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A boundary-layer theory for plates with initial stress

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Abstract. A theory is presented for small deformations of a plate which is initially stressed, the state of initial stress not necessarily being obtained by elastic deformation from some other unstressed state. The method used is an asymptotic integration of the full three-dimensional equations. The usual Kirchhoff boundary conditions for stretching and bending are deduced from the analysis.

1. *Introduction.* In recent years Friedrichs and Dressler ((1)) and Reiss and Locke ((2)) have given boundary-layer theories for isotropic elastic plates. The method used by these authors is a systematic asymptotic integration of the full three-dimensional equations of linear elasticity. In the first place they obtain successive approximations for the 'interior' problem, that is the region away from the edge of the plate. It is not possible to satisfy arbitrary specified boundary conditions on the edge surface of the plate with these equations. To deal with this situation a further asymptotic integration is employed. The classical theories of bending and stretching of plates are recovered as the first approximation to the 'interior' problems. Further they deduce the usual Kirchhoff boundary conditions which must be applied to the major terms of the interior stresses. Gol'denveizer ((8)) has given a similar analysis.

The purpose of this paper is to present a boundary-layer theory for vibrating plates with initial stress using the methods developed in (1) and (2). As an immediate corollary we obtain the classical theory of anisotropic plates. (Those whose sole concern is classical infinitesimal elasticity can obtain the relevant equations by setting the initial stress zero throughout.) The technique used here differs slightly from the previous approach in that it avoids using a 'matching' procedure between the interior and boundary-layer stresses. This simplification was first given by Green ((3)) in a discussion of the boundary layer in the linear theory of elastic shells.

2. *Basic theory of bodies with initial stress.* We use the theory of classically small deformations in the presence of initial stress given by Green ((4)). We do, however, make some changes in notation. (It is, perhaps, useful to list the changes in notation. The quantities $\epsilon\mathbf{w}$, $\mathbf{t}$, $\epsilon\mathbf{t}'$, λ'^{ij}, γ'_{ij} in (4) are denoted by $\mathbf{w}$, $\mathbf{T}$, $\mathbf{t}$, τ^{ij}, γ_{ij} respectively in the present work.) Points in the body are defined by a general curvilinear system of coordinates θ^i which move with the body as it deforms. The base vectors and metric tensors associated with θ^i coordinates in the initial body B_0 are $\mathbf{g}_i$, $\mathbf{g}^i$ and g_{ij}, g^{ij}, respectively, so that

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j, \quad g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j, \quad g^{ir} g_{rj} = \delta^i_j, \tag{2·1}$$

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where δ^i_j is the Kronecker delta. The displacement vector $\mathbf{w}$ is given by

$$\mathbf{w} = w^i \mathbf{g}_i = w_i \mathbf{g}^i. \tag{2·2}$$

The strain tensor is γ_{ij} where

$$\gamma_{ij} = \tfrac{1}{2}(\mathbf{g}_i . \mathbf{w}_{,j} + \mathbf{g}_j . \mathbf{w}_{,i}) = \tfrac{1}{2}(w_i\|_j + w_j\|_i). \tag{2·3}$$

In these formulae a comma denotes partial differentiation with respect to θ^i and the double line denotes covariant differentiation with respect to θ^i where the Christoffel symbols are formed from the metric tensors g_{ij}, g^{ij}. Latin indices take the values 1, 2, 3 and repeated indices are summed if they appear twice.

We suppose that the body B_0 is subjected to initial stress and is at constant temperature T_0. We may not know how the body B_0 acquired this stress; in particular it may not be by elastic deformation from some other unstressed state. We then assume that the body B_0 is subjected to displacements $\mathbf{w}$ and temperature changes T. For the initial stress distribution the stress vector $\mathbf{T}$, per unit area of B_0, associated with a surface in B_0 whose unit normal is $\mathbf{n}$, is

$$\left.\begin{aligned} \mathbf{T} &= n_i E^{ij} \mathbf{g}_j, \\ \mathbf{n} &= n_i \mathbf{g}^i = n^i \mathbf{g}_i. \end{aligned}\right\} \tag{2·4}$$

The tensor E^{ij}, which is symmetric, gives the six components of initial stress. They can be arbitrary functions of θ^i apart from satisfying the condition that B_0 is in equilibrium with zero body force, that is

$$E^{ij}\|_i = 0. \tag{2·5}$$

The stress vector $\mathbf{T}+\mathbf{t}$, associated with a surface in the deformed body whose unit normal in its position in B_0 is $\mathbf{n}$ and measured per unit area of B_0, is given by

$$\mathbf{T}+\mathbf{t} = n_i(E^{ij}+\tau^{ij})\,\mathbf{g}_j. \tag{2·6}$$

In the present work we restrict attention to isothermal deformations so that

$$\tau^{ij} = E^{ijrs}\gamma_{rs} + E^{ir}w^j\|_r, \tag{2·7}$$

where

$$E^{ijrs} = E^{jirs} = E^{ijsr} = E^{rsij}. \tag{2·8}$$

The coefficients E^{ij} and E^{ijrs}, which are derived from the free energy, are, in general, functions of the coordinates θ^i. If the initial body B_0 is homogeneous then

$$E^{ij}\|_m = 0, \quad E^{ijrs}\|_m = 0. \tag{2·9}$$

If body forces are neglected the equation of motion is

$$\tau^{ij}\|_i = \rho \frac{\partial^2 w^j}{\partial t^2}, \tag{2·10}$$

where ρ is the density of B_0.

3. *Surface coordinates.* We consider a plane surface M and we denote the position vector of any point of M by $\mathbf{r}$, referred to some origin O, where $\mathbf{r}$ is a function of curvilinear coordinates θ_1, θ_2 only, so that

$$\mathbf{r} = \mathbf{r}(\theta_1, \theta_2). \tag{3·1}$$

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At every point of M we erect the unit normal $\mathbf{a}_3$, and we denote the perpendicular distance of a point on $\mathbf{a}_3$ from M by θ_3, so that $\theta_3 = 0$ is the surface M. We restrict θ_3 to lie in the interval

$$-h \leqslant \theta_3 \leqslant h, \tag{3·2}$$

where h is constant. The plane surfaces $\theta_3 = \pm h$ are taken as the boundary surfaces of a body which includes the surface M. The body is called a plate, the boundaries $\theta_3 = \pm h$ are the faces of the plate, M is the middle surface and $2h$ the thickness of the plate. The generators of the edge surface S are taken to be perpendicular to M and intersect M in the 'smooth' curve C enclosing a simply connected region D.

