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Maximin: a direct approach to sustainability

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ABSTRACT. We solve directly a general maximin (sustainment, intergenerational-equity) problem. Because the shadow values of a maximin problem do not correspond to the shadow values from a general discounted-utility solution, they correspond to the prices of only a very special competitive economy. Virtual discount factors for the economy arise. They do not correspond to hyperbolic discount factors. Hartwick's rule is derived and generalized naturally to take into account non-autonomous and non-deterministic features of the economy. Under uncertainty, Hartwick's rule is the analytic expression of a form of precautionary principle. Hotelling's rule is a necessary condition, but may be more complex than has been appreciated in simple models. Some interpretations of strong sustainment are special cases of weak sustainment but, paradoxically, may be more difficult to solve.

Maximizing welfare or sustaining utility?

In this paper, we discuss the conditions for sustaining, primarily 'weakly' sustaining, an abstract economy, on the assumption that it is sustainable. An erudite survey by Pezzey and Toman (2002–3) sums up what they call the neoclassical treatment of weak sustainability. That treatment utilizes a welfare integral, which Pezzey and Toman call *general* because the force of interest (or instantaneous rate of discount) need not be constant. Let U denote current utility or felicity, a function of a vector of consumption goods C; $GPV(t)$ the general present value (welfare), as of time t; and $\theta(t)$ the factor that discounts from time t to time 0. Then

$$GPV(t) = \int_t^\infty U(C(s))\frac{\theta(s)}{\theta(t)}\,ds,$$

subject to various conditions including transition equations for capital stocks. The problem faced by a planner is a standard control problem, to maximize $GPV(t)$.

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The literature being reviewed holds that the planner is interested in *sustaining* this economy. Let $M(t)$ be the *maximin* utility at time t

$$M(t) = \max m \text{ s.t. } U(C(s)) \geq m \text{ for all } s \geq t.$$

The criterion for sustainability at time t is that

$$U(C(t)) \leq M(t). \tag{1}$$

We perceive two inconsistencies in this received formulation. (i) The planner is not consistent. The planner's objective is to maximize $GPV(t)$, but, in addition, some notion of sustaining utility is a consideration. Pezzey and Toman (2002–3: 176, 177) remark, 'Maximizing GPV in general has nothing to do with sustainability, and gives a complete and unique prescription for the time paths of every decision that ever has to be made in the economy. So there is no apparent motive . . . to measure sustainability on a GPV-optimal path.' Pezzey and Toman reason that the double objective is schizophrenic. We agree. Motivational inconsistency may well characterize a *political-economic* model of social decisions, but not the actions of a rational planner. (ii) Somehow the maximin utility $M(t)$ must be in the mind of the planner, but the planner is myopic as regards sustaining utility, since criterion (1) applies at only the current instant. Interest in the future is confined to the welfare integral, and in that integral future generations' utility is discounted. This literature does not 'reject the notion that individuals should be regarded differently just because of the time when they are alive' (Pezzey and Toman, 2002–03: 168).

Another perspective that is current in the literature assumes current utility at t to be given by $GPV(t)$ (with the force of interest constant) and defines a realized path to be a sustainable-development path if

$$dGPV(t)/dt \geq 0. \tag{2}$$

Condition (2) also applies only at the current instant.

Sustaining is *forever*, not just for the current moment. Suppose the path of welfare, for $t_1 > 0$ and $v > 0$, is

$$\begin{aligned} GPV(t) &= v \exp(t - t_1)^2 \quad \text{for } t < 10t_1; \\ &= v \exp 9t_1^2 \quad \text{for } t > 10t_1. \end{aligned} \tag{3}$$

For $t < t_1$, current welfare is not being sustained according to criterion (2). But *the economy's* welfare is being sustained; sustained welfare is v. (For $t > t_1$, sustained welfare is $GPV(t)$.) Now let us replace $GPV(t)$ in equation (3) by $U(C(t))$. Maximin felicity $M(t)$ is almost surely greater than v and very likely, given the degree of substitutability usually assumed, greater than $U(C(0))$. If so, then criterion (1) holds that $U(C(0))$ is being sustained, whereas $U(C(0))$ is not being maintained for $t \in (0, 2t_1)$.

These considerations lead us away from maximizing welfare integrals to an explicit examination of the necessary conditions for a maximin policy (Rawls, 1972; Solow, 1974; Hartwick, 1977), which we argue is in

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harmony with philosophic rationales for sustaining an economy.[1] We investigate the importance of Hotelling's rule and of Hartwick's rule and its converse. We further interpret the property that a regular maximin path corresponds to one in which utility is discounted using specific discount factors. We extend the analysis to uncertainty and interpret Hartwick's rule as a type of precautionary principle. We show that weak sustainment is an overwhelmingly difficult implementational problem. Some interpretations of strong sustainment can be formally accommodated in our method, but these interpretations are even more technically complicated.

A maximin problem

We assume that there exists, for the given initial conditions of the problem, a path on which the objective can be maintained and which is Pareto optimal (does not involve 'throwing away' stocks which could increase utility temporarily).[2] We begin with the type of problem usually studied, in which there is no 'pure state constraint'. That is to say, there is no condition which involves state variables without also involving control variables.

Suppose that at any time t an economy has at its disposal $n > 0$ stocks, $(S_1, S_2, \ldots, S_n) = S$, and $r > 0$ control variables, $(u_1, u_2, \ldots, u_r) = u$. The stocks may be interpreted as stocks of natural resources, of produced or human capital or knowledge, of ability of the environment to cleanse itself, etc. They are subject to transition equations

$$\dot{S}_i = f_i(S, u, t), \quad i = 1, \ldots, n. \tag{4}$$

The controls may be interpreted as levels of consumption of various goods, as levels of investment in various stocks (including ameliorating environmental stocks), as levels of use of various environmental or natural resources, etc.

The economy is subject to various mixed constraints

$$g_i(S, u, t) \geq 0, \quad i = 1, \ldots, l'; \tag{5}$$

$$g_i(S, u, t) = 0, \quad i = l' + 1, \ldots, l. \tag{6}$$

For example, these could represent capacity constraints on the utilization of stocks.

Instantaneous felicity in this economy is $W(S, u, t)$, a function of the *current* levels of the stocks and the controls, as well as time. A planner sets as the goal of the economy to maximize the *sustained* level of felicity, i.e. to

$$\max \bar{W}, \tag{7}$$

subject to the above conditions and to the *sustainment* constraint

$$W(S, u, s) \geq \bar{W} \text{ for all } s \geq t. \tag{8}$$

[1] For modifications of the maximin principle and approaches toward a just savings principle based on Rawls (1972: section 44), see Long (2005).

[2] Burmeister and Hammond (1977: 858–860) give sufficient conditions for the existence of a regular maximin path, but also point out that they are stringent. We discuss regularity below.

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Following Leonard and Long (1992) and Takayama (1974), we define $\bar{W}$ to be a *control parameter*. We depart from Leonard and Long's analysis by not introducing an arbitrary discount factor by which to obtain an integral of $\bar{W}$, discounted by that factor, over all future time periods. As in standard control theory, we introduce n adjoint variables $p_i(t)$, to be interpreted as shadow values of the state variables $S_i(t)$. The problem is unlike many problems in control theory in that the objective is not an integral. In some ways it is analogous to an optimal-time problem (Takayama, 1974: 614). In addition, unlike many problems, this problem is non-autonomous: time appears as an explicit argument of the functions introduced above. In order to make it *formally* autonomous, we introduce a new state variable, $S_{n+1}(t)$, which tracks time in the sense that $S_{n+1}(0)=0$ and $\dot{S}_{n+1}(t)=f_{n+1}(S,u,t)=1$ for all t. Then we define the Hamiltonian to be

$$H(S,u,p,p_{n+1},S_{n+1})=\sum_{i=1}^{n} p_i(S_{n+1})f_i(S,u,S_{n+1})+p_{n+1}.$$

Finally, we introduce l multipliers λ_i and a multiplier w in defining a Lagrangian

$$\begin{aligned} &L(S,u,p,p_{n+1},\bar{W},\lambda,S_{n+1},w)\\ &\quad=H(S,u,p,p_{n+1},S_{n+1})+\sum_{i=1}^{l}\lambda_i(S_{n+1})g_i(S,u,S_{n+1})\\ &\qquad+w(S_{n+1})[W(S,u,S_{n+1})-\bar{W}]. \end{aligned} \tag{9}$$

Because the objective is not an integral, current felicity $W(S,u,t)$ does not appear in the Hamiltonian and appears in the Lagrangian only because of the sustainment constraint (8).

