'$. It follows that $\mathcal{B} = \{A_1, \ldots, A_{r-1}, (a_r u)\}$ is a string such that $\#A_{r+1} < \#B_{r+1}$, a contradiction establishing (4.4).

Now (4.3) holds by (4.2), if $A_i \cap \|A_{r+1}\| \neq \emptyset$. So assume

$$A_i \cap \|A_{r+1}\| = \emptyset.$$

Put $K = \mathrm{St}(A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_r)$. Set $\mathcal{C} = \{A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_r, A_i\}$. Now (4.3) holds by (4.4) if $\mathcal{C}$ is a string. Suppose the contrary. Using Lemma 4.1 and (2.5), it follows that $a_i - b_i \in K$. By (2.3), $K = Q$. Choose $z \in \|A_{r+1}\|$. Let us prove that

$$a_i - z \in Q.$$

Suppose the contrary. It follows that $a_i - z \notin K$.

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Set $\mathcal{D} = \{A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_r, (a_i z)\}$. By (2.5), $\mathcal{D}$ is a chain. Now $\#A_{r+1} \leqslant \#D_{r+1}$ and $|A_1 + \cdots + A_r| < |A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_r + (a_i z)|$, contradicting maximality of $\mathcal{A}$. We have shown that $a_i - z \in Q$. By (4.2), $a_i - w \in Q$. The proof of (4.3) is now complete. □

We can now extend the result of Bollobás and Leader.

Theorem 4.4. *Let G be an abelian group of order n, and let q, m be positive integers such that $q \leqslant m \leqslant 2q - 1$. Let $\mu = \{x_i;\ 1 \leqslant i \leqslant m\}$ be a sequence of elements of G. Then one of the following conditions holds:*

(i) *there is a $1 \leqslant i \leqslant m$ such that $qx_i \in q \wedge \mu$,*
(ii) $|q \wedge \mu| \geqslant m - q + 1$.

Proof. Let F be the multisubset of G associated with the sequence μ. Choose a maximal F-string $\mathcal{A} = \{A_1, \ldots, A_{r+1}\}$. Take $w \in \|A_{r+1}\|$. Set $Q = \mathrm{St}\left(\sum_{i \leqslant r} A_i\right)$. We have $|A_{r+1}| = m - 2r$. First assume $r \geqslant m - q$. Thus

$$\begin{aligned} |q \wedge \mu| &\geqslant |(m-q) \wedge (A_1, \ldots, A_{m-q}) + (2q - m) \wedge (A_{m-q+1}, \ldots, A_r . A_{r+1})| \\ &\geqslant |A_1 + \cdots + A_{m-q}| \\ &\geqslant m - q + 1. \end{aligned}$$

So we may suppose $r \leqslant m - q - 1$. Clearly

$$|q \wedge \mu| \geqslant |A_1 + \cdots + A_r + ((q-r) \wedge A_{r+1})|.$$

So we may assume $|A_1 + \cdots + A_r| \leqslant m - q \leqslant m - r - 1$. By (4.2),

$$((k-r) \wedge A_{r+1}) \cap ((k-r)w + Q) \neq \emptyset.$$

By (4.3), $(A_1 + \cdots + A_r) \cap (rw + Q) \neq \emptyset$. Therefore

$$(A_1 + \cdots + A_r + (k-r) \wedge A_{r+1}) \cap (qw + Q) \neq \emptyset.$$

But $(A_1 + \cdots + A_r + (k-r) \wedge A_{r+1})$ is Q-periodic. Therefore $qw + Q \subset q \wedge F$. Hence $qw \in q \wedge F$. □

Theorem 4.4 reduces to the result of Bollobás and Leader [2] if $q = |G|$.

5. An addition theorem

Let H be a subgroup of an abelian group G, and let X be a subset of G. The family of non-empty sets of the form $(x + H) \cap X$ induces a partition of X. We shall call it the *canonical* H-decomposition of X.

Theorem 5.1. *Let G be an abelian group such that $\epsilon(G) \leqslant 2$. Let A be a subset of G such that $0 \in A$. Let X be a subset of G such that $X + A \neq X + \langle A \rangle$. Then*

$$\max(|X + A|, |X + 2 \wedge A|) \geqslant |X| + |A| - 1.$$

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Proof. Let $X = X_1 \cup \cdots \cup X_v$ be the canonical $\langle A\rangle$-decomposition of X. There exists i such that $X_i + A \neq X_i + \langle A\rangle$. It follows that we may assume without loss of generality that A generates G.