The position vector $\mathbf{R}$ of a point of the plate is defined by the equation

$$\mathbf{R} = \mathbf{r}(\theta_1, \theta_2) + \theta_3 \mathbf{a}_3, \tag{3·3}$$

where $\mathbf{a}_3$ is a constant unit vector. Thus

$$\mathbf{g}_\alpha = \frac{\partial \mathbf{R}}{\partial \theta^\alpha} = \mathbf{a}_\alpha, \quad \mathbf{g}_3 = \mathbf{a}_3, \tag{3·4}$$

Greek indices take the values 1, 2 and

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \theta^\alpha}.$$

It follows that

$$g_{\alpha\beta} = \mathbf{a}_\alpha . \mathbf{a}_\beta = a_{\alpha\beta}, \quad g_{\alpha 3} = 0, \quad g_{33} = 1. \tag{3·5}$$

In view of (3·4) and (3·5) the displacement vector $\mathbf{w}$ may be expressed in the form

$$w = w^\alpha \mathbf{a}_\alpha + w^3 \mathbf{a}_3 = w_\alpha \mathbf{a}^\alpha + w_3 \mathbf{a}_3, \tag{3·6}$$

where

$$w^3 = w_3, \quad w^\alpha = a^{\alpha\beta} w_\beta. \tag{3·7}$$

Hence from (2·3), (3·4) and (3·6)†

$$\left.\begin{aligned} \gamma_{\alpha\beta} &= \tfrac{1}{2}(w_{\alpha|\beta} + w_{\beta|\alpha}), \\ \gamma_{3\alpha} &= \frac{1}{2}\left(w_{3|\alpha} + \frac{\partial w_\alpha}{\partial \theta^3}\right), \\ \gamma_{33} &= \frac{\partial w_3}{\partial \theta^3}, \end{aligned}\right\} \tag{3·8}$$

where a single vertical line denotes covariant differentiation with respect to middle surface coordinates θ^α with the use of Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$, and

$$\Gamma^\alpha_{\beta\gamma} = \mathbf{a}^\alpha . \frac{\partial \mathbf{a}_\beta}{\partial \theta^\gamma} = \frac{a^{\alpha\lambda}}{2}\left(\frac{\partial a_{\beta\lambda}}{\partial \theta^\gamma} + \frac{\partial a_{\gamma\lambda}}{\partial \theta^\beta} - \frac{\partial a_{\beta\gamma}}{\partial \theta^\lambda}\right). \tag{3·9}$$

The order of covariant differentiation with respect to surface coordinates is immaterial.

4. *Basic equations.* In this section we give the complete set of equations which are essential for the subsequent work. For simplicity we only consider the case when the initial body B_0 is homogeneous and in a state of plane stress,

$$E^{\alpha\beta} \neq 0, \quad E^{\alpha 3} = E^{3\alpha} = 0, \quad E^{33} = 0. \tag{4·1}$$

† We note that $w_{3|\alpha} = \partial w_3 / \partial \theta^\alpha$.

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Because of the assumed homogeneity of B_0, we see from (2·5), (2·9) and the results of section 3 that

$$E^{\alpha\beta}|_\gamma = 0, \quad E^{ijrs}|_\beta = 0 \tag{4·2}$$

and that B_0 is in equilibrium. We impose the further restriction that $E^{\alpha\beta}$, E^{ijrs} are independent of θ^3. It is convenient to define a tensor F_{ijrs} with symmetry properties

$$F_{ijrs} = F_{jirs} = F_{ijsr} = F_{rsij}, \tag{4·3}$$

by the equations

$$E^{ijrs}F_{ijmn} = \tfrac{1}{2}(\delta^r_m \delta^s_n + \delta^r_n \delta^s_m). \tag{4·4}$$

Using the results of this and the preceding section we see that equations (2·7) may be rewritten in the form

$$F_{ijmn}\tau^{ij} = \gamma_{mn} + F_{\beta jmn}E^{\beta\alpha}w^j|_\alpha. \tag{4·5}$$

Hence from (3·8) and (4·5) we have

$$\left.\begin{aligned} \tfrac{1}{2}(w_{\lambda|\mu} + w_{\mu|\lambda}) + F_{\beta j\lambda\mu}E^{\beta\alpha}w^j|_\alpha &= F_{ij\lambda\mu}\tau^{ij}, \\ \frac{1}{2}\left(w_{3|\lambda} + \frac{\partial w_\lambda}{\partial\theta^3}\right) + F_{\beta j3\lambda}E^{\beta\alpha}w^j|_\alpha &= F_{ij3\lambda}\tau^{ij}, \\ \frac{\partial w_3}{\partial\theta^3} + F_{\beta j33}E^{\beta\alpha}w^j|_\alpha &= F_{ij33}\tau^{ij}. \end{aligned}\right\} \tag{4·6}$$

It is convenient to write down two further equations, obtained from (4·6) or directly from (2·7), namely

$$\tau^{\beta 3} = \tau^{3\beta} + E^{\beta\alpha}w_3|_\alpha, \tag{4·7}$$

$$\tau^{\alpha\beta} - \tau^{\beta\alpha} = E^{\alpha\lambda}w^\beta|_\lambda - E^{\beta\lambda}w^\alpha|_\lambda. \tag{4·8}$$

The equations of motion (2.10) become

$$\left.\begin{aligned} \tau^{\alpha\beta}|_\alpha + \frac{\partial\tau^{3\beta}}{\partial\theta^3} &= \rho\frac{\partial^2 w^\beta}{\partial t^2}, \\ \tau^{\alpha 3}|_\alpha + \frac{\partial\tau^{33}}{\partial\theta^3} &= \rho\frac{\partial^2 w_3}{\partial t^2}. \end{aligned}\right\} \tag{4·9}$$

We wish to examine the deformation of the plate when stresses are prescribed on the boundaries of the plate. In the course of the present work we examine several situations and it is more convenient to discuss boundary conditions separately for each case. In all cases, however, we require that the plane faces of the plate be free from applied shear stresses. This restriction is not essential and merely simplifies the calculations.

It has already been stated that the initial stresses $E^{\alpha\beta}$ are assumed to be independent of θ_3, thus the fully loaded plate problem can be resolved into two distinct parts. The first problem in which $\tau^{\alpha\beta}$, τ^{33}, w_α are even functions of θ_3 and $\tau^{\alpha 3}$, $\tau^{3\alpha}$, w_3 are odd functions of θ_3 describes the stretching of the plate. The second problem in which $\tau^{\alpha\beta}$, τ^{33}, w_α are odd functions of θ_3 and $\tau^{\alpha 3}$, $\tau^{3\alpha}$, w_3 are even functions of θ_3 describes the pure bending of the plate. In the ensuing analysis we shall make frequent use of the even or odd property of each of these quantities without explicit reference.

