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A STAR IS BORN: EXPLOSIVE CRUMP–MODE–JAGERS BRANCHING PROCESSES

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Abstract

We study a family of Crump–Mode–Jagers branching processes in a random environment that explode, i.e. that grow infinitely large in finite time with positive probability. Building on recent work of Iyer and the author ('On the structure of genealogical trees associated with explosive Crump–Mode–Jagers branching processes', arXiv:2311.14664, 2023), we weaken certain assumptions required to prove that the branching process, at the time of explosion, contains a (unique) individual with infinite offspring. We then apply these results to *super-linear preferential attachment models*. In particular, we fill gaps in some of the cases analysed in Appendix A of the work of Iyer and the author and study a large range of previously unattainable cases.

Keywords: Explosive Crump–Mode–Jagers branching processes; super-linear preferential attachment; preferential attachment with fitness; random recursive trees; condensation

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1. Introduction

In a Crump–Mode–Jagers (CMJ) branching process (named after [4, 5, 12]), an ancestral *root* individual produces offspring according to a collection of points on the non-negative real line. Each individual 'born' produces offspring according to an identically distributed collection of points, translated by its birth time. One is generally interested in properties of the population as a function of time. Classical work from the 1970s and 1980s related to this model generally deals with the *Malthusian case*, which, informally, refers to the fact that the population grows exponentially in time; see, e.g., [3, 13] and the references therein for an overview of classical CMJ theory.

Far fewer results exist for CMJ branching processes when a Malthusian parameter does not exist. One family of CMJ branching processes for which this is the case is the family of *explosive* CMJ branching processes. Here, the total progeny of the branching process grows infinitely large in *finite* time with positive probability. Early work on such CMJ branching processes by Sevast'yanov [19, 20], Grey [6], Grishechkin [7], and Vatutin [21] concerns necessary and sufficient conditions for (non-)explosion of Bellman–Harris processes

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(a special type of CMJ branching process). More recent work on this topic distinguishes between processes where individuals produce finitely many offspring in finite time almost surely and processes where individuals can produce infinite offspring themselves in finite time. In the former case, Komjáthy [14] provides general criteria for the explosion of CMJ branching processes in terms of solutions of a functional fixed point equation, and also extends the necessary and sufficient criteria for explosion in branching random walks in [1]. In the latter case, Oliveira and Spencer [16], and, more recently, Iyer [10] study a particular family of explosive CMJ branching processes with exponential inter-birth times, and Sagitov and Lindo [17, 18] consider Bellman–Harris processes where individuals, on death, may produce infinite offspring.

In a recent work, Iyer and the author [11] investigate a family of explosive CMJ branching processes in a *random environment*. Here, each individual born in the branching process is assigned an independent identically distributed (i.i.d.) random weight, and the distribution of the point process that governs the offspring of the individual depends on its weight. Sufficient conditions for the emergence of *infinite stars* (individuals that produce infinite offspring) and *infinite paths* (an infinite lineage of descendants) in the branching process, stopped at the (finite) time of explosion, are formulated in a general set-up. Other sufficient criteria for explosion for CMJ processes in a random environment are also studied in [9].

1.1. Overview of our contribution

In this paper, we study the emergence of *infinite stars* in explosive CMJ processes in a random environment. In particular, we weaken certain assumptions provided in previous work of Iyer and the author [11], under which we prove that an infinite star appears almost surely.

We use these results in an application to *super-linear preferential attachment* trees with fitness. This model consists of a sequence of discrete-time trees in which vertices are assigned vertex-weights and arrive one by one. A new vertex connects to a random vertex in the tree, selected with a preference for vertices with high degree or vertex-weight, or both. We largely extend the class of models for which such results can be proved. This significantly improves on the results in [11], and also adds to the recent work of Iyer [10] on persistence in (super-linear) preferential attachment trees (without fitness).

1.2. Model definition

We consider *individuals* in the process as being labelled by elements of the infinite *Ulam–Harris* tree $\mathcal{U}_\infty := \bigcup_{n\geq 0} \mathbb{N}^n$; where $\mathbb{N}^0 := \{\varnothing\}$ contains a single element, which we call the *root*. We denote elements $u \in \mathcal{U}_\infty$ as tuples, so that if $u = (u_1, \ldots, u_k) \in \mathbb{N}^k$, $k \geq 1$, we write $u = u_1 \cdots u_k$. An individual $u = u_1 u_2 \cdots u_k$ is to be interpreted recursively as the u_kth child of the individual $u_1 \cdots u_{k-1}$; for example, $1, 2, \ldots$ represent the offspring of $\varnothing$. Further, for individuals $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_\ell)$, we let $uv = (u_1, \ldots, u_k, v_1, \ldots, v_\ell)$ denote their concatenation. Suppose that $(\Omega, \Sigma, \mathrm{P})$ is a complete probability space and $(S, \mathcal{S})$ is a measure space. We also equip $\mathcal{U}_\infty$ with the sigma algebra generated by singleton sets. Then we fix random mappings $X : \Omega \times \mathcal{U}_\infty \to [0, \infty]$, $W : \Omega \times \mathcal{U}_\infty \to S$, and define $(X, W) : \Omega \times \mathcal{U}_\infty \to [0, \infty] \times S$, so that $(\omega, u) \mapsto ((X(u))(\omega), W_u(\omega))$. In general, for $u \in \mathcal{U}_\infty$ and $j \in \mathbb{N}$, we interpret W_u as a 'weight' associated with u, and $X(uj)$ the waiting time or *inter-birth* times between the birth of the $(j-1)$th and jth *child* of u.

