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A UNIFIED APPROACH FOR LARGE QUEUE ASYMPTOTICS IN A HETEROGENEOUS MULTISERVER QUEUE

MASAKIYO MIYAZAWA,* *Tokyo University of Science*

Abstract

We are interested in a large queue in a GI/G/k queue with heterogeneous servers. For this, we consider tail asymptotics and weak limit approximations for the stationary distribution of its queue length process in continuous time under a stability condition. Here, two weak limit approximations are considered. One is when the variances of the interarrival and/or service times are bounded, and the other is when they become large. Both require a heavy-traffic condition. Tail asymptotics and heavy-traffic approximations have been separately studied in the literature. We develop a unified approach based on a martingale produced by a good test function for a Markov process to answer both problems.

Keywords: Piecewise-deterministic Markov process; martingale; heterogeneous server; queue length process; stationary distribution; tail asymptotics; large variance; heavy-traffic approximation

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1. Introduction

We are interested in a large queue in a GI/G/k queue with FCFS (first-come–first-served) service discipline. This queueing model has a single queue and k servers who may have different service-time distributions. Throughout the paper, k is an arbitrary positive integer, but fixed. We refer to this model as a heterogeneous multiserver queue. As a queueing model, this queue is basic and classical, but it is known to be difficult to obtain analytical results for this model. Because of this, theoretical studies on it have been directed to asymptotic analysis. Among them, behavior of a large queue is one of their major interests.

In this paper we study the large queue behavior through tail asymptotics and weak limit approximations for the stationary distribution of the queue length process of the heterogeneous multiserver queue. We describe this queueing model by a continuous-time Markov process, and assume that it has a stationary distribution, that is, it is stable. Here, the weak limit approximation is meant to derive a limiting distribution for approximation from a sequence of the queueing models under appropriate scaling of their queue length. Such an approximation traditionally assumes that the variances of the interarrival and/or service times are bounded, and is obtained by increasing the traffic intensity to 1 from below, which is called a heavy-traffic condition. We also consider another approximation under the heavy-traffic condition by increasing the variances of the interarrival and/or service times, which is referred to as large variance approximation.

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* Postal address: Department of Information Sciences, Tokyo University of Science, 2641 Yamazaki, Noda, Chiba 278-8510, Japan. Email address: miyazawa@rs.tus.ac.jp

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The tail asymptotics and heavy-traffic approximations have been separately studied in the literature. For example, the tail asymptotics are obtained in Neuts and Takahashi [25] and Sadowsky and Szpankowski [30] for the stationary waiting time distributions and some other characteristics at arrival instants of customers, while the heavy-traffic approximation is obtained for the stationary waiting time distribution in a GI/G/k queue with homogeneous servers by Köllerström [19]. This heavy-traffic approximation is related to a diffusion approximation for a process limit, which has been widely studied. In particular, for the heterogeneous multiserver queue, a diffusion approximation is obtained for the queue length process in Chen and Ye [5]. Unlike those studies, we aim to give a unified approach to simultaneously answer both the tail asymptotic problem and the heavy-traffic approximation on the stationary distribution of the queue length process in continuous time.

For this unified approach, we describe the heterogeneous multiserver queue by a piecewise-deterministic Markov process due to Davis [8], [9], which is called a PDMP for short. A major difficulty in studying its stationary distribution comes from state transitions at arriving and departing instants of customers, which are particularly complicated when some of the servers are idle. In the PDMP, the effect of those transitions can be removed from system equations for its dynamics by the so-called boundary condition. Since the boundary in this terminology is different from boundary states for the queue length process, we refer to the boundary condition of [8] as a terminal condition in this paper. A key step of our approach is to find a good test function satisfying this terminal condition, which produces a martingale from the PDMP. This idea will be shown to be useful not only for the tail asymptotic problem but also for the weak limit approximations.

Obviously, the tail asymptotics of the stationary distribution are crucially influenced by the heaviness of the service-time distributions. Because of this, different techniques have been employed for the light and heavy tail cases in the literature (see, e.g. [12] and [20]). However, it may also be interesting to see what occurs when they are mixed. This question was answered for the stationary waiting time distribution by Sadowsky [29]. We revisit this question for the stationary queue length distribution in continuous time, and obtain a similar answer to it (see Theorem 3.1 and Corollary 3.1).

Our weak limit approximations for a large queue have the same spirit as Kingman's [16] two moment approximation for the stationary waiting time distribution of the GI/G/1 queue in heavy traffic, which is extended to the homogeneous multiserver queue by Köllerström [19]. Motivated by them, the author [21] recently studied heavy-traffic approximation of the stationary queue length distribution in continuous time in the GI/G/1 queue. The basic idea is similar to using the aforementioned terminal condition. In this paper we refine it using the PDMP, and attack various asymptotic problems for the heterogeneous multiserver queue, which have been not studied in the line of Kingman's [16] two moment approximation.

The present approach is flexible in modeling assumptions. For example, it allows arrivals and service completions of different servers to occur simultaneously. This considerably complicates analysis on state transitions, and is often excluded by assuming additional assumptions such that the service-time distributions have densities. We do not need such assumptions. We also demonstrate it to be applicable to the case that the arrival process is a superposition of the renewal processes. Furthermore, the approach may be amendable for some other queueing models. In particular, a queueing network is apparently more interesting for application. Braverman *et al.* [4] showed that its heavy-traffic approximation can be studied in the same spirit (see Section 5.2 for more discussion).

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This paper comprises five sections. In Section 2 we formally introduce the heterogeneous multiserver queue and the PDMP for describing it. We also discuss a martingale obtained from the PDMP. A representation of the PDMP using this martingale is used to derive a stationary equation with a certain good test function. Main results on the tail asymptotics and weak limit approximations are presented in Section 3. In this section we also consider an extension of the heterogeneous multiserver queue so that it has multiple renewal arrivals. Theorems are proved in Section 4, where we detail change of measure for the tail asymptotic because they may have their own interest. In Section 5 we discuss possible extensions of the present approach. Technical lemmas are proved in the appendices.

2. Heterogeneous multiserver queue

We introduce the heterogeneous multiserver queue in Section 2.1. We then introduce the Markov process and the related tools in Sections 2.2–2.4. These tools will be used to prove our asymptotic results (Theorems 3.1–3.4) that will be stated in Section 3. Developing the tools is an important part of the contribution of this paper, which may have independent interest. Readers who wish to read the statement of the theorems without focusing on their proofs can go to Section 3 directly after reading Section 2.1, picking up the definitions of functions $\eta(v, \cdot)$ and $\xi(v, \cdot)$ in Section 2.3.

2.1. Model description and PDMP

By a heterogeneous k-server queue for $k \geq 1$, we mean that customers arrive at a system according to a renewal process and are served in a FCFS manner by k servers, numbered as $1, 2, \ldots, k$, who have their own service-time distributions, which may be different. It is assumed that service times at each server are independent and identically distributed (i.i.d.) and independent of everything else. We denote the interarrival time distribution by F_0, which is assumed to have finite positive mean $1/\lambda_0$. Thus, λ_0 is the arrival rate. Denote the service-time distribution for the ith server by F_i, which is assumed to have a finite mean m_i. Let $\lambda_i = 1/m_i$, which is the service rate for $i \in K$, and let $\rho = \lambda_0/\sum_{i\in K} \lambda_i$, where $K = \{1, 2, \ldots, k\}$.

For convenience, we let $\overline{K} = \{0\} \cup K$, where 0 represents arrivals of customers.

Let T_i be a random variable subject to the distribution F_i for $i \in \overline{K}$. To exclude trivial cases, we assume throughout the paper that

(A1) $\mathbb{P}(T_i > 0) > 0$ for all $i \in \overline{K}$.

This condition is equivalent to that of $\mathbb{E}(T_i) > 0$ for all $i \in \overline{K}$. It is notable that $F_0(0)(= \mathbb{P}(T_i = 0)) > 0$ is possible under (A1).

Since servers are heterogeneous, we need to specify which idle server is chosen by the next customer to be served in a FCFS manner. Furthermore, if more than one server simultaneously completes service and if the number of waiting customers is less than the number of those servers, then we also need a rule to assign the waiting customers to available servers. This may not occur with a positive probability if service-time distributions have densities. However, we do not make such an assumption, and provide a server assigning rule to fully specify the model.

For this, we first put an arriving customer in the waiting line, and put the servers who complete service at the same moment into the set of available servers. We then select customers in the waiting line in a FCFS manner, and exclusively assign them to servers from this set. Let j be the number of waiting customers, and let A be the index set of available servers. If $j \geq |A|$, then all servers become busy, so we do not need any rule, where $|A|$ is the cardinality of A. If $1 \leq j < |A|$, we denote the index set of the servers who start new services by $D(A, j)$.

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For convenience, we let $D(A, j) = A$ if $j \geq |A|$. Thus, $D(A, j)$ is a set-valued random variable for each $A \in 2^K$ and $j \geq 1$. We assume that $D(A, j)$ has the same distribution for each fixed A and j and is independent of everything else. We refer to this server selection according to $D(A, j)$ as a server selection rule. Since we consider the large queue asymptotics, the majority of our results are independent of this rule.

Since $\mathbb{P}(T_0 = 0) > 0$ is possible, there is some arbitrariness when we apply the server selection rule. In this paper we number arriving customers one by one, and independently apply the server selection rule for them even when they arrive simultaneously. Similarly, if some servers have zero service times, then the server selection rule is repeatedly applied every time when they occur. This dynamics is detailed in the proof of Lemma 2.3 in Appendix A.

We describe this queueing model by a continuous-time Markov process. For this, we define the following characteristics at each time t, all of which are assumed to be right continuous.

- $L(t)$: the number of customers in system.
- $I(t)$: the index set of idle servers.
- $J(t) \equiv K \setminus I(t)$: the index set of busy servers.
- $\overline{J(t)} \equiv \overline{K} \setminus I(t)$: the index 0 is included.
- $R_0(t)$: the residual exogenous arrival time.
- $R_i(t)$: the residual service time at server $i \in J(t)$, and $R_i(t) = 0$ for $i \in I(t)$,

where T_i is independently sampled subject to F_i when $i \in \overline{J(t)}$ and $R_i(t-) = 0$. Here,

$$R(t-) = \lim_{\varepsilon \downarrow 0} R(t - \varepsilon), \qquad t > 0.$$

We denote the $(k+1)$-dimensional vector whose ith entry is $R_i(t)$ by $\boldsymbol{R}(t)$. Let

$$X(t) = (L(t), J(t), \boldsymbol{R}(t)), \qquad t \geq 0,$$

then $X(\cdot) \equiv \{X(t);\ t \geq 0\}$ is a Markov process, which describes the heterogeneous k-server queue. By the right-continuous assumption, $X(t)$ is right-continuous and has a left-hand limit for $t \geq 0$. Its state space is a subset of $\mathbb{Z}_+ \times 2^K \times \mathbb{R}_+^{k+1}$, where $\mathbb{Z}_+$ and $\mathbb{R}_+$ are the sets of nonnegative integers and numbers, respectively, and 2^K is the set of all subsets of K.

This Markov process can be considered as a PDMP, introduced by Davis [8]. To have an appropriate state space for the Markov process, we first define faces which are specified by idle servers. For $A \in 2^K$, let

$$S_A = \begin{cases} \{(\ell, K, \boldsymbol{y}) \in \mathbb{Z}_+ \times 2^K \times \mathbb{R}_+^{k+1};\ \ell \geq k\}, & A = \varnothing, \\ \{(k - |A|,\ K \setminus A, \boldsymbol{y}) \in \mathbb{Z}_+ \times 2^K \times \mathbb{R}_+^{k+1}\}, & A \neq \varnothing. \end{cases}$$

These S_A are referred to as faces. Thus, the state space of the PDMP $\{X(t)\}$ is given by $S = \bigcup_{A \in 2^K} S_A$. In some cases, it will be convenient to use other faces defined as

$$\Gamma_i = \{(\ell, U, \boldsymbol{y}) \in S;\ i \in K \setminus U\}, \qquad i \in K.$$

It is notable that Γ_i is different from $S_{\{i\}}$. Yet another useful notation is the boundary of S which represents the points where some residual times vanish. Similar to [8], we denote it by Γ. Namely,

$$\Gamma = \{(\ell, U, \boldsymbol{y}) \in S;\ y_0 = 0 \text{ or } K \setminus U \neq \varnothing\}.$$

We will refer to Γ as a terminal set of S rather than a boundary.

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In a PDMP, Γ is the set of states just before discontinuous state changes. However, Γ can not be identified by $L(t)$ and $\boldsymbol{R}(t)$. We here need their left-hand limits. This is the reason why we include $J(t)$ in $X(t)$.

Throughout the paper, we assume the stability condition that

(A2) the PDMP process $X(\cdot)$ has a stationary distribution, denoted by π.

Recall $\rho = \lambda_0/(\sum_{i\in K}\lambda_i)$. It is easy to see that (A2) implies that $\rho \le 1$. On the other hand,

$$\lambda_0 < \sum_{i\in K}\lambda_i, \quad \text{equivalently,} \quad \rho < 1, \tag{2.1}$$

is necessary for the stability of the fluid limit in Theorem 3.1 of [6], and, therefore, implies that (A2) under the spread-out condition on either the interarrival time distribution or the service time distributions, where a distribution on $\mathbb{R}_+$ is said to be spread-out if its n times convolution for some $n \ge 1$ has a absolute continuous component with respect to Lebesgue measure (see Section VII.1 of [3]). Condition (A2) is unlikely to be satisfied for $\rho = 1$ except for the case that the interarrival and service times are deterministic, but its verification is not known. So, we will just assume that (2.1) holds in addition to (A1) and (A2).

2.2. Extended generator of the PDMP

We first consider the dynamics of the Markov process $X(\cdot)$. Let $\mathcal{F}_t = \sigma(X(s);\ s \le t)$, where $\sigma(\cdot)$ stands for the minimal σ-field. We call $\{\mathcal{F}_t;\ t \ge 0\}$ a filtration. Then, $X(\cdot)$ is a strong Markov process with respect to $\{\mathcal{F}_t;\ t \ge 0\}$. We describe the dynamics of this Markov process using real-valued test functions. For this, we introduce useful classes for them. By $C^{(1,p)}_{k+1}(S)$, we denote the set of measurable functions f from S to $\mathbb{R}$ such that, for each fixed $(\ell, U) \in \mathbb{Z}_+ \times 2^K$, $f(\ell, U, \boldsymbol{y})$ is continuous for $\boldsymbol{y} \in \mathbb{R}^{k+1}_+$, and has continuous partial derivatives for $\boldsymbol{y} \in \mathbb{R}^{k+1}_+$ except for $y_i \ne 0$. The set of bounded functions in $C^{(1,p)}_{k+1}(S)$ is denoted by $C^{(1,p)}_{b,k+1}(S)$.

For $f \in C^{(1,p)}_{k+1}(S)$, define a partial differential operator $\mathcal{A}$ as

$$\mathcal{A}f(\ell, U, \boldsymbol{y}) = -\sum_{i\in\overline{U}}\frac{\partial f}{\partial y_i}(\ell, U, \boldsymbol{y}), \qquad (\ell, U, \boldsymbol{y}) \in S, \qquad y_j > 0 \quad \text{for all } j \in \overline{U}, \tag{2.2}$$

where $\overline{U} = \{0\} \cup U$, and recall that D_i is a finite subset of $\mathbb{R}_+$ such that the partial derivative of $f(\ell, U, \boldsymbol{y})$ with respect to $y_i > 0$.

For each $i \in \overline{K}$, let $T_i(\ell)$ be the ℓth interarrival time for $i = 0$ and service time for $i \in K$, let $t_{i,0} = 0$ and define $t_{i,n}$ as

$$T_i(1) + T_i(2) + \cdots + T_i(n), \qquad n = 1, 2, \ldots,$$

and define the counting process $N_i(\cdot) \equiv \{N_i(t);\ t \ge 0\}$ as

$$N_i(t) = \sum_{n=1}^{\infty}\mathbf{1}\{t_{i,n} \le t\}, \qquad t \ge 0,$$

where $\mathbf{1}$ is the indicator function.

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Thus, $N_0(\cdot)$ counts arrivals while $N_i(\cdot)$ for $i \in K$ counts departures from server i. By superposing them, define the counting process $N(\cdot) \equiv \{N(t);\ t \geq 0\}$ as

$$N(t) = N_0(t) + \sum_{i \in K} N_i(t), \qquad t \geq 0.$$

By the definitions, $N(0) = N_i(0) = 0$. It also is easy to see that

$$L(t) = L(0) + N_0(t) - \sum_{i \in K} N_i(t), \qquad t \geq 0.$$

Note that $N(\cdot)$ may have jumps with size greater than 1. If this is the case, an arrival and/or multiple departures from different servers occur simultaneously. This situation is already argued as for the server selection rule. For our arguments, it is convenient to define the simplified version of $N(\cdot)$, which counts the number of atoms of $N(\cdot)$ but ignores their magnitude. To do so, we let $\{t_n;\ n \geq 1\}$ be a sequence of time instants defined as

$$t_n = \inf\{s > t_{n-1};\ N(s) - N(t_{n-1}) > 0\}, \qquad n = 1, 2, \ldots,$$

where $t_0 = 0$. We then define $N^*(\cdot) \equiv \{N^*(t);\ t \geq 0\}$ as

$$N^*(t) = \sum_{n=1}^{\infty} \mathbf{1}\{t_n \leq t\}, \qquad t \geq 0.$$

Since $X(t)$ is absolutely continuous in $t \in \mathbb{R}_+$ with respect to the sum of the Lebesgue measure and the random measure generated by the counting process $N^*(\cdot)$, the integration formula yields

$$f(X(t)) = f(X(0)) + \int_0^t \mathcal{A} f(X(s))\, \mathrm{d}s + \int_0^t \Delta f(X(s))\, \mathrm{d}N^*(s), \qquad f \in C_{k+1}^{(1,p)}(S), \quad (2.3)$$

where $\Delta f(X(s)) = f(X(s)) - f(X(s-))$. Throughout the paper, operator Δ stands for such a difference. Note that $\Delta N^*(s) > 0$ if and only if $X(s-) \in \Gamma$, which causes a discontinuous state change, that is, $X(s-) \neq X(s)$. We refer to it as a jump. To describe this jump, we define the transition operator Q on $C_{k+1}^{(1,p)}(S)$ as

$$Qf(X(s-)) = \mathbb{E}(f(X(s)) \mid X(s-)), \qquad X(s-) \in \Gamma,\ f \in C_{k+1}^{(1,p)}(S), \quad (2.4)$$

where $X(0)$, $X(0-)$ should be considered as $X(0+)$ and $X(0)$, respectively. We refer to this Q as a jump kernel.

Our arguments will be based on the following fact.

Lemma 2.1. (A version of Proposition 4.3 of Davis [8].) *For a Borel-measurable function f from S to $\mathbb{R}$, let*

$$M_0(t) = \int_0^t (f(X(s)) - Qf(X(s-)))\, \mathrm{d}N^*(s), \qquad t \geq 0, \quad (2.5)$$

then $M_0(\cdot) \equiv \{M_0(t);\ t \geq 0\}$ is an $\mathcal{F}_t$-martingale if

$$\mathbb{E}\left(\int_0^t |f(X(s)) - Qf(X(s-))|\, \mathrm{d}N^*(s)\right) < \infty \quad \textit{for each } t > 0. \quad (2.6)$$

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This fact was proved in [8] using the compensator of the point process $N^*(\cdot)$, which was derived in [7]. However, its proof can be much simpler, which is given in Appendix A. Let $M(t) = M_0(t) + f(X(0))$, then (2.3) implies that, for $f \in C_{k+1}^{(1,p)}(S)$,

$$M(t) = f(X(t)) - \left(\int_0^t \mathcal{A} f(X(s))\,\mathrm{d}s + \int_0^t (Qf(X(s-)) - f(X(s-)))\,\mathrm{d}N^*(s) \right). \quad (2.7)$$

The following fact is immediate from Lemma 2.1.

Lemma 2.2. (A special case of Theorem 5.5 of [8].) *For $f \in C_{k+1}^{(1,p)}(S)$, if either (2.6) with $\mathbb{E}(|f(X(0))|) < \infty$ or $\mathbb{E}(|M(t)|) < \infty$ is satisfied, then $M(\cdot) \equiv \{M(t);\ t \geq 0\}$ is an $\mathcal{F}_t$-martingale. In particular, if f satisfies*

$$Qf(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \textit{for all } \boldsymbol{x} \in \Gamma, \quad (2.8)$$

then the $\mathcal{F}_t$-martingale $M(\cdot)$ is simplified to

$$M(t) = f(X(t)) - \int_0^t \mathcal{A} f(X(s))\,\mathrm{d}s. \quad (2.9)$$

Davis [8] referred to (2.8) as a boundary condition. Since the term boundary is used in a different sense in the queueing process context, we refer to (2.8) as a terminal condition. Recall that Γ is named as the terminal set, which corresponds to this terminology. Ideally, one would want to find a class of test functions such that its elements satisfy (2.8) and characterize the Markov process $X(\cdot)$ (see, e.g. Chapter 4 of [11]). However, no recipe is available for it. We attack this problem from a different viewpoint. Namely, we work with a smaller class of test functions whose elements satisfy (2.8), while it does not characterize $X(\cdot)$ but is still useful for asymptotic analysis.