5. *The interior problem.* We wish to obtain an approximate theory for plates, which are initially in a state of plane stress, in as systematic a manner as possible from the equations of section 4.

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Introduce the dimensionless coordinate

$$\zeta = \theta^3/h \quad (-1 \leqslant \zeta \leqslant 1). \tag{5·1}$$

The stress-displacement equations (4·6)–(4·8) and the equations of motion (4·9) become

$$\tfrac{1}{2}(w_{\lambda|\mu} + w_{\mu|\lambda}) + F_{\beta j\lambda\mu} E^{\beta\alpha} w^j|_\alpha = F_{ij\lambda\mu} \tau^{ij}, \tag{5·2}$$

$$\tfrac{1}{2}(h w_{3|\lambda} + w_{\lambda,3}) + h F_{\beta j3\lambda} E^{\beta\alpha} w^j|_\alpha = h F_{ij3\lambda} \tau^{ij}, \tag{5·3}$$

$$w_{3,3} + h F_{\beta j33} E^{\beta\alpha} w^j|_\alpha = h F_{ij33} \tau^{ij}, \tag{5·4}$$

$$\tau^{\beta 3} = \tau^{3\beta} + E^{\beta\lambda} w_{3|\lambda}, \tag{5·5}$$

$$\tau^{\alpha\beta} - \tau^{\beta\alpha} = E^{\alpha\lambda} w^\beta|_\lambda - E^{\beta\lambda} w^\alpha|_\lambda, \tag{5·6}$$

and

$$h\tau^{\alpha\beta}|_\alpha + \tau^{3\beta}{}_{,3} = \rho h \frac{\partial^2 w^\beta}{\partial t^2}, \tag{5·7}$$

$$h\tau^{\alpha 3}|_\alpha + \tau^{33}{}_{,3} = \rho h \frac{\partial^2 w_3}{\partial t^2}, \tag{5·8}$$

where we have introduced the notation $\partial/\partial\zeta \equiv {}_{,3}$.

We assume expressions for the displacements and stresses in the forms

$$\left.\begin{aligned} w_i &= \overset{(0)}{w}_i + h\overset{(1)}{w}_i + h^2\overset{(2)}{w}_i + \ldots, \\ \tau^{ij} &= \overset{(0)}{\tau}{}^{ij} + h\overset{(1)}{\tau}{}^{ij} + h^2\overset{(2)}{\tau}{}^{ij} + \ldots. \end{aligned}\right\} \tag{5·9}$$

These expansions are then substituted into the equations (5·2)–(5·8) and by systematically equating coefficients of like powers of h a sequence of systems of differential equations is obtained from which expressions for displacements and stresses can be derived in terms of ζ and arbitrary functions of θ^α and t.

5*a*. *Pure stretching superposed on plane stress.* In this subsection we suppose that the plate is deformed from its initial state B_0 by stresses applied to the edge surface S only. Thus from (2·6) and (3·4)

$$\tau^{33} = 0, \quad \tau^{3\beta} = 0 \quad (\zeta = \pm 1). \tag{5·10}$$

If we remember the even or odd property of the stresses and displacements, we see from (5·2) to (5·10) that

$$\overset{(0)}{w}_\lambda = W_\lambda, \quad \overset{(0)}{w}_3 = 0, \tag{5·11}$$

$$\overset{(0)}{\tau}{}^{33} = \overset{(1)}{\tau}{}^{33} = 0, \quad \overset{(0)}{\tau}{}^{3\beta} = \overset{(0)}{\tau}{}^{\beta 3} = \overset{(1)}{\tau}{}^{3\beta} = 0, \tag{5·12}$$

$$\tfrac{1}{2}(W_{\lambda|\mu} + W_{\mu|\lambda}) + F_{\beta\gamma\lambda\mu} E^{\beta\alpha} W^\gamma|_\alpha = F_{\alpha\beta\lambda\mu} \overset{(0)}{\tau}{}^{\alpha\beta}, \tag{5·13}$$

$$E^{\alpha\lambda} W^\beta|_\lambda - E^{\beta\lambda} W^\alpha|_\lambda = \overset{(0)}{\tau}{}^{\alpha\beta} - \overset{(0)}{\tau}{}^{\beta\alpha}, \tag{5·14}$$

$$\overset{(0)}{\tau}{}^{\alpha\beta}|_\alpha = \rho \frac{\partial^2 W^\beta}{\partial t^2}, \tag{5·15}$$

where W_λ are functions of θ^β and t. We now define the six independent coefficients C^{1111}, C^{1112}, C^{1122}, C^{1212}, C^{1222}, C^{2222} by the equations

$$C^{\rho\gamma\lambda\mu} F_{\alpha\beta\lambda\mu} = \tfrac{1}{2}(\delta^\rho_\alpha \delta^\gamma_\beta + \delta^\rho_\beta \delta^\gamma_\alpha), \tag{5·16}$$

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then equations (5·13) and (5·14) may be solved for $\overset{(0)}{\tau}{}^{\alpha\beta}$ in the form

$$\overset{(0)}{\tau}{}^{\alpha\beta} = C^{\alpha\beta\lambda\mu}W_{\lambda|\mu} + E^{\alpha\lambda}W^{\beta}|_{\lambda}. \tag{5·17}$$

From (5·15), (5·17) and (4·2) we immediately obtained the equations of motion for extensional vibrations of an initially stressed plate explicitly in terms of the displacements:

$$C^{\alpha\beta\lambda\mu}W_{\lambda|\mu\alpha} + E^{\alpha\lambda}W^{\beta}|_{\lambda\alpha} = \rho\frac{\partial^2 W^{\beta}}{\partial t^2}. \tag{5·18}$$

The remaining quantities which can be found in terms of the two independent functions W_λ are

$$\overset{(1)}{w}_3 = F_{\alpha\beta 33}C^{\alpha\beta\lambda\mu}W_{\lambda|\mu}\zeta, \tag{5·19}$$

$$\overset{(1)}{\tau}{}^{\beta 3} = E^{\beta\gamma}F_{\alpha\rho 33}C^{\alpha\rho\lambda\mu}W_{\lambda|\mu\gamma}\zeta, \tag{5·20}$$

and

$$\overset{(2)}{\tau}{}^{33} = \tfrac{1}{2}F_{\alpha\beta 33}C^{\alpha\beta\lambda\mu}\left(-\rho\frac{\partial^2}{\partial t^2}W_{\lambda|\mu} + E^{\rho\gamma}W_{\lambda|\mu\gamma\rho}\right)(1-\zeta^2). \tag{5·21}$$

It is now clear that extensional vibrations of a plate, with or without initial stress, do not, in general, give rise to a state of plane stress.

