ADDITIVE FUNCTIONALS ON GROUPS*

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The kernel of a non-trivial linear functional ϕ on a linear space E is a maximal proper linear subspace of E which determines ϕ up to a non-zero multiple. Does a similar result hold for homomorphisms of a group G into the additive group R of the real numbers?

It is easily seen (§ 1, Remarks (1) and (2)) that kernels and subgroups are of little use here and, following Fleischer (7) and Hoffman and Singer (10), we look instead at the semigroups in G. Those semigroups of the forms

$$\{x \in G\colon\ \phi(x) \geqslant 0\} \quad \text{and} \quad \{x \in G\colon\ \phi(x) > 0\}$$

for some non-trivial homomorphism ϕ of G into R are characterized and shown to determine the corresponding homomorphisms up to a positive multiple.

Applications are given to partially ordered and lattice ordered groups, and necessary and sufficient conditions are obtained in order that these be representable as groups of real-valued functions on some set.

Bonsall (2) has proved the Hahn–Banach Theorem using partially ordered linear spaces: in the same way we prove a Hahn–Banach Theorem for Abelian groups using partially ordered groups. (This is a particular case of a general result due to Halperin (8).)

The semigroup technique is similar to the semispace method of Hammer (9); in fact our results can be readily transferred to real linear spaces by using convex cones instead of semigroups.

Throughout the paper a subset A of a set B is said to be *properly contained in* B, or to be *proper*, if $A \neq B$. If $\mathscr{B}$ is a class of subsets of a set B, a *maximal* element of $\mathscr{B}$ is a maximal element under set inclusion. The empty set is denoted by $\varnothing$.

1. *Additive functionals on groups.* First a few definitions:

Let G be a group, written additively but not necessarily Abelian. A *semigroup* in G is a non-empty subset S of G with $S+S \subset S$: it is called *invariant* if $x+S = S+x$ for all $x \in G$, and *convex* if $x \in S$ whenever $x \in G$ and $nx \in S$ for some positive integer n. If $A \subset G$ we write $\mathrm{Sg}\,(A)$ for the least semigroup in G containing A. A *unit* for S is an $x \in S$ with $\mathrm{Sg}\,(\{-x\} \cup S) = G$.

An *additive functional* on G is a homomorphism ϕ of G into the additive group of the real numbers. We set $S(\phi) = \{x \in G\colon \phi(x) \geqslant 0\}$, and say that ϕ is *trivial* if $\phi(G) = \{0\}$.

A pair $\{G, G^+\}$, where G^+ is an invariant semigroup in a group G and $0 \in G^+$, is called a *quasi-ordered group*. We write $x \geqslant y$ $(y \leqslant x)$ if x and y are in G and $x-y \in G^+$, and

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$x > y$ ($y < x$) if $x \geqslant y$ ($y \leqslant x$) and $x \neq y$. If $G^+ \cap -G^+ = \{0\}$ then $\{G, \geqslant\}$ is a *partially ordered group* ((1), p. 214) and, if in addition $G^+ \cup -G^+ = G$, it is a *simply ordered group*. An *order unit* for a quasi-ordered group $\{G, G^+\}$ is just a unit for G^+. Two quasi-ordered groups $\{G, G^+\}$ and $\{H, H^+\}$ are said to be *isomorphic* if there is an isomorphism I of G onto H with $I(G^+) = H^+$: I is then called an *isomorphism* of $\{G, G^+\}$ onto $\{H, H^+\}$.

If S is an invariant semigroup containing 0 in a group G we construct the *quotient partially ordered group* G/S as follows:

For any $x \in G$ let $[x] = x + S$ and if $A \subset G$ let $[A] = \{[x]: x \in A\}$. Since S is invariant we can make $[G]$ into a group by defining $[x]+[y] = [x+y]$ for all $x, y \in G$. Now $[S]$ is a semigroup in $[G]$ and contains $[0]$, the zero of $[G]$. Furthermore, if $[x] \in [G]$ and $[y] \in [S]$ then $[x]+[y] = [x+y] = [y'+x]$ (where $y' \in S$) $= [y']+[x]$; so $[S]$ is invariant. Finally, if $[x] \in [S] \cap -[S]$ then $x+S \subset S$ and $-x+S \subset S$, whence

$$x+S \subset S \subset x+S \quad \text{and} \quad [x] = S = [0].$$

G/S is defined to be $\{[G], [S]\}$.