2.3. Good test functions for the terminal condition

Let us consider the following form of a test function for each fixed $\theta, \eta, \zeta_i \in \mathbb{R}$:

$$f(\ell, U, \boldsymbol{y}) = \mathrm{e}^{\theta(\ell \vee k)}\, \mathrm{e}^{\eta g_0(y_0)} \prod_{i \in U} \mathrm{e}^{\zeta_i g_i(y_i)}, \qquad (\ell, U, \boldsymbol{y}) \in S, \quad (2.10)$$

where g_i is a function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $g_i(0) = 0$, $g_i(y)$ is continuous and has a continuous derivative in y except for finitely many values, and $a \vee b = \max(a, b)$ for real numbers a and b. We also use the notation $\wedge$ as $a \wedge b = \min(a, b)$.

We will use the following two functions for these g_i for $i \in \overline{K}$:

$$g_i(y) = y, \quad \text{or} \quad g_i(y) = y \wedge v_i, \qquad v_i > 0. \quad (2.11)$$

Obviously, $g_i(y) = y$ is a natural choice for g_i, but this choice may not work well when the tail probability of T_i is heavy. This is the reason why we introduced the second choice in (2.11), where v_i is referred to as a truncation level. In what follows, a common v is used for the truncation level for i in a subset of K, while u is used for $i = 0$.

Our next task is to find η, ζ_i as functions of θ for the f of (2.10) to satisfy (2.8). The idea behind this is to make the expected values of $\Delta f(X(t))$ equal zero at the jump instants of $N^*(\cdot)$ to vanish. For this, we choose them as

$$\eta \text{ is the solution of the equation:} \quad \mathrm{e}^{\theta}\mathbb{E}(\mathrm{e}^{\eta(T_0 \wedge u)}) = 1, \qquad u > 0, \quad (2.12)$$

$$\zeta_i \text{ is the solution of the equation:} \quad \mathrm{e}^{-\theta}\mathbb{E}(\mathrm{e}^{\zeta_i(T_i \wedge v)}) = 1, \qquad v > 0, \quad (2.13)$$

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and denote these η and ζ_i by $\eta(u,\theta)$ and $\zeta_i(v,\theta)$, respectively. Clearly, these functions are well defined and finite for all θ in appropriate half lines or in $\mathbb{R}$ for finite u, v since $\mathbb{E}(\mathrm{e}^{x(T_i\wedge u)})$ is strictly increasing in $x \in \mathbb{R}$ for $i \in \overline{K}$. For simplicity, we denote $\eta(\infty,\theta)$ and $\zeta_i(\infty,\theta)$, respectively, by $\eta(\theta)$ and $\zeta_i(\theta)$, which may not exist for some $\theta \in \mathbb{R}$.

For $A \in 2^K$, we denote f of (2.10) with $g_0(y) = y \wedge u$, $g_i(y) = y$ for $i \in K \setminus A$, $g_i(y) = y \wedge v$ for $i \in A$ for each $u, v > 0$, and $\theta \in \mathbb{R}$ by $f_{u,A(v),\theta}$. Namely,

$$f_{u,A(v),\theta}(\ell, U, \boldsymbol{y}) = \exp\Bigg(\theta(\ell \vee k) + \eta(u,\theta)(y_0 \wedge u) + \sum_{i\in U\setminus A} \zeta_i(\theta) y_i + \sum_{i\in A\cap U} \zeta_i(v,\theta)(y_i \wedge v)\Bigg) \quad \text{for } (\ell, U, \boldsymbol{y}) \in S. \tag{2.14}$$

If $u = \infty$, this test function is denoted by $f_{A(v),\theta}$. If $A = \varnothing$, then it is denoted by $f_{u,\theta}$. If $u = \infty$ and $A = \varnothing$, then it is denoted by f_θ. For example,

$$f_{K(v),\theta}(\ell, U, \boldsymbol{y}) = \exp\Bigg(\theta(\ell \vee k) + \eta(\theta) y_0 + \sum_{i\in U} \zeta_i(v,\theta)(y_i \wedge v)\Bigg).$$

This notation system will be used throughout the paper. The next lemma will be proved in Appendix A.

Lemma 2.3. *Under the setting (2.12) and (2.13) and assumption (A1), the terminal condition (2.8) holds for $f = f_{u,A(v),\theta}$ for finite $u, v > 0$, and $A \subset K$ as long as $\zeta_i(\theta)$ are well defined for $i \in K \setminus A$ and $\eta(u,\theta)$ is well defined when $u = \infty$.*

We will note that $\eta(u,\theta)$ and $\zeta_i(v,\theta)$ have nice probabilistic interpretations. Because they are similar, we present their general properties using some new notation. Let T be a nonnegative random variable with a finite and positive expectation. Denote its distribution by F, and let $\widehat{F}(\theta) = \mathbb{E}(\mathrm{e}^{\theta T})$ for $\theta \in \mathbb{R}$. We define β_* and θ_* as

$$\beta_* = \sup\{\theta \in \mathbb{R};\ \widehat{F}(\theta) < \infty\}, \qquad \theta_* = -\sup\{\theta \in \mathbb{R};\ \mathrm{e}^\theta < \widehat{F}(\beta_*)\}. \tag{2.15}$$

Note that $\beta_* \geq 0 \geq \theta_*$. If $\beta_* = 0$, then the distribution F is said to have a heavy tail. Otherwise, it is said to have a light tail. For $v > 0$, we consider $T \wedge v$ instead of T, and define $F(v,\cdot)$, $\widehat{F}(v,\cdot)$, $\theta_*(v)$, and $\beta_*(v)$ in a similar manner. It is notable that

$$\beta_*(v) = +\infty, \qquad \theta_*(v) = -\infty,$$

since $T \wedge v$ is bounded by v. Let $\overline{\theta} = -\log F(0)$, where $\overline{\theta} = \infty$ if $F(0) = 0$. Similar to (2.12), we define the functions $\xi(v,\cdot)$ as the solution to

$$\mathrm{e}^\theta \widehat{F}(v, \xi(v,\theta)) = 1. \tag{2.16}$$

Then, for finite $v > 0$, $\xi(v,\theta)$ is well defined and finite for all $\theta < \overline{\theta}$, but this may not be the case for $\xi(\theta) \equiv \xi(\infty,\theta)$.

We note that $-\xi(v,\cdot)$ can be considered as a rate function for large deviations. To see this, let $\{N(v,t);\ t \geq 0\}$ be the renewal counting process with interval time distribution $F(v,\cdot)$. From (2.16) and Theorem 1 of [13], we have

$$\xi(v,\theta) = -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}(\mathrm{e}^{\theta N(v,t)}), \qquad \theta < \overline{\theta}. \tag{2.17}$$

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Intuitively, this fact is helpful to understand what the function $\xi(v,\cdot)$ is, but technically it is not so helpful. Because of this, we will not use the fact that (2.17) holds in proofs. We now present important properties of $\xi(v,\cdot)$.

Lemma 2.4. (i) *For each finite $v > 0$, $\xi(v,\theta)$ is decreasing, concave and infinitely differentiable in $\theta < \overline{\theta}$.*

(ii) *For each fixed $\theta \in (0,\overline{\theta})$ ($\theta < 0$, respectively), $\xi(v,\theta)$ is negative (positive) and increasing (decreasing) in v.*

(iii) *For each $\delta \in (0,\overline{\theta})$ and $v > 0$, we have*

$$|\xi(v,\theta)| \le \max\left(\frac{1}{\mathbb{E}(T\wedge v)},\ \frac{1}{\delta}|\xi(v,\delta)|\right)|\theta| \qquad \theta < \delta. \tag{2.18}$$

Although (i) is immediate from (2.17), we prove it in Appendix A as mentioned above. Using Lemma 2.4, we define $\xi(\triangle,\cdot)$ as

$$\xi(\triangle,\theta) = \lim_{v\uparrow\infty} \xi(v,\theta), \qquad \theta < \overline{\theta}. \tag{2.19}$$

It is notable that $\xi(\triangle,\theta)$ may be different from $\xi(\theta) \equiv \xi(\infty,\theta)$ as shown in the lemma below. From Lemma 2.4, we immediately have the following facts.

Lemma 2.5. (i) *$\xi(\triangle,\theta)$ is finite and concave for all $\theta < \overline{\theta}$.*

(ii) *$\xi(\triangle,\theta) \le \beta_*$ for all $\theta < \overline{\theta}$ with equality for $\theta \le \theta_*$, and $\xi(\triangle,\theta) = \xi(\theta)$ for $\theta \in (\theta_*,\overline{\theta})$.*

(iii) *$\widehat{F}(\xi(\triangle,\theta)) < \infty$ for all $\theta < \overline{\theta}$.*

Remark 2.1. Despite (iii), the derivative $\widehat{F}'(\xi(\triangle,\theta)) = \mathbb{E}(T_i\,\mathrm{e}^{\xi(\triangle,\theta)T_i})$ may not exist at $\theta = \theta_* > -\infty$, namely at $\theta < 0$ satisfying $\xi(\triangle,\theta) = \beta_*$. A typical example for this is that $F(x) = 1 - (1+x)^{-r}\,\mathrm{e}^{-\delta x}$ with $1 < r \le 2$ and $\delta \ge 0$, for which $\beta_* = \delta$, $\widehat{F}(\delta) < \infty$ and $\theta_* = -\log\widehat{F}(\delta) < 0$, and therefore $\xi(\triangle,\theta_*) = \delta$, but $\widehat{F}'(\delta) = \infty$.

We next apply these properties for T_0 and T_i for $i \in K$. For $T = T_0$, we obviously have $\eta = \xi$ and $\eta(v,\cdot) = \xi(v,\cdot)$, and therefore $\eta(\triangle,\cdot)$ is defined as $\xi(\triangle,\cdot)$. Similarly, for $T = T_i$ with $i \in K$, we have $\zeta_i(\theta) = \xi(-\theta)$ and $\zeta_i(v,\theta) = \xi(v,-\theta)$ by (2.13), and $\zeta_i(\triangle,\theta)$ is defined as $\xi(\triangle,-\theta)$. Similar to θ_* and β_* of (2.15), define, for $i \in K$,

$$\begin{aligned} \underline{\theta}_i &= \log F_i(0), \qquad \beta_i = \sup\{\theta \in \mathbb{R};\ \widehat{F}_i(\theta) < \infty\}, \\ \theta_i &= \sup\{\theta \in \mathbb{R};\ e^{\theta} < \widehat{F}_i(\beta_i)\}, \end{aligned} \tag{2.20}$$

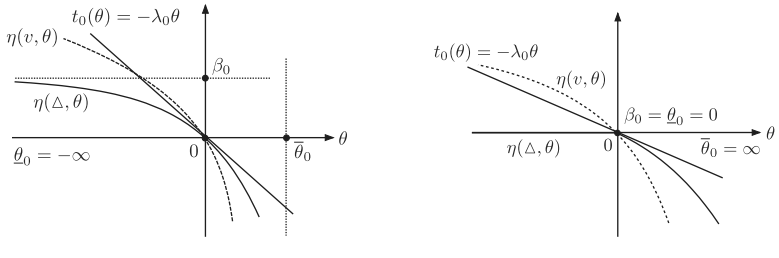


FIGURE 1: Plots of $\eta(v,\theta)$ and $\eta(\triangle,\theta)$ for light-tailed (*left*) and heavy-tailed (*right*) F_0.

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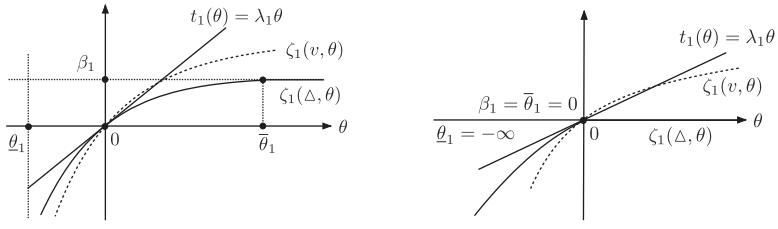


FIGURE 2: Plots of $\zeta_1(v,\theta)$ and $\zeta_1(\triangle,\theta)$ for light-tailed (*left*) and heavy-tailed (*right*) F_1.

while $\overline{\theta}_0 = -\log F_0(0)$, $\beta_0 = \beta_*$, and $\theta_0 = \theta_*$ for $T = T_0$, where $\widehat{F}_i$ is the moment generating function of the distribution F_i, namely, $\widehat{F}_i(\theta) = \mathbb{E}(\mathrm{e}^{\theta T_i})$, and, for $i \in \overline{K}$, $\log F_i(0) = -\infty$ if $F_i(0) = 0$. Based on Lemmas 2.4 and 2.5, typical shapes of the functions $\eta(v,\cdot)$, $\eta(\triangle,\cdot)$, $\zeta(v,\cdot)$, and $\zeta(\triangle,\cdot)$ are shown in Figures 1 and 2.

2.4. Stationary equations

Let $u = v$ and $A = K$ for $f_{u,A(v),\theta}$ of (2.14). We will use this test function $f_{v,K(v),\theta}$ for deriving the stationary equations. Let us introduce the following notation:

$$\Phi_i(v,\theta) = \mathbb{E}_\pi(f_{v,K(v),\theta}(X(0))\,\mathbf{1}\{R_i(0) < v\}), \qquad i \in \overline{K},$$
$$\Phi_{i,0}(v,\theta) = \mathbb{E}_\pi(f_{v,K(v),\theta}(X(0))\,\mathbf{1}\{i \notin J(0)\}), \qquad i \in K,$$

where $\mathbb{E}_\pi$ stands for the expectation conditioned on $X(0)$ subject to π. We define these functions also for $v \uparrow \infty$, and denote the limit of $\Phi_i(v,\theta)$ by $\Phi(\triangle,\theta)$, because it does not depend on i, while $\Phi_{i,0}(\triangle,\theta)$ is used as is. We wish to apply Lemma 2.2 to the function $f = f_{v,K(v),\theta}$ under the stationary distribution π. From the definition of $\mathcal{A}$ in (2.2), we have

$$\begin{aligned}\mathcal{A}f_{v,K(v),\theta}(\ell,U,\boldsymbol{y}) &= -\Bigg(\eta(v,\theta)\,\mathbf{1}\{y_0 < v\} + \sum_{i\in U}\zeta_i(v,\theta)\,\mathbf{1}\{y_i < v\}\Bigg) f_{v,K(v),\theta}(\ell,U,\boldsymbol{y}) \\ &= -\Bigg(\eta(v,\theta)\,\mathbf{1}\{y_0 < v\} + \sum_{i\in K}\zeta_i(v,\theta)\,\mathbf{1}\{y_i < v\} \\ &\qquad - \sum_{i\in K\setminus U}\zeta_i(v,\theta)\,\mathbf{1}\{y_i < v\}\Bigg) f_{v,K(v),\theta}(\ell,U,\boldsymbol{y}). \end{aligned} \tag{2.21}$$

Taking the expectation of (2.9) in Lemma 2.2 for $f = f_{v,K(v),\theta}$ and assuming that $\mathbb{E}_\pi(f(X(0)))$ is finite, we have the following lemma since $\Phi_i(v,\theta) < \infty$ implies that $\Phi_{i,0}(v,\theta)$ are finite for all $i \in \overline{K}$.

Lemma 2.6. *Under assumptions (A1) and (A2), for $v > 0$ and $\theta \in (\underline{\theta}_K, \overline{\theta}_0)$, where $\underline{\theta}_K = \vee_{i\in K}\underline{\theta}_i$,*

$$\eta(v,\theta)\Phi_0(v,\theta) + \sum_{i\in K}\zeta_i(v,\theta)\Phi_i(v,\theta) - \sum_{i\in K}\zeta_i(v,\theta)\Phi_{i,0}(v,\theta) = 0, \tag{2.22}$$

as long as $\Phi_i(v,\theta)$ is finite for all $i \in \overline{K}$. In particular, (2.22) always holds for $\theta \le 0$.

Let v go to ∞ in (2.22), then Lemmas 2.4 and 2.5 and the dominated convergence theorem yield the following lemma, recalling that $\Phi(\triangle,\theta) = \Phi_i(\triangle,\theta)$.

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Lemma 2.7. *Under the same assumptions as Lemma 2.6,*

$$\left(\eta(\triangle,\theta)+\sum_{i\in K}\zeta_i(\triangle,\theta)\right)\Phi(\triangle,\theta)-\sum_{i\in K}\zeta_i(\triangle,\theta)\Phi_{i,0}(\triangle,\theta)=0, \tag{2.23}$$

as long as $\Phi(\triangle,\theta)$ *is finite.*

We may consider (2.22) and (2.23) as stationary equations. For studying the asymptotic problems related to the stationary distribution of $L(\cdot)\equiv\{L(t);\ t\geq 0\}$, one may consider to apply the techniques developed in [18], [22], which are based on the stationary equation similar to (2.23) and its variant called a stationary inequality. However, one must be careful about it. First, $\Phi(\triangle,\theta)$ is not the moment generating function of L. Second, $\eta(\triangle,z)+\sum_{i\in K}\zeta_i(\triangle,z)$ may not be analytically extendable from the complex region $\{z\in\mathbb{C};\,\mathrm{Re}\,z<0\}$, since it is difficult to see their analytic properties. Therefore, it is difficult to apply the method of analytic extension in general, which is typically used for tail asymptotic problems (see, e.g. [17]).

3. Large queue asymptotics

In this section we present main results on tail asymptotics and weak limit approximations for the stationary distribution of $L(\cdot)$.

3.1. Tail asymptotics

We first consider the tail asymptotics. For simplicity, we assume that $F_0(0)=0$ in this subsection, so $\overline{\theta}_0=\infty$. As we argued at the end of the previous section, Lemma 2.7 may not be helpful for this purpose. Nevertheless, the coefficient of $\Phi(\theta)$ in it suggests that the tail decay rate would be given by

$$\alpha\equiv\sup\left\{\theta\geq 0;\ -\left(\eta(\theta)+\sum_{i\in K}\zeta_i(\triangle,\theta)\right)\leq 0\right\}, \tag{3.1}$$

where we have replaced $\eta(\triangle,\theta)$ by $\eta(\theta)$ using Lemma 2.5. Recall the definitions of β_i and θ_i for the distribution F_i (see (2.20)). If $\alpha<\theta_i$, Lemma 2.5 implies that we can also replace $\zeta_i(\triangle,\theta)$ in (3.1) by $\zeta_i(\theta)$. Let

$$\gamma_{K(\triangle)}(\theta)=-\left(\eta(\theta)+\sum_{i\in K}\zeta_i(\triangle,\theta)\right), \tag{3.2}$$

then $\gamma_{K(\triangle)}(\theta)$ is finite for $\theta\geq 0$ and a convex function by Lemma 2.5. This $\gamma_{K(\triangle)}$ corresponds to the rate function corresponding to $-\xi(v,\theta)$ of (2.17), which is the reason why minus signs are in (3.1) and (3.2). If $\alpha>0$ then $\gamma_{K(\triangle)}(\theta)<0$ for $\theta\in(0,\alpha)$ since $\gamma'_{K(\triangle)}(0)=\lambda_0-\sum_{i\in K}\lambda_i<0$ by (2.1). Furthermore, if $\zeta_i(\triangle,\alpha)<\beta_i$, equivalently, $\alpha<\theta_i$, for all $i\in K$, then $0<\gamma'_{K(\triangle)}(\alpha)<\infty$. Otherwise, $\gamma'_{K(\triangle)}(\alpha)$ may diverge by Remark 2.1.

We prove the following theorem in Section 4. It shows that α is indeed the decay rate in the logarithmic sense. For this, we need a further condition.

(A3) For all $i\in K$, the distribution of T_i is spread-out.

Theorem 3.1. *Assume that the PDMP* $\{(L(t),J(t),\boldsymbol{R}(t))\}$ *for the heterogeneous* k*-server queue satisfies (A1) with* $F_0(0)=0$*, (A2), (A3), and (2.1), and let* L *be a random variable subject to the stationary distribution of* $L(\cdot)$*.*

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(i) *If $\alpha < \theta_i$ for all $i \in K$, where θ_i is defined by (2.20), then $\alpha > 0$ and, for some $c > 0$,*

$$\lim_{x\to\infty} \mathrm{e}^{-\alpha x}\mathbb{P}(L > x) = c. \tag{3.3}$$

(ii) *For each $\alpha \geq 0$, we have*

$$\lim_{x\to\infty} \frac{1}{x} \log \mathbb{P}(L > x) = -\alpha. \tag{3.4}$$

Remark 3.1. For a heterogenous multiserver queue with batch arrivals, similar tail asymptotics are obtained for the stationary distributions of the waiting time, the queue length just after arrival instants and some related characteristics embedded at arrival instants in [29], [30]. Thus, Theorem 3.1 may be considered as the continuous-time counterparts of those discrete-time results for single arrivals. Its batch arrival version will be discussed in Section 5.1, which includes the case that $F_0(0) > 0$.