There are two technical problems associated with the fact that this problem has an infinite horizon. It is more transparent to address these problems after stating the necessary conditions that apply

$$\sum_{i=1}^{n} p_i\frac{\partial f_i}{\partial u_k}+\sum_{i=1}^{l}\lambda_i\frac{\partial g_i}{\partial u_k}+w\frac{\partial W}{\partial u_k}=0,\quad k=1,\ldots,r; \tag{10}$$

$$\dot{p}_k=-\frac{\partial L}{\partial S_k}=-\left[\sum_{i=1}^{n} p_i\frac{\partial f_i}{\partial S_k}+\sum_{i=1}^{l}\lambda_i\frac{\partial g_i}{\partial S_k}+w\frac{\partial W}{\partial S_k}\right],\quad k=1,\ldots,n; \tag{11}$$

$$\dot{p}_{n+1}=-\frac{\partial L}{\partial S_{n+1}}=-\sum_{i=1}^{n} p_i\frac{\partial f_i}{\partial S_{n+1}}-\sum_{i=1}^{l}\lambda_i\frac{\partial g_i}{\partial S_{n+1}}-w\frac{\partial W}{\partial S_{n+1}}; \tag{12}$$

$$\frac{dL}{dt}=\frac{\partial L}{\partial t}; \tag{13}$$

$$-\int_0^{\infty}\frac{\partial L}{\partial \bar{W}}\,ds=\int_0^{\infty} w(s)\,ds=1; \tag{14}$$

$$\lim_{t\to\infty} H(S,u,p,p_{n+1},S_{n+1})=0; \tag{15}$$

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$$\lim_{t\to\infty} p_i(t)S_i(t) = 0 \quad i = 1, \ldots, n; \tag{16}$$

$$(p, p_{n+1}, \lambda, w) \neq 0 \in R^{n+l+2}. \tag{17}$$

In addition, the usual complementary-slackness conditions hold: $w \geq 0$; $W - \bar{W} \geq 0$; $w(W - \bar{W}) = 0$; etc.

Condition (14) is called a transversality condition by Takayama (1974: 660) but can be viewed as an intertemporal first-order condition for maximizing with respect to the choice of the control parameter $\bar{W}$. Technically, in the expression of problem (7), we should write $\pi_0\bar{W}$ in place of the objective $\bar{W}$. Condition (14) must then be expressed as $\int_0^\infty w(s)\,ds = \pi_0 \geq 0$. In order to be sure that $\pi_0 > 0$, so that we may normalize to $\pi_0 = 1$, analysts typically invoke a sufficient condition called a *constraint qualification*. Long and Vousden (1977) show that in an infinite-horizon problem the constraint qualification for a problem with an integral objective is more stringent than the one used in static or finite-horizon problems. Our example of a simple fishery (see the appendix) indicates, however, that not even Long and Vousden's more stringent constraint qualification can assure that $\pi_0 > 0$ in a maximin problem.

The issue for a maximin problem is even deeper than whether $\pi_0 > 0$. It is possible to have $\int_0^\infty w(s)\,ds = \pi_0 > 0$ if $w(s)$ is positive on an interval but zero outside that interval. Whenever $w(s) = 0$, the sustainment constraint (8) is ineffective; we may have $W(S, u, s) > \bar{W}$. Having $\pi_0 > 0$ is not enough to ensure that the objective can be maintained without being able to throw away stocks (or to forgo utility) at some times, as we wished to ensure at the outset. A suitable constraint qualification (which would still be only a sufficient condition) has eluded us and instead we make the following assumption.

Assumption R: *Regularity*. For any interval $A = (t_1, t_2) \neq (0, \infty)$, constant utility $\bar{W}$ can be increased outside A by relaxing constraint (8) in A.

This assumption means that, from any interval, utility or felicity can be smoothly reallocated to the rest of the program. If not, we could end up with felicity $W(S, u, t) > \bar{W}$ on an interval and be able to forgo some felicity, $W(S, u, t) - \bar{W}$, while maintaining maximin felicity $\bar{W}$. This forgoing of felicity would not be Pareto efficient. It is intuitively clear that this assumption implies that $w(t) > 0$ everywhere (that to make the assumption is basically to assume this condition). It is also consistent with the intent of Burmeisterand Hammond (1977) in defining regularity.

The transversality condition (15) follows by an extension of the proof of a theorem due to Michel (1982). The proof is not informative and is placed in the appendix. Condition (16), another transversality condition, is related to the corollary to Michel's theorem. We note that our proof is quite general; these transversality conditions apply to any maximin problem, not just a regular one.

Because autonomous problems are the most commonly studied ones in the economics of sustainability, let us suppose first that the argument t does not appear explicitly in any of the functions f, g or W. We distinguish the autonomous program by tildes, eliminate reference to p_{n+1} and S_{n+1}, and

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show that the total value of net investment, evaluated at the shadow values of the various stocks, is zero.

Proposition 1: Hartwick's rule and converse
(*i*) Converse of Hartwick's rule. *In an autonomous maximin problem, sustaining utility implies that total net investment is zero; that is*

$$\tilde{H}(S, u, p) = \sum_{i=1}^{n} \tilde{p}_i(t) \tilde{f}_i(S, u) = 0, \text{ for all } t.$$

The state variables remain on a hypersurface $\tilde{\phi}(S) = \bar{W}$ *through time, and, if* $\tilde{\phi}$ *is differentiable,* $\tilde{p}_i(t) = [\partial \tilde{\phi}(S)/\partial S_i]|_t$, $i = 1, \ldots, n$.
(*ii*) Hartwick's rule. *If the total net investment, evaluated at the shadow prices, is nil, then social well being is constant.*

Proof. (i) Since $\partial \tilde{L}/\partial t = 0$

$$\tilde{H}(S, u, p)|_t = \tilde{L}(S, u, p, \bar{W}, \lambda, w)|_t = \tilde{L}(S, u, p, \bar{W}, \lambda, w)|_\infty = \tilde{H}(S, u, p)|_\infty = 0.$$

Also, the 'maximin utility' can be expressed as a function of the state variables, $\tilde{\phi}(S) = \bar{W}$, and (if they exist) the partial derivatives are equal to the shadow values. (ii) Burmeister and Hammond (1977) show that (if all the above functions are concave) the conditions we have derived are among the sufficient conditions. Transversality condition (15) is also a sufficient condition (Seierstad and Sydsaeter, 1987: 235). Cairns (1998) loosens the requirement of concavity somewhat. □

Proposition 1 depends on (a) the envelope condition (13) for the Lagrangian and (b) the transversality condition (15). The surface $\tilde{\phi}$ expresses the level of maximin utility in terms of the available stocks. All are standard properties. Hartwick's rule states that, if total investment, *evaluated at the shadow prices for this problem*, is nil, then utility is sustained. Now we generalize Hartwick's rule to the case in which underlying conditions in the economy do not remain constant for all time. For example, in reality there are short- and medium-term fluctuations and long-term trends in climate that are not controlled from within the economy.

Proposition 2: Hartwick's rule and converse in a non-autonomous problem
When the functions f, g, h, and W are not all autonomous, the Hamiltonian involves a shadow value of time. When felicity is constant, the levels of stocks lie on a hypersurface $\phi(S, t) = \bar{W}$. *The value of net investment exactly compensates for the value of exogenous change at any time. When the value of net investment compensates for the value of time, felicity is constant.*

Proof. The Hamiltonian is

$$H(S, u, p, p_{n+1}, S_{n+1}) = \sum_{i=1}^{n} p_i(S_{n+1}) f_i(S, u, S_{n+1}) + p_{n+1}(S_{n+1}),$$

and $p_{n+1}(S_{n+1})$ is the shadow value of time. In the appendix we show that $p_{n+1}(t) = \int_t^\infty (\partial L/\partial s)\, ds$ and that $H = 0$ for all $S_{n+1} = t$. The equation of the hypersurface is a function of the stocks and of time, $\phi(S, t) = \bar{W}$. Moreover,

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if $\phi(S, t)$ is differentiable

$$\sum_1^n p_i(t) f_i(S, u, S_{n+1}) + p_{n+1}(t) = \sum_1^n \frac{\partial \phi(S,t)}{\partial S_i} \dot{S}_i + \frac{\partial \phi(S,t)}{\partial t} = \frac{d\phi(S,t)}{\partial t} = 0.$$

Hartwick's rule holds by the argument used above. □

Time can be viewed literally as one of the state variables of this problem, having rate of growth equal to one. In this sense, Hartwick's rule can be said to be satisfied. Alternatively, the autonomous changes create a drift in the program. The value of net investment is equal to the negative of the value of this drift, $-\partial\phi(S, t)/\partial t = -p_{n+1}(t) = -\partial L|_t/\partial t$, i.e. to the negative of the shadow value of time.