We can now prove:

THEOREM 1. *Let G be a group.*

(1) *If ϕ is a non-trivial additive functional on G then $S(\phi)$ is an invariant convex maximal proper semigroup in G.*

(2) *If S is an invariant maximal proper semigroup in G there is a non-trivial additive functional ϕ on G with $S(\phi) = S$.*

(3) *If ϕ and ψ are non-trivial additive functionals on G then $S(\phi) = S(\psi)$ if and only if $\phi = \lambda\psi$ for some positive real number λ.*

Proof. (1) $S(\phi)$ is obviously a convex semigroup in G. If $y = x+z$, with $x \in G$ and $z \in S(\phi)$, then $y = (x+z-x)+x$ and $x+z-x \in S(\phi)$; therefore $S(\phi)$ is invariant. If $x \notin S(\phi)$ and $y \in G$ then, for a big enough positive integer n, $\phi(y-nx) > 0$; whence $y = (y-nx)+nx$ with $y-nx \in S(\phi)$, proving that $S(\phi)$ is a maximal proper semigroup in G.

(2) Since S contains 0 we may form G/S. This is simply ordered; for if $[x] \notin [S]$ then $x \notin S$ and for some non-negative integer n and some $y \in S$ we have $y+nx = -x$; whence $(n+1)(-x) \in S$, $-x \in S$ and $-[x] \in [S]$. Furthermore, any $[x] > 0$ is an order unit for G/S; for there is a $z \in S$ with $z \notin x+S$, that is $-x+z \notin S$. Therefore, for any $y \in G$ there is a non-negative integer n such that $y \in n(-x+z)+S$. The invariance of S gives $y \in -nx+S$; so $[y] \in n(-[x])+[S]$. By (1), pp. 225–226, there is an isomorphism I of G/S onto a subgroup of the additive group of the real numbers with the usual ordering. Clearly $x \to I([x])$ is the functional ϕ required.

(3) If $\phi = \lambda\psi$ for some positive real number λ it is obvious that $S(\phi) = S(\psi)$. If no such λ exists then either $\phi = \mu\psi$ for some negative real number μ, in which case $S(\phi) = -S(\psi) \neq S(\psi)$, or ϕ and ψ are linearly independent and there are $x, y \in G$ with

$$\begin{vmatrix} \phi(x) & \phi(y) \\ \psi(x) & \psi(y) \end{vmatrix} \neq 0$$

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and real numbers α, β with $\alpha\phi(x)+\beta\phi(y) = +1$ and $\alpha(x)+\beta(y) = -1$, whence, choosing rational numbers r/s and t/s so close to α and β respectively that

$$(r/s)\,\phi(x)+(t/s)\,\phi(y) > 0 \quad \text{and} \quad (r/s)\,\psi(x)+(t/s)\,\psi(y) < 0,$$

we have $rx+ty \in S(\phi) \setminus S(\psi)$.

Theorem 1 can be greatly improved when G is Abelian; for, calling a semigroup S in a group G *strict* when it is not a subgroup of G, we have:

THEOREM 2. *If S is a maximal proper convex semigroup or a maximal strict semigroup in an Abelian group G then there is a non-trivial additive functional ϕ on G with $S(\phi) = S$.*

Proof. If S is a maximal proper convex semigroup it is a maximal proper semigroup in G; for if $x \in G \setminus S$ and $y \in G$ there is a non-negative integer n and a positive integer m such that $my \in nx+S$. Taking $-x$ for y gives $-x \in S$; so

$$my \in nmx+(nm-n)\,(-x)+S \subset nmx+S, \quad m(y-nx) \in S \quad \text{and} \quad y \in nx+S.$$

Theorem 1 (2) gives ϕ as required.