We use the values of X to associate *birth times* $\mathcal{B}(u)$ to individuals $u \in \mathcal{U}_\infty$. In particular, we define $\mathcal{B} : \Omega \times \mathcal{U}_\infty \to [0, \infty]$ recursively, as

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$$\mathcal{B}(\varnothing) := 0 \quad \text{and,} \quad \text{for } u \in \mathcal{U}_\infty,\ i \in \mathbb{N}, \quad \mathcal{B}(ui) := \mathcal{B}(u) + \sum_{j=1}^{i} X(uj).$$

We note that a value of $X(ui) = \infty$ indicates that the individual u has stopped producing offspring, and produces at most $i-1$ children.

We introduce some notation related to elements $u \in \mathcal{U}_\infty$. We use $|\cdot|$ to measure the *length* of a tuple u, so that $|u| = 0$ when $u = \varnothing$ and $|u| = k$ when $u = u_1 \cdots u_k$. If, for some $x \in \mathcal{U}_\infty$, we have $x = uv$, we say u is an *ancestor* of x. Further, given $\ell \le |u|$, we write $u_{|_\ell} := u_1 \cdots u_\ell$ for the different ancestors of u. We equip $\mathcal{U}_\infty$ with the lexicographic total order $\le_L$. Given elements u, v, we say $u \le_L v$ if either u is an ancestor of v, or $u_\ell < v_\ell$ where $\ell = \min\{i \in \mathbb{N} : u_i \ne v_i\}$. For $u \in \mathcal{U}_\infty$, we let $\mathcal{P}_i(u)$ denote the time, after the *birth* of u, required for u to produce i offspring. That is,

$$\mathcal{P}_i(u) := \sum_{j=1}^{i} X(uj) \quad \text{and} \quad \mathcal{P}(u) := \mathcal{P}_\infty(u) = \sum_{j=1}^{\infty} X(uj). \tag{1.1}$$

We use the notation $\mathcal{P}_i$ and $\mathcal{P}$ to denote i.i.d. copies of $\mathcal{P}_i(u)$ and $\mathcal{P}(u)$, respectively.

For $t \ge 0$, we set $\mathscr{T}_t := \{x \in \mathcal{U}_\infty : \mathcal{B}(x) \le t\}$ and identify $\mathscr{T}_t$ as the *genealogical tree* of individuals with birth time at most t. For a given choice of X, W, we say $(\mathscr{T}_t)_{t\ge 0}$ is the genealogical tree process associated with an (X, W)-*CMJ* branching process; often, we refer to $(\mathscr{T}_t)_{t\ge 0}$ directly as an (X, W)-CMJ branching process, viewed as a stochastic process in t.

With regards to the process $(\mathscr{T}_t)_{t\ge 0}$, we define the stopping times $(\tau_k)_{k\in\mathbb{N}_0}$ such that

$$\tau_k := \inf\{t \ge 0 : |\mathscr{T}_t| \ge k\},$$

where we adopt the convention that the infimum of the empty set is ∞. One readily verifies that $(|\mathscr{T}_t|)_{t\ge 0}$ is right-continuous, and thus $|\mathscr{T}_{\tau_k}| \ge k$ (when $\tau_k < \infty$). For each $k \in \mathbb{N}$, we define the tree $\mathcal{T}_k$ as the tree consisting of the first k individuals in $\mathscr{T}_{\tau_k}$ ordered by birth time, breaking ties lexicographically. We call

$$\tau_\infty := \lim_{k\to\infty} \tau_k$$

the *explosion time* of the process, at which the branching process has grown infinitely large (given the process survives). We also define the tree $\mathcal{T}_\infty := \bigcup_{k=1}^{\infty} \mathcal{T}_k$.

Remark 1. With the more commonly used notation for CMJ branching processes, we assign a point process (denoted $\xi^{(u)}$) to each $u \in \mathcal{U}_\infty$, and refer to the points $\sigma_1^{(u)} \le \sigma_2^{(u)}, \ldots$ associated with this point process (in the notation used here, $\mathcal{B}(u1)$, $\mathcal{B}(u2)$, ...). We do not use this framework here, because this requires us to be able to write the measure $\xi^{(u)} = \sum_{i=1}^{\infty} \delta_{\sigma_i}$, which requires us to impose σ-finiteness assumptions on the point process (see, for example, [15, Corollary 6.5]). This σ-finiteness is implied by the classical Malthusian condition but, in this general setting, we believe that it is easier to have a framework where we can directly refer to the points $\mathcal{B}(u1)$, $\mathcal{B}(u2)$, ...

1.3. Notation

Throughout the paper we use the following notation. We let $\mathbb{N} := \{1, 2, \ldots\}$ and set $\mathbb{N}_0 := \{0, 1, \ldots\}$ and $\mathbb{R}_+ := [0, \infty)$. For $n \in \mathbb{N}$, we set $[n] := \{1, \ldots, n\}$. For $x \in \mathbb{R}$, we let $\lceil x \rceil := \inf\{n \in \mathbb{Z} : n \ge x\}$ and $\lfloor x \rfloor := \sup\{n \in \mathbb{Z} : n \le x\}$. For sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, such that b_n is positive for all n, we say that $a_n = o(b_n)$, $a_n = \mathcal{O}(b_n)$, and $a_n = \Theta(b_n)$ if $\lim_{n\to\infty} a_n/b_n = 0$, if there exists a constant $C > 0$ such that $|a_n| \le Cb_n$ for all $n \in \mathbb{N}$, and if $a_n = \mathcal{O}(b_n)$ and

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$b_n = \mathcal{O}(a_n)$, respectively. For random variables X, $(X_n)_{n\in\mathbb{N}}$, we let $X_n \xrightarrow{\text{d}} X$, $X_n \xrightarrow{\text{P}} X$, and $X_n \xrightarrow{\text{a.s.}} X$ denote convergence in distribution, probability, and almost sure convergence of X_n to X, respectively. For a non-negative real-valued random variable X and $\lambda > 0$, we let

$$\mathcal{M}_\lambda(X) := \mathrm{E}[\exp(\lambda X)] \quad \text{and} \quad \mathcal{L}_\lambda(X) := \mathrm{E}[\exp(-\lambda X)].$$

Finally, for real-valued random variables X, Y, we say that $X \preceq Y$ if Y stochastically dominates X.