The adjoint conditions (11) imply that a generalization of Hotelling's rule applies to each stock. The conditions may be complicated if they involve other shadow values as well; for example, the rate of extraction of oil from a reservoir may depend on the stock of gas in that reservoir. Such possibilities are neglected in many resource models. Simple problems are frequently solved by eliminating $\dot{p}_k$ using the adjoint condition and the time derivative of the first-order condition. Equations (10) and (11) indicate that this device is practicable only if there are few variables. Otherwise, it could be difficult to determine the shadow values. A non-autonomous extension of Solow's (1974) problem is presented as a worked example of the technique in the appendix.

The shadow value of equity

A shadow value of special interest is $w(t)$, the shadow value of the constraint $W(S, u, t) \geq \bar{W}$. Since this shadow value does not enter the Hamiltonian, it is not directly relevant to Hartwick's rule. In a regular program, there is sufficient malleability that there is no temporary period of extra felicity; the constraint is always effective: $w(t) > 0$ for all t.

By equation (14), the (improper) integral of the shadow value $w(t)$ is unity. It corresponds to a discount factor in a particular welfare maximization in the following sense.

Proposition 3: *The path* $[\bar{W}^*, S^*, u^*]$ *solves the maximin problem if and only if the path* $[S^*, u^*]$ *solves the following control problem*

$$\max \int_0^\infty w(s) W(S, u, s)\, ds, \tag{18}$$

subject to equations (4), (5), *and* (6) *and* $S|_0 = (S_1, \ldots, S_n)$ *and such that* $[w(t)]_{t=0}^\infty$, *a set of given discount factors, is equal to the set of shadow values of the constraint* (8) *of the maximin problem.*

Proof. It is well known that, in an autonomous discounted-utility problem, $\sum \tilde{p}_i \tilde{f}_i = 0$ if and only if $\tilde{W}(S, u)$ is constant. The extension to a non-autonomous problem is trivial. Therefore, under the normalization that $\int_0^\infty w(t)\, dt = 1$, the problems have the same necessary conditions and the same sufficient conditions. □

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There is a mapping from the solution of a regular maximin problem to the solution of a discounted-utility maximization, provided that the discount factors used in the latter are equal to the shadow values of the maximin constraint of the former. In equation (9), the term $wW(S, u, t)$ appears, as it would in the discounted-utility problem; the term $w\bar{W}$ is differentiated out of the first-order and adjoint conditions.

Unlike a discount factor, the shadow value $w(t)$ is not a measure of the relative values of increments to felicity at two different time periods. Such comparisons are not permissible under the maximin objective. Rather, it is the cost, in terms of the objective $\bar{W}^*$ (the sustained utility), resulting from a postulated unit deviation at some time from the constraint of maintaining $\bar{W}^*$. Consider a modification to our problem in which the constraint on felicity is that $W(S, u, t) \geq \bar{W} - \varepsilon(t)$, where $\varepsilon(t) = \delta$ for $t \in (t_1, t_1 + \Delta)$ and is zero elsewhere. Condition (16) implies that Leonard's (1987) envelope theorem can be extended to this infinite-horizon problem using his proof. That is to say

$$\frac{\partial \bar{W}^*}{\partial \delta} = \int_0^\infty \frac{\partial L(t)}{\partial \delta}\, dt = \int_0^\infty w(t) \frac{\partial \varepsilon(t)}{\partial \delta} = \int_{t_1}^{t_1+\Delta} w(t)\, dt \approx w(t_1)\Delta.$$

A two-period example can help to clarify the relationship between our analysis and that of Proposition 3. Let an economy have the transformation curve

$$c_2 = A - \pi c_1^2.$$

Given a utility function $U(c_t)$, can we have $c_2 = c_1 > 0$? Consider the problem

$$\max U(c_1) + wU(c_2).$$

In the solution, the following conditions hold

$$c_1 = A - \pi c_1^2, \text{ or } c_1 = \frac{(1 + 4\pi A)^{\frac{1}{2}} - 1}{2\pi};$$

$$U'(c_1) - w\pi c_1 U'\left(A - \pi c_1^2\right) = 0, \text{ or } w = 1/(\pi c_1) = 2/\left[(1 + 4\pi A)^{\frac{1}{2}} - 1\right].$$

The discount factor w that yields $c_1^* = c_2^*$ is the slope of the transformation curve at the point at which it intersects the ray $c_1 = c_2$. Once known, it can be plugged into the problem of maximizing discounted utility.

Now consider the maximin problem, max $\bar{c}$ such that $c_t \geq \bar{c}, t = 1, 2$. The multiplier w measures by how much $c_1 = \bar{c} = [(1 + 4\pi A)^{1/2} - 1]/[2\pi]$ is increased if the constraint $c_2 \geq \bar{c}$ is replaced by $c_2 \geq \bar{c} - \varepsilon$, for an arbitrarily small ε.

The comparison can be generalized to any number of periods $T < \infty$, with support prices $w_t, t = 2, 3, \ldots, T$. If $T = \infty$ it must be that $\sum_{t=2}^{\infty} w_t < \infty$. Dixit *et al.* (1980: 553) list, among others, the following condition for a *regular*

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maximin problem in continuous time

$$0 < \int_0^\infty w(t)\,dt < \infty.$$

Occasionally one sees for this condition a one-sided condition, $\int_0^\infty w(t) < \infty$. Our discussion in the appendix shows that, even though $w(t) \geq 0$, it is always true that $\int_0^\infty w(t) < \infty$. The conditions to check are (i) whether $\int_0^\infty w(t) > 0$ and hence whether $w(t) > 0$ on some interval(s) and (ii) whether $w(t) > 0$ for all t.

Proposition 3 states that an economy that follows a regular maximin path follows the same path as a hypothetical one that trades off utility at one time against utility at another according to discount factors. Unlike in some infinite-time models in growth theory, not admitting utility discounting is not the same as assuming an interest rate of zero in what may be called the virtual welfare integral. The virtual discount factors $w(t)$ are not social time preferences, however; social time preferences are expressed in the maximin objective. If time preferences do not coincide with $[w(t)]_0^\infty$ then, in a discounted-utility problem, utility is not sustained. If the objective is discounted utility, putting any significance at all on whether the economy is sustained is, in any case, inconsistent.

Associated with the virtual discount factor at time t is a virtual force of interest

$$\rho(t) = -\dot{w}(t)/w(t).$$

Integrating gives that

$$w(t) = k \exp\left[-\int_0^t \rho(s)\,ds\right],$$

and the constant k is chosen such that $w(t)$ integrates to unity. A constant virtual force of interest (where, for some constant r, $w(t) = re^{-rt}$) would be observed in a steady state. In Solow's original problem since $\dot{K} > 0$ and $\dot{R} < 0$, it must be that $\dot{\rho} = \dot{F}_K < 0$; discounting becomes progressively less severe than discounting at a constant force of interest.

A controversial proposal is the hyperbolic discount factor

$$w(t) = \frac{a}{a + bt},$$

for which the force of interest

$$\rho(t) = \frac{b}{a + bt},$$

is indeed declining. Unlike $w(t)$, the hyperbolic discount factor is not integrable over an infinite horizon. We can state the following immediately.

Proposition 4: *The virtual discount factor in a virtual maximin problem cannot be expressed as a hyperbolic discount factor.*

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The hyperbolic discount factor has the disadvantage of not being time consistent: the relative value of the two discount factors changes as time passes, since, for $s > t > 0$

$$\frac{w(s)}{w(t)} = \frac{a + bt}{a + st},$$

whereas at time $t/2$ the relative value is

$$\frac{w(s - t/2)}{w(t/2)} = \frac{a + bt/2}{a + b(s - t/2)} \neq \frac{a + bt}{a + st}.$$

Some authors use a virtual welfare integral to study regular sustainment problems indirectly. Virtual discount factors in a sustainment problem are unlikely to involve a constant force of interest, and may well involve a decreasing force of interest. They do not adhere to axioms postulated by Koopmans (1960). Pezzey (2004) solves a special case in which the discount factor takes a form that is akin to hyperbolic discounting and in which the virtual force of interest at any time is a function of the level of the stocks at that time. Pezzey's example emphasizes that, for sustainment, there is a specific set of virtual discount factors that 'works', rather than that just any welfare integral can be postulated.