From Figures 1 and 2, we can see that the heaviness of the interarrival time distribution F_0 has no significant influence on the tail decay rate α. This is intuitively clear. Let us consider how the heaviness of the service-time distributions influences the tail decay rate. Sadowsky [29] calls a server with a heavy-tailed service-time distribution lazy, and remarks that they have no contribution for the decay rate α (see the second last paragraph in Section 1 in [29]). In this paper, such a server is called slow. To see the influence of slow servers, define, for $A \in 2^K$,

$$\rho_A = \frac{\lambda_0}{\sum_{i\in A}\lambda_i}, \qquad \alpha_A = \sup\left\{\theta \geq 0;\ -\left(\eta(\theta) + \sum_{i\in A}\zeta_i(\triangle, \theta)\right) \leq 0\right\}.$$

Let $K_0 = \{i \in K;\ \beta_i = 0\}$, which is the index set of the heavy-tailed service-time distributions. Then, the following fact is immediate from Theorem 3.1 since $\zeta_i(\triangle, \alpha) = 0$ for $i \in K_0$ and $\gamma_{K(\triangle)}(\theta)$ is convex in $\theta \in \mathbb{R}$.

Corollary 3.1. *Under the assumptions of Theorem 3.1, $\alpha = \alpha_{K\setminus K_0}$, and $\alpha > 0$ if and only if $\rho_{K\setminus K_0} < 1$.*

As we mentioned in Section 1, our approach is flexible for modeling extensions. As an example, we here modify the arrival process. The arrival process has been assumed to be a renewal process for the heterogeneous multiserver queue. We replace it by the superposition of renewal processes. Assume that there are m arrival streams, indexed by $1, 2, \ldots, m$, which are renewal processes with the interval time distributions $F_{e,1}, F_{e,2}, \ldots, F_{e,m}$, respectively. We further assume that they are mutually independent and independent of everything else. Denote those renewal processes as counting processes by $N_{e,1}(\cdot), N_{e,2}(\cdot), \ldots, N_{e,m}(\cdot)$, then their superposition is given by

$$N_e(t) \equiv \sum_{i=1}^{m} N_{e.i}(t), \qquad t \geq 0.$$

We take this $N_e(\cdot) \equiv \{N_e(t);\ t \geq 0\}$ as an arrival process for the heterogeneous multiserver queue. We assume the same servers as in Section 2.

In general, superposing renewal processes is analytically not tractable for an arriving process of a queueing model. However, this is not the case in the present formulation because we only need to redefine $\eta(\theta)$ as

$$\eta(\theta) = \sum_{i=1}^{m} \eta_i(\theta), \qquad \theta \in \mathbb{R}, \tag{3.5}$$

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as long as all $\eta_i(\theta)$ exists, where $\eta_i(\theta)$ is obtained as the solution to

$$e^{\theta}\widehat{F}_{e,i}(\eta_i(\theta)) = 1, \qquad i = 1, 2, \ldots, m.$$

Note that $\eta_i(\theta) \le 0$ for $\theta \ge 0$, and therefore it exists and is finite for $\theta \ge 0$.

Theorem 3.1 can be extended for the superposed renewal arrivals. This extension is routine, so here we only present results without proof. Let $R_{e,i}(t)$ for $i \in J$ be the residual arrival time at time $t \ge 0$ for the ith renewal arrival process, and let $R_{s,j}(t)$ be the residual service time at time $t \ge 0$ for the jth server. Denote $\boldsymbol{R}_e(t)$ and $\boldsymbol{R}(t)$ be the vectors whose ith and jth entries are $R_{e,i}(t)$ and $R_{s,i}(t)$, respectively.

Theorem 3.2. *Assume that the PDMP* $\{(L(t), J(t), \boldsymbol{R}_e(t), \boldsymbol{R}(t))\}$ *for the heterogeneous k server queue with superposed renewal arrivals satisfies (A1) with* $F_{e,i}(0) = 0$ *for* $i = 1, 2, \ldots, m$, *(A2), (A3), and (2.1), where* λ *is the total arrival rate, and let L be a random variable subject to the stationary distribution of* $L(\cdot)$. *Then, (i) and (ii) of Theorem 3.1 hold for the* η *of (3.5).*

3.2. Weak limit approximations for a large queue

We next consider weak limit approximations for a sequence of the heterogeneous multiserver queues, where $F_0(0)$ may not necessarily vanish. For this queue, Chen and Yao [5] studied a diffusion approximation, that is, a process limit of those queues under diffusion scaling in heavy traffic. They use a bounding method, called a sandwich. As mentioned in Section 1, we do not deal with process limits, and directly consider the stationary queue length distribution. In this situation, we do not need to scale time. We consider two weak limit approximations for bounded and large variances under the heavy-traffic condition.

We consider a sequence of systems indexed by $n = 1, 2, \ldots$, and denote the characteristics of the nth system by superscript '$^{(n)}$' (for example, $X^{(n)}(t)$, $R_i^{(n)}(t)$, and $T_i^{(n)}$ for $i \in \overline{K}$). Throughout the indexed systems, we assume that the number k of servers is unchanged, while the server selection rule may change. The mean arrival rate and the mean service rate of server i are denoted by $\lambda_0^{(n)}$ and $\lambda_i^{(n)}$, respectively. Denote the variances of $T_i^{(n)}$ by $(\sigma_i^{(n)})^2$, which are assumed to be finite. Let

$$\rho^{(n)} = \lambda_0^{(n)} \left(\sum_{i \in K} \lambda_i^{(n)} \right)^{-1},$$

and let $L^{(n)}$ be a random variable subject to the stationary distribution of $L^{(n)}(\cdot) \equiv \{L^{(n)}(t);\ t \ge 0\}$.

We will use the stationary equations (2.22) and (2.23) for $\theta \le 0$ in order to study the weak limit of the stationary distribution of $L^{(n)}(\cdot)$ under a suitable scaling. Let $\eta^{(n)}(v, \theta)$ and $\zeta_i^{(n)}(v, \theta)$ be the truncated versions of $\eta^{(n)}(\theta)$ and $\zeta_i^{(n)}(\theta)$, respectively, for the nth system, where v is a truncation level. Of importance to us will be their Taylor expansions around the origin. Let $\lambda_i^{(n)}(v) = 1/\mathbb{E}(T_i^{(n)} \wedge v)$, and let $(\sigma_i^{(n)}(v))^2$ be the variance of $T_i^{(n)} \wedge v$ for $i \in \overline{K}$. For $v > 0$, we define the functions

$$\begin{aligned} \varepsilon_0^{(n)}(v, \theta) &= \eta^{(n)}(v, \theta) + \lambda_0^{(n)}(v)\theta + \tfrac{1}{2}(\lambda_0^{(n)}(v))^3(\sigma_0^{(n)}(v))^2\theta^2, \qquad \theta < \overline{\theta}_0^{(n)}, \\ \varepsilon_i^{(n)}(v, \theta) &= \zeta_i^{(n)}(v, \theta) - \lambda_i^{(n)}(v)\theta + \tfrac{1}{2}(\lambda_i^{(n)}(v))^3(\sigma_i^{(n)}(v))^2\theta^2, \qquad \theta > \underline{\theta}_i^{(n)},\ i \in K, \end{aligned} \tag{3.6}$$

which correspond to the error terms for the second-order Taylor expansions of $\eta^{(n)}(v, \theta)$ and $\zeta_i^{(n)}(v, \theta)$. We define these error functions also for $v = \triangle$ similarly to the definition (2.19), in which $\eta^{(n)}(\triangle, \theta) = \lim_{v \uparrow \infty} \eta^{(n)}(v, \theta)$, $\zeta_i^{(n)}(\triangle, \theta) = \lim_{v \uparrow \infty} \zeta_i^{(n)}(v, \theta)$, $\lambda_i^{(n)}(\triangle) = \lambda_i^{(n)}(\infty) = \lambda_i^{(n)}$, and $\sigma_i^{(n)}(\triangle) = \sigma_i^{(n)}(\infty) = \sigma_i^{(n)}$.

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We will study the weak convergence of $q_n L^{(n)}$ for a sequence of positive numbers $\{q_n; n = 1, 2, \ldots\}$ which vanishes as n goes to ∞. To this end, we will evaluate $\varepsilon_i^{(n)}(1/q_n, q_n\theta)$. We first assume the following mean rate condition:

$$\lim_{n\to\infty} \lambda_i^{(n)} = \lambda_i, \qquad i \in \overline{K} \tag{3.7}$$

for some finite $\lambda_i > 0$. Clearly, this condition is equivalent to the convergence of the first moments of $T_i^{(n)}$ to finite and positive constants. We need the following fact, which is essentially a special case of Lemma 4.4 of [4], but an extra condition is assumed in [4]. We show that it is not needed in Appendix B.

Lemma 3.1. *If (3.7) holds and if $\{T_i^{(n)}; n \geq 1\}$ is uniformly integrable, then, for a sequence $q_n > 0$ converging to 0, we have, for some positive constants $c_i(\delta)$ for each $\delta > 0$,*

$$\limsup_{n\to\infty} \left| \max\left(\frac{\eta^{(n)}(\triangle, q_n\theta)}{q_n}, \frac{\eta^{(n)}(1/q_n, q_n\theta)}{q_n} \right) \right| \leq c_0(\delta)|\theta|, \qquad |\theta| < \delta, \tag{3.8}$$

$$\limsup_{n\to\infty} \left| \max\left(\frac{\zeta_i^{(n)}(\triangle, q_n\theta)}{q_n}, \frac{\zeta_i^{(n)}(1/q_n, q_n\theta)}{q_n} \right) \right| \leq c_i(\delta)|\theta|, \qquad i \in K,\ |\theta| \leq \delta. \tag{3.9}$$

Up to now, we have only assumed the convergence of the first moments, and $q_n > 0$ can be arbitrary as long as it converges to 0. We are ready to consider two scenarios on the variances of interarrival and service-time distributions for this sequence.

We first consider the heavy-traffic approximation for bounded variances. That is, in addition to (3.7), we assume that there are nonnegative constants σ_i for $i \in \overline{K}$ such that

$$0 < \rho^{(n)} < 1 \quad \text{for all } n \geq 1, \qquad \lim_{n\to\infty} \rho^{(n)} = 1, \tag{3.10}$$

$$\lim_{n\to\infty} \sigma_i^{(n)} = \sigma_i \geq 0, \quad i \in \overline{K}, \qquad \sum_{i\in\overline{K}} \sigma_i > 0, \tag{3.11}$$

$$\lim_{a\to\infty} \sup_{n\geq 1} \mathbb{E}((T_i^{(n)})^2\, \mathbf{1}\{T_i^{(n)} > a\}) = 0, \qquad i \in \overline{K}, \tag{3.12}$$

where $(\sigma_i^{(n)})^2$ is the variance of $T_i^{(n)}$. The conditions (3.10) and (3.12) are so-called heavy traffic and uniform integrability conditions, respectively. Note that the λ_i in (3.7) must be positive under (3.10). Let $r_n = 1 - \rho^{(n)}$ and choose $q_n = r_n$. The next lemma is a key for our proof, which is proved in Appendix B.

Lemma 3.2. *Assume that (3.7) and the heavy-traffic conditions (3.10)–(3.12), then, for each $i \in \overline{K}$ and each $a > 0$,*

$$\limsup_{n\to\infty} \sup_{0<|\theta|<a} \frac{1}{r_n^2\theta^2} \left| \max\left(\varepsilon_i^{(n)}(\triangle, r_n\theta),\ \varepsilon_i^{(n)}\left(\frac{1}{r_n}, r_n\theta\right) \right) \right| = 0. \tag{3.13}$$

This lemma is essentially the same as Lemma 4.3 of [4]. We now present the heavy-traffic approximation. The following theorem is proved in Section 4.3.

Theorem 3.3. *For a sequence of the stable heterogeneous multiserver queues satisfying assumptions (A1), (A2), and (2.1), assume that (3.7) and the heavy-traffic conditions (3.10)–(3.12) hold, and let $L^{(n)}$ be a random variable subject to the stationary distribution of $L^{(n)}(\cdot)$, then we have*

$$\lim_{n\to\infty} \mathbb{P}((1-\rho^{(n)})L^{(n)} \leq x) = 1 - \exp\left(-\frac{2\lambda_0}{\sum_{i\in\overline{K}} \lambda_i^3 \sigma_i^2}\, x \right), \qquad x \geq 0. \tag{3.14}$$

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Remark 3.2. Theorem 3.3 is a direct extension of Theorem 2.1 of [21] for the GI/G/1 queue. For a multiserver queue, this theorem generalizes Theorem 2 of [19] which assumes homogeneous servers.

The second scheme assumes large variances. In addition to (3.7) and (3.10), we assume that there is constant $b_i \geq 0$ for $i \in \overline{K}$ and a sequence of positive numbers s_n for $n \geq 1$ satisfying the following conditions:

$$\lim_{n\to\infty} s_n = 0, \qquad \lim_{n\to\infty} s_n(\sigma_i^{(n)})^2 = b_i^2, \quad i \in \overline{K}, \qquad \sum_{i\in\overline{K}} b_i^2 > 0, \tag{3.15}$$

which is referred to as a large variance condition. Let us consider the condition that

$$\lim_{a\to\infty} \sup_{n\geq 1} s_n \mathbb{E}((T_i^{(n)})^2 \mathbf{1}\{T_i^{(n)} > a\}) = 0, \qquad i \in \overline{K}. \tag{3.16}$$

If this were to hold, then we can obtain a weak limit approximation for $q_n = s_n$, replacing (3.10) by

$$\rho^{(n)} > 0, \qquad \lim_{n\to\infty} \rho^{(n)} = \rho < 1.$$

However, (3.16) can not hold because, for each $a > 0$, (3.15) implies that

$$\lim_{n\to\infty} s_n \mathbb{E}((T_i^{(n)})^2 \mathbf{1}\{T_i^{(n)} > a\}) = b_i^2.$$

Hence, the scaling factor must be of the small order of s_n. From the stationary equation (2.23), we can see that the heavy-traffic condition (3.10) is still needed and the scaling factor must be proportional to $r_n s_n \equiv (1 - \rho^{(n)})s_n$. In this case, Lemma 3.2 changes to the following lemma.

Lemma 3.3. *Assume that (3.7), (3.10), and the large variance condition (3.15) hold, then, for each $i \in \overline{K}$ and each $a > 0$,*

$$\limsup_{n\to\infty} \sup_{0<|\theta|<a} \frac{1}{s_n r_n^2 \theta^2} \left| \max\left(\varepsilon_i^{(n)}(\triangle, r_n s_n \theta),\ \varepsilon_i^{(n)}\left(\frac{1}{(r_n s_n)}, r_n s_n \theta \right) \right) \right| = 0.$$

This lemma is proved in Appendix B. Using this lemma, we prove the following theorem in Section 4.4.

Theorem 3.4. *For a sequence of stable heterogeneous multiserver queues satisfying (A1), (A2), and (2.1), assume that (3.7) and the large variance condition (3.15) in addition to (3.10) hold, then we have*

$$\lim_{n\to\infty} \mathbb{P}((1 - \rho^{(n)})s_n L^{(n)} \leq x) = 1 - \exp\left(-\frac{2\lambda_0}{\sum_{i\in\overline{K}} \lambda_i^3 b_i^2} x \right), \qquad x \geq 0, \tag{3.17}$$

where $L^{(n)}$ is a random variable subject to the stationary distribution of $L^{(n)}(\cdot)$.

4. Proofs of theorems

We first prove Theorem 3.1. Throughout Sections 4.1 and 4.2, it is assumed that $F_0(0) = 0$. We will use a change of measure using a martingale obtained from Lemma 2.2, which starts from $L(0) = k$. However, we can not directly introduce such a martingale because we may require the truncations of $R_i(t)$ for $i \in K$. For this, we recall the definitions (2.20) for θ_i for $i \in K$ and α of (3.1), and let, for $\theta \in \mathbb{R}$,

$$K_\theta = \{i \in K;\ \theta_i \leq \theta\}.$$

Obviously, $K_\theta = \varnothing$ for $\theta < 0$, and $K_\alpha = \varnothing$ for Theorem 3.1(i).

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4.1. Martingale and change of measure

We will truncate the service-time distributions whose indexes are in $A \supset K_\theta$ for $\theta \in \mathbb{R}$. Thus, similarly to $M(t)$ of (2.9), we define $M_{u,A(v),\theta}(t)$ for this A and $u, v > 0$, as

$$M_{u,A(v),\theta}(t) = f_{u,A(v),\theta}(X(t)) - \int_0^t \mathcal{A} f_{u,A(v),\theta}(X(s))\,\mathrm{d}s, \qquad t \ge 0, \tag{4.1}$$

which is well defined for $\theta > \underline{\theta}_K$ because $A \supset K_\theta$. From $\mathcal{A} f_{v,K(v),\theta}$ of (2.21), we have

$$\begin{aligned}
&\mathcal{A} f_{u,A(v),\theta}(X(s)) \\
&\quad = \Bigg(\gamma_{u,A(v)}(\theta) + \eta(u,\theta)\,\mathbf{1}\{R_0(s) \ge u\} + \sum_{i\in A} \zeta_i(v,\theta)\,\mathbf{1}\{R_i(s) \ge v\} \\
&\qquad + \sum_{i\in K} (\zeta_i(\theta)\,\mathbf{1}\{i \notin A\} + \zeta_i(v,\theta)\,\mathbf{1}\{i \in A\})\,\mathbf{1}\{X(s) \in \Gamma_i\}\Bigg) f_{u,A(v),\theta}(X(s)),
\end{aligned} \tag{4.2}$$

where $\gamma_{u,A(v)}(\theta) = -(\eta(u,\theta) + \sum_{i\in K}(\zeta_i(\theta)\,\mathbf{1}\{i \notin A\} + \zeta_i(v,\theta)\,\mathbf{1}\{i \in A\}))$.

By Lemma 2.3, $f_{u,A(v),\theta}$ satisfies the terminal condition (2.8), and therefore we can apply Lemma 2.2 if $\mathbb{E}_\nu(|M_{u,A(v),\theta}(t)|) < \infty$ for each $t \ge 0$ and each $\theta \ge 0$ because $\underline{\theta}_K < 0$, where $\mathbb{E}_\nu$ stands for the expectation given that $X(0)$ has a distribution ν. If nothing is mentioned, this ν is assumed to have a compact support concerning $L(0)$. To verify the finiteness of $\mathbb{E}_\nu(|M_{u,A(v),\theta}(t)|)$, we will use the following fact.

Lemma 4.1. *For each fixed $t > 0$ and for any $a > 0$, we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_\nu(N_0(t) \ge n) \le -a. \tag{4.3}$$

This lemma is intuitively clear, but we prove it in Appendix C in order to make our mathematical arguments transparent. The following lemma is a key for our arguments. For this, let $\widehat{\pi}$ be the stationary distribution of the embedded process of $X(\cdot)$ just before arrival instants, that is, $X(t_{0,n}-)$, which is extended for $n \le 0$ with $t_{0,0} = 0$. Note that $\widehat{\pi}$ is well defined because $X(\cdot)$ has the stationary distribution π. We define its conditional distribution ν_k on $(L(0-),\, L(0),\, \boldsymbol{R}(0-))$ as $\nu_k(\{\ell\} \times \{\ell'\} \times \boldsymbol{B}) = \mathbf{1}\{\ell + 1 = \ell' = k\}\mathbb{P}_{\widehat{\pi}}(\boldsymbol{R}(0-) \in \boldsymbol{B} \mid L(0-) + 1 = L(0) = k)$ for $\ell, \ell' \in \mathbb{Z}_+$, and $\boldsymbol{B} \in \mathcal{B}(\mathbb{R}_+^{k+1})$.

Lemma 4.2. *Let $\delta_0 = \infty$ if $K_\alpha = K$, while $\delta_0 = \min\{\theta_i;\ i \in K \setminus K_\alpha\} - \alpha > 0$ otherwise. For $\theta \in [0, \alpha + \delta_0)$, we have the following facts.*

(i) *For $t \ge 0$, $\sup_{v>0, s\in[0,t]}(\mathbb{E}_{\nu_k}(f_{K_\alpha(v),\theta}(X(s))) + \mathbb{E}_{\nu_k}(|M_{K_\alpha(v),\theta}(s)|)) < \infty$.*

(ii) *$\{M_{K_\alpha(v),\theta}(t);\ t \ge 0\}$ is an $\mathcal{F}_t$-martingale under $\mathbb{P}_{\nu_k}$.*

(iii) *$M_{K_\alpha(v),\theta}(t)$ almost surely converges to $M_{K_\alpha(\triangle),\theta}(t)$ as $v \uparrow \infty$, and*

$$M_{K_\alpha(\triangle),\theta}(t) = f_{K_\alpha(\triangle),\theta}(X(t)) - \int_0^t \mathcal{A} f_{K_\alpha(\triangle),\theta}(X(s))\,\mathrm{d}s, \qquad t \ge 0,$$

(iv) *$\{M_{K_\alpha(\triangle),\theta}(t);\ t \ge 0\}$ is an $\mathcal{F}_t$-supermartingale under $\mathbb{P}_{\nu_k}$.*

Remark 4.1. This lemma also holds for $f_{u,K(v),\theta}$ and $M_{u,K(v),\theta}(t)$ for $u, v > 0$ and $\theta > \underline{\theta}_K$ because all $R_i(t)$ are truncated for $i \in \overline{K}$.