If S is a maximal strict semigroup in G,

$$S' = \{x \in G\colon nx \in S \text{ for some positive integer } n\}$$

is just S; for if not then S' is a subgroup of G and there is an $x \in S$ with $-x \in S' \setminus S$. But then $-nx \in S$ for some positive integer n; whence $-x = -nx+(n-1)\,x \in S$, contrary to hypothesis. Thus S is convex. Again, if $x \in G \setminus S$ then $\mathrm{Sg}\,(\{x\} \cup S)$ is a subgroup of G; whence $-x \in nx+S$ for some non-negative integer n and it follows that $-x \in S$. Therefore $G = S \cup -S \subset \mathrm{Sg}\,(\{x\} \cup S)$ and S is shown to be a maximal proper semigroup in G. Theorem 1 completes the proof.

COROLLARY 1. ((10), Lemma 4·3 (i).) *If S is a maximal proper semigroup in an Abelian group G and if $S \cup -S = G$ then there is a non-trivial additive functional ϕ in G with $S(\phi) = S$.*

Proof. S cannot be a group: so it is a maximal strict semigroup.

Unfortunately there are strict semigroups which are not contained in a maximal strict semigroup and proper convex semigroups which are not contained in maximal proper convex semigroups (see Remark (2) below). This difficulty can be avoided at the expense of using slightly more complicated semigroups. We say that a semigroup S in a group G is *open* if every $x \in S$ is a unit for S; *relatively open* if it is open when considered as a semigroup in the group $S-S$, and *zero-less* if $0 \notin S$. For any additive functional ϕ on a group G we write $S^0(\phi) = \{x\colon \phi(x) > 0\}$.

THEOREM 3. *Let G be an Abelian group.*

(1) *If ϕ is a non-trivial additive functional on G then $S^0(\phi)$ is a maximal zero-less relatively open semigroup in G.*

(2) *If S is a maximal zero-less relatively open semigroup in G there is a non-trivial additive functional ϕ on G with $S^0(\phi) = S$.*

(3) *If ϕ and ψ are non-trivial additive functionals on G then $S^0(\phi) = S^0(\psi)$ if and only if $\phi = \lambda\psi$ for some positive real number λ.*

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Proof. (1) Clearly $S^0(\phi)$ is open and zero-less. Suppose S is a relatively open semigroup in G which properly contains $S^0(\phi)$. Then there is an $x \in S$ with $\phi(x) \leqslant 0$, and if $y \in S^0(\phi)$ there is a positive integer n with $nx - y \in S$, but $y - nx \in S^0(\phi) \subset S$, so S is not zero-less.

(2) A short calculation shows that

$$S = \{x: nx \in S \text{ for some positive integer } n\}$$

is a relatively open semigroup; it must be zero-less since S is; so $S' = S$ and S is convex. Next, if $x \in G$ then $S_1 = \{nx+s: n \text{ any integer}, s \in S\}$ is a relatively open semigroup in G which contains S; therefore $S_1 = S$ and either $x+s \in S$, for some $s \in S$, or $0 \in S$ and $nx \in -S$, $n \neq 0$: in either case $x \in S-S$. Thus $-S$ is a proper convex semigroup in G and S is open in G; therefore no proper convex semigroup in G which contains $-S$ can meet S; consequently there is a maximal proper convex semigroup T in G with $T \cap S = \varnothing$. If ψ is an additive functional on G with $S(\psi) = T$ then $S^0(-\psi) \supset S$ and, by (1), $S^0(-\psi) = S$; so $-\psi$ is the ϕ required.

(3) This follows immediately from Theorem 1 (3).

COROLLARY 1. *If S is any zero-less relatively open semigroup in G there is a non-trivial additive functional ϕ on G with $S^0(\phi) \supset S$.*

Proof. By Zorn's Lemma there is a maximal zero-less relatively open semigroup in G containing S.

COROLLARY 2. *If ϕ_0 is an additive functional on a subgroup G_0 of G it can be extended to an additive functional on all G.*

Proof. If ϕ_0 is trivial the result is immediate. If not, then there is an additive functional ψ on G with $S^0(\psi) \supset S^0(\phi_0)$. So $S^0(\psi|G_0) = S^0(\psi) \cap G \supset S^0(\phi_0)$; whence $S^0(\psi|G_0) = S^0(\phi_0)$ and $\psi|G_0 = \alpha\phi_0$ for some positive real number α. Therefore $\phi = \alpha^{-1}\psi$ is as required.