Underlying Proposition 3 is a composition of two mappings which depend on the Pareto optimality of the solutions (and on other requirements such as convexity). By the two theorems of welfare economics, a regular maximin solution can be mapped to a competitive solution which in turn can be mapped to the solution to a discounted-utility maximization.[3] The *problems* are not so mapped, however. Indeed, they express incompatible economic values. In a maximin program, generation s's felicity is not *valued* at $w(s)/w(t)$ times generation t's, as it is in discounted-utility maximization both on and off the optimal path. Rather, the marginal unit of generation s's felicity has only the technical opportunity cost of $w(s)/w(t)$ times generation t's. If either generation actually does get an additional unit of felicity for some reason, that additional unit has value to the planner only in as much as it is redistributed to all. If the question is whether or not society is on the optimal path (whether society is being sustained), the distinction between the problems overshadows the parallels between the optimal paths.

This technical opportunity cost does raise a further consideration. If ever $w(t) = 0$ (if the problem is not regular), maximin felicity cannot be increased by any policy at time t. If felicity is constant over the program and the program is Pareto efficient, the optimal path must be a golden-rule path, and $w(t) = 0$ for all t. If felicity is not constant over the program or if the

[3] If the maximin solution is decentralized to a competitive economy, the planner must ensure that the particular, endogenous path of the associated virtual discount factors is decentralized. Only if the discount factors and other shadow values are right can a decentralized economy follow a constant-utility path.

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program is not Pareto efficient then (as always)

$$\bar{W} = \inf_t W(S, u, t).$$

Let Z be the set on which $W(S, u, t) = \bar{W}$ if $\bar{W} > 0$ is attained, or be $t = \infty$ if not. If $Z \neq (0, \infty)$, then $w(t) = 0$, $t \in (0, \infty) - Z$. There is some constraint preventing the smoothing of consumption from times $t \in (0, \infty) - Z$ to times in Z. A *regular* path, as discussed by Burmeister and Hammond (1977), is one for which felicity is constant for all time and which is Pareto efficient, but is not a golden-rule path. This implies that $w(t) > 0$ for all t in a regular program.

Our analysis applies to non-regular paths as well as the regular ones we have stressed. A non-regular path does not share the interpretation in terms of virtual discounted-utility maximization. Consider a regular economy, in which any amount of current felicity can be smoothly reallocated to the rest of the program from intervals centred on any date. Now consider, in the spirit of Leonard's (1987) proof, the bestowing of non-transferable manna on the economy, granting a far higher level of felicity on an interval without the use of any of the other stocks. On the interval, $w(t) = 0$ and the interpretation of $w(t)$ as a discount factor breaks down. The breakdown is related to Pareto's criterion. Maximin is not a customary problem in that it stresses equity as the fundamental value. There is no conflict with satisfying Pareto's criterion, but if $W(S, u, t) > \bar{W}$ the difference does not contribute to the objective.

Do you take your sustainment weak or strong?

Confining attention to constraints of the form (5), (6), and (8), which involve both the state and control variables, obviates some technical difficulties which may arise at times when a pure state constraint, such as $h(S, t) \geq 0$, becomes effective or ineffective. A natural such constraint is that the level of some stock be nonnegative. In an important class of problems, including Solow's (1974) problem, all stocks are essential for production; no nonnegativity constraint concerning a stock is ever effective on the optimal path. It would be imprudent, however, to neglect the possibility that, on the optimal path, certain stocks might be driven to zero in finite time, so that their nonnegativity constraints would be effective. For example, horses are no longer used as draft animals. A certain type of capital or resource could be useful but not essential. (Petroleum was not used until the mid-nineteenth century.) On the other hand, a new type of capital good could be brought onstream. (Computers did not always exist.) Under conditions which Seierstad and Sydsaeter (1987: ch 5, 6) elaborate, at the point when a particular stock begins or ceases to be constrained by a condition $h(S, t) \geq 0$, there can be a discontinuity in the shadow value of the particular stock. Whether there are such discontinuities and what their magnitudes are must be determined individually in each particular problem.

Hotelling's rules (11) may involve the shadow values of other stocks. Consequently, the rule for a state variable whose state constraints are never effective on the optimal path must take into account any jumps in the shadow values of other state variables. The shadow values for stocks

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may be discontinuous at points when the state constraints involving those stocks become or cease to be binding. This is a further complication of Hotelling-type rules. We remark in passing that such considerations would also apply in discounted-utility models, rendering Hotelling's rule in a market economy more difficult to express and to verify empirically.

The jumps in the shadow values could conceivably cause jumps in the Hamiltonian. The condition $dL/dt = \partial L/\partial t$, used in the proof of Hartwick's rule, holds *almost* everywhere, and not necessarily at the points of discontinuity. However, our economic intuition leads us to conjecture that before and after points of discontinuity Hartwick's rule continues to hold and the Hamiltonian remains at zero. If this conjecture is right, then, whenever there is a jump in a shadow value $p_i(t)$, there must be a jump in the level of net investment in at least one other stock $j \neq i$.

If we admit pure state constraints, our approach becomes formally applicable to analyzing stronger versions of sustainment. Consider the objective of maximizing a sustained index of environmental stocks, E

$$\max \bar{I} \text{ such that } I(E, t) \geq \bar{I}.$$

This is a special case of the objective of this paper. We remark that expressing the index implies that smooth trade-offs of stocks, giving rise to shadow prices, must be contemplated by the proponents. Furthermore, this objective has what most would consider an ethical defect in taking no note of the human condition. Alternatively, a set of constraints $I(E, t) \geq 0$ can be entertained as either constraining desired levels of stocks or expressing threshold values in problem (7). Such constraints must be compatible over infinite time; they also imply trade-offs and shadow prices. Pure state constraints, with their attendant analytical difficulties, appear to be a defining feature of strong sustainment.

Sustainment under uncertainty

Uncertainty arises from events which affect the evolution of the stocks, including environmental stocks. Since the number of controls, r, is not necessarily equal to the number of state variables, n, some stocks can be viewed as being the current levels of uncontrolled, background variables which evolve stochastically. One can even conceive of stochastic events, such as a meteor's striking the earth, which are so devastating as to render unsustainable any program for sustainment: Like the mouse's foresight in Robert Burns's *To a Mouse*, society's 'foresight may be vain'.[4] Unlike the mouse, a society aiming to sustain the value of its objective can try to avoid the possibility that its decisions might be overturned at some point. In general, no matter what society does, sustaining felicity cannot be guaranteed.

[4] Foresight may not all be vain, however. Suppose the stock of knowledge could be improved to permit deflecting or destroying the meteor before it caused harm. The level of uncertainty could be endogenous, too.

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To consider the implications of such uncertainty, let the stocks S_k evolve according to a stochastic process involving both Brownian motion and Poisson 'jumps'. Let there be M statistically independent sources of jumps that affect each stock[5] so that, for source of jump i, $i = 1, \ldots, M$, and for each stock S_k

$$\begin{aligned} dq_{ik} &= y_{ik}(S,t), \text{ with probability } \mu_i\, dt \\ &= 0, \text{ with probability } 1 - \mu_i\, dt, \end{aligned}$$

where $y_{ik}(S,t)$ has a known probability distribution. Let dz_k be the increment for Brownian motion, so that $Edz_k = 0$.[6] Let $\rho_{ij}(S,\ u,\ t)$ be the correlation between dz_i and dz_j. Then, we assume that stocks evolve according to the following equation

$$dS_k = f_k(S,u,t)\,dt + \sigma_k(S,u,t)\,dz_k + \sum_i dq_{ik}.$$

We also assume that the constraints (5) and (6) from the deterministic problem hold.

Since sustaining felicity cannot be guaranteed in an uncertain environment, the planner must update the control parameter (target) as events unfold, using the information available at the time. Let the selected value of the control parameter at time τ be written $\bar{W}(\tau)$. The constraint on the current generation is that it does not consume so much as to reduce the *expected* felicity that can be sustained in an uncertain world.[7] Let E_r be the expectations operator, applied at time τ. At any time $t \geq \tau$, we impose the condition that

$$E_\tau W[S(t), u(t), t] \geq \bar{W}(\tau). \tag{19}$$

The planner's objective depends on the levels of the stocks at time τ, $S(\tau)$, and is conditional on the control parameter put into effect at time τ, $\bar{W}(\tau)$.

What the planner aims to sustain through time is an expected value, revised at each future date τ in light of the stocks available. Let that expected value at time $t \geq \tau$ be $E_\tau \Phi[S(t), t] = \bar{W}(\tau)$. As before, we confine our attention to problems in which it is feasible to sustain this expectation through time. Then

$$\max_u E_\tau \Phi[S(t), t] = \max_u E_\tau \Phi[S(t) + dS, t + dt], \quad \text{or } \max_u E_\tau d\Phi[S(t), t] = 0. \tag{20}$$

[5] We view these random jumps as exogenous to rational decision making or control. They could be climatic, for example.