Put $H = \mathrm{St}(X + A)$. Let $A = A_1 \cup \cdots \cup A_u$ be the canonical H-decomposition of A. Choose $a_j \in A_j$. We shall assume

$$|A_1| \leqslant \cdots \leqslant |A_u|.$$

Since A generates G and $0 \in A$, we necessarily have $u \geqslant 2$.

Put $\epsilon = |X + H| - |H|$. Assume that

$$|X + A| \leqslant |X| + |A| - 2.$$

By Kneser's theorem, $|X| + |A| - 2 \geqslant |X + A| \geqslant |X + H| + |A + H| - |H|$. Therefore

$$|A| - 2 \geqslant |A + H| - |H| + \epsilon. \tag{5.1}$$

In particular, we have $|H| \geqslant 2$. Since $X + A \neq G$, we have $H \neq G$. Therefore H is a proper subgroup.

Let us prove that

$$2A + H = (2 \wedge A) + H. \tag{5.2}$$

By (5.1), we have $|A_i| + (u - 1)|H| \geqslant |A| \geqslant 2 + (u - 1)|H|$. Therefore $|A_i| \geqslant 2$. Choose $b_i \in A_i \setminus a_i$. We have $2A_i + H = a_i + b_i + H \subset (2 \wedge A_i) + H$. Now

$$(2 \wedge A) + H = \bigcup_{i,j} ((A_i \wedge A_j) + H).$$

It follows that

$$(2 \wedge A) + H = \bigcup_{i,j} (A_i + A_j + H) = 2A + H.$$

First assume

$$X + 2A + H = X + (2 \wedge A).$$

Since A generates G and since $X + A \neq G$, we necessarily have

$$|X + (2 \wedge A)| = |X + H + A + A| > |X + A|.$$

Since these sets are H-periodic, we have

$$|X + (2 \wedge A)| = |X + H + A + A| \geqslant |X + A| + |H|.$$

By Kneser's theorem,

$$|X + (2 \wedge A)| \geqslant |X + A| + |H| \geqslant |X + H| + |A + H|.$$

The result holds in this case.

So we may assume

$$X + 2A + H \neq X + (2 \wedge A).$$

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By (5.2), we have

$$X + 2 \wedge A \neq X + (2 \wedge A) + H. \tag{5.3}$$

Let $1 \leqslant j \leqslant u$ be such that $|A_j| \geqslant (|H| + 3)/3$. Then

$$A \wedge A_j = A + A_j + H. \tag{5.4}$$

Let $1 \leqslant i \leqslant u$. Observe that $A_i \wedge A_j$ is contained in the H-coset $A_i + A_j$. First assume $i \neq j$. By Lemma 2.4, we have $A_i \wedge A_j = A_i + A_j = A_i + A_j + H$.

Now assume $i = j$. By Lemma 2.4, we have $2 \wedge A_j = 2A_j + H$. It follows that

$$A + A_j = \bigcup_{1 \leqslant i \leqslant u} A_i \wedge A_j = \bigcup_{1 \leqslant i \leqslant u} A_i + A_j + H = A + A_j + H.$$

By (5.3), it follows that $X + (2 \wedge A)$ is not H-periodic. In particular $\epsilon \geqslant 1$. By (5.1), $|A + H| - |A| \leqslant |H| - 3$. It follows that

$$|A_2| \geqslant (|A_1| + |A_2|)/2 \geqslant (2|H| - (|A + H| - |A|))/2 \geqslant (|H| + 3)/2.$$

By (5.4),

$$A \wedge A_i = A + A_i + H, \quad \text{for all } i \geqslant 2.$$

It follows that $X + (A \setminus A_1) + A = X + H + (A \setminus A_1) + A$. Therefore

$$\begin{aligned} |X + (A \setminus A_1) + A| &\geqslant |X + H + A| \\ &\geqslant |X + H| + |A + H| - |\mathrm{St}(X + A + H)| \\ &= |X + H| + |A + H| - |H|. \end{aligned}$$

By (5.3), and since $X + (A \setminus A_1) + A$ is H-periodic, $X + (2 \wedge A)$ contains a non-H-periodic part of the form $(A_i \wedge A_j) + X_0$, for some $X_0 \subset X$. Hence, $X + (2 \wedge A)$ contains the two disjoint subsets $X + (A \setminus A_s) + A$ and $(A_i \wedge A_j) + X_0$. It follows that

$$\begin{aligned} |X + (2 \wedge A)| &\geqslant |X| + 1 + |A + H| - |H| + |A_i| - 1 \\ &\geqslant |X| + (u - 1)|H| + |A_1| \\ &\geqslant |X| + |A|. \end{aligned}$$