The above equations are not, in general, uniformly valid on to the edge surface S in the sense that it is not possible to satisfy arbitrary specified boundary conditions on S with such equations. We return to this problem later when we deduce the appropriate boundary conditions to be imposed on these equations.

5*b*. *Pure bending superposed on plane stress*. Here we suppose that the plate is deformed from the initial state B_0 by normal stresses on the faces together with stresses on the edge surface S. Thus from (2·6) and (3·4)

$$\tau^{33} = \mp p, \quad \tau^{3\beta} = 0 \quad (\zeta = \pm 1), \tag{5·22}$$

where p is a function of θ_1, θ_2. We assume that p can be expanded in the form†

$$p = \overset{(0)}{p} + h\overset{(1)}{p} + h^2\overset{(2)}{p} + \ldots. \tag{5·23}$$

From (5·2)–(5·9), (5·16), (5·22) and (5·23), a straightforward computation yields

$$\left.\begin{aligned}
&\overset{(0)}{w}_3 = W, \quad \overset{(0)}{w}_\lambda = 0, \quad \overset{(1)}{w}_\lambda = -W_{|\lambda}\zeta,\\
&\overset{(0)}{\tau}{}^{3\beta} = \overset{(1)}{\tau}{}^{3\beta} = 0, \quad \overset{(0)}{\tau}{}^{33} = \overset{(0)}{p} = 0,\\
&\overset{(0)}{\tau}{}^{\alpha\beta} = 0,\\
&\overset{(0)}{\tau}{}^{\beta 3} = E^{\beta\alpha}W|_{\alpha},\\
&\overset{(0)}{\tau}{}^{\beta 3}|_{\beta} - \overset{(1)}{p} = \rho\frac{\partial^2 W}{\partial t^2},\\
&\overset{(1)}{\tau}{}^{33} = -\overset{(1)}{p}\zeta,
\end{aligned}\right\} \tag{5·24}$$

† This expansion has a different significance from those in (5.9). It is simply a device for finding the appropriate scaling factor.

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where W is a function of θ^β and t. From (5·24) we obtain the equation of motion in terms of the transverse displacement W, namely

$$E^{\beta\alpha}W|_{\alpha\beta} - \overset{(1)}{p} = \rho\frac{\partial^2 W}{\partial t^2}, \tag{5·25}$$

where we have used (4·2). Equation (5·25) is simply the classical equation for the transverse displacement of a loaded membrane. This indicates that the initial stress $E^{\alpha\beta}$ is so large that the plate behaves like a membrane. We continue by examining situations when the initial stress is of a smaller order of magnitude (in h).

Consider the case when the initial stress $E^{\alpha\beta}$ is of order of magnitude h, thus

$$E^{\alpha\beta} = h\overset{(1)}{E}{}^{\alpha\beta}, \tag{5·26}$$

where $\overset{(1)}{E}{}^{\alpha\beta}$ is independent of h. In order that the ensuing deformation be compatible with the three-dimensional equations it is necessary to alter the time scale to t_1 by the relation†

$$t_1 = t\sqrt{(h/L)}, \tag{5·27}$$

where L is a typical length associated with the deformation pattern in the middle surface. This leads to a membrane theory similar to the one just given. Details are omitted.

Finally, we consider the initial stress $E^{\alpha\beta}$ to be of order of magnitude h^2, thus

$$E^{\alpha\beta} = h^2\overset{(2)}{E}{}^{\alpha\beta}, \tag{5·28}$$

where $\overset{(2)}{E}{}^{\alpha\beta}$ is independent of h. Again it is necessary to change the time scale. Here we need to put

$$t_2 = t(h/L). \tag{5·29}$$

If we substitute (5·24) and (5·25) into (5·2)–(5·8) we obtain

$$\left.\begin{aligned}
\tfrac{1}{2}(w_{\lambda|\mu} + w_{\mu|\lambda}) + h^2 F_{\beta j\lambda\mu}\overset{(2)}{E}{}^{\beta\alpha}w^j|_\alpha &= F_{ij\lambda\mu}\tau^{ij},\\
\tfrac{1}{2}(hw_{3|\lambda} + w_{\lambda,3}) + h^3 F_{\beta j3\lambda}\overset{(2)}{E}{}^{\beta\alpha}w^j|_\alpha &= hF_{ij3\lambda}\tau^{ij},\\
w_{3,3} + h^3 F_{\beta j33}\overset{(2)}{E}{}^{\beta\alpha}w^j|_\alpha &= hF_{ij33}\tau^{ij},\\
\tau^{\beta 3} &= \tau^{3\beta} + h^2\overset{(2)}{E}{}^{\beta\lambda}w_{3|\lambda},\\
\tau^{\alpha\beta} - \tau^{\beta\alpha} &= h^2(\overset{(2)}{E}{}^{\alpha\lambda}w^\beta|_\lambda - \overset{(2)}{E}{}^{\beta\lambda}w^\alpha|_\lambda),\\
h\tau^{\alpha\beta}|_\alpha + \tau^{3\beta}{}_{,3} &= h^3 K(\partial^2 w^\beta/\partial t_2^2),\\
h\tau^{\alpha 3}|_\alpha + \tau^{33}{}_{,3} &= h^3 K(\partial^2 w_3/\partial t_2^2),
\end{aligned}\right\} \tag{5·30}$$

where

$$K = \rho/L^2. \tag{5·31}$$

† Essentially the same idea is used by Green and Naghdi ((5)). The significance of this scaling appears to be that the surface loading must vary neither too quickly nor too slowly in time. Indeed, if we had put $t_1 = t(h/L)$, then the inertia terms would have been absent from the first approximation. The precise circumstances under which a particular time scaling is suitable cannot be determined until the whole question of the validity of the expansion method is resolved—one way or the other.