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Lemma 4.2 enables us to apply an exponential change of measure following Palmowski and Rolski [26]. Taking (4.1) into account, we let

$$E^{f_{K_\alpha(v),\theta}}(t) = \frac{f_{K_\alpha(v),\theta}(X(t))}{f_{K_\alpha(v),\theta}(X(0))} \exp\left(-\int_0^t \frac{\mathcal{A} f_{K_\alpha(v),\theta}(X(s))}{f_{K_\alpha(v),\theta}(X(s))}\,\mathrm{d}s\right),$$

then $E^{f_{K_\alpha(v),\theta}}(t)$ is an $\mathcal{F}_t$-martingale by Lemma 3.1 of [26]. It follows from (4.2) that

$$\begin{aligned} E^{f_{K_\alpha(v),\theta}}(t) = \frac{f_{K_\alpha(v),\theta}(X(t))}{f_{K_\alpha(v),\theta}(X(0))} \exp\Bigg(&-\gamma_{K_\alpha(v)}(\theta)t - \sum_{i\in K_\alpha} \zeta_i(v,\theta) \int_0^t \mathbf{1}\{R_i(s) \ge v\}\,\mathrm{d}s \\ &- \sum_{i\in K} (\zeta_i(\theta)\,\mathbf{1}\{i \notin K_\alpha\} + \zeta_i(v,\theta)\,\mathbf{1}\{i \in K_\alpha\}) \\ &\times \int_0^t \mathbf{1}\{X(s) \in \Gamma_i\}\,\mathrm{d}s\Bigg). \end{aligned} \tag{4.4}$$

Recall the conditional distribution ν_k. Since $E^{f_{K_\alpha(v),\theta}}(t) > 0$ and $E^{f_{K_\alpha(v),\theta}}(0) = 1$, we can define a probability measure $\mathbb{P}_{\nu_k}^{(K_\alpha(v),\theta)}$ by

$$\left.\frac{d\mathbb{P}_{\nu_k}^{(K_\alpha(v),\theta)}}{d\mathbb{P}_{\nu_k}}\right|_{\mathcal{F}_t} = E^{f_{K_\alpha(v),\theta}}(t), \qquad t \ge 0. \tag{4.5}$$

We denote expectations under $\mathbb{P}_{\nu_k}^{(K_\alpha(v),\theta)}$ by $\mathbb{E}_{\nu_k}^{(K_\alpha(v),\theta)}$. By Theorem 5.3 of [26], $X(\cdot)$ under $\mathbb{P}_{\nu_k}^{(K_\alpha(v),\theta)}$ is again a PDMP with generator $\mathcal{A}$ starting with the initial distribution ν_k. To find the jump kernel Q under the new measure $\mathbb{P}_{\nu_k}^{(K_\alpha(v),\theta)}$, we note the following fact, which is immediate from the definition (4.5) and the truncation arguments.

Lemma 4.3. (Proposition 3.8(a) of [15, Chapter III].) *Let Z be a nonnegative random variable with finite expectation with respect to $\mathbb{P}_{\nu_k}^{(K_\alpha(v),\theta)}$, then*

$$\mathbb{E}_{\nu_k}^{(K_\alpha(v),\theta)}(Z \mid \mathcal{F}_s) = \frac{1}{E^{f_{K_\alpha(v),\theta}}(s)} \mathbb{E}_{\nu_k}(E^{f_{K_\alpha(v),\theta}}(t) Z \mid \mathcal{F}_s), \qquad 0 \le s < t. \tag{4.6}$$

Letting $s \uparrow t$ in (4.6), we have

$$\mathbb{E}_{\nu_k}^{(K_\alpha(v),\theta)}(Z \mid \mathcal{F}_{t-}) = \frac{1}{f_{K_\alpha(v),\theta}(X(t-))} \mathbb{E}_{\nu_k}(f_{K_\alpha(v),\theta}(X(t)) Z \mid \mathcal{F}_{t-}). \tag{4.7}$$

Thus, we define the transition kernel $Q^{(K_\alpha(v),\theta)}$ as, for $g \in C_{b,k+1}^{(1,p)}(S)$,

$$\begin{aligned} &Q^{(K_\alpha(v),\theta)} g(X(t-)) \\ &\quad = \frac{1}{f_{K_\alpha(v),\theta}(X(t-))} \mathbb{E}_{\nu_k}(f_{K_\alpha(v),\theta}(X(t)) g(X(t)) \mid X(t-)), \qquad X(t-) \in \Gamma, \end{aligned}$$

then this is the jump transition kernel Q under $\mathbb{P}_{\nu_k}^{(K_\alpha(v),\theta)}$. One can check that this is identical with Equation (5.14) of [26] because $f_{K_\alpha(v),\theta}$ satisfies the terminal condition (2.8). Similarly, we define $\mathcal{F}_t$-supermartingale $E^{f_{K_\alpha(\triangle),\theta}}(t)$, possibly defective probability measure $\mathbb{P}_{\nu_k}^{(K_\alpha(\triangle),\theta)}$, and jump kernel $Q^{(K_\alpha(\triangle),\theta)}$. Obviously, those jump kernels do not depend on ν_k.

We next consider the distributions of T_i at the jump instant under $\mathbb{P}_{\nu_k}^{(K_\alpha(v),\theta)}$ for $v > 0$ and $v = \triangle$. They are assumed to be independent of the initial distribution of $(L(0-), \boldsymbol{R}(0-))$ under ν_k, and may be written as $T_i^{(K_\alpha(v),\theta)}$. However, this notation is quite cumbersome, so we keep them as they are, which causes no confusion as long as measures for their expectations are specified.

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Lemma 4.4. *For each $v > 0$ and $\theta \in [0, \alpha + \delta_0)$,*

$$\mathbb{E}_{\nu_k}^{(K_\alpha(v),\theta)}(\mathrm{e}^{sT_0}) = \mathrm{e}^{\theta}\widehat{F}_0(\eta(\theta)+s), \qquad s \le \beta_0 - \eta(\theta), \tag{4.8}$$

$$\mathbb{E}_{\nu_k}^{(K_\alpha(v),\theta)}(\mathrm{e}^{sT_i}) = \mathrm{e}^{-\theta}\widehat{F}_i(\zeta_i(\theta)+s), \qquad s \le \beta_i - \zeta_i(\theta),\ i \in K \setminus K_\alpha, \tag{4.9}$$

$$\mathbb{E}_{\nu_k}^{(K_\alpha(v),\theta)}(\mathrm{e}^{s(T_i\wedge v)}) = \mathrm{e}^{-\theta}\widehat{F}_i(v, \zeta_i(v,\theta)+s), \qquad s \le \beta_i - \zeta_i(v,\theta),\ i \in K_\alpha. \tag{4.10}$$

The same formulas also hold for $v = \triangle$.

Remark 4.2. (i) The right-hand terms of the inequalities for s in (4.8) and (4.9) are the convergence exponents of the corresponding distributions because of (2.20).

(ii) Since $\eta(\alpha) \le 0$ and $\zeta_i(\alpha) < \beta_i$ for $i \in K \setminus K_\alpha$, $\mathbb{E}_{\nu_k}^{(K_\alpha(\triangle),\alpha)}(T_i)$ is finite for $i = 0$ and $i \in K \setminus K_\alpha$. Otherwise, consider $i \in K_\alpha$. Since $\beta_i - \zeta_i(\triangle, \alpha) = 0$ in this case, we have

$$\mathbb{E}_{\nu_k}^{(K_\alpha(\triangle),\alpha)}(T_i) = \mathrm{e}^{-\alpha}\mathbb{E}_{\nu_k}(T_i\,\mathrm{e}^{\beta_i T_i}),$$

which may be infinite as noted in Remark 2.1. Hence, even if $X(t)$ is a PDMP under $\mathbb{P}_{\nu_k}^{(K_\alpha(\triangle),\theta)}$, the corresponding service time of the ith server may have an infinite mean.

Lemma 4.4 will be proved in Appendix C. Since the generator $\mathcal{A}$ is unchanged under $\mathbb{P}_{\nu_k}^{(K_\alpha(v),\theta)}$ and $\mathbb{P}_{\nu_k}^{(K_\alpha(\triangle),\theta)}$, the distributions in Lemma 4.4 uniquely determine jump kernels $Q^{(K_\alpha(v),\theta)}$ and $Q^{(K_\alpha(\triangle),\theta)}$. Note that the distributions of T_0 and T_i for $i \in K \setminus K_\alpha$ do not depend on v under those change of measures.

Let us consider the mean drift of $L(t)$ in the off-boundary states under $\mathbb{P}_{\nu_k}^{(K_\alpha(v),\theta)}$ for $v > 0$ and $v = \triangle$. For this, we compute the corresponding characteristics to $\gamma_{K_\alpha(v)}(\theta)$. It is easy to see that η, ζ_i for $i \in K \setminus K_\alpha$ and $\zeta_i(v,\cdot)$ for $i \in K_\alpha$ under $\mathbb{P}_{\nu_k}^{(K_\alpha(v),\theta)}$ are given by

$$\eta^{(K_\alpha(v),\theta)}(\widetilde{\theta}) = \eta(\theta+\widetilde{\theta}) - \eta(\theta), \qquad \theta, \widetilde{\theta}+\theta \ge 0, \tag{4.11}$$

$$\zeta_i^{(K_\alpha(v),\theta)}(\widetilde{\theta}) = \zeta_i(\theta+\widetilde{\theta}) - \zeta_i(\theta), \qquad \theta, \widetilde{\theta}+\theta \in [0, \alpha+\delta_0),\ i \in K \setminus K_\alpha, \tag{4.12}$$

$$\zeta_i^{(K_\alpha(v),\theta)}(v,\widetilde{\theta}) = \zeta_i(v,\theta+\widetilde{\theta}) - \zeta_i(v,\theta), \qquad \theta, \widetilde{\theta}+\theta > \underline{\theta}_K,\ i \in K_\alpha. \tag{4.13}$$

From (4.11)–(4.13), we have, for $v > 0$ and $v = \triangle$,

$$\begin{aligned}\gamma_{K_\alpha(v)}^{(K_\alpha(v),\theta)}(\widetilde{\theta}) &\equiv -\Bigg(\eta^{(K_\alpha(v),\theta)}(\widetilde{\theta}) + \sum_{i\in K\setminus K_\alpha}\zeta_i^{(K_\alpha(v),\theta)}(\widetilde{\theta}) + \sum_{i\in K_\alpha}\zeta_i^{(K_\alpha(v),\theta)}(v,\widetilde{\theta})\Bigg)\\ &= \gamma_{K_\alpha(v)}(\theta+\widetilde{\theta}) - \gamma_{K_\alpha(v)}(\theta).\end{aligned} \tag{4.14}$$

Since $\zeta_i(\alpha) = \zeta_i(\triangle,\alpha)$ for $i \in K \setminus K_\alpha$, we have

$$\gamma_{K_\alpha(\triangle)}(\alpha) = \gamma_{K(\triangle)}(\alpha) = 0. \tag{4.15}$$

Hence, letting $v = \triangle$ and $\theta = \alpha$ in (4.14),

$$\gamma_{K_\alpha(\triangle)}^{(K_\alpha(\triangle),\alpha)}(\widetilde{\theta}) = \gamma_{K_\alpha(\triangle)}(\alpha+\widetilde{\theta}) > 0, \qquad \widetilde{\theta} > 0,$$

since $\gamma_{K_\alpha(\triangle)}(\alpha+\cdot)$ is a convex function and $\gamma'_{K_\alpha(\triangle)}(0-) < 0$. They also imply that

$$(\gamma_{K_\alpha(\triangle)}^{(K_\alpha(\triangle),\alpha)})'(0-) = \gamma'_{K_\alpha(\triangle)}(\alpha-) > 0.$$

This means that $L(t)$ has a positive mean drift at off-boundary states under $\mathbb{P}_{\nu_k}^{(K_\alpha(\triangle),\alpha)}$. Since $\gamma_{K_\alpha(v)}(\theta) \uparrow \gamma_{K_\alpha(\triangle)}(\theta)$ as $v \uparrow \infty$ by Lemma 2.4, we have the following key facts.

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Lemma 4.5. (i) *For all $v > 0$ and $\theta \in [0, \alpha]$, $\gamma_{K_\alpha(v)}(\theta) \le 0$.*

(ii) *There are a $\theta > 0$ and $v_0 > 0$ such that*

$$(\gamma_{K_\alpha(v)}^{(K_\alpha(v),\theta)})'(0-) > 0 \qquad \text{for all } v \ge v_0. \tag{4.16}$$

(iii) *Either for $(v, \theta) = (\triangle, \alpha)$ or for each $v, \theta > 0$ satisfying (4.16), $L(t) \to \infty$ almost surely as $t \to \infty$ under $\mathbb{P}_{\nu_k}^{(K_\alpha(v),\theta)}$.*

We can get back $\mathbb{P}$ from $\mathbb{P}_{\nu_k}^{(K_\alpha(v),\theta)}$. For this, let $Z(t)$ be a process which is $\mathcal{F}_t$-adapted and has a finite expectation for each $t \ge 0$. From (4.4) and (4.5), we have

$$\begin{aligned}\mathbb{E}_{\nu_k}(Z(t)) = \mathbb{E}_{\nu_k}^{(K_\alpha(v),\theta)}\Bigg(& Z(t)\exp(-\theta(L(t)-L(0)) + \gamma_{K_\alpha(v)}(\theta)t - \eta(\theta)(R_0(t)-R_0(0)) \\ & - W_{K_\alpha(v),\theta}(\boldsymbol{R}(t)) + W_{K_\alpha(v),\theta}(\boldsymbol{R}(0))) \\ & \times \exp\Bigg(\sum_{i\in K_\alpha}\zeta_i(v,\theta)\int_0^t (\mathbf{1}\{R_i(s) > v\} + \mathbf{1}\{X(s)\in\Gamma_i\})\,\mathrm{d}s \\ & + \sum_{i\in K\setminus K_\alpha}\zeta_i(\theta)\int_0^t \mathbf{1}\{X(s)\in\Gamma_i\}\,\mathrm{d}s\Bigg)\Bigg) \quad \text{for } v > 0,\ v = \triangle, \end{aligned} \tag{4.17}$$

where

$$W_{K_\alpha(v),\theta}(\boldsymbol{y}) = \sum_{i\in K\setminus K_\alpha}\zeta_i(\theta)y_i + \sum_{i\in K_\alpha}\zeta_i(v,\theta)(y_i\wedge v), \qquad \boldsymbol{y} = (y_0, y_1, \dots, y_k) \in \mathbb{R}_+^{k+1}.$$

4.2. Proof of Theorem 3.1

The spirit of our proof is similar to that of Theorem 2.3 of [30], which is based on the change of measure technique. However, the time index as well as the formulations are quite different. So, we will detail our proof. In this subsection, we present basic facts, then we prove the cases (i) and (ii) in Sections 4.2.1 and 4.2.2, respectively.

Define random times σ_ℓ^+ and σ_k^- as

$$\sigma_\ell^+ = \inf\{t > 0;\ L(t-) < L(t) = \ell\}, \qquad \sigma_k^- = \inf\{t > 0;\ L(t-) = k > L(t)\},$$

then they are stopping times. By this definition, σ_ℓ^+ is $\mathcal{F}_{\sigma_\ell^+-}$-measurable. Recall that ν_k specifies the conditional distribution of $\boldsymbol{R}(t_{0,0}-) \equiv \boldsymbol{R}(0-)$ under the stationary distribution of the embedded process $\{X(t_{0,n}-);\ n \in \mathbb{Z}\}$ given that $L(0-)+1 = L(0-) = k$. Then, the stationary tail probability of L is given by the so-called cycle formula (see, e.g. Corollary 2.1 of [24]), i.e.

$$\mathbb{P}(L \ge \ell \mid L \ge k) = \frac{1}{\mathbb{E}_{\nu_k}(\sigma_k^-)}\mathbb{E}_{\nu_k}\left(\int_0^{\sigma_k^-}\mathbf{1}\{L(s)\ge\ell\}\,\mathrm{d}s\right), \qquad \ell \ge k. \tag{4.18}$$

From (4.17) with $t = \sigma_\ell^+-$ (see, e.g. Theorem 3.4 of [15, Chapter III]) and Lemma 4.5, which implies that $\mathbb{P}_{\nu_k}^{(K_\alpha(\triangle),\alpha)}(\sigma_\ell^+ < \infty) = 1$, we have, for $\ell \ge k$,

$$\begin{aligned}&\mathbb{E}_{\nu_k}\left(\int_0^{\sigma_k^-}\mathbf{1}\{L(s)\ge\ell\}\,\mathrm{d}s\right) \\ &\quad = \mathbb{E}_{\nu_k}\left(\int_{\sigma_\ell^+}^{\sigma_k^-}\mathbf{1}\{L(s)\ge\ell\}\,\mathrm{d}s\,\mathbf{1}\{\sigma_\ell^+ < \sigma_k^-\}\right)\end{aligned}$$

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$$
\begin{aligned}
&= \mathbb{E}_{\nu_k}\left(\mathbb{E}_{\nu_k}\left(\int_{\sigma_\ell^+\wedge\sigma_k^-}^{\sigma_k^-} \mathbf{1}\{L(s)\geq \ell\}\,\mathrm{d}s \;\middle|\; \mathcal{F}_{\sigma_\ell^+-}\right)\mathbf{1}\{\sigma_\ell^+<\infty\}\right)\\
&= \mathbb{E}_{\nu_k}^{(K_\alpha(\Delta),\alpha)}\\
&\quad\times\left(\mathbb{E}_{\nu_k}\left(\int_{\sigma_\ell^+\wedge\sigma_k^-}^{\sigma_k^-} \mathbf{1}\{L(s)\geq \ell\}\,\mathrm{d}s \;\middle|\; \mathcal{F}_{\sigma_\ell^+-}\right)\mathbf{1}\{\sigma_\ell^+<\infty\}\right.\\
&\qquad\left.\times\,\mathrm{e}^{-\alpha(L(\sigma_\ell^+-)-L(0))-\eta(\alpha)(R_0(\sigma_\ell^+-)-R_0(0))-W_{K_\alpha(\Delta),\alpha}(\boldsymbol{R}(\sigma_\ell^+-))+W_{K_\alpha(\Delta),\alpha}(\boldsymbol{R}(0))}\right)\\
&= \mathrm{e}^{-\alpha(\ell-k-1)}\mathbb{E}_{\nu_k}^{(K_\alpha(\Delta),\alpha)}\left(\mathbb{E}_{\nu_k}\left(\int_{\sigma_\ell^+\wedge\sigma_k^-}^{\sigma_k^-} \mathbf{1}\{L(s)\geq \ell\}\,\mathrm{d}s \;\middle|\; \mathcal{F}_{\sigma_\ell^+-}\right)\right.\\
&\qquad\left.\times\,\mathrm{e}^{-W_{K_\alpha(\Delta),\alpha}(\boldsymbol{R}(\sigma_\ell^+-))+W_{K_\alpha(\Delta),\alpha}(\boldsymbol{R}(0-))}\right),
\end{aligned}
\tag{4.19}
$$

where we have used the fact that $\gamma_{K_\alpha(\Delta)}(\alpha)=0$ and $R_0(0-)=0$.

Since $L(t)$ in (4.19) is never below k, we can replace it by $H(t)$ defined as

$$
H(t)=L(0)+\widetilde{N}_0(t)-\sum_{i\in K}\widetilde{N}_i(t),
$$

where we recall that $\widetilde{N}_0$ and $\widetilde{N}_i$ are the renewal counting processes with the interval times T_0 and T_i, respectively. We denote the residual time to the next point of N_i at time t by $\widetilde{R}_i(t)$, which is identical with $R_i(t)$ for $t<\sigma_k^-$ when $L(0)=k$. Since the increments $\Delta H(t)$ only depend on $H(t-)$ and $\widetilde{\boldsymbol{R}}(t-)\equiv\{\widetilde{R}_i(t-);\ i\in\overline{K}\}$,

$$
Y(t)\equiv(H(t),\widetilde{\boldsymbol{R}}(t))
$$

is a continuous-time Markov additive process with additive component $H(t)$ and background process $\{\widetilde{\boldsymbol{R}}(t);\ t\geq 0\}$. We assume that $Y(\cdot)\equiv\{Y(0-),Y(t);\ t\geq 0\}$ starts with $Y(0-)$ subject to ν_k. In particular, $H(0)=k$ under this distribution.

