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Piecewise isometries have zero topological entropy

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Abstract. We show that piecewise isometries, i.e. non-necessarily invertible maps defined on a finite union of polytopes and coinciding with an isometry on the interior of each polytope, have zero topological entropy in any dimension. This had been conjectured by a number of authors. The proof is by an induction on the dimension and uses a device (the differential of a piecewise linear map) introduced by M. Tsujii.

1. *Introduction*

We prove that piecewise isometries in arbitrary dimension have zero topological entropy. This was proven in dimension 2 by Gutkin and Haydn [**4**] for invertible but not-necessarily affine maps, who also obtained a new proof of a result of Katok [**5**] on billiards. But in higher dimensions it was only conjectured while much weaker results were proven (see Goetz [**3**] and the references therein).

Definition 1. $(X, \mathcal{Z}, T)$ is a piecewise isometry if

(1) $X \subset \mathbb{R}^d$, $d = 1, 2, \ldots$;
(2) $\mathcal{Z}$ a finite collection of pairwise disjoint open polytopes, i.e. finite intersections of open half-spaces, such that $Y = \bigcup_{Z \in \mathcal{Z}} Z$ is a dense subset of X;
(3) $T : Y \to X$ is such that any restriction to an element of $\mathcal{Z}$ is an isometry.

Definition 2. The topological entropy of $(X, \mathcal{Z}, T)$ is:

$$h_{\text{top}}(T) = \limsup_{n \to \infty} \frac{1}{n} \log N(T^n)$$

where $N(T^n) = \#\mathcal{Z}^n$ and $\mathcal{Z}^n$ is the set of non-empty intersections $A_1 \cap T^{-1}A_2 \cap \cdots \cap T^{-n+1}A_n$ with $A_k \in \mathcal{Z}$.

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In other words, $h_{\text{top}}(T)$ is the usual topological entropy (see [**2**]) of the symbolic dynamics of $(X, \mathcal{Z}, T)$, i.e. the subshift $\left\{A \in \mathcal{Z}^{\mathbb{N}} : \bigcap_{n\geq 0} \overline{[A_0 \dots A_n]} \neq \varnothing\right\}$. It is also easy to see that $h_{\text{top}}(T)$ bounds the (ϵ, n)-entropy of T, i.e.

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log r(\epsilon, n)$$

where $r(\epsilon, n)$ is the maximal number of pairwise (ϵ, n)-separated points (x, y are (ϵ, n)-separated if $d(T^k x, T^k y) \geq \epsilon$ for some $k < n$, see [**2**]).

Any ergodic invariant measure has zero exponents and therefore, by Ruelle's inequality, zero entropy. However, for T, being only partially defined on X, there is no variational principle (see again [**2**]) and therefore the nullity of the metric entropies does not tell us anything about $h_{\text{top}}(T)$.

THEOREM 3. *The topological entropy of any piecewise isometry is zero.*

Remark 4. This theorem and its proof apply to piecewise affine contracting maps, but fails if one tries to generalize it, say, to piecewise polynomial contracting maps.

Consider the map $(x, y) \mapsto (\frac{1}{2}x, \frac{1}{2}y - \text{sgn}(y)x^2)$ on $[0, 1] \times [-1, 1]$ ($\text{sgn}(y) = -1, 0, +1$ is the sign of y). One can check that for any $x \in]0, 1]$, the vertical interval $\{x\} \times [-x^2, x^2]$ is mapped in a 2-to-1 way onto $\{\frac{1}{2}x\} \times [-\frac{1}{4}x^2, \frac{1}{4}x^2]$. It is easy to see that $h_{\text{top}}(T) = \log 2$ (the (ϵ, n)-entropy is zero, but the example could be easily modified to have that entropy also equal to $\log 2$: take two copies and declare that if $y > 0$, you go to the other copy, and otherwise you stay in the same).

The proof of this theorem is based on the analysis of the local complexity of iterates of the map. The dimension 2 case is very simple and follows in fact from ideas in Keller's analysis [**6**] of piecewise conformal and expanding maps. This case is presented in Appendix A, even though it is not logically necessary for the above theorem. The general case is obtained by an induction on the dimension d, using ideas of Tsujii [**7**], again developed for piecewise expanding maps.

2. *Differentials of piecewise isometry*

This notion is due to Tsujii. It extracts from a map its local behavior around one point. It could be called a 'canonical representative of the germ of the map at the given point'. It turns out to be technically superior to a perhaps more natural alternative, which would be to define a 'projectivization' of the map at a given point. The key reason is that there are only *finitely many distinct* differentials of a given piecewise isometry (see Lemma 10 below).