1.4. Structure of the paper

We present the main results in Section 2 and discuss applications in Section 2.3. We prove the main results and applications in Section 3. Section 4 finally investigates a number of examples.

2. Results

Throughout this paper, we make the following general assumptions, which are common when studying (explosive) (X, W)-CMJ processes. We assume that the pairs $((X(uj))_{j\in\mathbb{N}}, W_u)$ with $u \in \mathcal{U}_\infty$ are i.i.d. For a given $w \in S$, we let $(X_w(uj))_{j\in\mathbb{N}}$ denote a sequence $(X(uj))_{j\in\mathbb{N}}$, conditionally on the weight $W_u = w$. We also assume throughout that, for any $w \in S$ and any $u \in \mathcal{U}_\infty$, the random variables $(X_w(ui))_{i\in\mathbb{N}}$ are mutually independent.

2.1. Sufficient criteria for infinite star in CMJ branching processes in random environment

Our main interest and aim is to understand *how* an explosive CMJ branching process explodes. We determine sufficient conditions under which the branching process explodes due to an individual giving birth to an infinite number of children in finite time (an *infinite star*). The assumptions are as follows.

Assumption 1.

(i) *There exist positive random variables $(Y_n)_{n\in\mathbb{N}_0}$ with finite mean such that, for any $w \in S$ and all $n \in \mathbb{N}_0$,*

$$\sum_{i=n+1}^{\infty} X_w(i) \preceq Y_n. \tag{2.1}$$

(ii) *There exists an increasing sequence $(\lambda_n)_{n\in\mathbb{N}} \subset (0, \infty)$ such that $\lim_{n\to\infty} \lambda_n = \infty$ and*

$$\sum_{n=1}^{\infty} \mathcal{M}_{\lambda_n}(Y_n)\mathcal{L}_{\lambda_n}(\mathcal{P}_n(\varnothing)) < \infty.$$

(iii) *We have $\mathrm{E}[\sup\{k : X(1) = \cdots = X(k) = 0\}] < 1$. In addition, for each $n \in \mathbb{N}$, almost surely,*

$$\sum_{i=n+1}^{\infty} X(i) > 0.$$

Remark 2. Note that under Condition (i) of Assumption 1, we have $\mathrm{P}(\tau_\infty < \infty) = 1$, since, for example, the root $\varnothing$ produces an infinite number of children in finite time, almost surely.

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Remark 3. In Assumption 1, we can consider Condition (i) as a uniform explosivity condition: regardless of its weight, the distribution of the time until an individual produces infinite offspring, after having already produced n children, is dominated by Y_n. This assumption is essential to work around dependencies that arise because of the random environment (i.e. the weights of individuals).

Condition *(ii)* is used in the proof of Proposition 1, and provides an upper bound for the expected number of children that produce infinite offspring before their parent does.

Condition *(iii)* is a technical assumption, necessary to rule out certain trivial cases. Indeed, if, for example, $\mathrm{E}\left[\sup\{k : X(k) = 0\}\right] > 1$, the tree consisting of all the individuals born *instantaneously* at time 0 is a supercritical Bienaymé–Galton–Watson branching process. Hence, with positive probability, this tree is infinitely large, whilst it may not contain an individual with infinite offspring.

Remark 4. Condition (ii) weakens and combines two conditions in Assumption 2.2 in [11]. There, it was assumed, with $\lambda_n := c\mathrm{E}\,[Y_n]^{-1}$ and $c < 1$, that λ_n increases and diverges, and that

$$\limsup_{n\to\infty} \mathcal{M}_{\lambda_n}(Y_n) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \mathcal{L}_{\lambda_n}\left(\mathcal{P}_n(\varnothing)\right) < \infty.$$

It is clear that these assumptions imply Condition (ii).

We then have our main result.

Theorem 1 (Infinite star). *Under Assumption 1, the infinite tree $\mathcal{T}_\infty$ almost surely contains a node of infinite degree (an infinite star).*

2.2. Application

We apply the result of Theorem 1 when the inter-birth times are *exponentially* distributed. In particular, this allows us to relate the results for (X, W)-CMJ branching processes to a family of recursively grown discrete trees, known as *super-linear* preferential attachment trees with *fitness*, introduced in [11]. This is a sequence of trees where vertices are introduced one by one and are assigned vertex-weights (fitness values). When a new vertex is introduced, it connects to one of the vertices already in the tree, where vertices with high degree or high fitness are more likely to make connections with new vertices.

We generally consider trees as being rooted, with edges directed away from the root; hence, the number of 'children' of a node corresponds to its *out-degree*. More precisely, given a vertex labelled v in a directed tree T, we let $\deg^+ v, T$ denote its out-degree in T. We now define the preferential attachment with fitness model.

Definition 1. Suppose that $(W_i)_{i\in\mathbb{N}}$ are i.i.d. copies of a random variable W that takes values in S, and let $f\colon \mathbb{N}_0 \times S \to (0, \infty)$ denote the fitness function. A preferential attachment tree with fitness is the sequence of random trees $(T_i)_{i\in\mathbb{N}}$ such that T_1 consists of a single node 1 with weight W_1; for $n \geq 2$, T_n is constructed, conditionally on T_{n-1}, as follows.

(i) Sample a vertex $j \in T_{n-1}$ with probability

$$\frac{f(\deg^+ j, T_{n-1}, W_j)}{\sum_{i=1}^{n-1} f(\deg^+ i, T_{n-1}, W_i)}.$$

(ii) Connect j with an edge directed outwards to a new vertex n with weight W_n.