Remarks. (1) If M is an invariant maximal proper subgroup of a group G then G/M is a cyclic group of prime order; so M cannot be the kernel of a homomorphism into R.

(2) Let P_1 be the additive group of all polynomials with integral coefficients on the real line R, addition being defined pointwise. For any $x \in R$ let $\tilde{x}$ be the non-trivial additive functional $f \to f(x)$ for all $f \in P_1$. If x is transcendental then $\tilde{x}(f) = 0$ if and only if f is identically zero. Thus the kernels of additive functionals do not serve to distinguish them.

(3) If I is the additive group of the integers then $2I$ is a maximal proper semigroup and a maximal proper relatively open semigroup in I, but no non-trivial additive ϕ on I has $S(\phi) = 2I$ or $S^0(\phi) = 2I$.

(4) Let S_I be the group of all sequences of integers with only a finite number of non-zero terms, addition being defined term by term. The semigroup S of all elements of S_I with at least one non-zero term and whose last non-zero term is positive is strict and convex and also a maximal zero-less semigroup, yet there is no additive functional ϕ on S_I with $S(\phi) = S$.

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(5) Theorems 1 and 2 show that there is a unique real-valued function l (the natural logarithm) on the multiplicative group P of the positive real numbers such that $l(e) = 1$, $l(xy) = l(x) + l(y)$ for all $x, y \in P$ and $l(x) \geqslant 0$ whenever $x \in P$ and $x \geqslant 1$.

(6) A point x in a convex cone C in the real linear space E is a unit for C if and only if it is in the interior of C with respect to the finest locally convex topology on E. The analogue of Theorem 3 derived from this observation gives the usual separation theorems in linear spaces. Hammer (9) uses maximal zero-less convex cones.

2. *Quasi-ordered groups.* If $\{G, G^+\}$ is a quasi-ordered group we are particularly interested in the *positive additive functionals* on $\{G, G^+\}$; these are the non-trivial additive functionals on G which are non-negative on G^+.

A quasi-ordered group $\{G_0, G_0^+\}$ is called a *subquasi-ordered group* of $\{G, G^+\}$ if G_0 is a subgroup of G and $G_0^+ = G_0 \cap G^+$. Any subgroup G_0 of G becomes a subquasi-ordered group of $\{G, G^+\}$ if we define $G_0^+ = G_0 \cap G^+$.

The following theorem is an extension to Abelian groups of (5), Theorem 1, p. 20 and (3), Corollary to Theorem 2.

THEOREM 4. *If ϕ_0 is a positive additive functional on a subquasi-ordered group $\{G_0, G_0^+\}$ of a quasi-ordered Abelian group $\{G, G_0^+\}$ and if $G_0 + G^+$ is a subgroup of G, then ϕ_0 extends to a positive additive functional on $\{G, G^+\}$.*

Proof. $S_1 = S^0(\phi_0) + G^+$ is a zero-less open semigroup in $G_1 = S_1 - S_1 = G_0 + G^+$; so there is an additive functional ϕ_1 on G with $S^0(\phi_1) \supset S_1$.

$$S^0(\phi_1 | G_0) = S^0(\phi_1) \cap G_0 \supset S^0(\phi_0);$$

therefore $\phi_1 | G_0 = \alpha\phi_0$ for some positive real number α. Suppose $g \in G^+$ and $\phi_1(g) < 0$. If $s \in S^0(\phi_0)$ then $s + ng \in S^0(\phi_1)$ and $\phi_1(s + ng) > 0$ for all positive integers n, a contradiction. Thus ϕ_1 is a positive additive functional on $\{G, G^+\}$ and $\alpha^{-1}\phi_1$ is the ϕ required.

COROLLARY 1. *ϕ_0 extends to a positive additive functional on $\{G, G^+\}$ if either (1) $G_0 + G^+ = G$, or (2) G_0 contains an order unit for $\{G, G^+\}$.*

Proof. (2) implies $G_0 + G^+ = G$.