Taking Remark 4.1 into account, we introduce the following $\mathcal{F}_t$-supermartingale for $Y(\cdot)$ with the initial distribution ν_k and for $\theta\in[0,\alpha+\delta_0)$:

$$
\begin{aligned}
\widetilde{E}^{\widetilde{f}_{u,K_\alpha(\Delta),\theta}}(t) &= \frac{\widetilde{f}_{u,K_\alpha(\Delta),\theta}(Y(t))}{\widetilde{f}_{u,K_\alpha(\Delta),\theta}(Y(0))}\\
&\quad\times\exp\left(-\gamma_{u,K_\alpha(\Delta)}(\theta)t-\eta(u,\theta)\int_0^t\mathbf{1}\{\widetilde{R}_0(s)\geq u\}\,\mathrm{d}s\right),
\end{aligned}
\tag{4.20}
$$

where $\widetilde{f}_{u,K_\alpha(\Delta),\theta}(Y(t))$ is defined as

$$
\widetilde{f}_{u,K_\alpha(\Delta),\theta}(Y(t))=\mathrm{e}^{\theta H(t)+\eta(u,\theta)\widetilde{R}_0(t)\wedge u+W_{K_\alpha(\Delta),\theta}(\widetilde{\boldsymbol{R}}_s(t))}.
$$

Similar to $\mathbb{P}_{\nu_k}^{(K_\alpha(\Delta),\theta)}$, we define the probability measure $\widetilde{\mathbb{P}}_{\nu_k}^{(u,K_\alpha(\Delta),\theta)}$ on $(\Omega,\mathcal{F})$ by

$$
\left.\frac{d\widetilde{\mathbb{P}}_{\nu_k}^{(u,K_\alpha(\Delta),\theta)}}{d\mathbb{P}_{\nu_k}}\right|_{\mathcal{F}_t}=\widetilde{E}^{\widetilde{f}_{u,K_\alpha(\Delta),\theta}}(t),\qquad t\geq 0.
\tag{4.21}
$$

We denote expectation under $\widetilde{\mathbb{P}}_{\nu_k}^{(u,K_\alpha(\Delta),\theta)}$ by $\widetilde{\mathbb{E}}_{\nu_k}^{(u,K_\alpha(\Delta),\theta)}$. We prove the following lemma in Appendix C.

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Lemma 4.6. *For $u > 0$ and $\theta \in [0, \alpha + \delta_0)$ satisfying that $\gamma_{u,K_\alpha(\triangle)}(\theta) < 0$ and $\gamma'_{u,K_\alpha(\triangle)}(\theta) > 0$, there is a positive constant C such that, for $\ell \geq k + 1$,*

$$\mathbb{E}_{\nu_k}\left(\mathrm{e}^{-W_{K_\alpha(\triangle),\theta}(\widetilde{\boldsymbol{R}}(\sigma_\ell^+-))}\int_{\sigma_\ell^+}^{\infty}\mathbf{1}\{H(s)\geq\ell\}\,\mathrm{d}s\;\middle|\;\mathcal{F}_{\sigma_\ell^+-}\right) < C. \tag{4.22}$$

Define $h_+(\ell)$ and $h_-(\ell)$ for $t \geq 0$ as

$$h_+(\ell) = \mathbb{E}_{\nu_k}\left(\int_{\sigma_\ell^+}^{\infty}\mathbf{1}\{H(s)\geq\ell\}\,\mathrm{d}s\;\middle|\;\mathcal{F}_{\sigma_\ell^+-}\right)\mathrm{e}^{-W_{K_\alpha(\triangle),\theta}(\widetilde{\boldsymbol{R}}(\sigma_\ell^+-))},$$

$$h_-(\ell) = \mathbb{E}_{\nu_k}\left(\int_{\sigma_k^-}^{\infty}\mathbf{1}\{H(s)\geq\ell\}\,\mathrm{d}s\,\mathbf{1}\{\sigma_\ell^+<\sigma_k^-\}\;\middle|\;\mathcal{F}_{\sigma_\ell^+-}\right)\mathrm{e}^{-W_{K_\alpha(\triangle),\theta}(\widetilde{\boldsymbol{R}}(\sigma_\ell^+-))}.$$

By Lemma 4.6, $h_+(\ell)$ and $h_-(\ell)$ are uniformly and almost surely bounded by C over all $\ell \geq 0$. Then, (4.19) can be written as,

$$\begin{aligned}&\mathrm{e}^{\alpha(\ell-k-1)}\mathbb{E}_{\nu_k}\left(\int_0^{\sigma_k^-}\mathbf{1}\{L(s)\geq\ell\}\,\mathrm{d}s\right)\\&\quad=\widetilde{\mathbb{E}}_{\nu_k}^{(K_\alpha(\triangle),\alpha)}([h_+(\ell)-h_-(\ell)]\\&\qquad\qquad\times\mathrm{e}^{-(W_{K_\alpha(\triangle),\alpha}(\widetilde{\boldsymbol{R}}(\sigma_\ell^+-))-W_{K_\alpha(\triangle),\theta}(\widetilde{\boldsymbol{R}}(\sigma_\ell^+-)))}\,\mathrm{e}^{W_{K_\alpha(\triangle),\alpha}(\boldsymbol{R}(0-))}).\end{aligned} \tag{4.23}$$

We need two more lemmas. The first one concerns the Markov additive process $Y(\cdot)$ under $\widetilde{\mathbb{P}}_{\nu_k}^{(K_\alpha(\triangle),\alpha)}$. Since $\widetilde{R}_i(t)$ for $i \in \overline{K}$ are the residual times for the next counting times of independent renewal processes, $\{\widetilde{\boldsymbol{R}}(t)\}$ is Harris positive recurrent by the spread-out condition (A3), and therefore the embedded Markov process $\{\widetilde{\boldsymbol{R}}(t_{0,n}-)\}$ also is Harris positive recurrent under $\widetilde{\mathbb{P}}_{\nu_k}^{(K_\alpha(\triangle),\alpha)}$. Since this process can be considered as a background process of $\{Y(t_{n,0}-)\}$ and $H(t_{0,n}-)$ has a positive drift under $\widetilde{\mathbb{P}}_{\nu_k}^{(K_\alpha(\triangle),\alpha)}$, Theorem 1 of [1] (see also Theorem 4.1 of [2]) yields the following fact.

Lemma 4.7. *Under the assumptions of Theorem 3.1, $\{\widetilde{\boldsymbol{R}}(\sigma_\ell^+-);\ \ell = 1, 2, \ldots\}$ is Harris positive recurrent under $\widetilde{\mathbb{P}}_{\nu_k}^{(K_\alpha(\triangle),\alpha)}$, so it has the stationary distribution, which is denoted by $\nu_\infty(Y)$.*

The second lemma concerns the distribution ν_k. Recall that the embedded process $\{X(t_{0,n})\}$ has the stationary distribution because $X(\cdot)$ does also. Then, by the spread-out condition (A3), $\{\boldsymbol{R}(t_{0,n}-);\ n = 1, 2, \ldots\}$ is Harris positive recurrent. Thus, the following fact was obtained in Lemma 4.3 of [30], which assumed Harris positive recurrence for $\{X(t_{0,n}-)\}$, but its proof only requires it for $\{\boldsymbol{R}(t_{0,n}-)\}$.

Lemma 4.8. *Under the assumptions of Theorem 3.1, $\mathbb{E}_{\nu_k}(\mathrm{e}^{W_{K_\alpha(v),\theta}(\boldsymbol{R}(0-))}) < \infty$ for $v > 0$ or $v = \triangle$ and $\theta \in [0, \alpha + \delta_0)$.*

4.2.1. *Proof of Theorem 3.1(i) for exact decay rate.* By the assumptions in Theorem 3.1(i), $K_\alpha = \varnothing$, so $\mathbb{P}_{\nu_k}^{(K_\alpha(\triangle),\alpha)}$ is simply denoted by $\mathbb{P}^{(\alpha)}$. This also implies that there exist $u > 0$ and $\theta \in (0, \alpha)$ satisfying the conditions of Lemma 4.6. Hence, from Lemma 4.6 and the fact that $\zeta_i(\alpha) - \zeta_i(\theta) > 0$ for $\alpha > \theta$, we have

$$\begin{aligned}&\lim_{\ell\to\infty}\mathbb{E}_{\nu_k}^{(\alpha)}\left(h_+(\ell)\exp\left(-\sum_{i\in K}(\zeta_i(\alpha)-\zeta_i(\theta))R_i(\sigma_\ell^+-)\right)\right)\\&\qquad=\mathbb{E}_{\nu_\infty(Y)}^{(\alpha)}\left(h_+(0)\exp\left(-\sum_{i\in K}(\zeta_i(\alpha)-\zeta_i(\theta))R_i(0-)\right)\right).\end{aligned} \tag{4.24}$$

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Note that $0 \le h_-(\ell) \le h_+(\ell)$. Furthermore, $\mathbf{1}\{H(s) \ge \ell\}\mathbf{1}\{\sigma_\ell^+ < \sigma_k^-\}$ almost surely converges to 0 as $\ell \to \infty$ because $\mathbb{P}_{\nu_k}(\sigma_k^- < \infty \mid \mathcal{F}_{\sigma_\ell^+-}) = 1$ for each $\ell \ge k+1$. Hence, $h_-(\ell)$ also converges to 0 as $\ell \to \infty$. Thus, by Lemma 4.8, it follows from (4.23), (4.24), and the bounded convergence theorem that

$$\begin{aligned}
&\lim_{\ell\to\infty} \mathrm{e}^{\alpha(\ell-k-1)}\mathbb{E}_{\nu_k}\left(\int_0^{\sigma_k^-} \mathbf{1}\{L(s) \ge \ell\}\,\mathrm{d}s\right) \\
&\quad = \mathbb{E}^{(\alpha)}_{\nu_\infty(Y)}\left(h_+(0)\exp\left(-\sum_{i\in K}(\zeta_i(\alpha)-\zeta_i(\theta))R_i(0-)\right)\right)\mathbb{E}_{\nu_k}\left(\exp\left(\sum_{i\in K}\zeta_i(\alpha)R_i(0-)\right)\right) \\
&\quad < \infty.
\end{aligned}$$

Thus, (3.3) is obtained from (4.18), which proves (i). □

4.2.2. *Proof of Theorem 3.1(ii) for logarithmic decay rate.* To prove (ii), we first consider an upper bound of the limit in (3.4). For this, we apply Lemma 4.6 to (4.23), then we have

$$\mathrm{e}^{\alpha(\ell-k-1)}\mathbb{E}_{\nu_k}\left(\int_0^{\sigma_k^-} \mathbf{1}\{L(s) \ge \ell\}\,\mathrm{d}s\right) \le C\widetilde{\mathbb{E}}^{(K_\alpha(\triangle),\alpha)}_{\nu_k}(\mathrm{e}^{W_{K_\alpha(\triangle),\alpha}(\boldsymbol{R}(0-))}).$$

Hence, from (4.18) and Lemma 4.8 we conclude that

$$\limsup_{\ell\to\infty} \frac{1}{\ell}\log \mathbb{P}(L > \ell) \le -\alpha.$$

We next consider a lower bound. For this, we first choose θ satisfying the conditions that, for some $v > 0$, $\theta > \alpha$,

$$\gamma_{K_\alpha(v)}(\theta) < 0, \qquad \gamma'_{K_\alpha(v)}(\theta) > 0,$$

which is possible because $\gamma_{K_\alpha(v)}(\theta) \uparrow \gamma_{K_\alpha(\triangle)}(\theta)$ as $v \uparrow \triangle$ and $\gamma_{K_\alpha(\triangle)}(\alpha) = 0$ by (4.15). Then, it is easy to see that we can find a sufficiently large $u > 0$ by which the conditions of Lemma 4.6 are satisfied. We now use (4.17) for $Z(t) = \mathbb{E}_{\nu_k}(\int_t^{\sigma_k^-} \mathbf{1}\{L(s) \ge \ell\}\,\mathrm{d}s|\mathcal{F}_{t-})\,\mathbf{1}\{\sigma_k^- > t\}$. Since $L(\sigma_\ell^+-) = \ell - 1$, $L(0) = k$, $\widetilde{R}_0(\sigma_\ell^+-) = \widetilde{R}_0(0-) = 0$, $\zeta_i(v,\theta) > 0$, and $\widetilde{\mathbb{P}}^{(K_\alpha(v),\theta)}_{\nu_k}(\sigma_\ell^+ < \infty) = 1$, we have, similar to (4.19),

$$\begin{aligned}
&\mathrm{e}^{(\ell-k-1)\theta}\mathbb{E}_{\nu_k}\left(\int_0^{\sigma_k^-} \mathbf{1}\{L(s) \ge \ell\}\,\mathrm{d}s\right) \\
&\quad \ge \widetilde{\mathbb{E}}^{(K_\alpha(v),\theta)}_{\nu_k}\left(\mathbb{E}_{\nu_k}\left(\int_{\sigma_\ell^+}^{\sigma_k^-} \mathbf{1}\{H(s) \ge \ell\}\,\mathrm{d}s \;\middle|\; \mathcal{F}_{\sigma_\ell^+-}\right)\mathrm{e}^{-W_{K_\alpha(v),\theta}(\boldsymbol{R}(\sigma_\ell^+-))}\right) \\
&\quad = \widetilde{\mathbb{E}}^{(K_\alpha(v),\theta)}_{\nu_k}((h_+(\ell)-h_-(\ell))\,\mathrm{e}^{W_{K_\alpha(\triangle),\theta}(\boldsymbol{R}(\sigma_\ell^+-))-W_{K_\alpha(v),\theta}(\boldsymbol{R}(\sigma_\ell^+-))}). \qquad (4.25)
\end{aligned}$$

The last term of this inequality converges to

$$\widetilde{\mathbb{E}}^{(K_\alpha(v),\theta)}_{\nu_\infty(Y)}(h_+(0)\,\mathrm{e}^{W_{K_\alpha(\triangle),\theta}(\boldsymbol{R}(0-))-W_{K_\alpha(v),\theta}(\boldsymbol{R}(0-))}) > 0$$

as $\ell \to \infty$ by the following facts. First, $\mathrm{e}^{W_{K_\alpha(\triangle),\theta}(\boldsymbol{R}(0))-W_{K_\alpha(v),\theta}(\boldsymbol{R}(0))}$ is bounded by 1 because $W_{K_\alpha(\triangle),\theta}(\boldsymbol{R}(0-)) - W_{K_\alpha(v),\theta}(\boldsymbol{R}(0-)) \le 0$. Second, $h_-(\ell)$ is not greater than $h_+(\ell)$ and

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vanishes as $\ell \to \infty$. Thus, we can apply Lemma 4.6, the $\mathbb{P}^{(K_\alpha(v),\theta)}_{\nu_\infty(Y)}$-version of Lemma 4.7 and the bounded convergence theorem. Hence, from (4.25), we have

$$\liminf_{\ell\to\infty} \frac{1}{\ell} \log \mathbb{P}(L \geq \ell) \geq -\theta.$$

Since θ can be chosen to go to α as $v \uparrow \infty$, we arrive at the lower bound, which completes the proof of (ii). □

4.3. Proof of Theorem 3.3

We consider the stationary equation of the nth system corresponding to (2.22) for $\theta \leq 0$. Similar to $f_{v,K(v),\theta}$ of (2.14), we define $f^{(n)}_{v,K(v),\theta}$ for each $n \geq 1$ as

$$f^{(n)}_{v,K(v),\theta}(\ell, \boldsymbol{y}) = \mathrm{e}^{\theta \max(\ell,k)}\, \mathrm{e}^{\eta^{(n)}(v,\theta)(y_0\wedge v)+\sum_{i\in K}\zeta_i^{(n)}(v,\theta)(y_i\wedge v)}, \qquad (\ell, \boldsymbol{y}) \in S.$$

Define $\Phi^{(n)}$ as

$$\Phi^{(n)}(v,\theta) = \mathbb{E}(f^{(n)}_{v,K(v),\theta}(X^{(n)})), \qquad v > 0,\ \theta \leq 0,$$

where $X^{(n)}$ is a random variable subject to the stationary distribution of the PDMP $X^{(n)}(\cdot) \equiv \{X^{(n)}(t);\ t \geq 0\}$, which describes the nth system. Obviously, $\Phi^{(n)}(v,\theta)$ is finite for $\theta \leq 0$. Similarly, we define, for $v > 0$ and $\theta \leq 0$,

$$\Phi^{(n)}_i(v,\theta) = \mathbb{E}(f^{(n)}_{v,K(v),\theta}(X^{(n)})\, \mathbf{1}\{R^{(n)}_i < v\}), \qquad i \in \overline{K},$$
$$\Phi^{(n)}_{j,0}(v,\theta) = \mathbb{E}(f^{(n)}_{v,K(v),\theta}(X^{(n)})\, \mathbf{1}\{R^{(n)}_j = 0\}), \qquad j \in K.$$

Then, the stationary equation of the nth system corresponding to (2.22) is given by

$$\eta^{(n)}(v,\theta)\Phi^{(n)}_0(v,\theta) + \sum_{i\in K}\zeta_i^{(n)}(v,\theta)\Phi^{(n)}_i(v,\theta) - \sum_{j\in K}\zeta_i^{(n)}(v,\theta)\Phi^{(n)}_{j,0}(v,\theta) = 0 \tag{4.26}$$

for $v > 0$ and $\theta \leq 0$.

Let $r_n = 1 - \rho^{(n)}$. By the heavy-traffic condition (3.10), r_n vanishes as $n \to \infty$. We scale $L^{(n)}(t)$ as $r_n L^{(n)}(t)$. To realize this, we let

$$\widetilde{\Phi}^{(n)}(\theta) = \Phi^{(n)}\left(\frac{1}{r_n}, r_n\theta\right), \qquad \widetilde{\Phi}^{(n)}_i(\theta) = \Phi^{(n)}_i\left(\frac{1}{r_n}, r_n\theta\right), \qquad \widetilde{\Phi}^{(n)}_{j,0}(\theta) = \Phi^{(n)}_{j,0}\left(\frac{1}{r_n}, r_n\theta\right).$$

We correspondingly define $\widetilde{\eta}^{(n)}(\theta)$ and $\widetilde{\zeta}^{(n)}_i(\theta)$ as

$$\widetilde{\eta}^{(n)}(\theta) = \eta^{(n)}\left(\frac{1}{r_n}, r_n\theta\right), \qquad \widetilde{\zeta}^{(n)}_i(\theta) = \zeta^{(n)}_i\left(\frac{1}{r_n}, r_n\theta\right), \qquad i \in K.$$

Then, (4.26) can be written as

$$\widetilde{\eta}^{(n)}(\theta)\widetilde{\Phi}^{(n)}_0(\theta) + \sum_{i\in K}\widetilde{\zeta}^{(n)}_i(\theta)\widetilde{\Phi}^{(n)}_i(\theta) - \sum_{j\in K}\widetilde{\zeta}^{(n)}_j(\theta)\widetilde{\Phi}^{(n)}_{j,0}(\theta) = 0. \tag{4.27}$$

We claim that

$$\lim_{n\to\infty} \widetilde{\Phi}^{(n)}(\theta) = \frac{2\lambda_0}{2\lambda_0 - \sum_{j\in\overline{K}}\lambda_j^3\sigma_j^2\theta}, \qquad \theta \leq 0. \tag{4.28}$$

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Suppose this claim holds, then the proof of Theorem 3.3 is completed if we can show that

$$\lim_{n\to\infty} |\widetilde{\Phi}^{(n)}(\theta) - \mathbb{E}(\mathrm{e}^{r_n L^{(n)}\theta})| = 0.$$

Let $\widetilde{f}_\theta^{(n)}(X^{(n)}) = f^{(n)}_{1/r_n, K(1/r_n), r_n\theta}(X^{(n)})$ for simplicity, then this is obtained by recalling the definition

$$\widetilde{\Phi}^{(n)}(\theta) = \mathbb{E}(\widetilde{f}_\theta^{(n)}(X^{(n)})),$$

and applying the bounded convergence theorem to

$$\begin{aligned}
&|\widetilde{\Phi}^{(n)}(\theta) - \mathbb{E}(\mathrm{e}^{r_n L^{(n)}\theta})| \\
&\quad \le \mathbb{E}\Bigg(\widetilde{f}_\theta^{(n)}(X^{(n)}) \Bigg| 1 - \exp(-\theta r_n(k - |J^{(n)}|)) \exp\Bigg(-\widetilde{\eta}^{(n)}(\theta)(R_0^{(n)} \wedge 1/r_n) \\
&\qquad\qquad - \sum_{i\in K} \widetilde{\zeta}_i^{(n)}(\theta)(R_i^{(n)} \wedge v) \Bigg) \Bigg| \Bigg)
\end{aligned}$$

because $\widetilde{\eta}^{(n)}(\theta)(R_0^{(n)} \wedge 1/r_n)$ and $\widetilde{\zeta}_i^{(n)}(\theta)(R_i^{(n)} \wedge 1/r_n)$ vanish in probability as $n \to \infty$ by (3.8) and $|r_n(R_i^{(n)} \wedge 1/r_n)| \le 1$.

Thus, it remains to prove (4.28). For this, we will use (4.26) and Lemma 3.2. We first specialize this lemma in the following way.