Definition 5. (Tsujii [**7**]) The *differential* of a piecewise isometry $(X, \mathcal{Z}, T)$ at the point $x \in X$ is the piecewise isometry $(X_x, \mathcal{Z}_x, T_x)$ defined as follows.

(1) $X_x = \mathbb{R}^d$.

(2) $\mathcal{Z}_x = \{Z_x : Z \in \mathcal{Z}\} \setminus \{\varnothing\}$ with

$$Z_x = \{z \in \mathbb{R}^d : \exists \epsilon_0 > 0 \text{ s.t. } \forall 0 < \epsilon < \epsilon_0, x + \epsilon \cdot z \in Z\}.$$

Set $Y_x = \bigcup_{Z\in\mathcal{Z}_x} Z$.

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(3) $T_x : Y_x \to X_x$ is defined by

$$T_x(z) = \lim_{\epsilon \to 0^+} \frac{T(x + \epsilon \cdot z) - \lim_{\delta \to 0^+} T(x + \delta \cdot z)}{\epsilon}.$$

Remark 6. If $(X, \mathcal{Z}, T)$ is the differential of some piecewise isometry, then:

(1) $X = \mathbb{R}^d$;

(2) each $Z \in \mathcal{Z}$ is a positive cone, i.e. is invariant by positive homotheties centered at $0 \in \mathbb{R}^d$.

In particular, if the point of differentiation belongs to the interior of a piece of the original map, then the differential is the usual differential: it is just a linear map. Non-trivial situations occur when the point of differentiation is on the boundary of the pieces.

Remark 7. What is happening to T around a point x coincides with what is happening to T_x around 0. More precisely, if $(X, \mathcal{Z}, T)$ is a piecewise isometry and if $(X_x, \mathcal{Z}_x, T_x)$ is its differential at some point $x \in X$ then there exists $\epsilon > 0$ such that

$$\{Z \cap B(x, \epsilon) : Z \in \mathcal{Z}\} \setminus \{\varnothing\} = \{Z_x \cap B(0, \epsilon) : Z_x \in \mathcal{Z}_x\} + x.$$

We shall iterate differentiation, i.e. we shall consider piecewise isometries of the form:

$$(X_{x_1,x_2,\ldots,x_n}, \mathcal{Z}_{x_1,x_2,\ldots,x_n}, T_{x_1,x_2,\ldots,x_n})$$

for $x_1 \in X$, $x_2, \ldots, x_n \in \mathbb{R}^d$ where $T_{x_1,x_2,\ldots,x_n} = (T_{x_1,x_2,\ldots,x_{n-1}})_{x_n}$, etc.

Observe that $T_{x,0} = T_x$. More generally, we have the following.

LEMMA 8. *For any $x_1, \ldots, x_n$, there exists $y \in X$ such that $T_{x_1 \ldots x_n} = T_y$.*

Proof. The proof is by induction on n. For $n = 1$, there is nothing to prove. If $n \geq 2$, then by induction, $T_{x_1,\ldots,x_{n-1}} = T_z$ for some $z \in X$. Thus $T_{x_1,\ldots,x_n} = T_{z,x_n}$. Therefore it coincides with T_z around x_n. Obviously it also coincides with T_z around τx_n for any $\tau > 0$. But for $\tau > 0$ small enough, τx_n is in the neighborhood of 0 where T_z coincides with (some translation) of the original map T. Thus, $T_{x_1,\ldots,x_n} = T_{z+\tau x_n}$. □

LEMMA 9. (Tsujii [7]) *All $Z \in \mathcal{Z}_{x_1,x_2,\ldots,x_n}$ are invariant by translations along the vector space generated by $x_2, x_3, \ldots, x_n$, i.e.*

$$Z + \mathrm{vect}(x_2, x_3, \ldots, x_n) = Z.$$

We say that $T_{x_2,x_3,\ldots,x_n}$ is vect$(x_2, x_3, \ldots, x_n)$*-cylindrical.*

Proof. For $n = 1$, the claim is empty. Let $n \geq 2$. Assume the claim for smaller n. Let $z \in Z_{x_1,\ldots,x_n}$, an element of $\mathcal{Z}_{x_1,\ldots,x_n}$ defined by some $Z \in \mathcal{Z}$, and $t_2, \ldots, t_n \in \mathbb{R}$. We must prove that $z + t_2x_2 + \cdots + t_nx_n \in Z_{x_1,\ldots,x_n}$. This will hold if $x_n + \tau(z + t_2x_2 + \cdots + t_nx_n) \in Z_{x_1,\ldots,x_{n-1}}$ for all $\tau > 0$ small enough. But $Z_{x_1,\ldots,x_{n-1}}$ is a positive cone and this is equivalent, for $\tau < |t_n|^{-1}$, to $x_n + [\tau/(1 + \tau t_n)](z + t_2x_2 + \cdots + t_{n-1}x_{n-1}) \in Z_{x_1,\ldots,x_{n-1}}$. By the inductive hypothesis, this is equivalent to $x_n + [\tau/(1 + \tau t_n)]z \in Z_{x_1,\ldots,x_{n-1}}$ for all $\tau > 0$ small enough. But this holds as $z \in Z_{x_1,\ldots,x_n}$. □