The next result shows that there are in general plenty of positive additive functionals on a quasi-ordered group; it extends (13), Theorem 4·1.

THEOREM 5. *If x is a point in a quasi-ordered Abelian group $\{G, G^+\}$ then there is a positive additive functional ϕ on $\{G, G^+\}$ with $\phi(x) < 0$ if and only if there is an open semigroup U in G with $x \in U$ and $U \cap G^+ = \varnothing$.*

Proof. If such a ϕ exists take $U = -S^0(\phi)$.

If U exists as stated then $U_1 = G^+ - U$ is a zero-less open semigroup in G and there is an additive functional ϕ on G with $S^0(\phi) \supset U_1$. Clearly $\phi(x) < 0$. Moreover if $y \in G^+$ and $\phi(y) < 0$ then for every positive integer n, $ny - x \in S^0(\phi)$ and $n\phi(y) - \phi(x) > 0$, a contradiction. Thus ϕ is a positive additive functional on $\{G, G^+\}$.

Remark. A pair $\{E, E^+\}$, where E^+ is a convex cone containing as its vertex the zero of a linear space E, is called a *quasi-ordered linear space.* A *positive linear functional* on $\{E, E^+\}$ is a non-trivial linear functional on E which is non-negative on E^+. Quasi-

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ordered linear spaces can be investigated using convex cones with vertex 0 in place of semigroups. Sometimes, however (e.g. Theorem 9, below), a linear functional can be deduced from the results for Abelian groups and the following theorem:

If $\{E, E^+\}$ is a quasi-ordered linear space with $E = E^+ - E^+$ then every positive additive functional on $\{E, E^+\}$ is a linear functional.

The proof is exactly the same as the proof of (4), Prop. 1, p. 33.

3. *Lattice-ordered groups.* A *lattice-ordered group* is a partially ordered group $\{G, G^+\}$ such that for each $x, y \in G$ there is a $z \in G$ with $z + G^+ = (x + G^+) \cap (y + G^+)$: this element z is unique and is denoted by $x \vee y$. For each $x, y \in G$ we define

$$x \wedge y = -(-x \vee -y),$$

$$x^+ = x \vee 0, \quad x^- = -x \vee 0, \quad |x| = -x \vee x.$$

The reader is referred to (1), pp. 219–221 for manipulative results.

Any subgroup of the additive group of the real numbers with the usual ordering is a lattice-ordered group and for each x, y in it $x \vee y$ is just $\max(x, y)$.

Two lattice-ordered groups are called *isomorphic* if they are isomorphic as partially ordered groups. Any isomorphism preserves the operations $\vee$ and $\wedge$.

A subset A of a lattice-ordered group $\{G, G^+\}$ is called $\vee$-*closed* ($\wedge$-*closed*) if it contains $x \vee y$ ($x \wedge y$) whenever it contains x and y; *lattice-closed* if it is both $\vee$- and $\wedge$-closed, and *solid* if it is $\wedge$-closed and contains G^+.

If $x, y \in G$ and S is an invariant solid semigroup in G then

$$([x] + [S]) \cap ([y] + [S]) = [x + S] \cap [y + S] = \{[z]\colon\ z \in (x + S) \cap (y + S)\}$$
$$= \{[z]\colon\ z \in (x \vee y) + S\} = [(x \vee y) + S] = [x \vee y] + [S].$$

Therefore G/S is a lattice-ordered group and $[x] \vee [y] = [x \vee y]$ for all $x, y \in G$.

A *lattice-additive functional* on $\{G, G^+\}$ is an additive functional ϕ on G with $\phi(x \vee y) = \phi(x) \vee \phi(y)$ for all $x, y \in G$.

LEMMA 1. *Any lattice-ordered group $\{G, G^+\}$ for which G^+ is a maximal proper solid semigroup is isomorphic to a subgroup of the additive group of the real numbers with the usual ordering.*

Proof. Any $x > 0$ is an order unit for $\{G, G^+\}$; for $\bigcup_{n=1}^{\infty} \{-nx + G^+\}$ is a solid semigroup in $\{G, G^+\}$ which properly contains G^+ and is therefore all G. Further, G is simply ordered; for otherwise there is an $x \in G$ with $x^+ > 0$ and $x^- > 0$; whence $nx^+ > x^-$ for some positive integer n and $0 = nx^+ \wedge x^- = x^- > 0$, a contradiction. The standard result ((1), pp. 225–226) completes the proof.