Lemma 4.9. *Assume (3.7) and the heavy-traffic conditions (3.10)–(3.12) hold, and let, for each* $n \ge 1$ *and* $r_n\theta \in (\underline{\theta}_K, \overline{\theta}_0)$,

$$\begin{aligned}
\widetilde{\varepsilon}_0^{(n)}(\theta) &= \widetilde{\eta}^{(n)}(\theta) + \lambda_0^{(n)} r_n\theta + \tfrac{1}{2}(\lambda_0^{(n)})^3(\sigma_0^{(n)})^2 r_n^2\theta^2, \\
\widetilde{\varepsilon}_i^{(n)}(\theta) &= \widetilde{\zeta}_i^{(n)}(\theta) - \lambda_i^{(n)} r_n\theta + \tfrac{1}{2}(\lambda_i^{(n)})^3(\sigma_i^{(n)})^2 r_n^2\theta^2,
\end{aligned} \tag{4.29}$$

then, for each $i \in \overline{K}$ *and* $c > 0$, *there is a sequence* $\{\widetilde{a}_{i,n}(c)\}$ *vanishing as* n *goes to* ∞ *such that*

$$\frac{1}{r_n^2}|\widetilde{\varepsilon}_i^{(n)}(\theta)| \le \widetilde{a}_{i,n}(c)|\theta|, \qquad n \ge 1,\ |\theta| < c. \tag{4.30}$$

We defer the proof of this lemma to Appendix C. We rewrite (4.27) as

$$\begin{aligned}
&\Bigg(\widetilde{\eta}^{(n)}(\theta) + \sum_{i\in K} \widetilde{\zeta}_i^{(n)}(\theta)\Bigg)\widetilde{\Phi}^{(n)}(\theta) - \sum_{i\in K} \widetilde{\zeta}_i^{(n)}(\theta)\widetilde{\Phi}_{i,0}^{(n)}(\theta) \\
&\quad = \widetilde{\eta}^{(n)}(\theta)(\widetilde{\Phi}^{(n)}(\theta) - \widetilde{\Phi}_0^{(n)}(\theta)) + \sum_{i\in K} \widetilde{\zeta}_i^{(n)}(\theta)(\widetilde{\Phi}^{(n)}(\theta) - \widetilde{\Phi}_i^{(n)}(\theta)).
\end{aligned} \tag{4.31}$$

We will divide both sides of this formula by $r_n^2\theta$, and let n go to ∞. For this computation, we will use the following facts, which are obtained from Lemma 4.9, (3.7), (3.10), and (3.11).

$$\lim_{n\to\infty} \frac{1}{r_n^2\theta}\Bigg(\widetilde{\eta}^{(n)}(\theta) + \sum_{i\in K} \widetilde{\zeta}_i^{(n)}(\theta)\Bigg) = \lambda_0 - \frac{1}{2}\sum_{i\in\overline{K}} \lambda_i^3\sigma_i^2\theta, \tag{4.32}$$

$$\lim_{n\to\infty} \frac{1}{r_n\theta}\widetilde{\eta}^{(n)}(\theta) = -\lambda_0, \tag{4.33}$$

$$\lim_{n\to\infty} \frac{1}{r_n\theta}\widetilde{\zeta}_i^{(n)}(\theta) = \lambda_i. \tag{4.34}$$

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As is well known, it follows from the renewal theorem that

$$\mathbb{P}(R_i^{(n)} > v) = \lambda_i^{(n)} \mathbb{P}(R_i^{(n)} > 0) \int_v^\infty \mathbb{P}(T_i^{(n)} > x)\,\mathrm{d}x, \qquad i \in \overline{K}.$$

This and the uniformly integrable condition (3.12) yield that

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{r_n} \mathbb{P}\left(R_i^{(n)} > \frac{1}{r_n}\right) &\le \limsup_{n\to\infty} \frac{1}{r_n} \lambda_i^{(n)} \int_{1/r_n}^\infty \mathbb{P}(T_i^{(n)} > x)\,\mathrm{d}x \\
&\le \limsup_{n\to\infty} \lambda_i^{(n)} \mathbb{E}\left(\int_{1/r_n}^{T_i^{(n)}} x\, \mathbf{1}\left\{T_i^{(n)} > \frac{1}{r_n}\right\} \mathrm{d}x\right) \\
&\le \limsup_{n\to\infty} \frac{1}{2} \lambda_i^{(n)} \mathbb{E}\left((T_i^{(n)})^2\, \mathbf{1}\left\{T_i^{(n)} > \frac{1}{r_n}\right\}\right) \\
&= 0. \qquad (4.35)
\end{aligned}$$

Similarly, we have

$$\limsup_{n\to\infty} \mathbb{E}\left(R_i^{(n)}\, \mathbf{1}\left\{R_i^{(n)} > \frac{1}{r_n}\right\}\right) \le \limsup_{n\to\infty} \frac{1}{2} \lambda_i^{(n)} \mathbb{E}\left((T_i^{(n)})^2\, \mathbf{1}\left\{T_i^{(n)} > \frac{1}{r_n}\right\}\right) = 0. \qquad (4.36)$$

Since $\widetilde{f}_\theta^{(n)}(X^{(n)})$ is uniformly bounded in n, (4.35) implies that

$$0 \le \limsup_{n\to\infty} \frac{1}{r_n} (\widetilde{\Phi}^{(n)}(\theta) - \widetilde{\Phi}_i^{(n)}(\theta)) \le \limsup_{n\to\infty} \mathbb{E}\left(\widetilde{f}_\theta^{(n)}(X^{(n)}) \frac{1}{r_n}\, \mathbf{1}\left\{R_i^{(n)} > \frac{1}{r_n}\right\}\right) = 0.$$

This, (4.33), and (4.34) yield that the right-hand side of (4.31) divided by $r_n^2\theta$ converges to 0 as $n \to \infty$. Hence, if we can show that

$$\lim_{n\to\infty} \frac{1}{r_n^2\theta} \sum_{i\in K} \widetilde{\zeta}_i^{(n)}(\theta) \widetilde{\Phi}_{i,0}^{(n)}(\theta) = \lambda_0, \qquad (4.37)$$

then dividing both sides of (4.31) by $r_n^2\theta$ and letting $n \to \infty$ yields (4.28) by (4.32). This is equivalent to (3.14), and therefore the proof is completed. Thus, it remains to prove (4.37). By (4.34), (4.37) is immediate if we have

$$\lim_{n\to\infty} \frac{1}{r_n} \sum_{i\in K} \lambda_i \widetilde{\Phi}_{i,0}^{(n)}(\theta) = \lambda_0.$$

To prove this, we first note that dividing (4.31) by $r_n^2\theta$ and letting $\theta \uparrow 0$ yields

$$\lim_{n\to\infty} \frac{1}{r_n} \sum_{i\in K} \lambda_i \widetilde{\Phi}_{i,0}^{(n)}(0) = \lambda_0. \qquad (4.38)$$

We remark that this also follows from Little's law for customers in service at each server and the heavy-traffic condition (3.10). Thus, (4.37) is obtained if we can show that

$$\lim_{n\to\infty} \frac{1}{r_n} |\widetilde{\Phi}_{i,0}^{(n)}(\theta) - \widetilde{\Phi}_{i,0}^{(n)}(0)| = 0, \qquad \theta \le 0. \qquad (4.39)$$

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To prove this, let

$$h_n(\theta) = \theta L^{(n)} + r_n^{-1}\widetilde{\eta}^{(n)}(\theta)\left(R_0^{(n)} \wedge \frac{1}{r_n}\right) + \sum_{j\in K} r_n^{-1}\widetilde{\zeta}_i^{(n)}(\theta)\left(R_j^{(n)} \wedge \frac{1}{r_n}\right),$$

then $\widetilde{\Phi}_{i,0}^{(n)}(\theta) = \mathbb{E}(\mathrm{e}^{r_n h_n(\theta)}\,\mathbf{1}\{R_i^{(n)} = 0\})$. Thus, for $\theta < 0$, we have

$$\begin{aligned}\frac{1}{r_n}|\widetilde{\Phi}_{i,0}^{(n)}(\theta) - \widetilde{\Phi}_{i,0}^{(n)}(0)| &\le \mathbb{E}\left(\left|\frac{\mathrm{e}^{r_n h_n(\theta)} - 1}{r_n}\right| \mathbf{1}\{R_i^{(n)} = 0\}\right)\\ &= \mathbb{E}\left(\left|\frac{\mathrm{e}^{r_n h_n(\theta)} - 1}{r_n h_n(\theta)} h_n(\theta)\right| \mathbf{1}\{R_i^{(n)} = 0\}\right).\end{aligned}$$

Since $\lambda_i > 0$ for all $i \in \overline{K}$, it follows from (4.38) that

$$\lim_{n\to\infty} \frac{\mathbb{P}(R_i^{(n)} = 0)}{\sqrt{r_n}} = 0, \qquad i \in K. \tag{4.40}$$

On the other hand,

$$\begin{aligned}&\mathbb{E}\left(\left(R_j^{(n)} \wedge \frac{1}{r_n}\right)\mathbf{1}\{R_i^{(n)} = 0\}\right)\\ &\quad= \mathbb{E}\left(\left(R_j^{(n)} \wedge \frac{1}{r_n}\right)\mathbf{1}\{R_i^{(n)} = 0\}\left(\mathbf{1}\left\{R_j^{(n)} > \frac{1}{\sqrt{r_n}}\right\} + \mathbf{1}\left\{R_j^{(n)} \le \frac{1}{\sqrt{r_n}}\right\}\right)\right)\\ &\quad\le \mathbb{E}\left(R_j^{(n)}\left(\mathbf{1}\left\{R_j^{(n)} > \frac{1}{\sqrt{r_n}}\right\}\right)\right) + \frac{1}{\sqrt{r_n}}\mathbb{P}(R_i^{(n)} = 0).\end{aligned}$$

Hence, (4.36) and (4.40) imply that $\mathbb{E}(h_n(\theta)\,\mathbf{1}\{R_i^{(n)} = 0\})$ converges to 0 as $n \to \infty$. Thus, (4.39) is obtained if $(\mathrm{e}^{r_n h_n(\theta)} - 1)/r_n h_n(\theta)$ is uniformly bounded in n for each $\theta < 0$, which follows from that $r_n h_n(\theta)$ is uniformly bounded in n. This uniform boundedness is implied by the facts that $L^{(n)} \le k - 1$ on $\{R_i^{(n)} = 0\}$, $r_n^{-1}\widetilde{\eta}^{(n)}(\theta)$ and $r_n^{-1}\widetilde{\zeta}_i^{(n)}(\theta)$ are uniformly bounded in n for $r_n\theta \in (\underline{\theta}_K, 0)$ by Lemma 3.1. Consequently, (4.39) is obtained, and we complete the proof. □

The above inequality is one of the key techniques in the present approach, and has been used in the proof of Lemma 3.2 in Appendix B. It also can be found in the last part of the proof of Lemma 4.7 of [4].

4.4. Proof for Theorem 3.4

Up to (4.27), all the computations in Section 4.3 are also valid for this case if we replace r_n by $r_n s_n$ with $r_n = 1 - \rho_n$. To distinguish the notation in Section 4.3, we change $\widetilde{\eta}^{(n)}$, $\widetilde{\zeta}_i^{(n)}$, and $\widetilde{\varepsilon}_i^{(n)}$ to $\underline{\widetilde{\eta}}^{(n)}$, $\underline{\widetilde{\zeta}}_i^{(n)}$, and $\underline{\widetilde{\varepsilon}}_i^{(n)}$, respectively. Similar to Lemma 4.9, we have the following lemma.

Lemma 4.10. *Assume that (3.7), (3.10), and the large variance condition (3.15) hold, and let, for each $n \ge 1$ and $\theta \in \mathbb{R}$,*

$$\begin{aligned}\underline{\widetilde{\varepsilon}}_0^{(n)}(\theta) &= \underline{\widetilde{\eta}}^{(n)}(\theta) + \lambda_0^{(n)} r_n s_n \theta + \tfrac{1}{2}(\lambda_0^{(n)})^3 r_n^2 s_n (\sigma_0^{(n)})^2 s_n \theta^2,\\ \underline{\widetilde{\varepsilon}}_i^{(n)}(\theta) &= \underline{\widetilde{\zeta}}_i^{(n)}(\theta) - \lambda_i^{(n)} r_n s_n \theta + \tfrac{1}{2}(\lambda_i^{(n)})^3 r_n^2 s_n (\sigma_i^{(n)})^2 s_n \theta^2,\end{aligned}$$

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then, for each $i \in \overline{K}$ *and* $c > 0$*, there is a sequence* $\{\widetilde{a}_{i,n}(c)\}$ *vanishing as* n *goes to* ∞ *such that*

$$\frac{1}{r_n^2 s_n}|\widetilde{\varepsilon}_i^{(n)}(\theta)| \le \widetilde{a}_{i,n}(c)|\theta|, \qquad n \ge 1,\ |\theta| < c.$$

This lemma yields similar formulas to (4.32)–(4.34), in which the r_n are replaced by $r_n s_n$. Thus, we have (3.17) in the same way as Theorem 3.3. □

5. Concluding remarks

We here consider potential of the present approach for more general models and some other asymptotics.

5.1. More general arrival processes

We have assumed that the arrival process is renewal or its independent superposition. However, the renewal process is not essential. As long as it is independent of the state of system, a more general class of arrival processes can be used in our framework. In this case, the terminal condition (2.8) needs to be changed. Suppose that N_0 is a stationary point process with finite intensity λ_0. Then, (2.12) is changed to

$$\mathrm{e}^{\theta}\mathbb{E}_0(\mathrm{e}^{\eta(\theta)T_0} \mid \mathcal{F}_{0-}) = 1,$$

where $\mathbb{E}_0$ stands for the expectation by Palm distribution $\mathbb{P}_0$ concerning N_0, and T_0 is the time to the next counting instant measured from time 0. Namely,

$$\mathbb{E}_0(\mathrm{e}^{\eta(\theta)T_0}) = \frac{1}{\lambda_0}\mathbb{E}\left(\sum_{i=1}^{\infty} \mathrm{e}^{\eta(\theta)(t_{0,i+1}-t_{0,i})}\,\mathbf{1}\{0 < t_{0,i} < 1\}\right),$$

where $t_{0,i} = \inf\{u > t_{0,i-1};\ N_0(\{u\}) > 0\}$ for $i \ge 1$ with $t_{0,0} = 0$. A typical example for this counting process is a Markov arrival process with finitely many background states. See $\gamma^{(i,a)}(\theta)$ of Equation (4.15) in [22], which can be shown to be $-\eta(\theta)$.

We next consider batch arrivals, which includes the case that $F_0(0) > 0$ as a special case (see Remark 3.1). Denote the nth arrival batch size by $A_{b,n}$. We assume that $\{A_{b,n};\ n = 1, 2, \ldots\}$ are positive, i.i.d. Furthermore, they are independent of everything else. Denote their common distribution by F_b, and let A_b be a random variable subject to F_b. We also need to verify the terminal condition (2.8). This can be done similarly to the proof of Lemma 2.3 in Appendix B, where ℓ' in the proof of this lemma should be changed to

$$\ell' = \ell + \mathbf{1}\{y_0 = 0\}A_b - \sum_{i \in U \setminus U(\boldsymbol{y})} \mathbf{1}\{y_i = 0\},$$

and the definition of $\eta(\theta, v)$ should be appropriately updated. For this, we need to truncate not only T_0 but also A_b. So, we define $\eta(m, v, \theta)$ for positive integer m and positive real number v as the solution of the following equation:

$$\mathbb{E}(\mathrm{e}^{\theta(A_b \wedge m)})\mathbb{E}(\mathrm{e}^{\eta(m,v,\theta)(T_0 \wedge v)}) = 1. \tag{5.1}$$

Similar to $\eta(\triangle, \theta)$, we define

$$\eta(\triangle, \triangle, \theta) = \lim_{v \to \infty} \lim_{m \to \infty} \eta(m, v, \theta).$$

It is not difficult to see that these limiting operations are exchangeable. In particular, $\eta(\triangle, \triangle, \theta) = \eta(\triangle, \infty, \theta)$ for $\theta \ge 0$, and $\eta(\triangle, \triangle, \theta) = \eta(\infty, \triangle, \theta)$ for $\theta < 0$, where $\eta(m, v, \theta)$ is defined for $m = \infty$ or/and $v = \infty$ by (5.1).

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Using this notation, all the results in Sections 2 and 3 can be extended to the batch arrival case under appropriate conditions. For example, α of (3.1) is redefined as

$$\alpha \equiv \sup\left\{\theta \geq 0;\ -\left(\eta(\triangle, \infty, \theta) + \sum_{i \in K} \zeta_i(\triangle, \theta)\right) \leq 0\right\}.$$

For the weak limit approximations, let $\eta^{(n)}(\infty, v, \theta)$ be the $\eta(\infty, v, \theta)$ of the nth system, then define the error function of its second-order Taylor expansion for $\theta \leq 0$ as

$$\varepsilon_0^{(n)}(\infty, v, \theta) = \eta^{(n)}(\infty, v, \theta) - \frac{\partial \eta^{(n)}}{\partial s}(\infty, v, s)\bigg|_{s=0} \theta - \frac{1}{2}\frac{\partial^2 \eta^{(n)}}{\partial s^2}(\infty, v, s)\bigg|_{s=0} \theta^2,$$

since no truncation on A_b is needed for $\theta \leq 0$, where, for $\lambda_0^{(n)}(v) = \mathbb{E}(T_0^{(n)} \wedge v)$,

$$\frac{\partial \eta^{(n)}}{\partial s}(\infty, v, s)\bigg|_{s=0} = -\lambda_0^{(n)}(v)\mathbb{E}(A_b),$$

$$\frac{\partial^2 \eta^{(n)}}{\partial s^2}(\infty, v, s)\bigg|_{s=0} = -\lambda_0^{(n)}(v)\{(\sigma_{A_b}^{(n)})^2 + (\lambda_0^{(n)}(v)\mathbb{E}(A_b)\sigma_0^{(n)}(v))^2\},$$

where $(\sigma_{A_b}^{(n)})^2$ is the variance of $A_b^{(n)}$. Thus, we can apply exactly the same arguments to obtain the weak limits for the heavy-traffic and large variance approximations.

5.2. Some other problems and limitations

In this paper we only consider a single queue. However, the present approach is potentially applicable to systems with multiple queues and queueing networks.

For the tail asymptotic problem, this approach may relax phase-type assumptions in queues and their networks. For example, those assumptions were used for a two-node generalized Jackson network in [22] and for the joint shortest queue in [31]. They may be replaced by generally distributed assumptions for interarrival and service times.

For the weak limit approximations, a similar approach was studied by Braverman *et al.* [4]. They used a slightly different test function, and considered the heavy-traffic approximation of the stationary distribution for a generalized Jackson network, whose process limit was originally obtained by Reiman [28]. As discussed in [4], there are several issues to be overcome in the network case.

The tools of the present approach are based on the terminal condition (2.8). However, this condition may cause serious drawbacks. For example, it will be hard to apply for a multiclass queueing network with FCFS service discipline because we need to have information on the order of customer classes in a waiting line. Another difficult problem is the heavy-traffic approximation for the Halfin–Whitt regime (see, e.g. [14], [27]). We can describe the Halfin–Whitt regime by the present framework, but the stationary equations (2.22) and (2.23) may not be so useful because they do not adequately capture the number of busy servers.

Yet another problem of the present approach concerns a process limit under the heavy-traffic condition. We have only considered the stationary distribution of the PDMP. However, our starting formula (2.7) describes sample path evolutions. This suggests that the present approach may be applicable for a process limit.

Thus, there remain many challenging problems for future study.

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Appendix A. Lemmas in Section 2

Proof of Lemma 2.1. Recall the definition (2.5) of $M_0(t)$. For a stopping time τ, let $\mathcal{F}_{\tau-} = \sigma(\mathcal{F}_0 \cup \{A \cap \{t < \tau\};\ A \in \mathcal{F}_t\})$, then τ is $\mathcal{F}_{\tau-}$-measurable (see Section 1.1.14 of [15]). Note that t_i is $\mathcal{F}_{t_i-}$-measurable since t_i is a stopping time, and $X(t_i-)$ is $\mathcal{F}_{t_i-}$-measurable. Hence, for $0 < s < t$,

$$\begin{aligned}
\mathbb{E}(M_0(t) \mid \mathcal{F}_s) &= M_0(s) + \sum_{i=1}^{\infty} \mathbb{E}(\mathbb{E}(f(X(t_i)) - Qf(X(t_i-)) \mid \mathcal{F}_{t_i-})\, \mathbf{1}\{s < t_i \le t\} \mid \mathcal{F}_s) \\
&= M_0(s) + \sum_{i=1}^{\infty} \mathbb{E}(\{\mathbb{E}(f(X(t_i) \mid \mathcal{F}_{t_i-}) - Qf(X(t_i-))\}\, \mathbf{1}\{s < t_i \le t\} \mid \mathcal{F}_s) \\
&= M_0(s) + \sum_{i=1}^{\infty} \mathbb{E}(\{\mathbb{E}(f(X(t_i) \mid X(t_i-)) - Qf(X(t_i-))\}\, \mathbf{1}\{s < t_i \le t\} \mid \mathcal{F}_s) \\
&= M_0(s),
\end{aligned}$$

where the third equality follows from the strong Markov property of $X(\cdot)$, and the last equality is obtained from the definition (2.4) of Q. This concludes that $M_0(\cdot) \equiv \{M_0(t);\ t \ge 0\}$ is an $\mathcal{F}_t$-martingale since $\mathbb{E}(|M_0(t)|) < \infty$ by (2.6). □

Proof of Lemma 2.3. We prove only (2.8) for $f = f_{u,K(v),\theta}$ with finite $v > 0$ since the proof is similar for $u, v = \infty$ if $\eta(\theta)$ and $\zeta_i(\theta)$ are finite. For this, we recall that

$$\Gamma = \{(\ell, U, \boldsymbol{y}) \in S;\ y_0 = 0 \text{ or } U \ne K\}$$

and $X(t-) \in \Gamma$ if and only if $\Delta N^*(t) = 1$. For $\boldsymbol{x} \equiv (\ell, U, \boldsymbol{y}) \in \Gamma$, let

$$V(\boldsymbol{x}) = \{i \in \{0\} \cup U;\ y_i = 0\}.$$

This $V(\boldsymbol{x})$ specifies an arrival and/or departure which occurs at time t when $X(s)$ goes to $\boldsymbol{x} \in \Gamma$ as $s \uparrow t$.