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In particular, if $\dim \mathrm{vect}(x_2, x_3, \ldots, x_n) = d$, then $\mathcal{Z}_{x_1,x_2,\ldots,x_n} = \{\mathbb{R}^d\}$. Also, if $\dim \mathrm{vect}(x_2, x_3, \ldots, x_n) = d - 1$, then $\mathcal{Z}_{x_1,x_2,\ldots,x_n}$ is a pair of half-spaces.

$\dim \mathrm{vect}(x_2, x_3, \ldots, x_n)$, or simply $\dim(x_2, \ldots, x_3)$, is called the *order* of the differentiation.

LEMMA 10. *Starting from a given piecewise isometry and differentiating an arbitrary number of times only finitely many maps are obtained.*

Proof. Observe that a piecewise isometry has only finitely many distinct differentials T_x, $x \in X$. In fact their number is bounded by the number of simplexes (open faces of all dimensions) involved in the complexes $\bar{Z}$ for $Z \in \mathcal{Z}$. This finiteness extends to iterated differentials by Lemma 8. □

3. *Multiplicity*

Definition 11. The *multiplicity* of $(X, \mathcal{Z}, T)$ at $x \in X$ is

$$M(T, x) = \#\{Z \in \mathcal{Z} : x \in \overline{Z}\}.$$

Set $M(T) = \sup_{x \in X} M(T, x)$.

Observe that the pieces of differentials being cones, $M(T_x) = M(T_x, 0) = N(T_x) = M(T, x)$ for any piecewise isometry.

LEMMA 12. [**1**, Proposition 4] *For a piecewise isometry T,*

$$h_{\mathrm{top}}(T) = \limsup_{n \to \infty} \frac{1}{n} \log M(T^n).$$

Proof. We give an adapted version of the proof of [**1**] for the convenience of the reader.

Let $\epsilon > 0$. Pick m such that $m^{-1} \log M(T^m) \leq \limsup_{n \to \infty} n^{-1} \log M(T^n) + \epsilon$. Pick $\delta > 0$ so small that, for all $x \in X$, $\#\{Z \in \mathcal{Z}^m : Z \cap B(x, \delta) \neq \varnothing\} \leq M(T^m)$.

Let $n(T^n)$ be the minimal cardinality of a partition of X into sets each included in an element of $\mathcal{Z}^n$ and with diameter at most δ. Of course, $N(T^n) \leq n(T^n)$.

As the sequence $n \mapsto N(T^n)$ is submultiplicative, $h_{\mathrm{top}}(T) = \inf_{n \geq 1} n^{-1} \log N(T^n) \leq (1/qm) \log n(T^{qm})$ for all $q \geq 1$.

We estimate $n(T^{qm})$. First $n(T^0)$ is finite. Then,

$$n(T^{(q+1)m}) \leq n(T^{qm}) M(T^m).$$

Indeed, consider a partition associated with $n(T^{qm})$. It is enough to subdivide each set S of that partition according to $T^{-qm}\mathcal{Z}^m$, or equivalently subdivide each $T^{qm}S$ according to $\mathcal{Z}^m$. But $\mathrm{diam}(T^{qm}S) = \mathrm{diam}(S) \leq \delta$. Hence, $T^{qm}S$ can meet only $M(T^m)$ elements of $\mathcal{Z}^m$.

Thus, $n(T^{qm}) \leq C(T^m) \cdot M(T^m)^q$ for all $q \geq 1$. Setting q so large that $\log C(T^m)/qm \leq \epsilon$, we obtain

$$h_{\mathrm{top}}(T) \leq \frac{1}{m} \log M(T^m) + \epsilon \leq \limsup_{n \to \infty} \frac{1}{n} \log M(T^n) + 2\epsilon.$$

But $\epsilon > 0$ was arbitrary. □

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PROPOSITION 13. *Let T be a piecewise isometry and $r \geq 0$. There exists $C < \infty$ such that for all $(x_1, \ldots, x_r) \in X \times (\mathbb{R}^d)^{r-1}$,*

$$M((T^n)_{x_1,\ldots,x_r}) \leq C \cdot \Big(\sup_{y_1,\ldots,y_{r+1}} M(T_{y_1,\ldots,y_{r+1}}) \Big)^n \quad \forall n \geq 0,$$