There is a simple relationship between non-trivial lattice additive functionals and solid semigroups:

THEOREM 6. *If ϕ is a non-trivial lattice-additive functional on a lattice-ordered group $\{G, G^+\}$ then $S(\phi)$ is an invariant maximal proper solid semigroup in $\{G, G^+\}$. Conversely, if S is an invariant maximal proper solid semigroup in $\{G, G^+\}$ then there is a non-trivial lattice additive functional ϕ on $\{G, G^+\}$ with $S(\phi) = S$.*

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Proof. The first part is immediate. Conversely, since $[S]$ is a maximal proper solid semigroup in g/S there is an isomorphism I of G/S onto a subgroup of the additive group of the real numbers with the usual ordering and $x \to I([x])$ is the functional ϕ required.

We next prove analogues for lattice-ordered groups of Theorems 4 and 5.

THEOREM 7. *If ϕ_0 is a lattice additive functional on a lattice-closed subgroup $\{G_0, G_0^+\}$ of a lattice-ordered Abelian grovp $\{G, G^+\}$ and $G_0 + G^+ = G$, then ϕ_0 extends to a lattice additive functional ϕ on $\{G, G^+\}$.*

Proof. If ϕ_0 is trivial the result is immediate. If not, then $S(\phi_0) + G^+$ is a solid semigroup in $\{G, G^+\}$ and $(S(\phi_0) + G^+) \cap G_0 = S(\phi_0)$. Further, if S is any proper solid semigroup in $\{G, G^+\}$ which contains $S(\phi_0) + G^+$ then $S_1 \cap G_0 = S(\phi_0)$; otherwise S_1 contains G_0 and is all of G, contrary to hypothesis. Zorn's Lemma and Theorem 6 complete the proof.

THEOREM 8. *If x is a point in a lattice-ordered Abelian group $\{G, G^+\}$ then there is a lattice additive functional ϕ on $\{G, G^+\}$ with $\phi(x) < 0$ if and only if there is a $\vee$-closed semigroup U in G with x a unit for U and $U \cap G^+ = \varnothing$.*

Proof. If such a ϕ exists take $U = -S^0(\phi) = \{x \in G: \phi(x) < 0\}$.

If U exists as stated then $U_1 = G^+ \cup (G^+ - U)$ is a solid semigroup in $\{G, G^+\}$ with $U_1 \cap U = \varnothing$. If S_1 is a proper solid semigroup in $\{G, G^+\}$ which contains U_1 then $x \notin S_1$; for otherwise S_1 is all G. Zorn's Lemma shows that there is a maximal solid semigroup S in G which does not contain x and Theorem 6 completes the proof.

Theorem 8 may appear more reasonable if we give the corresponding result for *lattice-ordered linear spaces.* By a lattice-ordered linear space we mean a partially ordered linear space $\{E, E^+\}$ which is a lattice-ordered group. A *lattice linear functional* on $\{E, E^+\}$ is a linear functional on E which is a lattice additive functional on the lattice-ordered group $\{E, E^+\}$. The result stated in the remark at the end of § 2 is of particular importance here since $E^+ - E^+ = E$ for all lattice linear spaces. We have:

THEOREM 9. *If x is a point in a lattice-ordered linear space $\{E, E^+\}$ then there is a lattice linear functional ϕ on $\{E, E^+\}$ with $\phi(x) < 0$ if and only if there is a $\vee$-closed absorbent convex set U such that $(U + x) \cap G^+ = \varnothing$.*

Proof. The necessity is obvious. To prove the converse let $C_1 = \bigcup_{\alpha > 0} \alpha(x + U)$ and let

$$C = \bigcup \left\{ \bigvee_{i=1}^{n} x_i : x_i \in C_1, i = 1, \ldots, n \text{ and } n = 1, 2, \ldots \right\}.$$