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A systematic examination of (5·30) together with (4·2), (5·9), (5·16) and (5·23) yields the following results:

$$\left.\begin{aligned} &\overset{(0)}{w}_3 = W, \quad \overset{(0)}{w}_\lambda = 0, \\ &\overset{(0)}{\tau}{}^{\alpha\beta} = 0, \\ &\overset{(0)}{\tau}{}^{3\beta} = \overset{(0)}{\tau}{}^{\beta 3} = 0, \quad \overset{(0)}{\tau}{}^{33} = \overset{(0)}{p} = 0, \end{aligned}\right\} \tag{5·32}$$

$$\left.\begin{aligned} &\overset{(1)}{w}_\lambda = -W|_\lambda \zeta, \\ &\overset{(1)}{\tau}{}^{\alpha\beta} = \overset{(1)}{\tau}{}^{\beta\alpha} = -C^{\alpha\beta\lambda\mu} W|_{\lambda\mu} \zeta, \\ &\overset{(1)}{\tau}{}^{3\beta} = \overset{(1)}{\tau}{}^{\beta 3} = 0, \quad \overset{(1)}{\tau}{}^{33} = \overset{(1)}{p} = 0, \end{aligned}\right\} \tag{5·33}$$

$$\left.\begin{aligned} &\overset{(2)}{\tau}{}^{\beta 3} = \overset{(2)}{E}{}^{\beta\lambda} W|_\lambda - \tfrac{1}{2}(1-\zeta^2) C^{\alpha\beta\lambda\mu} W|_{\lambda\mu\alpha}, \\ &\overset{(2)}{\tau}{}^{3\beta} = -\tfrac{1}{2}(1-\zeta^2) C^{\alpha\beta\lambda\mu} W|_{\lambda\mu\alpha}, \\ &\overset{(2)}{\tau}{}^{33} = \overset{(2)}{p} = 0, \end{aligned}\right\} \tag{5·34}$$

$$\overset{(3)}{\tau}{}^{33} = -\overset{(3)}{p}\zeta + \tfrac{1}{6}\zeta(1-\zeta^2) C^{\alpha\beta\lambda\mu} W|_{\alpha\beta\lambda\mu}, \tag{5·35}$$

and

$$\overset{(2)}{E}{}^{\alpha\lambda} W|_{\alpha\lambda} - \tfrac{1}{3} C^{\alpha\beta\lambda\mu} W|_{\alpha\beta\lambda\mu} - \overset{(3)}{p} = K(\partial^2 W/\partial t_2^2). \tag{5·36}$$

Perhaps the significance of these results is most easily seen by substituting in (5·36) for $\overset{(3)}{p}$, $\overset{(2)}{E}{}^{\alpha\lambda}$, K and t_2, which are defined in (5·23), (5·28), (5·31) and (5·29), respectively. Thus

$$hE^{\alpha\lambda} W|_{\alpha\lambda} - \tfrac{1}{3} h^3 C^{\alpha\beta\lambda\mu} W|_{\alpha\beta\lambda\mu} - p = \rho h(\partial^2 W/\partial t^2), \tag{5·37}$$

and we recover the equation for transverse vibrations of a plate in the usual form.

In the three cases investigated in this section, the equations governing the first approximation are not, in general, uniformly valid on to the edge surface S in the sense already described. To determine the boundary conditions (on S) which may be imposed on such equations we now develop a boundary-layer analysis.

6. *The boundary-layer problem.* We wish to examine the boundary-layer equations in the neighbourhood of the edge surface S. For our present purpose it is convenient to choose the coordinates θ^1, θ^2 to be tangential and normal to C. We take θ^1 to be the arc length measured around C in an anticlockwise sense from some fixed point. The coordinate θ^2 is taken to be the distance from C measured along an inwardly drawn normal to C. Fox ((6)) shows that for each point in the boundary layer to correspond to a unique set $(\theta^1, \theta^2, \theta^3)$ it is necessary to restrict the range of θ^2 so as to exclude regions where normals to C intersect. This restriction will not be made here as it would make little difference to the analysis.

For the choice of coordinates indicated above, the line element ds in the middle surface is given by

$$(ds)^2 = a_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta, \tag{6·1}$$

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where
$$a_{11}=(1-\kappa\theta^2)^2, \quad a_{21}=a_{12}=0, \quad a_{22}=1. \tag{6·2}$$

In (6·2) κ is the curvature of C. Hence from (3·9) and (6·2) we see that the only non-zero Christoffel symbols are

$$\left.\begin{aligned} \Gamma^1_{11} &= -\frac{\theta^2}{1-\kappa\theta^2}\frac{d\kappa}{d\theta^1}, \quad \Gamma^2_{11}=\kappa(1-\kappa\theta^2), \\ \Gamma^1_{12} &= \Gamma^1_{21} = -\frac{\kappa}{1-\kappa\theta^2}. \end{aligned}\right\} \tag{6·3}$$

The equations of motion and the stress-strain relations can now be written down in the special coordinate system chosen in this section. Before doing this, however, we introduce a scaling of the coordinates which is suitable for our discussion of the boundary layer equations. Let

$$\xi^2=\frac{\theta^2}{h}, \quad \xi^3=\frac{\theta^3}{h}. \tag{6·4}$$

With the help of (6·3) and (6·4) the equations of motion (4·9) become†

$$h\frac{\partial\tau^{11}}{\partial\theta^1}+\frac{\partial\tau^{21}}{\partial\xi^2}+\frac{\partial\tau^{31}}{\partial\xi^3}-\frac{2h^2\xi^2}{1-h\kappa\xi^2}\frac{d\kappa}{d\theta^1}\tau^{11}-\frac{2h\kappa}{1-h\kappa\xi^2}\tau^{21}-\frac{h\kappa}{1-h\kappa\xi^2}\tau^{12}=\rho h\frac{\partial^2 w^1}{\partial t^2}, \tag{6·5}$$

$$h\frac{\partial\tau^{12}}{\partial\theta^1}+\frac{\partial\tau^{22}}{\partial\xi^2}+\frac{\partial\tau^{32}}{\partial\xi^3}-\frac{h^2\xi^2}{1-h\kappa\xi^2}\frac{d\kappa}{d\theta^1}\tau^{12}-\frac{h\kappa}{1-h\kappa\xi^2}\tau^{22}+h\kappa(1-h\kappa\xi^2)\tau^{11}=\rho h\frac{\partial^2 w^2}{\partial t^2}, \tag{6·6}$$

$$h\frac{\partial\tau^{13}}{\partial\theta^1}+\frac{\partial\tau^{23}}{\partial\xi^2}+\frac{\partial\tau^{33}}{\partial\xi^3}-\frac{h\kappa}{1-h\kappa\xi^2}\tau^{23}-\frac{h^2\xi^2}{1-h\kappa\xi^2}\frac{d\kappa}{d\theta^1}\tau^{13}=\rho h\frac{\partial^2 w_3}{\partial t^2}. \tag{6·7}$$

At this stage we do not write down the stress-displacement relations because of their excessive length.