$y_1, \ldots, y_{r+1}$ ranging over $X \times (\mathbb{R}^d)^r$ under the condition, if $r \geq 2$, that

$$\dim(y_2, \ldots, y_{r+1}) = \dim(x_2, \ldots, x_r) + 1.$$

This is the analogue in our setting of Tsujii's estimate [**7**, Theorem 12]. We remark that the multiplicity of a differential of order d or $d-1$ is at most 2. The multiplicity grows with the iteration only at lower orders.

The theorem follows from this proposition.

Proof of the theorem. Consider

$$\mu(r) = \limsup_{n\to\infty} \frac{1}{n} \log \sup_{\substack{y_1,\ldots,y_r \\ \dim(y_2,\ldots,y_r)=r-1}} M((T^n)_{y_1,\ldots,y_r}).$$

By Lemma 12, $h_{\text{top}}(T) = \mu(0)$. On the other hand $\mu(d+1) = 0$, as the multiplicity of a differentiation of order d is 1.

To conclude, observe that the Proposition, applied to a sufficiently high iterate of T, implies that $\mu(r) \leq \mu(r+1)$. □

Proof of the proposition. Observe first that if $r = 0$ then

$$M(T^n) = \sup_{x\in X} M(T^n, x) \leq \Big(\sup_{x\in X} M(T, x) \Big)^n \leq \Big(\sup_{x\in X} M(T_x) \Big)^n$$

hence the proposition is obvious in this case.

So we assume $r \geq 1$ and let $x_1, \ldots, x_r \in X \times (\mathbb{R}^d)^{r-1}$. We have to bound $M((T^n)_{x_1,\ldots,x_r})$ which is equal to $M((T^n)_{x_1,\ldots,x_r}, 0)$.

Let $S(x_2, \ldots, x_r)$ be the unit sphere of $\text{vect}(x_2, \ldots, x_r)^\perp$ centered at 0. Any element of $\mathcal{Z}^n_{x_1,\ldots,x_r}$ is $\text{vect}(x_2, \ldots, x_r)$-cylindrical and therefore must meet $S(x_2, \ldots, x_r)$. Hence it is sufficient to bound the minimal cardinality $n(\mathcal{Z}^n_{x_1,\ldots,x_r}, \epsilon)$ of a cover of $S(x_2, \ldots, x_r)$ by subsets included in an element of $\mathcal{Z}^n_{x_1,\ldots,x_r}$ and with diameter at most $\epsilon > 0$, for some $\epsilon > 0$.

Observe that

$$\mathcal{Z}^{n+1}_{x_1,\ldots,x_r} = \{Z_{x_1,\ldots,x_r} \cap T_Z^{n'-1} Z'_{y_1,\ldots,y_r} \neq \varnothing : Z_{x_1,\ldots,x_r} \in \mathcal{Z}^n_{x_1,\ldots,x_r}, \\ y_1 = T^n_Z(x_1),\ y_2 = T^{n'}_Z(x_2), \ldots, Z'_{y_1,\ldots,y_r} \in \mathcal{Z}_{y_1,\ldots,y_r}\}$$

where $Z \in \mathcal{Z}^n$ corresponds to $Z_{x_1,\ldots,x_r}$ and $T^{n'}_Z$ is the linear part of T^n_Z. Therefore

$$n(\mathcal{Z}^{n+1}_{x_1,\ldots,x_r}, \epsilon) \leq n(\mathcal{Z}^n_{x_1,\ldots,x_r}, \epsilon) \cdot \sup_{y_1,\ldots,y_r} m(\mathcal{Z}_{y_1,\ldots,y_r}, \epsilon)$$

where $y_1, \ldots, y_r$ ranges over $X \times (\mathbb{R}^d)^{r-1}$ under the condition $\dim(y_2, \ldots, y_r) = \dim(x_2, \ldots, x_r)$ and $m(\mathcal{Z}_{y_1,\ldots,y_r}, \epsilon)$ is the maximal number of elements of $\mathcal{Z}_{y_1,\ldots,y_r}$ that a subset of $S(y_2, \ldots, y_r)$ with diameter ϵ can meet.