[6] Since the sum of normal variates with mean zero is normal with mean zero, the process dz_k may be taken to be the sum of a number of processes with mean zero and affecting stock k.

[7] Any risk aversion is, as usual, represented in the degree of concavity of the felicity function.

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Let $o(dt)$ gather all terms of order smaller than dt. We use Taylor's Theorem and Ito's Lemma to expand equation (20) as follows

$$\begin{aligned}
0 &= \max_u E_\tau d\Phi|_t \\
&= \max_u E_\tau \left\{ \Phi_t\, dt + \sum_k \Phi_k d S_k + \frac{1}{2}\sum_k \Phi_{kt} d S_k\, dt \right. \\
&\quad \left. + \frac{1}{2}\sum_j \sum_k \Phi_{kj} d S_k d S_j + o(dt) \right\} \\
&= \max_u E_\tau \left\{ \Phi_t\, dt + \sum_k \Phi_k d S_k + \frac{1}{2}\sum_j \sum_k \Phi_{kj} d S_k d S_j + o(dt) \right\}.
\end{aligned}$$

Recalling that $E dz_k = 0$, we have

$$\begin{aligned}
E_\tau \sum_k \Phi_k d S_k &= E_\tau \sum_k \Phi_k \left[f_k(S,u,t)\, dt + \sigma_k(S,u,t)\, dz_k + \sum_i dq_{ik} \right] \\
&= \sum_k \Phi_k \left[f_k(S,u,t)\, dt + \sum_i E_\tau dq_{ik} \right] \\
&= \sum_k \Phi_k f_k(S,u,t)\, dt + \sum_k \sum_i \Phi_k E_\tau dq_{ik},
\end{aligned}$$

where $S + y_{ik}\,(S, t)$ has $S_k + y_{ik}\,(S, t)$ as its kth component but otherwise is equal to S. Then

$$E\left[\frac{1}{2}\sum_j \sum_k \Phi_{kj} d S_k d S_j \right] = \frac{1}{2}\sum_j \sum_k \Phi_{kj} \rho_{kj} \sigma_k \sigma_j .$$

After substitution of the maximizing value of u, we obtain

$$\Phi_t + \sum_k \Phi_k f_k + \sum_k \sum_i \Phi_k E dq_{ik} + \frac{1}{2}\sum_j \sum_k \Phi_{kj} \rho_{kj} \sigma_k \sigma_j = o(dt). \tag{21}$$

At the optimum, Hartwick's rule holds

$$\begin{aligned}
&H(\Phi_t, \Phi_k, \Phi_{kj}, S, u, t) \\
&\quad = \Phi_t + \sum_k \Phi_k f_k + \sum_k \sum_i \Phi_k E dq_{ik} + \frac{1}{2}\sum_j \sum_k \Phi_{kj} \rho_{kj} \sigma_k \sigma_j = 0.
\end{aligned} \tag{22}$$

The planner maximizes the Hamiltonian, $H(\Phi_t, \Phi_k, \Phi_{kj}, S, u, t)$, over choices of $u(t)$. For economy of notation, let the set of controls that obey the constraints (5), (6), and (8) be represented by $U(t)$. Then there is a function

$$L(\Phi_t, \Phi_k, \Phi_{kj}, S, u, t, \lambda, w) = H + \sum_i \lambda_i(t) g(S,u,t) + w(t)[W(S,u,t) - \bar{W}(\tau)],$$

for which $\max_u L\ (\Phi_t,\ \Phi_k,\ \Phi_{kj},\ S,\ u,\ t,\ \lambda,\ w) = \max_{u\in U} H\ (\Phi_t,\ \Phi_k,\ \Phi_{kj},\ S, u, t) = 0$.

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We assume that the matrix Φ_{jk} is negative semi-definite. The shadow values Φ_k correspond to the shadow prices p_k in the deterministic problem. The term $\Sigma_k \Phi_k f_k$ represents the deterministic part of the chosen changes in stock levels, evaluated at their shadow values Φ_k, and the term Φ_t represents the drift of the program. There are two additional terms which sum to the value of the total variance affecting the economy

$$\sum_k \sum_i \Phi_k E dq_{ik} + \frac{1}{2} \sum_j \sum_k \Phi_{kj} \rho_{kj} \sigma_k \sigma_j .$$

A referee puts to us what we consider a succinct definition of the precautionary principle: 'In deference to fears of asymmetric risk, we should take mitigating measures beyond those justified by expected welfare considerations.' Our objective is not an expected welfare integral, but the maximin level of utility. The solution is a level of permitted current utility that is the best estimate of what may be expected to hold for all time. The method mediates among present and future utility levels, using all current information to try to achieve intergenerational equity. The permitted utility level is updated after the realization of stochastic events and incorporates the risk aversion inherent in the function $W(S, u, t)$. The drift and the variance are forms of generalized depreciation (appreciation) which society must make up (can draw down) through net investments (disinvestments) in the stocks in order to pursue uncertain sustainment. By adjusting Hartwick's rule for the uncertainty facing the economy, equation (22) expresses a type of precautionary principle.

Proposition 5: Precautionary principle. *On an uncertain path of sustained felicity, the value of net investment, evaluated at the shadow values from the optimal program, exactly compensates for the sum of the value of any drift of the program and the value of the total variance of the program.*

The stochastic analogs to the adjoint conditions, which we interpret as Hotelling's rules, are, for each stock k

$$d\Phi_k = -\mathcal{L}_k^* dt + \sum_j \frac{\partial \Phi_k}{\partial S_j} \sigma_j dz_j + \sum_j \frac{\partial \Phi_k}{\partial S_j} (dq_{jk} - E dq_{jk}). \tag{23}$$

Although $E d\Phi_k = -\mathcal{L}_k^* dt$, observed prices evolve as stochastic events are realized and the program is updated.

Proposition 6: Hotelling's rule. *Shadow prices of stocks adjust according to equation (23) and are equal to the marginal values of those stocks as assets to be used in producing value for the economy in future periods.*

We interpret Hotelling's rule as being primarily about equilibrium price rather than about rent. As Ricardo argued, 'price causes rent'.

Decreasing $\bar{W}$ slightly increases Φ by $\partial \Phi / \partial \bar{W}$. This change allows a small change of $W(S, u, t)$ at each time $t \geq \tau$, and that change has a value, $w(t)$. The total gain is $\int_\tau^\infty w(t)\, dt$. As under certainty, this integral is finite and under

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a normalization the multipliers $w(t)$ can be interpreted as virtual discount factors.

Conclusion

Our method of solving the problem of weak sustainment under conditions of certainty is simple and synthetic, even though in a complex world finding the solution itself would be formidable. Hartwick's rule is robust in the sense that new terms are added in a simple, intuitive way. The method is flexible enough to handle formally, not just weak sustainment, but also some notions of strong sustainment. The mathematical difficulty expresses the difficulty of determining (let alone implementing) a sustained path, even for a presumed omniscient and omnipotent planner.

All assets obey a form of Hotelling's rule as a necessary condition. In contrast to Hartwick's rule, Hotelling's rule is much more complicated than in simple, partial-equilibrium models. The complexity arises from extending the model to greater numbers of types of capital.

Dixit *et al.* (1980) have argued that the sustained path is Pareto optimal and hence corresponds to the path of a specific, competitive economy. Although we have eschewed the discounting of future utilities, the problem obtains the same solution path as would a hypothetical society maximizing discounted utility using *virtual discount factors* corresponding to the shadow value of equity, i.e., of constraint (8). The virtual discount factors are time consistent; they are not hyperbolic. In some problems, however, the virtual force of interest may decrease through time. If there are many types of capital, Hotelling's rule may be complicated. Although the competitive economy discussed by Dixit *et al.* is a virtual one, the fact that Hotelling's rule is so complicated in that economy suggests that in a real, market economy, the rule may not be readily verified empirically.

In that virtual, competitive economy, Hartwick's rule applies at each instant to investment valued at the shadow prices from the solution to the maximin problem. Only if the prices are equal to the shadow prices emanating from the maximin solution can one apply Hartwick's rule to determine whether a competitive economy is following a sustained path.

If society is pursuing the goal of sustainment under uncertainty, Hartwick's rule is the analytic expression of a type of precautionary principle.