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The correspondence between preferential attachment trees with fitness and the (X, W)-CMJ process with exponential inter-birth times is as follows. For each $u \in \mathcal{U}_\infty$, set $X(ui) \sim \text{Exp}(f(i-1, W_u))$ for $i \in \mathbb{N}$. The trees $(\mathcal{T}_i)_{i\in\mathbb{N}}$ associated with the (X, W)-CMJ process then satisfy

$$\{T_i : i \in \mathbb{N}\} \stackrel{d}{=} \{\mathcal{T}_i : i \in \mathbb{N}\}.$$

In addition, with $T_\infty := \cup_{n=1}^\infty T_n$, we also have $T_\infty \stackrel{d}{=} \mathcal{T}_\infty$. As a result, the structural properties of $\mathcal{T}_\infty$ can be translated to T_∞. The equality in distribution is a consequence of the memory-less property and the fact that the minimum of exponential random variables is also exponentially distributed, with a rate given by the sum of the rates of the corresponding variables; see, for example, [8, Section 2.1]. The use of continuous-time embeddings of combinatorial processes was pioneered by Athreya and Karlin [2].

We require the following assumptions on the fitness function f:

$$\exists w^* \in S : \forall w \in S, j \in \mathbb{N} : f(j, w) \geq f(j, w^*) \quad \text{and} \quad \sum_{j=0}^{\infty} \frac{1}{f(j, w^*)} < \infty. \tag{w*}$$

That is, there exists a minimizer $w^* \in S$ that, uniformly in $j \in \mathbb{N}$, minimizes $f(j, \cdot)$, and the reciprocals of $f(j, w^*)$ are summable. Whilst the latter condition is necessary to have an (X, W)-CMJ branching process where individuals can produce an infinite number of children in finite time (i.e. where an infinite star can appear), the former assumption is to avoid certain technicalities only. We also define the quantities

$$\mu_n^w := \sum_{i=n}^{\infty} \frac{1}{f(i, w)}, \quad w \in \mathbb{R}_+, \ n \in \mathbb{N}, \quad \text{and set } \mu_n := \mu_n^{w^*}. \tag{2.2}$$

Define μ_x for $x \in \mathbb{R}_+ \setminus \mathbb{N}_0$ by linear interpolation. We then have the following corollary.

Corollary 1 (Infinite star in super-linear preferential attachment trees). *Let $(T_i)_{i\in\mathbb{N}}$ be a preferential attachment tree with fitness function f that satisfies Assumption (w*). Let $(\lambda_n)_{n\in\mathbb{N}} \subset [0, \infty)$ be increasing and tend to infinity with n, such that there exists $N \in \mathbb{N}$ with $\lambda_n < f(i, w^*)$ for all $i \geq n \geq N$. If*

$$\sum_{n=N}^{\infty} \left(\prod_{i=n}^{\infty} \frac{f(i, w^*)}{f(i, w^*) - \lambda_n} \right) \mathrm{E}\left[\prod_{i=0}^{n-1} \frac{f(i, W)}{f(i, W) + \lambda_n} \right] < \infty, \tag{2.3}$$

then the tree T_∞ contains a unique node of infinite degree and no infinite path, almost surely.

Remark 5. Though we present a corollary for the specific choice of exponentially distributed inter-birth times, similar results can be proved to hold when considering other distributions for the inter-birth times (e.g. beta, gamma, Rayleigh distributions).

The corollary follows by showing that the conditions in Assumption 1 are met. It is readily verified that Condition (iii) is satisfied, as the exponential distribution does not have an atom at zero. Setting

$$Y_n := \sum_{i=n+1}^{\infty} \widetilde{X}_{w^*}(i), \tag{2.4}$$

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where $\widetilde{X}_{w^*}(i)$ is an independent copy of $X_{w^*}(i)$, and applying Assumption (w*), Condition (i) follows. Finally, Condition (ii) is equivalent to (2.3) when the inter-birth times are exponentially distributed and Y_n is as in (2.4). The uniqueness of the node of infinite degree and absence of an infinite path follows from [11, Theorem 2.12] and is true, in general, for inter-birth time distributions that have no atoms.

We juxtapose Corollary 1 with the following result from [11].

Theorem 2 ([11, Theorem 3.4].) *Recall μ_n^w from (2.2). Recall μ_n^w from (2.2). If, for some $c > 1$ and all $w \geq 0$, we have*

$$\sum_{n=1}^{\infty} \mathrm{E}\Big[\prod_{i=0}^{\infty} \frac{f(i, W)}{f(i, W) + c(\mu_n^w)^{-1} \log n} \Big] = \infty,$$

then T_∞ contains a unique infinite path and no node of infinite degree, almost surely.

In the following subsection, we provide a range of examples for which the conditions in Corollary 1 can be satisfied, for which we use $\lambda_n = \delta \mu_n^{-1} \log n$, where $\delta > 0$ is a sufficiently small constant and μ_n is as in (2.2). For this choice of λ_n, we see that Corollary 1 and Theorem 2 are close to being converse results. Indeed, we confirm this in a range of examples in the following subsection.

2.2.1. *Super-linear preferential attachment with fitness.* We proceed by studying a family of preferential attachment trees (with fitness) for which we can verify the conditions provided in Corollary 1. In doing so, we extend the number of models for which it is known that T_∞ contains a unique node of infinite degree from a handful of examples to larger, but more particular, class. This class consists of a family of *multiplicative* fitness functions. That is, vertex-weights take values in $S = \mathbb{R}_+$, and $f(i, w) = g(w)s(i)$ for some functions $g\colon \mathbb{R}_+ \to (0, \infty)$ and $s\colon \mathbb{N}_0 \to (0, \infty)$. To satisfy Assumption (w*), we assume that

$$\exists w^* \in \mathbb{R}_+ \ \forall w \in \mathbb{R}_+ : g(w) \geq g(w^*) > 0 \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{1}{s(i)} < \infty. \tag{2.5}$$

Our main interest is in functions s that grow *barely faster than linear*, for which the summability condition in (2.5) is only just satisfied. For example, $s(n) = n \log (n + 2) \log \log (n + 3)^\sigma$, with $\sigma > 1$. Earlier work by Iyer and the author in [11] is only able to deal with cases for which $s(n)$ grows fast enough, that is, faster than $n(\log n)^\alpha$ with $\alpha > 2$. Here, we extend this to a much wider range of functions that grow more slowly.