In Section 5 equations for the 'interior' problem were obtained in which the conditions on the faces of the plate were satisfied. These equations were not, in general, uniformly valid on to the edge surface S. We now look for solutions of the three-dimensional equations which, together with the interior equations, enable us to satisfy arbitrary edge conditions on S. Green((3)) has pointed out that since the equations are linear in the dependent variables, solutions of different types may be added. It is therefore sufficient to seek boundary-layer solutions which have zero applied stresses on the faces $\theta_3=\pm h$, so that

$$\tau^{33}=0, \quad \tau^{3\beta}=0 \quad (\xi^3=\pm 1). \tag{6·8}$$

We also require the boundary-layer solutions to reduce rapidly to zero as we go away from S, thus we look for solutions of the three-dimensional equations which are such that the boundary-layer stresses tend to zero as $\xi^2\to\infty$.

6*a*. *Pure stretching*. The appropriate scaling for w_i which becomes apparent from the analysis is

$$w_i=hv_i. \tag{6·9}$$

† We use the notation for boundary-layer quantities that has been used previously for interior quantities, but this need cause no confusion. It is implicitly assumed here that no time scaling is necessary. For certain types of (time-dependent) loading, this assumption may not be justified.

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Then the first approximation to the equations of motion (6·5)–(6·7) is

$$\frac{\partial \overset{(0)}{\tau}{}^{21}}{\partial \xi^2} + \frac{\partial \overset{(0)}{\tau}{}^{31}}{\partial \xi^3} = 0, \tag{6·10}$$

$$\frac{\partial \overset{(0)}{\tau}{}^{22}}{\partial \xi^2} + \frac{\partial \overset{(0)}{\tau}{}^{32}}{\partial \xi^3} = 0, \tag{6·11}$$

$$\frac{\partial \overset{(0)}{\tau}{}^{23}}{\partial \xi^2} + \frac{\partial \overset{(0)}{\tau}{}^{33}}{\partial \xi^3} = 0. \tag{6·12}$$

Using equations (6·10), (6·11) and (6·8), and integrating over the region $\xi^2 \geqslant 0$, $-1 \leqslant \xi^3 \leqslant 1$, it may be shown that†

$$\int_{-1}^{1} \overset{(0)}{\tau}{}^{21} d\xi^3 = 0, \quad \int_{-1}^{1} \overset{(0)}{\tau}{}^{22} d\xi^3 = 0 \quad (\xi^2 = 0), \tag{6·13}$$

provided $\overset{(0)}{\tau}{}^{21}$, $\overset{(0)}{\tau}{}^{22}$ vanish uniformly with respect to ξ^3 as $\xi^2 \to \infty$.

To discuss the edge conditions on S we suppose that the surface stresses τ^{21}, τ^{22}, τ^{23} in the coordinate system corresponding to (6·1) and (6·2) take the specified values t^{21}, t^{22}, t^{23} say, when $\xi^2 = 0$, being given as functions of ξ^3, ξ^1 and t. We denote the major terms of the interior stresses evaluated in boundary-layer coordinates by Σ^{ij}. When $\xi^2 = 0$, we equate the total stress to the boundary values. Thus remembering (5·12) and (5·17),

$$\left.\begin{aligned} t^{21} &= \Sigma^{21} + \overset{(0)}{\tau}{}^{21} + \ldots, \\ t^{22} &= \Sigma^{22} + \overset{(0)}{\tau}{}^{22} + \ldots, \\ t^{23} &= \qquad \overset{(0)}{\tau}{}^{23} + \ldots, \end{aligned}\right\} \tag{6·14}$$

when $\xi^2 = 0$. From (6·13) and (6·14) we obtain the boundary conditions for the interior problem discussed in section 5a:

$$\left.\begin{aligned} \int_{-1}^{1} \overset{(0)}{t}{}^{21} d\xi^3 &= \int_{-1}^{1} \Sigma^{21} d\xi^3, \\ \int_{-1}^{1} \overset{(0)}{t}{}^{22} d\xi^3 &= \int_{-1}^{1} \Sigma^{22} d\xi^3, \end{aligned}\right\} \tag{6·15}$$

when $\xi^2 = 0$. Using these integral conditions the interior problem may now be solved.

If we require the complete first approximation, we must solve (6·10)–(6·12), together with the first approximation to the stress-displacement equations and edge conditions (6·14). This is a formidable task since the appropriate stress-displacement equations are rather complicated. Here we indicate a possible method of solution by considering the case when there is no initial stress. In these special circumstances, the stress-dis-

† The analysis now proceeds in the same way as in (3).

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placement relations (4·6)–(4·8), together with (6·3), (6·4) and (6·9) yield in the first approximation

$$\overset{(0)}{F}_{ij11}\overset{(0)}{\tau}^{ij} = 0, \tag{6·16}$$

$$\overset{(0)}{F}_{ij12}\overset{(0)}{\tau}^{ij} = \frac{1}{2}\frac{\partial \overset{(0)}{v}_1}{\partial \xi^2}, \tag{6·17}$$

$$\overset{(0)}{F}_{ij13}\overset{(0)}{\tau}^{ij} = \frac{1}{2}\frac{\partial \overset{(0)}{v}_1}{\partial \xi^3}, \tag{6·18}$$

$$\overset{(0)}{F}_{ij22}\overset{(0)}{\tau}^{ij} = \frac{\partial \overset{(0)}{v}_2}{\partial \xi^2}, \tag{6·19}$$

$$\overset{(0)}{F}_{ij23}\overset{(0)}{\tau}^{ij} = \frac{1}{2}\left(\frac{\partial \overset{(0)}{v}_3}{\partial \xi^2}+\frac{\partial \overset{(0)}{v}_2}{\partial \xi^3}\right), \tag{6·20}$$

$$\overset{(0)}{F}_{ij33}\overset{(0)}{\tau}^{ij} = \frac{\partial \overset{(0)}{v}_3}{\partial \xi^3}, \tag{6·21}$$

$$\overset{(0)}{\tau}^{\beta 3} = \overset{(0)}{\tau}^{3\beta}, \tag{6·22}$$

$$\overset{(0)}{\tau}^{\alpha\beta} = \overset{(0)}{\tau}^{\beta\alpha}, \tag{6·23}$$

where F_{ijrs} are to be evaluated at $\xi^2 = 0$.