Then C is a convex cone since

$$\bigvee_{i=1}^{n} x_i + \bigvee_{j=1}^{m} y_j = \bigvee_{i=1}^{n} \bigvee_{j=1}^{m} (x_i + y_j).$$

Also $C \cap G^+ = \varnothing$; for if

$$\bigvee_{i=1}^{n} \alpha_i(x + u_i) \geq 0,$$

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where $\alpha_i > 0$ and $u_i \in U$ for $i = 1, \ldots, n$, then

$$\bigvee_{i=1}^{n} \alpha_i \left(x + \bigvee_{i=1}^{n} u_i \right) \geqslant 0.$$

Putting $y = x + \bigvee_{i=1}^{n} u_i$ we have $\bigvee_{i=1}^{n} \alpha_i y \geqslant 0$,

so $\left(\bigvee_{i=1}^{n} \alpha_i y \right) \wedge 0 \geqslant 0$ and $\bigvee_{i=1}^{n} \alpha_i (y \wedge 0) = \left(\bigwedge_{i=1}^{n} \alpha_i \right) (y \wedge 0) \geqslant 0.$

Therefore $y \wedge 0 \geqslant 0$ and $y \geqslant 0$, which is impossible since $y \in x + U$. Finally, since x is clearly a unit for C, Theorem 8 and the remark preceding the present theorem complete the proof.

Remark. A subgroup H of a lattice-ordered group $\{G, G^+\}$ is called *solid* if $z \in H$ whenever $x \wedge y \leqslant z \leqslant x \vee y$ for some $x, y \in H$. Invariant solid subgroups are called *ideals* by Birkhoff (1) and are completely adequate for dealing with lattice-ordered groups. The mappings $H \to H + G^+$ and $S \to S \cap -S$ establish a (1-1) order-preserving relationship between the invariant solid semigroups S and the invariant solid subgroups H of $\{G, G^+\}$ (7). We make the quotient group G/H of G with respect to an invariant solid subgroup H into a lattice-ordered group by defining

$$(G/H)^+ = \{\langle x \rangle : x \in G^+\},$$

where $\langle x \rangle$ is the coset of x. Having done this the mapping $[x] \to \langle x \rangle$ is an isomorphism of the lattice-ordered group $G/(H + G^+)$ onto the lattice-ordered group G/H. Thus the two approaches are equivalent.

4. *Representations.* Throughout this section $G(X)$ stands for the class of all real-valued functions on a set X made into a partially ordered Abelian group by the definitions:

(i) For $f, g \in G(X)$, $f + g$ is the element of $G(X)$ with $(f + g)(x) = f(x) + g(x)$ for all $x \in X$.

(ii) $G(X)^+ = \{f \in G(X) : f(x) \geqslant 0 \text{ for all } x \in X\}$.

Notice that $G(X)$ is in fact lattice-ordered and that for each f and g in $G(X)$,

$$(f \vee g)(x) = f(x) \vee g(x).$$

Let us say that a semigroup S in an Abelian group G is *closed* if for each $x \notin S$ there is a zero-less open semigroup $U \subset G$ with $x \in U$ and $U \cap S = \varnothing$. Then we have the following result:

THEOREM 10. *A partially ordered Abelian group $\{G, G^+\}$ is isomorphic to a subpartially ordered group of $G(X)$ for some X if and only if G^+ is closed.*

Proof. (1) *If.* Take X to be the class of all positive additive functionals on $\{G, G^+\}$ and let I be the mapping of G into $G(X)$ defined by $I(g)(x) = x(g)$ for all $x \in X$. Then Theorem 5 shows I to be an isomorphism of G onto $I(G)$.

(2) *Only if.* If $G \subset G(X)$ and $f \notin G^+$ pick $x \in X$ with $f(x) < 0$ and consider

$$\{g \in G : g(x) < 0\}.$$

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For a lattice-ordered Abelian group $\{G, G^+\}$ with G^+ closed we could use Theorem 10 to find an isomorphism I of G onto $I(G) \subset G(X)$. But, though $I(G)$ is always a lattice-ordered group according to the definitions, it is not in general a lattice-closed subgroup of $G(X)$, and so in general we do not have

$$I(f \vee g)(x) = I(f)(x) + I(g)(x)$$

for all $f, g \in G$ and all $x \in X$.