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Appendix

Constraint qualifications

It is instructive to study the shadow price π_0 in a finite-horizon problem. We define $W(S, u, t)$ to be a new state variable with a free initial value $W(S_0, u, 0)$ and transition equation

$$\dot{W}(S, u, t) = F(S, u, t) \tag{24}$$

where $F(., ., .)$ is identically zero. This problem is a standard one,[8] namely to

$$\max \left\{ W(S_0, u, 0) + \int_0^T F(S, u, t)\, dt \right\}$$

[8] See Hestenes (1966, Theorem 11.1), Long and Vousden (1977: 14), Takayama (1974: 656 ff.) or Leonard and Long (1992, Theorem 7.8.1).

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subject to the same conditions as in problem (7) and with $S_i(0) = S_{i0}$ fixed and $w(0)$ free, and $S_i(T)$ free and $w(T)$ free. The Lagrangian becomes

$$L_F(S,u,p,q,t,\lambda,w) = \sum_{i=1}^{n} p_i(t) f_i(S,u,t) + q(t)F(S,u,t) + \sum_{j=1}^{l+1} \lambda_j(t) g_j(S,u,t) + w(t)[W(S,u,t) - \bar{W}]$$

To simplify notation we define

$$g_{l+1}(S,u,t) = W(S,u,t) - \bar{W} \geq 0 \quad \text{and} \quad \lambda_{l+1}(t) = w(t), \tag{25}$$

and assume that the following is satisfied.

Constraint qualification: At any point along the optimal path, the matrix

$$(\text{M K}) = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} & \cdots & \frac{\partial g_1}{\partial u_r} & g_1^* & 0 & \cdots & 0 \\ \frac{\partial g_2}{\partial u_1} & & & & 0 & g_2^* & \cdots & 0 \\ \cdots & & & & 0 & 0 & \cdots & 0 \\ \frac{\partial g_{l+1}}{\partial u_1} & & & \frac{\partial g_{l+1}}{\partial u_r} & 0 & 0 & \cdots & g_{l+1}^* \end{bmatrix} \text{ has rank } l+1. \tag{A}$$

In addition to the necessary conditions in the text, we have

$$\dot{q}(t) = -\frac{\partial L_F}{\partial \bar{W}} = w(t); \tag{26}$$

$$\dot{W}(S,u,t) = \frac{\partial L_F}{\partial q} = F(S^*,u^*,t) = 0; \tag{27}$$

$$dL_F/dt = \partial L_F/\partial t; \tag{28}$$

$$\pi_0 \bar{W} + q(0) = 0; \tag{29}$$

$$q(T) = 0; \tag{30}$$

$$p_i(T) = 0 \text{ for } i = 1,2,\ldots,n; \tag{31}$$

$$H_F|_T = 0; \tag{32}$$

$$H_F|_t = -\int_t^T \frac{\partial L_F}{\partial s} ds; \tag{33}$$

$$\int_0^T w(t)\,dt = 1. \tag{34}$$

Proof. See any of the references in the footnote above. Condition (30) holds because $\bar{W}|_T$ is free and condition (31) holds because $S_i(T)$ is free. Conditions (32) and (28) imply condition (33) since

$$H_F|_t = L_F|_t = L_F|_T - \int_t^T \frac{dL_F}{ds} ds = H_F|_T - \int_t^T \frac{\partial L_F}{\partial s} ds = -\int_t^T \frac{\partial L_F}{\partial s} ds.$$

We get from condition (29) that

$$q(0) = -1. \tag{35}$$

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Integrate equation (26) to get, in light of condition (30)

$$\int_0^T w(t)\,dt = q(T) - q(0) = -q(0). \tag{36}$$

Equations (35) and (36) imply condition (34). □

In an infinite-horizon problem, without the imposition of additional assumptions, condition (34) may not be met. The problem is *not* nonconvergence of the integral; the integral always converges, because condition (30) still holds as $T \to \infty$ and hence condition (36) holds

$$\lim_{T\to\infty} \int_0^T w(t)\,dt = \lim_{T\to\infty} q(T) - q(0) = -q(0). \tag{37}$$

Rather, the problem is that, if $q(0) = 0$, $w(t)$ is equal to zero for all t. To sharpen our understanding of this issue, we consider the following example.

Example: Failure of the extended constraint qualification

A fishery with a large initial stock. Let S be a fish stock, with the transition equation

$$\dot{S} = S(1 - S) - u, \tag{38}$$

where u is the harvest rate. Let $W(S\ u,\ t) = u$. The optimal control problem is to

$$\max_u \bar{W}$$

subject to (38), with the initial condition

$$S(0) = S_0$$

and the constraint

$$u(t) - \bar{W} \geq 0.$$

Since $\partial[u - \bar{W}]/\partial u = 1$, the constraint qualification (A) is satisfied. The golden-rule stock level is the stock S_G that maximizes consumption at all times. This stock gives rise to the maximum sustainable yield

$$S_G = \arg\max S(1 - S) = \frac{1}{2}.$$

Clearly, if $S_0 \geq S_G$, then the maximin solution is $\bar{W} = 1/4$, with $u^*(t) \geq 1/4$ for all t. Suppose, for example, that $S_0 = 3/4$. Consider a feasible maximin path with $u^*(t) = 1$ as long as $S(t) > 1/2$, and $u^*(t) = 1/4$ as soon as $S(t) = S_G = 1/2$ and remains at that value thereafter. Let t_1 be the time S reaches S_G from above. Then $w(t) = 0$ for all $t \in (0, t_1)$. The Lagrangian is

$$L = p(t)[S(t)(1 - S(t)) - u(t)] + w(t)[u(t) - \bar{W}]$$

Consider the conditions

$$\frac{\partial L}{\partial u(t)} = -[p(t) - w(t)] = 0 \tag{39}$$

$$\dot{p}(t) = -\frac{\partial L}{\partial S(t)} = -p(t)[1 - 2S^*(t)]. \tag{40}$$

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For the path $u^*(t)$ described above, $p(t) = w(t) = 0$ for all $t \in (0, t_1)$. By equation (40), $p(t) = 0$ forever. Hence $w(t) = 0$ forever, and $\int_0^\infty w(t)\,dt = 0$. Condition (34) is not met.

Maximin felicity is given by $\bar{W}(S) = S(1 - S)$ for $S \in (0, 1/2]$, and $\bar{W}(S) = 1/2$ for $S \geq 1/2$. At time t, if $S(t) \in (0, 1/2)$, then

$$w(t) = p(t) = (\partial \bar{W}(S)/\partial S)_t = 1 - 2S(t).$$

By condition (40), for $s \geq t$

$$w(s) = p(s) = [1 - 2S(t)] \exp[-(1 - 2S(t))(t - s)] > 0,$$

and $\int_t^\infty w(s)\,ds = 1$.

For $S \geq 1/2$, however

$$w(t) = \frac{\partial \bar{W}(S)}{\partial S}|_t = 0 \text{ for all } t.$$

The argument leading to equation (36) applies only for finite T, even in cases where transversality conditions such as $\lim_{t\to\infty} p_i(t)S_i(t) = 0$ or $\lim_{t\to\infty} p_i(t) = 0$ are satisfied. In the fishery example, when $S_0 \geq 1/2$, $\pi_0 = 0$, and hence, by condition (29), $q(0) = 0$. Since $\dot{q}(t) = w(t)$, and $\lim_{t\to\infty} q(t) = 0$

$$\int_0^\infty w(t)\,dt = \int_0^\infty \dot{q}(t)\,dt = q(\infty) - q(0) = 0 - 0.$$

We must make additional assumptions to ensure that $q(0) = -1$.

Long and Vousden (1977: 17) show that, for infinite-horizon problems, to prove that $\pi_0 = 1$ a stronger constraint qualification is needed:

Constraint qualification: The matrix

$$\bar{Q} = \begin{bmatrix} \text{M} & \text{K} \\ \text{N} & 0 \end{bmatrix} \text{ has rank } n + l + 1, \tag{B}$$

where

$$(\text{M K}) = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} & \cdots & \frac{\partial g_1}{\partial u_r} & g_1^* & 0 & \cdots & 0 \\ \frac{\partial g_2}{\partial u_1} & & & & 0 & g_2^* & \cdots & 0 \\ \cdots & & & & 0 & 0 & \cdots & 0 \\ \frac{\partial g_{l+1}}{\partial u_1} & & & \frac{\partial g_{l+1}}{\partial u_r} & 0 & 0 & \cdots & g_{l+1}^* \end{bmatrix} \text{ and } N = \left(\frac{\partial f_i}{\partial u_j}\right).$$

In the fishery example above

$$\bar{Q} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

and the constraint qualification (B) is violated. We cannot assert that $\pi_0 > 0$. The reader can easily check that the constraint qualification is violated in the same way even when $S_0 > 1/2$.