Equations (6·10)–(6·12), (6·22) and (6·23) admit the existence of two stress functions ϕ, ψ such that

$$\overset{(0)}{\tau}^{A1} = \overset{(0)}{\tau}^{1A} = \epsilon^{AB}\phi_{,B}, \tag{6·24}$$

$$\overset{(0)}{\tau}^{AB} = \overset{(0)}{\tau}^{BA} = \epsilon^{AC}\epsilon^{BD}\psi_{,CD}, \tag{6·25}$$

where Latin majuscules take the values 2, 3 and indices which occur twice are summed. Also $\phi_{,B} = \partial\phi/\partial\xi^B$ and

$$\epsilon^{23} = 1, \quad \epsilon^{32} = -1, \quad \epsilon^{22} = \epsilon^{33} = 0. \tag{6·26}$$

Using these expressions we see, from (6·16), that

$$-\overset{(0)}{F}_{1111}\overset{(0)}{\tau}^{11} = 2F_{1A11}\epsilon^{AB}\phi_{,B} + F_{AB11}\epsilon^{AC}\epsilon^{BD}\psi_{,CD}. \tag{6·27}$$

If we now use the fact that

$$\epsilon^{AB}\overset{(0)}{v}_{1,AB} = 0, \tag{6·28}$$

then from (6·17), (6·18), (6·24)–(6·28) we may show that

$$2(F_{1C1A}F_{1111} - F_{1C11}F_{111A})\epsilon^{AB}\epsilon^{CD}\phi_{,BD} + (F_{1111}F_{DC1A} - F_{111A}F_{CD11}) \times \epsilon^{AB}\epsilon^{CH}\epsilon^{DJ}\psi_{,BHJ} = 0. \tag{6·29}$$

Similarly, we may use (6·24)–(6·27), together with the fact that

$$\epsilon^{AC}\epsilon^{BD}(\overset{(0)}{v}_{A,B} + \overset{(0)}{v}_{B,A})_{,CD} = 0,$$

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to eliminate $\overset{(0)}{v}_2$ and $\overset{(0)}{v}_3$ from (6·19)–(6·21). The result is

$$2(F_{1HAB}F_{1111}-F_{11AB}F_{1H11})\,\epsilon^{AC}\epsilon^{BD}\epsilon^{HL}\phi_{,CDL} + (F_{HJAB}F_{1111}-F_{11AB}F_{HJ11})\,\epsilon^{AC}\epsilon^{BD}\epsilon^{HK}\epsilon^{JL}\psi_{,KLCD}=0. \quad (6\cdot30)$$

To solve the boundary-layer problem we should then need to solve (6·29) and (6·30). Difficulties remain in finding ϕ and ψ.

We conclude this discussion by noting two cases in which (6·29) and (6·30) become a second-order equation for ϕ and a fourth-order equation for ψ, respectively. If the initial body B_0 is symmetrical about the surface $\theta^1=0$ (in the sense of Green and Zerna ((7))), then all coefficients F_{ijrs} containing the index 1 once or three times vanish. Secondly, if B_0 is symmetrical about $\theta^2=0$ and $\theta^3=0$, then all coefficients F_{ijrs} containing the index 2 or 3 once or three times vanish. In both cases desired reduction is achieved.

6*b*. *Membrane.* Here we also require

$$w_i = hv_i.$$

Thus the basic boundary-layer equations for the first approximation are as before. From (6·11) and (6·12) we may show that

$$\int_{-1}^{1} \overset{(0)}{\tau}{}^{23}\,d\xi = 0, \quad (6\cdot31)$$

$$\int_{-1}^{1} \xi^3 \overset{(0)}{\tau}{}^{22}\,d\xi^3 = 0, \quad (6\cdot32)$$

when $\xi^2=0$, provided $\overset{(0)}{\tau}{}^{23}$, $\xi^3\overset{(0)}{\tau}{}^{22}$, $\xi^2\overset{(0)}{\tau}{}^{23}$ tend to zero uniformly in ξ^2 as $\xi^2\to\infty$.

Using the notation for the applied stresses and the major terms of the interior stresses introduced in section 6*a*, we now have

$$\left.\begin{aligned} t^{21} &= \overset{(0)}{\tau}{}^{21}+\dots \\ t^{22} &= \overset{(0)}{\tau}{}^{22}+\dots \\ t^{23} &= \Sigma^{23}+\overset{(0)}{\tau}{}^{23}+\dots \end{aligned}\right\} \quad (\xi^2=0), \quad (6\cdot33)$$

for the membrane theory of section 5*b*. Thus from (6·31) we obtain the only boundary condition for the interior problem,

$$\int_{-1}^{1} \overset{(0)}{t}{}^{23}\,d\xi^3 = \int_{-1}^{1} \Sigma^{23}\,d\xi^3 \quad (\xi^2=0). \quad (6\cdot34)$$

However, from (6·32) and (6·33), we also see that any applied stress $\overset{(0)}{t}{}^{22}$ must have zero bending moment about $\xi^3=0$.

The solution of the boundary-layer problem is evidently similar to the solution of the corresponding equations discussed in section 6*a*.

6*c*. *Pure bending.* In this case the appropriate scalings for τ^{ij} and w_i are

$$\tau^{ij} = hf^{ij}, \quad w_i = h^2u_i. \quad (6\cdot35)$$

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If we use (5·28) and (6·1)–(6·4) then the first approximation to the stress-displacement relations (4·6)–(4·8) reduces to the form (6·16)–(6·23) with v_i, $\overset{(0)}{\tau}{}^{ij}$ replaced by u_i, $\overset{(0)}{f}{}^{ij}$, respectively. We now substitute (6·35) into the equations of motion (6·5)–(6·7) and in the first approximation we recover (6·10)–(6·12) with $\overset{(0)}{\tau}{}^{ij}$ replaced by $\overset{(0)}{f}{}^{ij}$. To complete the discussion of the boundary conditions for the interior problem we require the second approximation to (6·7), that is

$$\frac{\partial \overset{(0)}{f}{}^{13}}{\partial \theta_1} + \frac{\partial \overset{(1)}{f}{}^{23}}{\partial \xi^2} + \frac{\partial \overset{(1)}{f}{}^{33}}{\partial \xi^3} - \kappa \overset{(0)}{f}{}^{23} = 0. \tag{6·36}$$

From these equations we may show that

$$\int_{-1}^{1} \overset{(0)}{f}{}^{23}\, d\xi^3 = 0, \quad \int_{-1}^{1} \xi^3 \overset{(0)}{f}{}^{22}\, d\xi^3 = 0, \tag{6·37}$$

and

$$\int_{-1}^{1} \left(\xi^3 \frac{\partial \overset{(0)}{f}{}^{21}}{\partial \theta^1} + \overset{(1)}{f}{}^{23} \right) d\xi^3 = 0 \tag{6·38}$$

when $\xi^2 = 0$, provided $\overset{(0)}{f}{}^{23}, \overset{(1)}{f}{}^{23}, \xi^3 \overset{(0)}{f}{}^{22}, \xi^3 \overset{(0)}{f}{}^{12}, \xi^2 \overset{(0)}{f}{}^{23}$ tend to zero uniformly in ξ^2 as $\xi^2 \to \infty$.