□

Lemma 5.2. *Let G be a finite abelian group, let A be subset of G containing 0, and let X be a non-empty subset of G. Assume $A \setminus 0$ is a minimal generating set and that $X + A \neq G$. Then*

$$|(X + A) \setminus X| \geqslant |A| - 1.$$

Proof. Put $H = \mathrm{St}(X + A)$. By Kneser's theorem, $|X + A| \geqslant |X + H| + |A + H| - |H|$. We may assume clearly that H is proper subgroup. Consider the canonical H-decomposition $A = A_1 \cup \cdots \cup A_u$. Choose $a_j \in A_j$. Without loss of generality we may assume $a_1 = 0$. Since A generates G and $0 \in A$, we necessarily have $u \geqslant 2$. By the minimality of $A \setminus 0$, $0 \notin (A_1 \setminus 0) + (A_2 - a_2)$. By Lemma 2.2, $|A_1| + |A_2| - 1 \leqslant |H|$. It follows that $|A + H| - |A| \geqslant |H| - 1$. By Kneser's theorem, $|X + A| \geqslant |X + H| + |A + H| - |H| \geqslant |X| + |A| - 1$. Therefore $|(X + A) \setminus X| \geqslant |A| - 1$. □

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6. Distinct n-subsequence sums

Let us begin with a simple lemma.

Lemma 6.1. *Let $A_1,\dots,A_v$ with cardinality 2. Let X,Y be multisets such that $|Y|\geqslant n-v-j$. Then*

$$|n\wedge(XYA_1\cdots A_v)|\geqslant|A_1+\cdots+A_v+(|X|-j)\wedge X|.$$

Proof. We clearly have

$$n\wedge(XYA_1\cdots A_v)\supset A_1+\cdots+A_v+(|X|-j)\wedge X+(n-j-v)\wedge Y.$$

The result now holds, since $(|X|-j)\wedge X$ is a translate of $j\wedge X$ and since $(n-j-v)\wedge Y\neq\emptyset$. □

Theorem 6.2. *Let G be an abelian group of order n. Assume G is either cyclic or $|G|$ is odd. Let μ be a multisubset of G with a maximal multiplicity $\leqslant n-\#\mu+2$ such that $n+1\leqslant|\mu|$.*

Then, one of the following conditions holds:

(i) *there is a non-null subgroup H contained in $n\wedge\mu$,*

(ii) $|n\wedge\mu|\geqslant|\mu|-n+\#\mu-2$.

Proof. Put $u=|\mu|-n$ and $k=\#\mu$. The validity of the result for $u=n-k+2$ implies its validity for all u. So we may assume

$$u\leqslant n-k+2.$$

Take $\mu=EF$, where $|E|=\#E=k$. Choose a maximal F-string $\mathcal{A}=\{A_1,\dots,A_r\}$. Take $w\in A_{r+1}$. Since $\mathcal{A}-w$ is a maximal string with respect to $F-w$, $\mu-w=(F-w)(E-w)$, and $n\wedge\mu=n\wedge(\mu-w)$, we may assume without loss of generality that $w=0$. Let G_0 be the subgroup generated by E. Set $Q=\mathrm{St}\left(\sum_{i\leqslant r}A_i\right)$.

First assume $r\geqslant u-1$. We have

$$|A_u\cdots A_{r+1}|=n+u-k-2(u-1)=n-k-u+2\geqslant 2.$$

We may assume

$$A_1+\cdots+A_{u-1}+E\neq A_1+\cdots+A_{u-1}+G_0,$$

since otherwise $A_1+\cdots+A_{u-1}+E=G_0$, and (i) holds by Lemma 6.1. By Lemma 6.1 and Theorem 5.1,

$$\begin{aligned}|n\wedge\mu|&\geqslant\max(|A_1+\cdots+A_{u-1}+E|,|A_1+\cdots+A_{u-1}+2\wedge E|)\\&\geqslant u-1+k.\end{aligned}$$

So we may assume $r\leqslant u-2$. It follows that

$$|A_{r+1}|\geqslant n+u-2r\geqslant 4.$$

We have

$$|Q|\geqslant 2,$$

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since otherwise, by (4.3), $0 \in A_i$ for all $1 \leqslant i \leqslant r$ and $\|A_{r+1}\| = \{0\}$. It follows that $F(0) \geqslant r + (n-k+u-2r) \geqslant n-k+2$. Therefore $\mu(0) = E(0) + F(0) \geqslant n-k+3$, a contradiction.