However:

THEOREM 11. *A lattice-ordered Abelian group $\{G, G^+\}$ is isomorphic to a lattice-closed subgroup of $G(X)$ for some X if and only if for each $g \in G \setminus G^+$ there is a $\vee$-closed open semigroup U in G with $g \in U$ and $U \cap G^+ = \varnothing$.*

Proof. (1) *If.* Take X to be the class of all non-trivial lattice additive functionals on $\{G, G^+\}$ and define the mapping I as in the proof of Theorem 10. Theorem 8 completes the proof.

(2) *Only if.* See the corresponding part of the proof of Theorem 10.

The analogue of Theorem 11 for lattice-ordered linear spaces, which may readily be proved using Theorem 9, was proved by Kuller in (12) using Kakutani's results on M-spaces (11).

It may be shown that, in the notation of (7), if $\{G, G^+\}$ is a completely integrally closed partially ordered Abelian group which has an order unit, then G^+ is closed; and that any completely integrally closed lattice-ordered Abelian group with an order unit satisfies the condition of Theorem 11. These observations relate Theorems 10 and 11 to the representation theorems of Fan (6), Fleischer (7) and Stone (14).

5. *The Hahn–Banach Theorem for Abelian groups.* Halperin (8) has proved a very general Hahn–Banach type theorem: this does not seem to be widely known, so the following proof of a special case may be of interest. It is based on Bonsall's proof of the Hahn–Banach Theorem (2). We say that a real-valued functional p on a group G is *subadditive* if $p(x+y) \leqslant p(x)+p(y)$ for all $x, y \in G$.

THEOREM 12. *Let G be an Abelian group, p a subadditive functional on G and ϕ_0 an additive functional on a subgroup G_0 of G which satisfies $\phi_0(x) \leqslant p(x)$ for all $x \in G_0$. Then ϕ_0 can be extended to an additive functional ϕ on G with $\phi(x) \leqslant p(x)$ for all $x \in G$.*

Proof. Form the direct sum $G \oplus R$, R being the additive group of the real numbers, and define

$$(G \oplus R)^+ = \{(x, \lambda) \in (G \oplus R) : \lambda > p(-x) \text{ or } \lambda = 0 \text{ and } x = 0\}.$$

This gives a partially ordered Abelian group with $(0, 1)$ for an order unit. ϕ_0 can be extended to an additive functional ϕ_1 on $G_0 \oplus R$ by defining $\phi_1(x, \lambda) = \phi_0(x) + \lambda$ for all $x \in G_0$ and $\lambda \in R$. A simple calculation shows that ϕ_1 is positive on $G_0 \oplus R$ considered as a subpartially-ordered group of $\{G \oplus R, (G \oplus R)^+\}$; therefore ϕ_1 extends to a positive additive functional ϕ_2 on $\{G \oplus R, (G \oplus R)^+\}$.

Now if ϵ is any positive real number and $x \in G$ then $(-x, p(x)+\epsilon) > 0$; so

$$p(x) + \epsilon - \phi_2(x) \geqslant 0,$$

and, since ϵ is arbitrary, $p(x) \geqslant \phi_2(x)$. Thus $\phi = \phi_2 | G$ is as required.

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Note added in proof. Dr R. Giles of Glasgow University has communicated to the author a direct proof of an independently arrived at extension of Theorem 12, in which p is allowed to take $+\infty$ as a value. This will appear in Appendix B to his forthcoming book *The Mathematical Foundations of Thermodynamics* in which additive functionals on Abelian groups play an important rôle.

The following slight extension of his result can be proved like Theorem 12, using Theorem 4 instead of its Corollary:

THEOREM 12*a*. *Theorem* 12 *is still true if* p *is allowed to take* $+\infty$ *as a value provided that* $G_0 + F(p)$ *is a subgroup of* G, *where* $F(p) = \{x \in G: p(x) < \infty\}$.

This is in fact equivalent to Theorem 4: define $p(x) = +\infty$ if $x \notin G^+$ and $p(x) = 0$ if $x \in G^+$.

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