Extension of Michel's theorem

The problem of this paper has the following differences from the one analysed by Michel (1982). (1) Ours is a problem of Bolza–Hestenes

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(Takayama, 1974: 656 ff.) with a control parameter and without an integral objective. Michel's problem is expressed as an integral of a function discounted at a constant interest rate. (2) There are mixed constraints. (3) The problem is non-autonomous. (4) There is a failure of Michel's corollary.

The idea of Michel's proof is to consider a finite-horizon problem that is the original problem truncated to a finite time $T > 0$, with the rest of the program (for $t > T$) summarized as a terminal value. He introduces a concept of artificial time, $z(t)$, which passes at a rate $\dot{z}(t) = v(t) > 0$ that can be controlled by the planner, and he redefines the paths of state and control variables consistently with the passage of artificial time. 'Joining' the integral part with the terminal value is done at $z(T)$, where the state variables are constrained to reach the values they attain in the original problem at time T. Subject to this condition, $z(T)$ can be controlled smoothly and affects the values of the integral and terminal parts smoothly. The Hamiltonian of the new problem is H_t from the original problem multiplied by v (t). Optimal control of v (t) implies that $H_t = 0$ at any $t \in [0, T]$. This finding is true for any truncation point $T > 0$, and Michel shows that $\lim_{T\to\infty} H_T = 0$.

In our proof we make the following changes: (1) We change the problem into a maximin problem to be *split* (not truncated) at time T. (2) We introduce the constraints in writing the Lagrangian L in place of the Hamiltonian H. (3) We add an argument to Michel's function f and write $f(Y(t), U(t), z(t))$. (4) We note that Michel's functions g and h are identically zero. For ease of comparison we use his notation, writing X in place of the vector S of the text and C in place of u. His multiplier a (our π_0) does not appear explicitly but is behind the scenes.

Proof. (1) *Lemma (optimality of a problem split at a finite time).* Our original problem is to maximize $\bar{W}$ subject to the vector of transition equations, $\dot{X} = f(X(t), C(t), t)$, with $X(0) = X_0$; the vector of constraints $g(X(t), C(t), t) \geq 0$; and the sustainment constraint $W(X(t), C(t), t) - \bar{W} \geq 0$. Let the optimal trajectory for this problem be $(\bar{X}(t), \bar{C}(t))_{t=0}^{\infty}$. For any $T > 0$, divide the problem into two parts, a finite-horizon problem defined on the interval $[0, T]$ and an infinite-horizon problem defined on (T, ∞), with stocks at T being $\bar{X}(T)$ in both parts. The finite-horizon problem is to max $\tilde{W}(0, T, X_0)$ subject to the same conditions, except that (i) the sustainment constraint becomes $W(X(t), C(t), t) \geq \tilde{W}(0, T, X_0)$ for $t \in [0, T]$ and (ii) $X(T) = \bar{X}(T)$. Also let $\tilde{W}\left(T, \infty, \bar{X}(T)\right)$ represent the maximin felicity on the interval (T, ∞), with initial stocks $\bar{X}(T)$.

Following Michel, we define

$$\dot{Y}(t) = v(t) f(Y(t), U(t), t), \quad Y(0) = X_0 \text{ and } Y(T) = \bar{X}(T);$$
$$\dot{z}(t) = v(t) \text{ and } z(0) = 0.$$

As in the proof of Michel's lemma (pp. 981–982), for any admissible trajectory on $[0, T]$, $z(t) = \int_0^t v(s)\, ds$ is a continuous, strictly increasing and invertible function. Let $C(s) = U(z^{-1}(s))$, $X(s) = Y(z^{-1}(s))$ and $S = z(T)$.

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Since $Y(0) = X_0$

$$X(z(t)) = Y(t) = X_0 + \int_0^t \dot{Y}(s)\,ds = X_0 + \int_0^t \dot{z}(s) f(X[z(s)], C[z(s)], z(s))\,ds$$

$$= X_0 + \int_0^{z(t)} f(X(u), C(u), u) du.$$

$$X(s) = X_0 + \int_0^s f(X(u), C(u), u)\,du. \tag{41}$$

Therefore, $X(t)$ is differentiable almost everywhere, with derivative $f(X(t), C(t), t)$. The terminal condition $Y(T) = \bar{X}(T)$ gives that

$$X(S) = X(z(T)) = Y(T) = \bar{X}(T).$$

For $s \geq S$, the trajectory $(X(s), C(s))$ is defined by

$$X(s) = \bar{X}(s - S + T) \text{ and } C(s) = \bar{C}(s - S + T).$$

Trajectory $(X(s), C(s))$ is continuous and satisfies the transition equations and initial conditions. This is shown by equation (41) for $s \leq S$. For $s \geq S$, differentiation yields

$$\dot{X}(s) = \frac{d}{ds}\bar{X}(s - S + T) = f(\bar{X}(s - S + T), \bar{C}(s - S + T), s)$$

$$= f(X(s), C(s), s).$$

Since the path $(X(s), C(s))_{s=0}^{\infty}$ is feasible, by definition of $\bar{W}$ we have

$$\min\{\tilde{W}(0, T, X_0), \tilde{W}(T, \infty, \bar{X}(T))\} \leq \bar{W},$$

and so the optimal trajectory for the split problem satisfies

$$(\bar{Y}(t), \bar{U}(t), \bar{z}(t), \bar{v}(t))_{t=0}^{\infty} = (\bar{X}(t), \bar{C}(t), t, 1)_{t=0}^{\infty}.$$

(2) *Theorem (converse of Hartwick's Rule*: $H_t = 0$ *everywhere*). In the finite-horizon part of the program, the Lagrangian $\tilde{L}_T$ includes an additional term $p_{n+1} f_{n+1}(Y, U, z(t)) = p_{n+1} v(t) = q_T(t) v(t)$. The $(r + 1)$th constraint, $W(Y, U, z) - \bar{W} \geq 0$, is subsumed into the inner product $\lambda_T \cdot g(Y, U, t)$.

$$\tilde{L}_T = v p_T \cdot f(Y, U, z) + v \lambda_T \cdot g(Y, U, z) + q_T v.$$

Maximizing $\tilde{L}_T$ with respect to v gives that

$$L_T = p_T \cdot f(\bar{Y}, \bar{U}, \bar{z}) + \lambda_T \cdot g(\bar{Y}, \bar{U}, \bar{z}) + q_T = 0 \text{ for all } t \in [0, T].$$

By the lemma, the Lagrangian is maximized at $\bar{v}(t) = 1$ and is also maximized with respect to $U(t)$. Since $\lambda_T \cdot g(\bar{Y}, \bar{U}, \bar{z}) = 0$

$$H_T = 0$$

and $q_T = -p_T \cdot f(\bar{Y}, \bar{U}, \bar{z}) = -p_T \cdot f(\bar{X}, \bar{C}, t)$. At any time $t \in [0, T]$

$$\dot{p}_T = -p_T \cdot f_X - \lambda_T \cdot g_X; \text{ and}$$

$$\dot{q}_T = -\frac{\partial L_T}{\partial t}.$$

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Therefore

$$q_T(t) = L_T(T) + \int_t^T \frac{\partial L_T}{\partial s}\, ds = \int_t^T \frac{\partial L_T}{\partial s}\, ds.$$

The limiting properties of $p_T(t)$ and $q_T(t)$ discussed by Michel (1982: 983–984) carry over to this problem and so we have proved the converse of Hartwick's rule. For an autonomous problem, $\partial \tilde{L}_T/\partial z = 0$ and so $q_T = 0$ for all $t \in [0, T]$.

(3) *Corollary (limiting prices zero)*. Michel's corollary gives conditions for which $\lim_{t\to\infty} p(t) = 0$. It requires that the possible values of changes in the state variables, $f(X, C, t)$, be contained in a ball centred on 0. This condition does not always hold. For example, the stock of an exhaustible resource cannot be increased. Entropy cannot be decreased. (Entropy, however, would seem to render meaningless any notion of sustaining, even weakly, a society forever.) In many cases a threshold of some resource would cause the corollary to fail. If the only stocks are renewable resources (e.g. a fishery) and manufactured capital, the condition holds.

Application: A Modification of Solow's Problem

In this section we use a common representation of exogenous technical progress to illustrate our method of analysis in a simple example (cf. Withagen 1990: 383–400; Aronsson *et al.* 1997). In Solow's (1974) model, let there be exogenous technical progress at rate $\gamma > 0$. An exhaustible resource with flow rate q and a capital good K are both inputs to a Cobb–Douglas production function with outputs being consumption c and investment $\dot{K}$. Let instantaneous utility be $U(c)$. Output is

$$Y(t) = c + \dot{K} = F(K, q, t) = e^{\gamma t} K^{\alpha} q^{\beta}, \tag{42}$$

where $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Let the initial stocks of resource and capital be R_0 and K_0. The input of the resource is given by $q = -\dot{R}$.