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Clearly

$$n(\mathcal{Z}_{x_1,\dots,x_r},\epsilon) \le \left(\frac{2\sqrt{d}}{\epsilon}\right)^d \cdot \#\mathcal{Z}.$$

Observe also that $m(\mathcal{Z}_{y_1,\dots,y_r},\epsilon) \le \sup_{y\in S(y_2,\dots,y_r)} \#\{Z \in \mathcal{Z}_{y_1,\dots,y_r} : Z \cap B(y,\epsilon) \neq \varnothing\}$.

Thus, the proposition will follow from the following.

CLAIM. *There exists* $\epsilon > 0$ *depending only on* T *such that, for all* $(y_1,\dots,y_r)$ *in* $X \times (\mathbb{R}^d)^{r-1}$, *all* $y \in S(y_2,\dots,y_r)$, *there exists* $y' \in \mathbb{R}^d$ *such that* $\dim(y_2,\dots,y_r,y') = 1 + \dim(y_2,\dots,y_r)$ *satisfying*

$$\#\{Z \in \mathcal{Z}_{y_1,\dots,y_r} : Z \cap B(y,\epsilon) \neq \varnothing\} \le M(T_{y_1,\dots,y_r,y'}). \tag{1}$$

By Remark 7, for each point $y \in S(y_2,\dots,y_r)$, there exists $\epsilon(y) > 0$ such that $T_{y_1,\dots,y_r}$ coincides with $T_{y_1,\dots,y_r,y}$ on $B(y,\epsilon(y))$.

By compactness of $S(y_2,\dots,y_r)$, there exist finitely many points, say $y^1,\dots,y^p$, in $S(y_2,\dots,y_r)$ such that $S(y_2,\dots,y_r)$ is covered by the balls $B(y^i,\epsilon(y^i))$, $i = 1,\dots,p$. Take $\epsilon > 0$ a Lebesgue number for this cover (i.e. such that any ϵ-ball centered at some point of $S(y_2,\dots,y_r)$ is contained in some $B(y^i,\epsilon(y^i))$). Such an ϵ satisfies (1) for all $y \in S(y_2,\dots,y_r)$, $(y_1,\dots,y_r)$ being fixed. We are going to see that one can chose $\epsilon > 0$ uniformly.

Observe that the previous analysis generalizes immediately to a given differential Δ (instead of $T_{y_1,\dots,y_r}$) and a vector space V of given dimension such that Δ is V-cylindrical (instead of vect$(y_2,\dots,y_r)$-cylindrical). Now, if V is close to vect$(y_2,\dots,y_r)$, then (1) also holds, with the same y' values, for $z_1,\dots,z_r$ instead of $y_1,\dots,y_r$ with a slightly smaller $\epsilon > 0$.

Thus, the best $\epsilon > 0$ is a lower semi-continuous function of the vector space V. As the set of vector subspaces of $\mathbb{R}^d$ such that Δ is V-cylindrical is compact, ϵ is positively lower-bounded for all such vector spaces V.

Thus, $\epsilon > 0$ can be chosen uniformly for a given differential. But there are only finitely many differentials of T (Lemma 10), so one can find an $\epsilon > 0$ which is adequate for *all* $y_1,\dots,y_r$. Claim (1) and therefore the proposition follow immediately. □

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A. *Appendix. The two-dimensional case*

By Lemma 12, we only need to bound $M(T^n)$ (see Definition 11). Observe that $M(T^n,x)$ can be interpreted as the number of arcs into which a small circle around x is cut by $\mathcal{Z}^n$. We call these arcs the n-arcs.

Observe that each n-arc around x is sent by the piece of T^n which contains it to a small arc around some other point y.

As $\mathcal{Z}^{n+1}$ is the intersection of $\mathcal{Z}^n$ with the pullback of $\mathcal{Z}$ by T^n, the $(n+1)$-arcs around x are the intersection of the n-arcs with the pullbacks of the 1-arcs around the T^n-images

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of x (given one n-arc, it defines one piece of T^n and therefore one image of x). Therefore, when n is increased by one, the number of n-arcs is multiplied at most by the maximal number of arcs cut by $\mathcal{Z}$ around a point that the T^n-image of an n-arc can meet.

Now, if one starts with very small arcs, they will only get smaller as one cuts them by the partitions $\mathcal{Z}^n$. As T is a piecewise isometry, the T^n-images of the arcs cut by $\mathcal{Z}^n$ will also be very small. One can assume that they define angles which are smaller than the minimal angle at any vertex of $\mathcal{Z}$. Therefore, a very small n-arc gives rise to at most 2 $(n+1)$-arcs.

We have shown that the number of n-arcs around any point grows like 2^n. Replacing T by a high-order iterate, we obtain an arbitrarily small exponential growth of the number of n-arcs and therefore of $M(T^n)$. This proves the theorem in dimension 2. □

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