We introduce the following assumption on the function s.

Assumption 2. *Suppose s satisfies (2.5). Moreover, suppose there exist $\beta \in (0, 1)$ and $p \in (1, 1 + \beta)$ such that*

$$\lim_{n\to\infty} \frac{s(n)}{n^\beta} = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{s(n)}{n^p} = 0,$$

and there exist $C > 0$ and $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$s(n) \leq C \frac{n^{1+\beta-p}}{\log n} \inf_{i \geq n} s(i).$$

Remark 6. Though the first limit in Assumption 2 may seem unnecessary when s satisfies (2.5), one can construct examples of s with a sufficiently sparse summable subsequence that

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grows slowly. For example, with $\epsilon > 0$ small,

$$s(n) = \begin{cases} n^{(1+\epsilon)/2} & \text{if } \sqrt{n} \in \mathbb{N}, \\ (n+2)(\log(n+2))^2 & \text{otherwise.} \end{cases}$$

Though this choice of s satisfies (2.5), the subsequence $s(n_k)$ with $n_k = k^2$ grows relatively slowly compared with the values s(n) when n is not a perfect square, so that the first limit in Assumption 2 is satisfied only for $\beta < \frac{1+\epsilon}{2}$. The inequality in Assumption 2 quantifies how much slower such sparse slowly growing subsequences are allowed to grow. Overall, Assumption 2 thus allows us to deal with some choices of s with sparse slowly growing subsequences.

We then have the following result.

Theorem 3 (Barely super-linear preferential attachment with fitness). *Suppose that $f(i, w) = g(w)s(i)$, where s satisfies Assumption 2 and g and s satisfy (2.5). Suppose there exists a sequence $(k_n)_{n\in\mathbb{N}}$ and constants $\epsilon > 0$, $\delta \in (0, \epsilon)$, and $n_0 \in \mathbb{N}$, such that both*

$$\mathrm{P}(g(W) > k_n) \le n^{-(1+\epsilon)}, \quad \text{for all } n \ge n_0,$$

and

$$\lim_{n\to\infty} \frac{\mu_{(\delta \log(n)/(\mu_n k_n))^{1/\beta}}}{\mu_n k_n} = \infty \tag{2.6}$$

are satisfied, where β is as in Assumption 2. Then, the limiting infinite tree T_∞ contains a unique infinite star and no infinite path, almost surely.

Remark 7. The conditions in Assumption 2 allow us to construct a sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lambda_n < f(i, w) = g(w)s(i)$ for all $i \ge n$ and all n large, whilst (2.6) ensures that the summability condition in (2.3) is met.

2.3. Examples

To conclude the section, we present a (non-exhaustive) list of examples of functions g and s and vertex-weight distributions, such that we can verify the conditions of Theorem 3. Without loss of generality, we can take $g(w) = w + 1$, as one can change vertex-weight distribution accordingly for other choices of g. For each example, we have the following assumptions for the function s and the tail distribution of the vertex-weights. With $\sigma > 1$, $\kappa > 0$, $\nu \in (0, 1)$, $\gamma > 1$, and $\alpha \in (1/2, 1]$,

$$\begin{aligned}
&(i) \quad s(i) = (i+1)(\log(i+2))^\sigma, && \mathrm{P}(W \ge x) = \Theta\big(\mathrm{e}^{-x^\kappa}\big), \\
&(ii) \quad s(i) = (i+1)\log(i+2)\exp((\log\log(i+3))^\nu), && \mathrm{P}(W \ge x) = \Theta\big(\exp\big(-\mathrm{e}^{(\log x)^\gamma}\big)\big), \\
&(iii) \quad s(i) = (i+1)\log(i+2)(\log\log(i+3))^\sigma, && \mathrm{P}(W \ge x) = \Theta\big(\exp\big(-\mathrm{e}^{x^\kappa}\big)\big), \\
&(iv) \quad s(i) = \begin{cases} i^\alpha & \text{if } \sqrt{i} \in \mathbb{N}, \\ (i+1)(\log(i+2))^\sigma & \text{otherwise,} \end{cases} && \mathrm{P}(W \ge x) = \Theta\big(\mathrm{e}^{-x^\kappa}\big).
\end{aligned}$$

We then have the following result.

Theorem 4. *Consider the four examples (i)–(iv). The infinite tree T_∞ almost surely contains a unique infinite star and no infinite path when*

$$(i)\ (\sigma-1)\kappa > 1; \quad (ii)\ \nu\gamma > 1; \quad (iii)\ (\sigma-1)\kappa > 1; \quad (iv)\ (\sigma-1)\kappa > 1.$$

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When these inequalities are reversed (i.e. when changing the $>$ to a $<$), almost surely, T_∞ contains a unique infinite path and no infinite star.

Remark 8. For case (i), the condition for the existence of an infinite path was already known [11]. A sufficient condition for the existence of an infinite path was $(\sigma-1)\kappa > 1+\frac{1}{\kappa}$, which we sharpen to $(\sigma-1)\kappa > 1$ here to close the gap in the phase transition. For the other cases, the conditions for the existence of both an infinite star and an infinite path are novel. For the existence of an infinite path, we verify conditions from [11].