Consider a μ-chain $\mathcal{B} = \{B_1, \ldots, B_s\}$, where $B_i = a_i b_i$, satisfying the following conditions:

(i) $B_i = A_i$, for all $1 \leqslant i \leqslant r$,
(ii) $s \leqslant u-1$,
(iii) $a_{r+1} \cdots a_s \leqslant A_{r+1}$ and $b_{r+1} \cdots b_s \leqslant E$, for all $r+1 \leqslant i \leqslant s$,
(iv) $0 \in A_{r+1}(a_{r+1} \cdots a_s)^{-1}$,
(v) $M + \langle E(b_{r+1} \cdots b_s)^{-1} \rangle = G_0$, where $M = \mathrm{St}\big(\sum_{i \leqslant s} B_i\big)$.

Note that $\mathcal{A}$ is such a chain. So we may choose such a chain with a maximal s. Put $V = A_{r+1}(a_1 \cdots a_s)^{-1}$ and $T = E(b_1 \cdots b_s)^{-1}$ By (2.2), $Q \subset M$. Since $B_1 + \cdots + B_r$ is Q-periodic, we have, for all $J \geqslant r$,

$$\begin{aligned} |B_1 + \cdots + B_J| &\geqslant |B_1 + \cdots + B_r| + (J-r)|Q| \\ &\geqslant r + 1 + 2J - 2r. \end{aligned}$$

In particular,

$$|B_1 + \cdots + B_s| \geqslant 2s - r + 1. \tag{6.1}$$

Case 1: $s = u-1$.
We may assume that $B_1 + \cdots + B_{u-1} + T \neq B_1 + \cdots + B_{u-1} + \langle T \rangle$, since otherwise

$$\begin{aligned} B_1 + \cdots + B_{u-1} + T &= B_1 + \cdots + B_{u-1} + M + T \\ &= B_1 + \cdots + B_{u-1} + \langle T \rangle + M \\ &= G_0, \end{aligned}$$

and the result follows by Lemma 6.1.

By Lemma 6.1, (6.1) and Theorem 5.1,

$$\begin{aligned} |n \wedge \mu| &\geqslant \min(|B_1 + \cdots + B_{u-1} + T|, |B_1 + \cdots + B_{u-1} + 2 \wedge T|) \\ &\geqslant |B_1 + \cdots + B_{u-1}| + |T| - 1 \\ &\geqslant u + k - 1. \end{aligned}$$

Case 2: $s \leqslant u-2$.
Let us observe that

$$\begin{aligned} |V| &= n + u - k - 2s \geqslant n + u - k - 2u + 4 \\ &= n - u - k + 4 \geqslant 2. \end{aligned}$$

Choose $v \in V \setminus 0$.

Let $\phi : G \longrightarrow G/M$. We shall write $\phi(x) = \overline{x}$. We claim that $\{\overline{x}; x \in (T \setminus M)\}$ is a minimal generating sequence of G/M. Suppose on the contrary that, for some $z \in (T \setminus M)$, $\{\overline{x}; x \in (T \setminus (M \cup \{z\}))\}$ generates G/M. Put $B_{s+1} = \{v, x\}$.

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Since $Q \subset M$, we have, by (4.2), $v \in M$. Since $z \notin M$, we have, by (2.5),

$$\left|\sum_{i\leqslant s+1} B_i\right| = \left|\sum_{i\leqslant s} B_i + \{v,z\}\right| > \left|\sum_{i\leqslant s} B_i\right|.$$

It follows that $B_{r+1},\ldots,B_{s+1}$ verifies conditions (i)–(v), contradicting the maximality of s. This proves our claim. In particular, for all $x, y \in (T \setminus M)$, $\overline{x} = \overline{y}$ holds only if $x = y$. Thus,

$$|\overline{T \setminus M}| = |T \setminus M|.$$

We have

$$|T| = k - s + r.$$

Assume first that $|V| + |T \cap M| \geqslant n - s$. We may suppose

$$|\Sigma A_i| \geqslant |F| - r + 1,$$

since otherwise $|\Sigma A_i + E| = n$, by Lemma 2.2. The result then holds, by Lemma 6.1.