We proceed to utilize our method and in so doing we show that $w(t) > 0$ for all t. (The problem is regular.) Efficiency requires that

$$\int_0^{\infty} q(t)\, dt = R_0. \tag{43}$$

To find an efficient constant-consumption path, we consider the maximin problem

$$\max \bar{W}$$

subject to $U(c) \geq \bar{W}$ and the other conditions above.

Let the state variable that tracks time be $S(t) = t$. The Lagrangian for this problem is

$$L = -p_1 q + p_2[e^{\gamma S} K^{\alpha} q^{\beta} - c] + p_3 + w[U(c) - \bar{W}].$$

The necessary conditions include

$$-p_1 + p_2 F_q = L_q = 0; \tag{44}$$

$$-p_2 + wU' = L_c = 0; \tag{45}$$

$$\dot{p}_1 = -L_R = 0; \tag{46}$$

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$$\dot{p}_2 = -L_K = -p_2 F_K; \tag{47}$$

$$\dot{p}_3 = -L_S = -\gamma p_2 F; \tag{48}$$

$$-p_1 q + p_2[F(K,q) - c] + p_3 = H = 0; \tag{49}$$

$$(p_1, p_2, p_3, w) \neq (0,0,0,0); \tag{50}$$

$$\int_0^\infty w(t)\,dt = 1. \tag{51}$$

By condition (45), if $w = 0$, then $p_2 = 0$. But then $p_1 = 0$ by condition (44). Hence, $p_3 = 0$ by condition (49). This contradiction of condition (50) implies that $w > 0$ for all t, and hence $U(c) = \bar{W}$ for all t. The problem is regular.

Condition (46) is Hotelling's rule. Since $\dot{p}_1 = 0$, we have that $p_1(t) = \bar{p}_1$, a constant. Definition (42) implies that $qF_q = \beta F$ and $F_s = -\gamma F$, and so, from conditions (48) and (44), we have

$$\dot{p}_3 = -\gamma p_2 F = -\gamma p_2 q F_q/\beta = (\gamma/\beta)\bar{p}_1 \dot{R}. \tag{52}$$

Now, by condition (43), $\lim_{t\to\infty} q(t) = 0$. Also, by conditions (45) and (51), $\lim_{t\to\infty} p_2(t) = \lim_{t\to\infty} (w/U') = 0$. By condition (49), $\lim_{t\to\infty} p_3(t) = 0$. Thus

$$p_3(t) = \frac{\gamma}{\beta}\bar{p}_1 R(t). \tag{53}$$

The shadow value of time is proportional to the resource stock.

Another condition that can be readily derived from the necessary conditions is that

$$F_K = \frac{\dot{F}_q}{F_q} = -\frac{\dot{p}_2}{p_2} = -\frac{\dot{w}}{w}. \tag{54}$$

In solutions to Solow's problem, the equality $F_K = \dot{F}_q/F_q$ is sometimes identified as Hotelling's rule.

We proceed with the analysis by proving the following special case:

Proposition 7: *If R_0 and K_0 are such that*

$$R_0 = \left(\frac{1}{\alpha}\right)^{1/\beta}\left(\frac{\gamma}{\beta}\right)^{(1-\beta)/\beta} K_0^{(1-\alpha)/\beta}, \tag{55}$$

then the efficient level of constant consumption is

$$c(t) = c_0 = \left(\frac{\gamma}{\beta}\right)^{\beta} K_0^{\alpha} R_0^{\beta}.$$

Along the optimal path, the capital stock is constant: $K(t) = K_0$. The stock of the resource falls at rate γ/β: $R(t) = R_0 e^{-(\gamma/\beta)t}$.

Proof. Suppose that condition (55) holds. We construct a candidate solution and check all of the necessary conditions. To this end, let $K(t) = K_0$ for all t, so that $\dot{K}(t) = 0$ and $Y(t) = c(t)$. Also, let $q(t) = (\gamma/\beta)\ R_0 e^{-(\gamma/\beta)t}$.

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Then $\int_0^\infty q(t)\,dt = R_0$: condition (43) holds. Also

$$c(t) = Y(t) = e^{\gamma t} K_0^\alpha \left(\frac{\gamma}{\beta}\right)^\beta R_0^\beta e^{-\gamma t} = K_0^\alpha \left(\frac{\gamma}{\beta}\right)^\beta R_0^\beta = c_0.$$

By condition (54)

$$\frac{\dot{w}}{w} = \frac{\dot{p}_2}{p_2} = -F_K = -\frac{\alpha F}{K} = -\frac{\alpha c_0}{K_0},$$

so that, in view of condition (51)

$$w(t) = \frac{\alpha c_0}{K_0} \exp\left(-\frac{\alpha c_0}{K_0} t\right).$$

In conjunction with condition (45) this implies that

$$p_2(t) = U'(c_0)\frac{\alpha c_0}{K_0} \exp\left(-\frac{\alpha c_0}{K_0} t\right).$$

Moreover, since $\dot{K} = 0$, we have from conditions (49) and (46) that

$$p_3(t) = \bar{p}_1 q(t) = \bar{p}_1(\gamma/\beta) R_0 e^{-(\gamma/\beta)t}.$$

Differentiation yields

$$\dot{p}_3 = -\bar{p}_1(\gamma/\beta)^2 R_0 e^{-(\gamma/\beta)t}.$$

On the other hand, using equation (52), we obtain that

$$\dot{p}_3 = -\gamma p_2 F = -\gamma p_2 c_0 = -\gamma U'(c_0)\frac{\alpha c_0^2}{K_0} \exp\left(-\frac{\alpha c_0}{K_0} t\right).$$

These equations are consistent if and only if

$$\frac{\gamma}{\beta} = \frac{\alpha c_0}{K_0} \tag{56}$$

and

$$\gamma U'(c_0)\frac{\alpha c_0^2}{K_0} = \bar{p}_1 \left(\frac{\gamma}{\beta}\right)^2.$$

The first of these is satisfied because of condition (55). The second is met by setting $\bar{p}_1 = (\beta^2/\gamma)\,U'(c_0)(\alpha c_0^2/K_0)$. □

Equation (55) gives the locus of points (K, R) which are steady states of a virtual system in which technical change is viewed as being embodied in the resource, so that effective resource input is $q(t)\,e^{(\gamma/\beta)t} = q_0$ and effective remaining resources are $R(t)\,e^{(\gamma/\beta)t} = R_0$. Since $\alpha + \beta < 1$, the locus is strictly convex in the positive orthant of the space (K, R). Equation (56) indicates that consumption is proportional to the steady-state capital stock and to the rate of technical progress.

Away from the steady state, equation (53) still holds. Substituting for p_3 and p_2 in Hartwick's rule (49) and then manipulating using the relation, $F = \dot{K} + c$, yield

$$\dot{K} = \frac{\beta - \gamma R/q}{1 - (\beta - \gamma R/q)}\bar{c}.$$

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We can make the following argument by general reasoning. Suppose the initial stocks lie north-west of the locus of virtual steady states. Since $\dot{R} < 0$, the path in the plane (K, R) must eventually meet that locus. Since a point directly south could be attained by throwing away some resource, the eventual steady state must lie south-east of the initial point, with a higher value of K and therefore of $\bar{c}$ than the point due south. Throwing away any of the resource is ruled out, because in that case we would have $\bar{p}_1 = 0$, and hence all of the shadow values equal to zero. Moreover, in approaching the steady state, we must have $\dot{K} \to 0$, for at the steady state $q/R = \gamma/\beta$.

Suppose now that the initial point is south-east of the locus of virtual steady states. Then it would be possible to achieve a steady state by throwing away capital. This is again suboptimal, for then $p_2 = 0$ and hence all the shadow values are zero. To maintain consumption constant, we need to increase capital without bound as the resource is exhausted.

Our reasoning suggests that, when $\gamma > 0$, there is always a solution with strictly positive consumption for any positive initial values of the stocks. In a virtual steady state, $c = \gamma K/(\alpha\beta)$. If R_0 is at least the value given by condition (55), then for a given value of $\alpha + \beta$ consumption is maximized when $\alpha = \beta$. As γ decreases, the locus of virtual steady states rotates clockwise and eventually approaches the K axis. When $\gamma = 0$, we have Solow's original case. There is no steady state. As is well known, in Solow's problem there is a maximin solution only if $\alpha > \beta$.

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