In view of (5·32)–(5·34) we denote the interior stresses by $h\sigma^{ij}$ (referred to boundary layer coordinates). Proceeding as before, we have

$$\left.\begin{aligned} t^{21} &= h(\overset{(0)}{\sigma}{}^{21} + \overset{(0)}{f}{}^{21}) + \ldots, \\ t^{22} &= h(\overset{(0)}{\sigma}{}^{22} + \overset{(0)}{f}{}^{22}) + \ldots, \\ t^{23} &= h \overset{(0)}{f}{}^{23} + h^2(\overset{(1)}{\sigma}{}^{23} + \overset{(1)}{f}{}^{23}) + \ldots. \end{aligned}\right\} \tag{6·39}$$

From (6·37) to (6·39) we see that the applied edge stresses must be of order of magnitude h (or smaller) and

$$\int_{-1}^{1} \overset{(1)}{t}{}^{23}\, d\xi^3 = 0, \tag{6·40}$$

$$\int_{-1}^{1} \xi^3 \overset{(1)}{t}{}^{22}\, d\xi^3 = \int_{-1}^{1} \xi^3 \overset{(0)}{\sigma}{}^{22}\, d\xi^3, \tag{6·41}$$

$$\int_{-1}^{1} \left(\xi^3 \frac{\partial \overset{(1)}{t}{}^{21}}{\partial \theta^1} + \overset{(2)}{t}{}^{23} \right) d\xi^3 = \int_{-1}^{1} \left(\xi^3 \frac{\partial \overset{(0)}{\sigma}{}^{21}}{\partial \theta^1} + \overset{(1)}{\sigma}{}^{23} \right) d\xi^3, \tag{6·42}$$

when $\xi^2 = 0$. It follows from (6·40) that any applied shear stress of the type $\overset{(1)}{t}{}^{23}$ must have zero resultant over the edge of the plate. Equations (6·41) and (6·42) are the usual Kirchhoff boundary conditions for the interior problem. Again the determination of the boundary-layer stresses requires the solution of the appropriate equations in section 6*a*.

7. *Conclusion.* We conclude by collecting the relevant formulae and equations needed in the first approximation to the interior problem. From the boundary conditions obtained in section 6 it is evidently convenient to introduce stress resultants and

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couples. Following Green ((3)), we define stress resultants and couples, in any general curvilinear system of coordinates θ^α, by the equations

$$n^{\alpha\beta} = \int_{-h}^{h} \tau^{\alpha\beta}\, d\theta^3, \tag{7·1}$$

$$q^\alpha = \int_{-h}^{h} \tau^{\alpha 3}\, d\theta^3, \tag{7·2}$$

$$m^{\alpha\beta} = \int_{-h}^{h} \tau^{\alpha\beta}\theta^3\, d\theta^3. \tag{7·3}$$

To re-cast the boundary conditions for the interior stresses in a general system of coordinates, we let ν_α be the covariant components of the unit normal vector to the curve C, then the shearing force n, the bending moment G and the twisting moment H are defined by

$$\left.\begin{aligned} n &= q^\alpha \nu_\alpha, \\ G &= m^{\alpha\beta}\nu_\alpha \nu_\beta, \\ H &= \epsilon_{\beta\lambda} m^{\alpha\beta}\nu_\alpha \nu^\lambda. \end{aligned}\right\} \tag{7·4}$$

(*a*) *Pure stretching.* From (5·11), (5·15), (5·17), (5·19) and (7·1) it follows that

$$w_\lambda = W_\lambda, \quad w_3 = F_{\alpha\beta 33} C^{\alpha\beta\lambda\mu} W_{\lambda|\mu}\theta^3,$$

$$n^{\alpha\beta}|_\alpha = 2\rho h \frac{\partial^2 W^\beta}{\partial t^2},$$

$$n^{\alpha\beta} = 2hC^{\alpha\beta\lambda\mu}W_{\lambda|\mu} + 2hE^{\alpha\lambda}W^\beta|_\lambda.$$

The boundary conditions are obtained from (6·15) and become in a general system of coordinates

$$n^{\alpha\beta}\nu_\alpha = \bar{n}^{\alpha\beta}\nu_\alpha,$$

where a bar denotes the value of externally applied resultants.

(*b*) *Membrane.* From (5·23), (5·24) and (7·2) we have

$$w_\lambda = -W|_\lambda \theta^3, \quad w_3 = W,$$

$$q^\beta|_\beta - 2p = 2\rho h \frac{\partial^2 W}{\partial t^2},$$

$$q^\beta = 2hE^{\beta\lambda}W|_\lambda.$$

If we write (6·34) in a general system of coordinates we see that the condition to be satisfied at C is

$$n = \bar{n}.$$

(*c*) *Bending.* From (5·32)–(5·37) we have

$$w_3 = W, \quad w_\lambda = -W|_\lambda \theta^3,$$

$$q^\beta = m^{\alpha\beta}|_\alpha + 2hE^{\beta\lambda}W|_\lambda,$$

$$q^\beta|_\beta - 2p = 2\rho h \frac{\partial^2 W}{\partial t^2},$$

$$m^{\alpha\beta} = -\tfrac{2}{3}h^3 C^{\alpha\beta\lambda\mu}W|_{\lambda\mu}.$$

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The appropriate boundary conditions, obtained from (6·41), (6·42) and (7·4), are

$$G = \bar{G}, \quad n - \frac{\partial H}{\partial s} = \bar{n} - \frac{\partial \bar{H}}{\partial s},$$

where $\partial/\partial s$ denotes differentiation along an arc s of the boundary C.

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