2.3.1. *Super-linear preferential attachment without fitness.* To conclude, we consider the case $g \equiv 1$, so that $f(i, w) = s(i)$. That is, we consider a model where the evolution of the tree does not depend on the vertex-weights. Here, recent work of Iyer [10] shows that the limiting tree T_∞ almost surely contains a unique infinite star when

$$\sum_{i=0}^{\infty} \frac{1}{s(i)} < \infty \quad \text{and} \quad \exists \kappa > 0 \; \forall n \in \mathbb{N}_0 : \max_{i \le n} \frac{s(i)}{i+1} \le \kappa \frac{s(n)}{n+1}. \tag{2.7}$$

In the following result, we allow for functions s that do not meet the second condition, thus extending the family of models for which we know that a unique infinite star arises.

Corollary 2 (Barely super-linear preferential attachment). *Let s satisfy (2.5) and Assumption 2, and suppose that*

$$\lim_{n\to\infty} \frac{\mu_{(\log(n)\mu_n^{-1})^{1/\beta}}}{\mu_n} = \infty, \tag{2.8}$$

where β is as in Assumption 2. Then the limiting infinite tree T_∞ contains a unique vertex with infinite degree and no infinite path, almost surely.

Remark 9. The condition in (2.8) implies the result by applying Theorem 3 to $g \equiv 1$.

Though not entirely general, the conditions in Corollary 2 encompass a large class of functions s. The four cases in Theorem 4 (ignoring the condition on the vertex-weights) satisfy the conditions, for example. In fact, we believe that the condition in (2.8) is satisfied by *any* function s that satisfies Assumption 2. As we were unable to prove this, we included it as a condition in the Corollary (as well as in Theorem 3).

Corollary 2 provides, to some extent, more general conditions under which T_∞ contains a unique infinite star, compared with (2.7). Indeed, we can construct examples such that the second condition in (2.7) is not satisfied but for which the weaker conditions in Corollary 2 are satisfied (see Case (iv) in Section 4). At the same time, the conditions in (2.7) can deal with a range of functions s that are not within the range of Corollary 2.

3. Proofs of Main Results

This section is dedicated to proving the results presented in Section 2. In Section 3.1, we focus on the main result; Theorem 1. Section 3.2 is dedicated to proving Theorem 3.

3.1. Proof of Theorem 1

We introduce the following terminology, used in the remainder of the section. Recall $\mathcal{P}_k$ and $\mathcal{P}$ from (1.1). For $a, b \in \mathcal{U}_\infty$, we say,

$$\text{‘}a \text{ has at least } k \text{ children before } b \text{ explodes’} \quad \text{if} \quad \mathcal{B}(a) + \mathcal{P}_k(a) < \mathcal{B}(b) + \mathcal{P}(b).$$

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We also say,

$$\text{'}a \text{ explodes before all of its ancestors'} \quad \text{if} \quad \forall \ell < |a| : \mathcal{B}(a) + \mathcal{P}(a) < \mathcal{B}(a_{|\ell}) + \mathcal{P}(a_{|\ell}).$$

Finally, for $a \in \mathcal{U}_\infty$ with $|a| \geq 1$, we say that $a = a_1 \cdots a_m$ is a_1-*conservative* if, for each $j \in \{2, \ldots, m\}$, we have $a_j \leq a_1$. We then have the following result.

Lemma 1. *Under Assumption 1, there exist $\zeta < 1$ and $K = K(\zeta) > 0$ such that, for all $a_1 > K(\zeta)$ and all integers $m \in \mathbb{N}$,*

$$\sum_{\substack{a:|a|=m \\ a \text{ is } a_1\text{-conservative}}} \mathrm{P}\,(a \text{ has at least } a_1 \text{ children before } \varnothing \text{ explodes}) \leq \zeta^{m-1} \mathcal{M}_{\lambda_{a_1}}(Y_{a_1}) \mathcal{L}_{\lambda_{a_1}}(\mathcal{P}_{a_1}).$$

Proof. Suppose that $a = a_1 \cdots a_m \in \mathcal{U}_\infty$. For a to have at least a_1 children before the explosion of $\varnothing$, all the ancestors of a (including a) need to be born, and then a needs to produce a_1 children. All this needs to happen before $\varnothing$ produces infinitely many children, starting to count from its $(a_1 + 1)$st child. That is,

$$\mathrm{P}\,(a \text{ has at least } a_1 \text{ children before } \varnothing \text{ explodes}) = \mathrm{P}\left(\mathcal{P}_{a_1}(a) + \sum_{j=2}^{m} \mathcal{P}_{a_j}\left(a_{|_{j-1}}\right) \leq \sum_{k=a_1+1}^{\infty} X(k)\right). \tag{3.1}$$

Using Assumption 1(i) and a Chernoff bound, with $\lambda_{a_1} > 0$, we arrive at the upper bound:

$$\mathrm{P}\left(\mathcal{P}_{a_1}(a) + \sum_{j=2}^{m} \mathcal{P}_{a_j}\left(a_{|_{j-1}}\right) \leq Y_{a_1}\right) \leq \mathcal{M}\left(\lambda_{a_1}\left(Y_{a_1}\right)\right) \mathcal{L}_{\lambda_{a_1}}\left(\mathcal{P}_{a_1}\right) \prod_{j=2}^{m} \mathcal{L}_{\lambda_{a_1}}\left(\mathcal{P}_{a_j}\right),$$

where the last line follows from the fact that the sequence $(\mathcal{P}_j(u))_{j\in\mathbb{N}}$ is independent and distributed like $(\mathcal{P}_j(\varnothing))_{j\in\mathbb{N}}$ for any $u \in \mathcal{U}_\infty$. When we sum over the possible conservative sequences a that are a_1-conservative, each a_j takes values between 1 and a_1, for $j = 2, \ldots, m$. Thus,