Suppose that $X(t-) = \boldsymbol{x}$ at time t. Denote this $X(t-)$ by $X(t, 0)$. Let $X(t, 1)$ be the state which is changed from $X(t, 0)$ caused by the arrivals and/or departures of $V(\boldsymbol{x})$, where the server selection rule is applied at this stage independent of everything else. If $V(X(t, 1)) = \varnothing$ then $X(t) = X(t, 1)$, and the state transition at time t is completed. Otherwise, namely the next interarrival time and/or some of new service times are zero, then we continue to define $X(t, 2)$ using $V(X(t, 1))$ and the server selection rule independent of everything else. In this way, we have the sequence of the states $X(t, n)$ for $n = 0, 1, 2, \ldots$. Since the numbers of external arrivals and service completions at the same time instants are finite almost surely, there is a finite number n for each sample path such that $V(X(t, n)) = \varnothing$ and $X(t, n) = X(t)$ almost surely. Denote the minimum of such n by τ then $\mathbb{P}(\tau < \infty) = 1$. To describe these state transitions, we define $X(t, m) = X(t, \tau)$ for $m \ge \tau + 1$. Then, we have, $X(t, \tau) = X(t)$, and, therefore,

$$X(t) = \lim_{n \to \infty} X(t, n). \tag{A.1}$$

We next consider the transition kernel Q redefined as

$$Qf(\boldsymbol{x}) = \mathbb{E}(f(X(t, 1)) \mid X(t, 0) = \boldsymbol{x}), \qquad \boldsymbol{x} \in \Gamma,\ f \in C^{(1,p)}_{b,k+1}(S).$$

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Clearly, we have, for all $n = 1, 2, \ldots,$

$$Qf(\boldsymbol{x}) = \mathbb{E}(f(X(t,n)) \mid X(t,n-1) = \boldsymbol{x}), \qquad \boldsymbol{x} \in \Gamma,\ f \in C^{(1,p)}_{b,k+1}(S).$$

However, this Q can not fully capture the transition from $X(t,n-1)$ to $X(t,n)$ because $X(t,n-1)$ may not be in Γ. To capture it, we extend Q to the transition kernel $\widehat{Q}$ as

$$\widehat{Q}f(\boldsymbol{x}) = \begin{cases} Qf(\boldsymbol{x}), & \boldsymbol{x} \in \Gamma, \\ f(\boldsymbol{x}), & \boldsymbol{x} \in S \setminus \Gamma. \end{cases}$$

Then, the domain of $\widehat{Q}$ is S, and we have, for $n = 1, 2, \ldots,$

$$\widehat{Q}^{n-1} f(\boldsymbol{x}) = \mathbb{E}(f(X(t,n)) \mid X(t,1) = \boldsymbol{x}), \qquad \boldsymbol{x} \in S,\ f \in C^{(1,p)}_{b,k+1}(S),$$
$$\widehat{Q}^{n} f(\boldsymbol{x}) = \mathbb{E}(f(X(t,n)) \mid X(t,0) = \boldsymbol{x}), \qquad \boldsymbol{x} \in \Gamma,\ f \in C^{(1,p)}_{b,k+1}(S).$$

Since these fs are bounded functions, it follows from (A.1) and the dominated convergence theorem that, for $\boldsymbol{x} \in \Gamma$,

$$\begin{aligned} \mathbb{E}(f(X(t)) \mid X(t-) = \boldsymbol{x}) &= \mathbb{E}\Big(\lim_{n\to\infty} f(X(t,n)) \mid X(t,0) = \boldsymbol{x}\Big) \\ &= \lim_{n\to\infty} \mathbb{E}(f(X(t,n)) \mid X(t,0) = \boldsymbol{x}) \\ &= \lim_{n\to\infty} \widehat{Q}^n f(\boldsymbol{x}). \end{aligned}$$

We now choose $f_{u,K(v),\theta}$ for f, and note that $\eta(u,\theta)$ and $\zeta_i(v,\theta)$ for $i \in K$ are well defined. If we show that

$$Qf_{u,K(v),\theta}(\boldsymbol{x}) = f_{u,K(v),\theta}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Gamma, \tag{A.2}$$

then we have $\widehat{Q} f_{u,K(v),\theta}(\boldsymbol{x}) = f_{u,K(v),\theta}(\boldsymbol{x})$ for $\boldsymbol{x} \in S$, and, therefore,

$$\mathbb{E}(f_{u,K(v),\theta}(X(t)) \mid X(t-) = \boldsymbol{x}) = \lim_{n\to\infty} \widehat{Q}^n f_{u,K(v),\theta}(\boldsymbol{x}) = f_{u,K(v),\theta}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Gamma,$$

which proves (2.8) for $f = f_{u,K(v),\theta}$.

Thus, it remains to prove (A.2) to complete the proof. Recall that U specifies busy servers. Suppose that, for given $(\ell, U, \boldsymbol{y}) \in \Gamma$, $X(t,0) \equiv (\ell, U, \boldsymbol{y})$ is changed to $X(t,1) \equiv (\ell', U', \boldsymbol{y}')$, where $\ell', U', \boldsymbol{y}'$ may be random. Let $U(\boldsymbol{y}) = \{i \in U;\ y_i > 0\}$, which is the index set of servers who continue service at this transition. Since U is the index set of busy servers and the residual interarrival and/or service times which are going to vanish at step 0, we have

$$\ell' = \ell + 1(y_0 = 0) - \sum_{i \in U \setminus U(\boldsymbol{y})} \mathbf{1}\{y_i = 0\} = \ell + \mathbf{1}\{y_0 = 0\} - |U| + |U(\boldsymbol{y})|. \tag{A.3}$$

Hence, the server index set which starts new service is $D(K \setminus U(\boldsymbol{y}),\ \ell' - |U(\boldsymbol{y}|)$, where we recall that $D(A, j)$ is the set of servers selected from A for starting service of $\min(j, |A|)$ customers. Thus, U' and $\boldsymbol{y}' \equiv \{y'_i;\ i \in \overline{K}\}$ are given by

$$U' = U(\boldsymbol{y}) \cup D(K \setminus U(\boldsymbol{y}), \ell' - |U(\boldsymbol{y}|), \qquad y'_0 = y_0 + \mathbf{1}\{y_0 = 0\} T_0,$$
$$y'_i = y_i + \mathbf{1}\{y_i = 0,\ i \in U' \setminus U(\boldsymbol{y})\} T_i, \qquad i \in K,$$

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where the T_i are independently chosen subject to the F_i. Note that $y_i' = y_i > 0$ and $i \in U$ if and only if $i \in U(\boldsymbol{y})$. Since there are $(k - \ell') \vee 0$ servers are idle in the state $X(t, 1)$ and $|U' \setminus U(\boldsymbol{y})| = (k - |U(\boldsymbol{y})|) \wedge (\ell' - |U(\boldsymbol{y})|)$, the definitions of $\eta(u, \theta)$ and $\zeta_i(v, \theta)$ yield

$$\begin{aligned}
&Qf_{u,A(v),\theta}(\boldsymbol{x}) \\
&\quad = \mathbb{E}(f_{u,A(v),\theta}(X(t, 1)) \mid X(t, 0) = (\ell, U, \boldsymbol{y})) \\
&\quad = \mathbb{E}\left(\exp\left(\theta(\ell' \vee k) + \eta(u, \theta)(T_0 \wedge u)\, \mathbf{1}\{y_0 = 0\} + \sum_{i \in U' \setminus U(\boldsymbol{y})} \zeta_i(v, \theta)(T_i \wedge v)\right)\right) \\
&\qquad \times \exp\left(\eta(u, \theta)(y_0 \wedge u) + \sum_{j \in U(\boldsymbol{y})} \zeta_j(v, \theta)(y_j \wedge v)\right) \\
&\quad = \mathbb{E}(\exp(\theta(\ell' \vee k))) \exp(-\mathbf{1}\{y_0 = 0\}\theta + ((k - |U(\boldsymbol{y})|) \wedge (\ell' - |U(\boldsymbol{y})|))\theta) \\
&\qquad \times \exp\left(\eta(u, \theta)(y_0 \wedge u) + \sum_{j \in U(\boldsymbol{y})} \zeta_j(v, \theta)(y_j \wedge v)\right).
\end{aligned}$$

Thus, we have (A.2) if we can show that

$$\ell' \vee k - 1(y_0 = 0) + ((k - |U(\boldsymbol{y})|) \wedge (\ell' - |U(\boldsymbol{y})|)) = \ell \vee k. \tag{A.4}$$

Since $|U' \setminus U(\boldsymbol{y})| = k \wedge \ell' - |U(\boldsymbol{y}|$ implies that

$$\ell' \vee k + ((k - |U(\boldsymbol{y})|) \wedge (\ell' - |U(\boldsymbol{y})|)) = k + \ell' - |U(\boldsymbol{y})|$$

and $|U| = \ell \wedge k$, from (A.3), it follows that the left-hand side of (A.4) is equal to

$$\ell' - 1(y_0 = 0) + k - |U(\boldsymbol{y})| = \ell - |U| + k = \ell - (\ell \wedge k) + k = \ell \vee k.$$

Thus, we have proved Lemma 2.3. □

Proof of Lemma 2.4. (i) Fix $v > 0$. Obviously, $\xi(v, \theta)$ is decreasing in θ by the definition of (2.16). We note that $\log \widehat{F}(v, s)$ is convex in $s \in \mathbb{R}$ (see, e.g. Lemma 2.2.5 of [10]). Hence, we have, for $p_1, p_2 > 0$ such that $p_1 + p_2 = 1$ and $\theta_1, \theta_2 < \overline{\theta}$,

$$\begin{aligned}
\log \widehat{F}(v, p_1\xi(v, \theta_1) + p_2\xi(v, \theta_2)) &\le p_1 \log \widehat{F}(v, \xi(v, \theta_1)) + p_2 \log \widehat{F}(v, \xi(v, \theta_2)) \\
&= -p_1\theta_1 - p_2\theta_2 \\
&= \log \widehat{F}(v, \xi(v, p_1\theta_1 + p_2\theta_2)),
\end{aligned}$$

where the last two equalities are obtained from (2.16). Since $\log \widehat{F}(v, s)$ is increasing in $s \in \mathbb{R}$, the above inequality implies that

$$p_1\xi(v, \theta_1) + p_2\xi(v, \theta_2) \le \xi(v, p_1\theta_1 + p_2\theta_2).$$

This proves the concavity of $\xi(v, \theta)$ for $\theta \in \mathbb{R}$. It remains to prove for $\xi(v, \theta)$ to be infinitely differentiable. For this claim, we first consider the case that $\mathbb{P}(T > v) = 1$. In this case, $\xi(v, \theta) = -\theta/v$, and the claim obviously holds. Otherwise, $\widehat{F}(v, s)$ is analytic for all complex number s, and its derivative $\widehat{F}'(v, s) \neq 0$ for all real number s because $\mathbb{E}(T) > 0$. Hence, $\xi(v, s)$ is analytic in a neighborhood around the real axis by the implicit function theorem. This proves the claim.

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(ii) These facts are easily inspected through (2.16) because $\widehat{F}(v,\theta)$ is increasing in v for $\theta > 0$ and decreasing in v for $\theta < 0$.

(iii) By (i) and (ii), we have, for $\theta \le 0$,

$$0 \le \xi(v,\theta) \le \frac{\partial}{\partial x}\xi(v,x)\bigg|_{x=0}\theta = -\frac{1}{\mathbb{E}(T \wedge v)}\theta, \qquad v \ge 1,$$

while, noting the fact that $\xi(v,\delta) < 0$ for $\delta > 0$, for $0 < \theta \le \delta$,

$$\frac{1}{\delta}\xi(v,\delta)\theta \le \xi(v,\theta) \le 0, \qquad v \ge 1.$$

Thus, combining these two inequalities, we have (2.18). □

Appendix B. Lemmas in Section 3

Proof of Lemma 3.1. The proof of (3.9) is similar to that of (3.8), so we only show (3.8). We first note that

$$\eta^{(n)}\left(\frac{1}{q_n}, q_n\theta\right) \le \eta^{(n)}(\triangle, q_n\theta) < 0, \qquad 0 < \theta < \overline{\theta}_0,$$
$$0 \le \eta^{(n)}(\triangle, q_n\theta) \le \eta^{(n)}\left(\frac{1}{q_n}, q_n\theta\right), \qquad \theta \le 0,$$

by Lemma 2.4. Hence, we only need to prove that

$$\limsup_{n\to\infty}\left|\frac{\eta^{(n)}(1/q_n, q_n\theta)}{q_n}\right| \le c_0(\delta)|\theta|, \qquad |\theta| < \delta.$$

However, this was already obtained in Lemma 4.4 of [4] assuming the uniformly integrable condition (3.12), tacitly assuming that $\overline{\theta}_0^{(n)} > a$ for all $n \ge 1$ for some constant $a > 0$, which implies that $q_n\delta < a$ for sufficiently large n. This tacit assumption is easily checked because $\mathbb{E}(T_0^{(n)})$ converges to positive constant $1/\lambda_0$. Thus, we here show that the proof of Lemma 4.4 of [4] is valid without (3.12).

It is easy to see that, in [4], condition (3.12) is only used to show Equation (A.9) and the fact that, for a given y_1, there exists a $y_2 > y_1$ for any $\varepsilon > 0$ such that

$$\sup_{n\ge1}\mathbb{P}(T_0^{(n)} \ge y_2) < \tfrac{1}{2}\varepsilon,$$

which is Equation (A.10) of [4], where $T_{e,i}^{(n)}$ is replaced by $T_0^{(n)}$ of our notation. Suppose that this claim does not hold, then, for some $\varepsilon > 0$ and any $y_2 > y_1$, there is a $n_0 \ge 1$ such that $\mathbb{P}(T_i^{(n_0)} \ge y_2) \ge \varepsilon/2$, which implies that

$$\mathbb{E}(T_i^{(n_0)}) \ge y_2\mathbb{P}(T_i^{(n_0)} \ge y_2) \ge \tfrac{1}{2}y_2\varepsilon.$$

Hence, we can choose y_2 and n_0 such that $y_2\varepsilon$ is arbitrarily large. This contradicts (3.7). Similarly, Equation (A.9) of [4] is verified using the uniform integrability of $\{T_i^{(n)};\ n \ge 1\}$ in addition to (3.7) . Thus, (3.7) implies the claim, and, therefore, (A.10) of [4] is obtained without (3.12). □

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Proof of Lemma 3.2. We prove only (3.13) for $i = 0$ since the proof is similar for $i \in K$. Fix $n \geq 1$ and $v > 0$. For simplicity, we denote $\eta^{(n)}(v, \theta)$ by f in this subsection. By a Taylor expansion of $f(\theta)$ around $\theta = 0$, we have, for some $\delta \in [0, 1]$,

$$\eta^{(n)}(v, \theta) = f'(0)\theta + \tfrac{1}{2} f''(0)\theta^2 + \tfrac{1}{2}(f''(\delta\theta) - f''(0))\theta^2,$$

since $f(0) = 0$. Taking derivatives of (2.16) with $F = F^{(n,v)}$ and $\xi = f$, we have

$$f''(\theta) = -\frac{(\widehat{F}_0^{(n,v)})''(f(\theta))(\widehat{F}_0^{(n,v)}(f(\theta)))^2}{((\widehat{F}_0^{(n,v)})'(f(\theta)))^3} + \frac{\widehat{F}_0^{(n,v)}(f(\theta))}{(\widehat{F}_0^{(n,v)})'(f(\theta))}, \tag{B.1}$$

$$f''(0) = -(\lambda_0^{(n)}(v))^3(\sigma_0^{(n)}(v))^2.$$

Hence, from (3.6), we have

$$\varepsilon_0^{(n)}(v, \theta) = \tfrac{1}{2}(f''(\delta\theta) - f''(0))\theta^2, \qquad v > 0, \text{ there exists } \delta \in [0, 1].$$

For $\theta = 0$, obviously $\varepsilon_i^{(n)}(v, \theta) = 0$, and (3.13) trivially holds. In what follows, we assume that $0 < \theta < \overline{\theta}_0$ since the case $\theta < 0$ is similarly proved. Note that

$$|\varepsilon_0^{(n)}(v, \theta)| \leq \tfrac{1}{2}\theta^2 \max_{0 \leq x \leq \theta} |f''(x) - f''(0)|. \tag{B.2}$$

By letting $v \uparrow \infty$, we can see that this inequality is valid for $v = \triangle$ because $\eta^{(n)}(\triangle, \theta) = \eta^{(n)}(\theta) < 0$ for $\theta > 0$ by Lemma 2.5. In the computations below, we assume that $0 < v < \infty$, but they are easily adapted for $v = \triangle$ by letting $v \uparrow \infty$, and lead to the same conclusion.

To simplify notation, let $g(s) = \widehat{F}_0^{(n,v)}(s)$. Then, $g'(0) = \mathbb{E}(T_0^{(n)} \wedge v)$, and (B.1) can be written as

$$f''(x) = -\frac{g''(f(x))(g(f(x)))^2}{(g'(f(x)))^3} + \frac{g(f(x))}{g'(f(x))},$$

and, therefore, using the fact that $f(0) = 0$,

$$f''(x) - f''(0) = \frac{g(f(x))g'(0) - g'(f(x))g(0)}{g'(0)g'(f(x))} + \frac{g''(0)(g'(f(x)))^3 - g''(f(x))(g'(0))^3(g(f(x)))^2}{(g'(0)g'(f(x)))^3}.$$

We evaluate the numerators as

$$g(f(x))g'(0) - g'(f(x))g(0) = g(0)(g'(0) - g'(f(x))) + g'(0)(g(f(x)) - g(0)),$$

and

$$\begin{aligned} &g''(0)(g'(f(x)))^3 - g''(f(x))(g'(0))^3(g(f(x)))^2 \\ &\quad = g''(0)((g'(f(x)))^3 - (g'(0))^3) + (g'(0))^3(g''(0) - g''(f(x))) \\ &\qquad + (g'(0))^3 g''(f(x))(1 - (g(f(x)))^2). \end{aligned}$$

Hence, it is important to evaluate the differences $g(f(x)) - g(0)$, $g'(f(x)) - g'(0)$ and $g''(f(x)) - g''(0)$. Since $f(x)$ is decreasing in x and $g(s), g'(s), g''(s)$ are all increasing

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in s, we have, using Taylor expansions again,

$$\max_{0\le x\le\theta}|g(f(x))-g(f(0))| = |g(f(0))-g(f(\theta))| \le |f(\theta)|g'(f(0)) = |f(\theta)|\mathbb{E}(T_0^{(n)}),$$

$$\begin{aligned}\max_{0\le x\le\theta}|g'(f(x))-g'(f(0))| &= |g'(f(0))-g'(f(\theta))|\\ &\le |f(\theta)|g''(f(0))\\ &= |f(\theta)|\mathbb{E}((T_0^{(n)})^2),\end{aligned}$$

and

$$\begin{aligned}&\max_{0\le x\le\theta}|g''(f(x))-g''(f(0))| = g''(f(0))-g''(f(\theta))\\ &\quad= |\mathbb{E}((T_0^{(n)}\wedge v)^2(1-\mathrm{e}^{(T_0^{(n)}\wedge v)f(\theta)}))|\\ &\quad\le \mathbb{E}((T_0^{(n)}\wedge v)^2\,\mathrm{e}^{(T_0^{(n)}\wedge v)|f(\theta)|}(1-\mathrm{e}^{-(T_0^{(n)}\wedge v)|f(\theta)|}))\\ &\quad\le |f(\theta)|\,\mathrm{e}^{v|f(\theta)|}\mathbb{E}((T_0^{(n)})^2 v(v^{-1}T_0^{(n)}\wedge 1)))\\ &\quad\le v|f(\theta)|\,\mathrm{e}^{v|f(\theta)|}\mathbb{E}((T_0^{(n)})^2(v^{-1}T_0^{(n)}\wedge 1)(\mathbf{1}\{T_0^{(n)}>\sqrt{v}\}+\mathbf{1}\{T_0^{(n)}\le\sqrt{v}\}))\\ &\quad\le v|f(\theta)|\,\mathrm{e}^{v|f(\theta)|}\mathbb{E}((T_0^{(n)})^2(\mathbf{1}\{T_0^{(n)}>\sqrt{v}\}+\sqrt{v}^{-1})).\end{aligned}$$

Similarly, we have

$$\max_{0\le x\le\theta}\frac{1}{g'(0)g'(f(x))}\le\frac{1}{(g'(0))^2}\,\mathrm{e}^{|vf(\theta)|}.$$

Since $g'(0)=\mathbb{E}(T_0^{(n)}\wedge v)$ and $g''(0)=\mathbb{E}((T_0^{(n)}\wedge v)^2)$, we have

$$\begin{aligned}\mathbb{E}(T_0^{(n)})-\mathbb{E}(T_0^{(n)}\,\mathbf{1}\{T_0^{(n)}>v\})\le g'(0)\le\mathbb{E}(T_0^{(n)}),\\ \mathbb{E}((T_0^{(n)})^2)-\mathbb{E}((T_0^{(n)})^2\,\mathbf{1}\{T_0^{(n)}>v\})\le g''(0)\le\mathbb{E}((T_0^{(n)})^2).\end{aligned}$$