We have $|V| = n + u - 2s - |T| \geqslant n - s + 2 - |T|$. Clearly

$$n \wedge \mu \supset P = B_1 + \cdots + B_s + (n-s) \wedge (V(T \cap M)).$$

By (4.2) and (4.3), $B_i \cap M \neq \emptyset$ for all $i \leqslant s$. Similarly $(n-s) \wedge (V(T \cap M)) \subset M$. It follows that $M \cap P \neq \emptyset$. On the other side $P + M = M$. It follows that $M \subset n \wedge \mu$. So we may assume $|V| + |T \cap M| \leqslant n - s - 1$. It follows that $n + u - 2s - |T| + |T \cap M| \leqslant n - s - 1$. Therefore

$$|T \setminus M| \geqslant u - s + 1.$$

We clearly have

$$\begin{aligned} |n \wedge \mu| &\geqslant |B_1 + \cdots + B_s + T + (n - s - 1) \wedge V| \\ &\geqslant |B_1 + \cdots + B_s + T|. \end{aligned}$$

The result clearly holds if $B_1 + \cdots + B_s + T = G$. So we may assume the contrary. Since $B_1 + \cdots + B_s + T$ is M-periodic, we have

$$\overline{B_1 + \cdots + B_s + T} \neq G/M.$$

By Lemma 5.2, we have

$$|(\overline{(B_1 + \cdots + B_s + T)}) \setminus (\overline{(B_1 + \cdots + B_s)}| \geqslant |T \setminus M|.$$

Therefore

$$\begin{aligned} |n \wedge \mu| &\geqslant |B_1 + \cdots + B_s + T| \\ &\geqslant |B_1 + \cdots + B_s| + |T \setminus M|\,|M| \\ &\geqslant 2s - r + 1 + |T \setminus M|\,|M| \\ &\geqslant 2s - r + 1 + |M| + 2(|T \setminus M| - 1) \\ &\geqslant 2s - r + |T| + |T \setminus M| - 1 \end{aligned}$$

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$$\geqslant 2s - r + |T| + u - s$$
$$\geqslant k + u.$$

The proof is complete. □

References

[1] Bialostocki, A. and Lotspeich, M. (1991) Some developments of the Erdős–Ginzburg–Ziv theorem I. *Colloquia Mathematica Societatis János Bolyai* **60** 97–117.
[2] Bollobás, B. and Leader, I. (1999) The number of k-sums modulo k. *J. Number Theory* **78** 27–35.
[3] Brakemeier, W. (1973) Ein Beitrag zur Additiven Zahlentheorie. Thesis, Braunschweig.
[4] Brakemeier, W. (1978) Eine Anzahlformel von Zahlen modulo n. *Monatsh. Math.* **85** 277–282.
[5] Caro, Y. (1996) Zero-sum Ramsey problems: A survey. *Discrete Math.* **152** 93–113.
[6] Eggleton, R. B. and Erdős, P. (1972) Two combinatorial problems in group theory. *Acta Arith.* **21** 112–116.
[7] Erdős, P., Ginzburg, A. and Ziv, A. (1961) Theorem in the additive number theory. *Bull. Res. Council Israel* **10**.
[8] Gao, W. and Geroldinger, A. (1999) On long minimal zero sequences in abelian groups. *Period. Math. Hungar.* **38** 179–211.
[9] Gao, W. and Hamidoune, Y. O. (1998) Zero sums in abelian groups. *Combin. Probab. Comput.* **7** 261–263.
[10] Gallardo, L., Grekos, G. and Pinko, J. (1999) On a variant of the Erdős–Ginzburg–Ziv problem. *Acta Arith.* **89** 331–336.
[11] Gallardo, L. and Grekos, G. (2000) On Brakemeier's variant of the Erdős–Ginzburg–Ziv problem. In *Number Theory* (Liptovský Ján, 1999), *Tatra Mt. Math. Publ.* **20** 91–98.
[12] Hamidoune, Y. O. (1995) On weighted sequence sums. *Combin. Probab. Comput.* **4** 363–367.
[13] Hamidoune, Y. (1998) Adding distinct congruence classes. *Combin. Probab. Comput.* **7** 81–87.
[14] Hamidoune, Y. O., Ordaz, O. and Ortunio, A. (1998) On a combinatorial theorem of Erdős, Ginzburg and Ziv. *Combin. Probab. Comput.* **7** 403–412
[15] Mann, H. B. (1965) *Addition Theorems*, Wiley, New York.
[16] Mann, H. B. (1967) Two addition theorems. *J. Combin. Theory* **3** 233–235.
[17] Nathanson, M. B. (1996) *Additive Number theory, 1: Inverse Theorems and the Geometry of Sumsets*, Vol. 165 of *Graduate Texts in Mathematics*, Springer, New York.
[18] Olson, J. E. (1976) A combinatorial problem of Erdős–Ginzburg–Ziv. *J. Number Theory* **8** 52–57.

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