$$\begin{aligned} &\sum_{\substack{a:|a|=m \\ a\ a_1\text{-conservative}}} \mathrm{P}(a \text{ has at least } a_1 \text{ children before } \varnothing \text{ explodes}) \\ &\qquad \leq \sum_{a_2=1}^{a_1} \sum_{a_3=1}^{a_1} \cdots \sum_{a_m=1}^{a_1} \mathcal{M}(\lambda_{a_1}(Y_{a_1})) \mathcal{L}_{\lambda_{a_1}}(\mathcal{P}_{a_1}) \prod_{j=2}^{m} \mathcal{L}_{\lambda_{a_1}}(\mathcal{P}_{a_j}) \\ &\qquad = \mathcal{M}(\lambda_{a_1}(Y_{a_1})) \mathcal{L}_{\lambda_{a_1}}(\mathcal{P}_{a_1}) \left(\sum_{n=1}^{a_1} \mathcal{L}_{\lambda_{a_1}}(\mathcal{P}_n)\right)^{m-1}. \end{aligned} \tag{3.2}$$

It thus remains to show that, for a_1 sufficiently large,

$$\sum_{n=1}^{a_1} \mathcal{L}_{\lambda_{a_1}}(\mathcal{P}_n) < \zeta.$$

As $\lambda_n > 0$ for all $n \in \mathbb{N}$, we have $\mathcal{M}_{\lambda_n}(Y_n) \geq 1$ for all $n \in \mathbb{N}$. Hence, by Assumption 1(ii),

$$\sum_{n=1}^{\infty} \mathcal{L}_{\lambda_n}(\mathcal{P}_n) < \infty$$

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also holds. As a result, for any $\eta > 0$ there exists $N = N(\eta) \in \mathbb{N}$ such that, for all $a_1 > N$,

$$\sum_{n=N}^{a_1} \mathcal{L}_{\lambda_{a_1}}(\mathcal{P}_n) < \sum_{n=N}^{\infty} \mathcal{L}_{\lambda_n}(\mathcal{P}_n) < \frac{\eta}{2}, \tag{3.3}$$

where the inequality uses the fact that λ_n is increasing in n. Conversely, since λ_n diverges with n, bounded convergence (bounding the integrand by 1) yields

$$\lim_{a_1 \to \infty} \sum_{n=1}^{N-1} \mathcal{L}_{\lambda_{a_1}}(\mathcal{P}_n) = \sum_{n=1}^{N-1} \mathrm{P}(\mathcal{P}_n = 0) = \sum_{n=1}^{N-1} \mathrm{P}(X(1) = \cdots = X(n) = 0). \tag{3.4}$$

We note that, by Assumption 1(iii), there exists $\xi > 0$, such that

$$\mathrm{E}[\sup\{k : X(1) = \cdots = X(k) = 0\}] = \sum_{n=1}^{\infty} \mathrm{P}(X(1) = \cdots = X(n) = 0) < 1 - \xi.$$

Hence, the right-hand side of (3.4) is at most $1 - \xi$ for any $N \in \mathbb{N}$. Thus, we first take $\eta < \xi$ and N large enough that we have the upper bound in (3.3). Then we take $K \geq N$ sufficiently large, so that, for all $a_1 \geq K$,

$$\sum_{n=1}^{N-1} \mathcal{L}_{\lambda_{a_1}}(\mathcal{P}_\ell) < 1 - \xi/2.$$

Combined, we thus arrive at

$$\sum_{n=1}^{a_1} \mathcal{L}_{\lambda_{a_1}}(\mathcal{P}_\ell) < 1 - \xi/2 =: \zeta.$$

Using this in (3.2), we conclude the proof. □

This lemma provides an upper bound for the probability of the event that a vertex a explodes before the root of $\mathcal{U}_\infty$, in the case that a is a_1-*conservative*. When a does not satisfy this condition, we can view a as a concatenation of a number of conservative sequences. That is, we write $a = \overline{b}_1 \cdots \overline{b}_\ell$, where $\overline{b}_i = b_{i,1} \ldots b_{i,m_i}$ for each $i \in [\ell]$ and for some $\ell \in \mathbb{N}$, $(m_i)_{i\in[\ell]} \in \mathbb{N}^\ell$, and $(b_{i,j})_{i\in[\ell],j\in[m_i]}$, such that $\overline{b}_i$ is $b_{i,1}$-conservative for each $i \in [\ell]$. By the independence of birth processes of distinct individuals (or, in fact, the independence of disjoint subtrees), we are able to apply Lemma 1 to each conservative sequence in the concatenation to arrive at a bound for the expected number of individuals that explode before all their ancestors.

Proposition 1. *Under Assumption 1, there exists $K' > 0$ sufficiently large, such that*

$$\mathrm{E}[|\{a \in \mathcal{U}_\infty : a_1 > K',\ a \text{ explodes before all its ancestors}\}|] < \infty.$$

Proof. As explained before the proposition statement, we think of sequences $a \in \mathcal{U}_\infty$ as a concatenation of conservative sequences. Let $a = a_1 \ldots a_m$ be a sequence of length $m \in \mathbb{N}$, and assume that there exist $k \in [m]$ and indices $1 := I_1 < I_2 < \ldots < I_k$ such that $a_1 =: a_{I_1} < a_{I_2} < \ldots < a_{I_k}$ and $a_i \leq a_{I_j}$ for all $i \in \{I_j + 1, \ldots, I_{j+1} - 1\}$ and $j \in [k]$ (with $I_{k+1} := m + 1$). That is, the I_j are the indices of the running maxima of a. We also set $I_{k+1} := m + 1$ and $a_0 = \varnothing$. We think of a as a concatenation of the conservative sequences $a_{I_j} \cdots a_{I_{j+1}-1}$, with $j \in [k]$. These

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subsequences can be seen as corresponding to an a_{I_j}-conservative individual, rooted at a_{I_j-1}, for each $j \in [k]$. We can thus apply Lemma 1 to these subsequences.