We now let $v=1/r_n$ and replace θ by $r_n\theta$, then $(1/r_n)f(r_n\theta)$ is uniformly bounded by the linear function of $\lambda_0|\theta|$ as $n\to\infty$ by Lemma 3.1. Hence, recalling $f(\theta)=\eta^{(n)}(v,\theta)$ and letting $f_n(x)=\eta^{(n)}(1/r_n,r_nx)$,

$$\limsup_{n\to\infty}\max_{0\le x\le\theta}|f_n''(x)-f_n''(0)|=0.$$

Thus, in (B.2), let either $v=1/r_n$ or letting $v\uparrow\infty$ and substitute $r_n\theta$ to θ, then applying the above limit completes the proof. □

Proof of Lemma 3.3. The proof is essentially the same as the proof of Lemma 3.2. Because of the conditions (3.10) and (3.11), we need to divide the error functions by $s_nr_n^2$ instead of r_n^2. Except for this manipulation, all the arguments in the proof of Lemma 3.2 go through. □

Appendix C. Lemmas in Section 4

Proof of Lemma 4.1. Recall that $F_0(0)=0$ is assumed. It implies that $\mathbb{P}(N_0(t)\ge n)=\mathbb{P}(T_0(1)+T_0(2)+\cdots+T_0(n)\le t)$, so there is a sufficiently large $n\ge 1$ for any $\delta>0$ and $t\ge 0$ such that $t\le n\delta$, which implies that

$$\mathbb{P}(N_0(t)\ge n)\le\mathbb{P}(T_0(1)+T_0(2)+\cdots+T_0(n)\le n\delta).$$

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Hence, (4.3) is obtained if we can find $\delta > 0$ for any $a > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n}(T_0(1) + T_0(2) + \cdots + T_0(n)) \le \delta \right) \le -a. \tag{C.1}$$

To prove this inequality, we use the Cramér theorem on large deviations (see, e.g. Theorem 2.2.3 of [10]) that, for any $\delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n}(T_0(1) + T_0(2) + \cdots + T_0(n)) \le \delta \right) \le -\inf_{x\le\delta} \Lambda^*(x), \tag{C.2}$$

where $\Lambda^*(x) = \sup_{\eta\in\mathbb{R}}(\eta x - \log \widehat{F}_0(\eta))$. Since $\mathbb{E}(T_0) > 0$ and $\widehat{F}_0(\eta)$ is finite for $\eta \le 0$ by Lemma 2.4, it follows from Lemma 2.2.5 of [10] that, for $\delta < \mathbb{E}(T_0)$,

$$\inf_{x\le\delta} \Lambda^*(x) = \Lambda^*(\delta) = \sup_{\eta<0}(\eta\delta - \log \widehat{F}_0(\eta)). \tag{C.3}$$

Hence, the proof is completed if $\Lambda^*(\delta)$ can be arbitrarily large for some $\delta \in (0, \mathbb{E}(T_0))$. Let $\theta = -\log \widehat{F}_0(\eta)$, then $\eta = \xi(\theta)$ for ξ of (2.16) with $v = \infty$ and $T = T_0$. Then, by Lemma 2.4, $\xi(\theta) < 0$ if and only if $\theta > 0$, and, therefore, $\Lambda^*(\delta)$ can be represented as

$$\Lambda^*(\delta) = \sup_{\theta>0}(\xi(\theta)\delta + \theta).$$

For each $\theta > 0$, we can choose a sufficiently small $\delta \in (0, \mathbb{E}(T_0))$ such that $\Lambda^*(\delta) \ge \frac{1}{2}\theta$. Hence, letting $a = 2\theta$, (C.1) follows from (C.2) and (C.3). □

Proof of Lemma 4.2. (i) Since $\theta \ge 0$, we can omit $\eta(\theta)R_0(t)$ can be omitted from the exponent in $f_{K_\alpha(v),\theta}(X(t))$ to prove the finiteness of its expectation because $\eta(\theta) \le 0$ for $\theta \ge 0$ by Lemma 2.4. Since

$$L(t) \le L(0) + N_0(t), \qquad t \ge 0,$$

Lemma 4.1 and $\mathbb{P}_{\nu_k}(L(0) = k) = 1$ lead to

$$\mathbb{E}_{\nu_k}(\mathrm{e}^{\theta L(t)}) \le \mathrm{e}^{\theta k}\mathbb{E}_{\nu_k}(\mathrm{e}^{\theta N_0(t)}) < \infty, \qquad \theta \ge 0. \tag{C.4}$$

We next check the finiteness of $\mathbb{E}_{\nu_k}(\mathrm{e}^{W_{K_\alpha(v),\theta}(\boldsymbol{R}(t))})$ for $\theta \in [0, \alpha + \delta_0]$. Let us consider $R_i(0-)$ on ν_k. Let $U_t(0) = \{i \in K;\ R_i(0-) \le t\}$. On ν_k, one of $R_i(0-)$ vanishes for $i \in K$, so $U_t(0) \in 2^K \setminus \{\varnothing\}$. Hence, it is enough to verify that $\mathbb{E}_{\nu_k}(\mathrm{e}^{W_{K_\alpha(v),\theta}(\boldsymbol{R}(t))}\mathbf{1}\{U_t(0) = A\}) < \infty$ for each $A \in 2^K \setminus \{\varnothing\}$.

We now fix $A \in 2^K \setminus \{\varnothing\}$, and consider $\boldsymbol{R}(t)$ on $U_t(0) = A$ under ν_k. If $i \in K \setminus A$, then $R_i(t) = R_i(0-) - t > 0$. Otherwise, if $i \in A$ then server i may start service in the time interval $[R_i(0-), t]$. We apply similar arguments as in the proof of Lemma 4.3 of [30]. For this, let $t_s(m)$ be the mth time when service is started and let $t_{s,i}(n)$ be the nth time when server i starts service, where if more than one server simultaneously starts service, we count them separately. Then, there is a unique n for each $m \ge 1$ such that $t_{s,i}(n) \le t_s(m) < t_{s,i}(n+1)$. Denote this n by $J_i(m)$. For $\boldsymbol{y} \in \mathbb{R}_+^k$, let

$$g_{A_\alpha(v),\theta}(\boldsymbol{y}) = \mathbf{1}\{U_t(0) = A,\ y_i \ge 0,\ i \in A\} \prod_{i\in A\setminus K_\alpha} \mathbb{E}(\mathrm{e}^{\zeta_i(\theta)(T_{s,i}-y_i)}\mathbf{1}\{T_{s,i} > y_i\})$$
$$\times \prod_{i\in A\cap K_\alpha} \mathbb{E}(\mathrm{e}^{\zeta_i(v,\theta)(v\wedge(T_{s,i}-y_i))}\mathbf{1}\{T_{s,i} > y_i\}),$$

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then, from the fact that $N_0(t)+1$ customers arrive in $[0,t]$, we have

$$\mathbb{E}_{\nu_k}(\mathrm{e}^{W_{K_\alpha(v),\theta}(\boldsymbol{R}(t))}\,\mathbf{1}\{U_t(0)=A\}) \le \mathbb{E}_{\nu_k}\Bigg(\sum_{m=1}^{N_0(t)+1}\mathrm{e}^{W_{A_\alpha^c(v),\theta}(\boldsymbol{R}(0-)-t\mathbf{1})}\,g_{A_\alpha(v),\theta}(t-t_{s,1}(J_1(m)),\dots,t-t_{s,k}(J_k(m)))\Bigg), \tag{C.5}$$

where $W_{A_\alpha^c(v),\theta}(\boldsymbol{y})=\sum_{i\in K\setminus(A\cup K_\alpha)}\zeta_i(\theta)y_i+\sum_{i\in K_\alpha\cap(K\setminus A)}\zeta_i(v,\theta)(y_i\wedge v)$. Hence, recalling the definitions of $\zeta_i(\theta)$ and $\zeta_i(v,\theta)$, Lemma 4.8 and (C.5) yield

$$\begin{aligned}\mathbb{E}_{\nu_k}(\mathrm{e}^{W_{K_\alpha(v),\theta}(\boldsymbol{R}(t))}\,\mathbf{1}\{U_t(0)=A\}) &\le \mathbb{E}_{\nu_k}(\mathrm{e}^{W_{A_\alpha^c(v),\theta}(\boldsymbol{R}(0-))})\,\mathrm{e}^{|A|\theta}\mathbb{E}_{\nu_k}(N_0(t)+1)\\ &\le \mathbb{E}_{\nu_k}(\mathrm{e}^{W_{K_\alpha(v),\theta}(\boldsymbol{R}(0-))})\,\mathrm{e}^{|A|\theta}\mathbb{E}_{\nu_k}(N_0(t)+1)\\ &<\infty.\end{aligned}$$

Since $|2^A-\{\varnothing\}|=2^k-1<2^k$, (C.4) implies that

$$\mathbb{E}_{\nu_k}(f_{K_\alpha(v),\theta}(X(t)))<\mathrm{e}^{\theta k}\mathbb{E}_{\nu_k}(\mathrm{e}^{\theta N_0(t)})\mathbb{E}_{\nu_k}(\mathrm{e}^{w_{K_\alpha(v),\theta}(\boldsymbol{R}(0-))})2^k\,\mathrm{e}^{(k-1)\theta}\mathbb{E}_{\nu_k}(N_0(t)+1),$$

which is uniformly bounded in $v>0$ by Lemma 4.8 because $w_{K_\alpha(v),\theta}(\boldsymbol{y})$ is decreasing in v. Since it follows from (4.1) and (4.2) for $u=\infty$ and $A=K_\alpha(v)$ that, for $\theta\ge 0$,

$$|M_{K_\alpha(v),\theta}(t)|\le f_{K_\alpha(v),\theta}(X(t))+\Bigg(|\eta(\theta)|+\sum_{i\in K}\zeta_i(v,\theta)\Bigg)\int_0^t f_{K_\alpha(v),\theta}(X(s))\,\mathrm{d}s, \tag{C.6}$$

$\mathbb{E}_{\nu_k}(|M_{K_\alpha(v),\theta}(t)|)$ also is uniformly bounded in $v>0$ for $\theta\in[0,\alpha+\delta_0)$ and fixed $t\ge 0$. This proves (i), and therefore Lemmas 2.2 and 2.3 imply (ii).

(iii) Since $\mathrm{e}^{\zeta_i(v,\theta)R_i(t)\wedge v}$ almost surely converges to $\mathrm{e}^{\zeta_i(\triangle,\theta)R_i(t)}$ as $v\to\infty$ by the definition of $\zeta_i(\triangle,\theta)$, $f_{K_\alpha(v),\theta}(X(u))$ for $u\in[0,t]$ and, therefore, $M_{K_\alpha(v),\theta}(t)$ almost surely converges to $f_{K_\alpha(\triangle),\theta}(X(u))$ and $M_{K_\alpha(\triangle),\theta}(t)$, respectively, as $v\to\infty$.

(iv) By (ii), $M_{K_\alpha(v),\theta}(t)$ is an $\mathcal{F}_t$-martingale under $\mathbb{P}_{\nu_k}$ for each $v>0$, which implies that

$$\mathbb{E}_{\nu_k}(M_{K_\alpha(v),\theta}(t)\,\mathbf{1}_D)=\mathbb{E}_{\nu_k}(M_{K_\alpha(v),\theta}(s)\,\mathbf{1}_D),\qquad D\in\mathcal{F}_s,\ 0\le s<t. \tag{C.7}$$

Since $|M_{K_\alpha(v),\theta}(s)|$ is uniformly bounded in expectation and $M_{K_\alpha(v),\theta}(s)$ almost surely converges to $M_{K_\alpha(\triangle),\theta}(s)$ as $v\uparrow\infty$ for each $s\ge 0$, we can apply the dominated convergence theorem to (C.7), which concludes (iii). □

Proof of Lemma 4.4. We only prove (4.8) because (4.9) and (4.10) are similarly proved. Since, at arrival instant t, for $L(t-)\ge k$,

$$\frac{f_{K_\alpha(v),\theta}(X(t))}{f_{K_\alpha(v),\theta}(X(t-))}=\mathrm{e}^{\theta}\,\mathrm{e}^{\eta(\theta)T_0},$$

and, for $L(t-)\le k-1$,

$$\frac{f_{K_\alpha(v),\theta}(X(t))}{f_{K_\alpha(v),\theta}(X(t-))}=\mathrm{e}^{\theta}\,\mathrm{e}^{\eta(\theta)T_0}\prod_{i\in\Delta J(t)\setminus K_\alpha}\mathrm{e}^{-\theta}\,\mathrm{e}^{\zeta_i(\theta)T_i}\prod_{i\in\Delta J(t)\cap K_\alpha}\mathrm{e}^{-\theta}\,\mathrm{e}^{\zeta_i(v,\theta)(T_i\wedge v)},$$

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where $\Delta J(t) = J(t) - J(t-)$ and either $\Delta J(t) \setminus K_\alpha$ or $\Delta J(t) \cap K_\alpha$ is an empty set, it follows from (4.7) and the definitions of ζ_i and $\zeta_i^{(n)}$ that

$$\begin{aligned}\mathbb{E}_{\nu_k}^{(K_\alpha(v),\theta)}(\mathrm{e}^{uT_0} \mid R_0(t-)=0) &= \mathbb{E}_{\nu_k}\left(\frac{f_{K_\alpha(v),\theta}(X(t))}{f_{K_\alpha(v),\theta}(X(t-))}\,\mathrm{e}^{uT_0} \;\middle|\; R_0(t-)=0\right)\\ &= \mathrm{e}^{\theta}\mathbb{E}_{\nu_k}(\mathrm{e}^{(\eta(\theta)+u)T_0})\\ &= \mathrm{e}^{\theta}\widehat{F}_0(\eta(\theta)+u),\end{aligned}$$

and $\eta(\theta)+u \le \beta_0$. □

Proof of Lemma 4.6. This lemma is partly a continuous-time version of Lemma 4.5 of [30], and we will use the same idea for our proof. Let $\widetilde{\tau}_0(t) = \inf\{s \ge t;\, H(s) \ge \ell\}$, then $\int_0^\infty \mathbf{1}\{H(s) \ge \ell\}\,\mathrm{d}s \ge t$ implies that $\widetilde{\tau}_0(t) < \infty$. Let $A_t = \{\widetilde{\tau}_0(t) < \infty\}$. To evaluate $\mathbb{P}_{\nu_k}(A_t \mid \mathcal{F}_{\sigma_\ell^+-})$, we will use the probability measure $\widetilde{\mathbb{P}}_{\nu_k}^{(u,K_\alpha(\Delta),\theta)}$ of (4.21). Since $\gamma'_{u,K_\alpha(\Delta)}(\theta) > 0$, we have $\widetilde{\mathbb{P}}_{\nu_k}^{(u,K_\alpha(\Delta),\theta)}(A_t \mid \mathcal{F}_{\sigma_\ell^+-}) = 1$. Furthermore, $\eta(u,\theta) \le 0$ for $\theta \ge 0$. Thus, we have, from (4.20),

$$\begin{aligned}&\mathbb{E}_{\nu_k}(\exp(-W_{K_\alpha,\theta}(\widetilde{\boldsymbol{R}}(\sigma_\ell^+-)))1_{A_t} \mid \mathcal{F}_{\sigma_\ell^+-})\\ &\quad= \widetilde{\mathbb{E}}_{\nu_k}^{(u,K_\alpha(\Delta),\theta)}\left(\exp(-W_{K_\alpha,\theta}(\widetilde{\boldsymbol{R}}(\sigma_\ell^+-)))\frac{\widetilde{f}_{u,K_\alpha(\Delta),\theta}(Y(\sigma_\ell^+-))}{\widetilde{f}_{u,K_\alpha(\Delta),\theta}(Y(\widetilde{\tau}_0(t)))}\right.\\ &\qquad\qquad\left.\times\exp\left(\gamma_{u,K_\alpha(\Delta)}(\theta)\widetilde{\tau}_0(t) + \eta(u,\theta)\int_0^{\widetilde{\tau}_0(t)}\mathbf{1}\{\widetilde{R}_0(s)\ge u\}\,\mathrm{d}s\right) \;\middle|\; \mathcal{F}_{\sigma_\ell^+-}\right)\\ &\quad\le \widetilde{\mathbb{E}}_{\nu_k}^{(u,K_\alpha(\Delta),\theta)}(\exp(-\theta(H(\widetilde{\tau}_0(t)) - (\ell-1)) - \eta(u,\theta)(\widetilde{R}_0(\widetilde{\tau}_0(t))\wedge u)\\ &\qquad\qquad - W_{K_\alpha,\theta}(\widetilde{\boldsymbol{R}}(\widetilde{\tau}_0(t))) + \gamma_{u,K_\alpha(\Delta)}(\theta)t) \mid \mathcal{F}_{\sigma_\ell^+-})\\ &\quad\le \widetilde{\mathbb{E}}_{\nu_k}^{(u,K_\alpha(\Delta),\theta)}(\exp(-\eta(u,\theta)u + \gamma_{u,K_\alpha(\Delta)}(\theta)t) \mid \mathcal{F}_{\sigma_\ell^+-}),\end{aligned}$$

where the last inequality uses the fact that $H(\widetilde{\tau}_0(t)) \ge \ell$ and $\zeta_i(\Delta,\theta) \ge 0$ for $\theta \ge 0$. Integrating both sides of this inequality for $t \in [0,\infty)$, we have (4.22) for $C = (-\gamma_{u,K_\alpha(\Delta)}(\theta))^{-1}\,\mathrm{e}^{-\eta(u,\theta)u}$ because $\int_0^\infty \mathbf{1}\{H(s)\ge\ell\}\,\mathrm{d}s \le \int_0^\infty \mathbf{1}_{A_s}\,\mathrm{d}s$. □

Proof of Lemma 4.9. We only need to prove (4.30) for $i = 0$ because the proof is similar for $i \in K$. We introduce the following notation for the nth system:

$$\widetilde{\lambda}_0^{(n)} = \left(\mathbb{E}\left(T_0^{(n)}\,\mathbf{1}\left\{T_0^{(n)} \le \frac{1}{r_n}\right\}\right)\right)^{-1}, \qquad \widetilde{\sigma}_0^{(n)} = \mathbb{E}\left((T_0^{(n)})^2\,\mathbf{1}\left\{T_0^{(n)} \le \frac{1}{r_n}\right\}\right) - (\widetilde{\lambda}_0^{(n)})^{-2}.$$

Comparing (3.6) with (4.29), we can see that Lemma 4.9 follows from Lemma 3.2 if we have,

$$\limsup_{n\to\infty}\frac{1}{r_n}|\lambda_0^{(n)} - \widetilde{\lambda}_0^{(n)}| = 0, \tag{C.8}$$

$$\limsup_{n\to\infty}|(\sigma_0^{(n)})^2 - (\widetilde{\sigma}_0^{(n)})^2| = 0. \tag{C.9}$$

We will use the uniform integrability condition (3.12) to verify them. We first verify (C.8) in

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the following way.

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{r_n}|\lambda_0^{(n)}-\widetilde{\lambda}_0^{(n)}| &= \limsup_{n\to\infty}\frac{1}{r_n}\lambda_0^{(n)}\widetilde{\lambda}_0^{(n)}\mathbb{E}\left(T_0^{(n)}\left(1-\mathbf{1}\left\{T_0^{(n)}\le\frac{1}{r_n}\right\}\right)\right)\\ &= \limsup_{n\to\infty}\frac{1}{r_n}\lambda_0^{(n)}\widetilde{\lambda}_0^{(n)}\mathbb{E}\left(T_0^{(n)}\,\mathbf{1}\left\{T_0^{(n)}>\frac{1}{r_n}\right\}\right)\\ &\le \limsup_{n\to\infty}\lambda_0^{(n)}\widetilde{\lambda}_0^{(n)}\mathbb{E}\left((T_0^{(n)})^2\,\mathbf{1}\left\{T_0^{(n)}>\frac{1}{r_n}\right\}\right)\\ &= 0,\end{aligned}$$

where the last equality is obtained by (3.7) and (3.12). Similarly, (C.9) is verified by

$$\begin{aligned}&\limsup_{n\to\infty}|(\sigma_0^{(n)})^2-(\widetilde{\sigma}_0^{(n)})^2|\\ &\quad\le \limsup_{n\to\infty}\left(\mathbb{E}\left((T_0^{(n)})^2\,\mathbf{1}\left\{T_0^{(n)}>\frac{1}{r_n}\right\}\right)+(\lambda_0^{(n)}\widetilde{\lambda}_0^{(n)})^{-2}|(\lambda_0^{(n)}-\widetilde{\lambda}_0^{(n)})(\lambda_0^{(n)}+\widetilde{\lambda}_0^{(n)})|\right)\\ &\quad= 0.\end{aligned}$$

This completes the proof of Lemma 4.9. □

In this paper, we omitted proofs of Lemmas 2.5 and 4.8. Furthermore, the proofs of Lemmas 3.1 and 4.2 are not fully detailed. Their complete proofs can be found in a supplemented version of this paper [23].

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