Since, by definition, we have $a_{I_{\ell+1}} > a_{I_\ell}$, applying a similar logic to (3.1), we have the inclusion

$$\begin{aligned} E_a &:= \{a \text{ explodes before any of its ancestors explodes}\} \\ &\subseteq \bigcap_{\ell=1}^{k} \{a_1 \cdots a_{I_{\ell+1}-1} \text{ gives birth to at least } a_{I_\ell} \text{ children before } a_1 \cdots a_{I_\ell - 1} \text{ explodes}\} \\ &= \bigcap_{\ell=1}^{k} \left\{ \mathcal{P}_{a_{I_\ell}}\left(a_{|I_{\ell+1}-1}\right) + \sum_{j=I_\ell}^{I_{\ell+1}-2} \mathcal{P}_{a_{j+1}}\left(a_{|j}\right) \le \sum_{i=a_{I_\ell}+1}^{\infty} X\left(a_1 \cdots a_{I_\ell - 1} i\right) \right\} =: \bigcap_{\ell=1}^{k} E_{a,\ell}. \end{aligned}$$

Now, note that the events $(E_{a,\ell}, \ell \in [k])$ are not independent, since, for a given ℓ, the term $\mathcal{P}_{a_{I_\ell}}(a_{|I_{\ell+1}-1})$ in $E_{a,\ell}$ may be correlated with the summands $X(a_1 \cdots a_{I_{\ell+1}-1} i)$ appearing in $E_{a,\ell+1}$ (via the weight $W_{a_{I_{\ell+1}-1}}$). However, as we assume that, for any $u \in \mathcal{U}_\infty$, the random variables $(X(uj))_{j\in\mathbb{N}}$, conditionally on W_u, are independent, it follows that these events are conditionally independent, given the weights of a and all its ancestors, $W_\varnothing, W_{a_1}, W_{a_1 a_2}, \ldots, W_a$. Thus,

$$\begin{aligned} &\mathrm{P}(E_a \mid W_\varnothing, W_{a_1}, W_{a_1 a_2}, \ldots, W_a) \\ &\le \prod_{\ell=1}^{k} \mathrm{P}\left(\mathcal{P}_{a_{I_\ell}}\left(a_{|I_{\ell+1}-1}\right) + \sum_{j=I_\ell}^{I_{\ell+1}-2} \mathcal{P}_{a_{j+1}}\left(a_{|j}\right) \le \sum_{i=a_{I_\ell}+1}^{\infty} X\left(a_1 \cdots a_{I_\ell - 1} i\right) \,\middle|\, W_\varnothing, W_{a_1}, W_{a_1 a_2}, \ldots, W_a \right) \\ &\overset{(2.1)}{\le} \prod_{\ell=1}^{k} \mathrm{P}\left(\mathcal{P}_{a_{I_\ell}}\left(a_{|I_{\ell+1}-1}\right) + \sum_{j=I_\ell}^{I_{\ell+1}-2} \mathcal{P}_{a_{j+1}}\left(a_{|j}\right) \le Y_{a_{I_\ell}}^{(a_1 \cdots a_{I_\ell - 1})} \,\middle|\, W_\varnothing, W_{a_1}, W_{a_1 a_2}, \ldots, W_a \right) \\ &= \prod_{\ell=1}^{k} \mathrm{P}(\widetilde{E}_{a,\ell} | W_\varnothing, W_{a_1}, W_{a_1 a_2}, \ldots, W_a), \end{aligned}$$

where each $Y_{a_{I_\ell}}^{(a_1 \cdots a_{I_\ell - 1})}$ is independent and distributed like $Y_{a_{I_\ell}}$ and does depend on the vertex-weights. Now, each of the events $\widetilde{E}_{a,\ell}$ is independent, as they each depend on different weights. Hence, so is each of the terms appearing in the product, so that taking expectations on both sides yields

$$\mathrm{P}(E_a) \le \prod_{\ell=1}^{k} \mathrm{P}\left(\widetilde{E}_{a,\ell}\right). \tag{3.5}$$

We now let $d_j := I_{j+1} - I_j - 1$ for $j \in [k-1]$ and $d_k := m - I_k$ denote the number of entries between the running maxima of a. We can then define, for $(d_j)_{j\in[k]} \in \mathbb{N}_0^k$ (with $[0] := \emptyset$),

$$\begin{aligned} &\mathscr{P}_k(a_{I_1}, a_{I_2}, \ldots, a_{I_k}, d_1, \ldots, d_k) \\ &\qquad := \{a \in \mathcal{U}_\infty : \text{ For all } j \in \{1, \ldots, k\} \text{ and all } i \in [d_j],\ a_{I_j+i} \in [a_{I_j}]\} \end{aligned}$$

as the set of all sequences a with running maxima $a_1 = a_{I_1}, \ldots, a_{I_k}$, and d_j many entries between the jth and $(j+1)$th maximum. For brevity, we omit the arguments of $\mathscr{P}_k$. We then

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write the expected value in the proposition statement as

$$\sum_{\substack{a\in\mathcal{U}_\infty\\ a_1>K'}} \mathrm{P}(E_a) = \sum_{m=1}^{\infty} \sum_{\substack{a\,:\,|a|=m\\ a_1>K'}} \mathrm{P}\,(E_a) \le \sum_{m=1}^{\infty}\sum_{k=1}^{m} \sum_{a_{I_k}>\ldots>a_{I_1}>K'} \sum_{\substack{(d_\ell)_{\ell\in[k]}\in\mathbb{N}_0^k\\ \sum_{\ell=1}^k d_\ell = m-k}} \sum_{a\in\mathscr{P}_k} \prod_{\ell=1}^{k} \mathrm{P}(\widetilde{E}_{a,\ell}). \tag{3.6}$$

In the first step, we introduce a sum over all sequence lengths m. In the second step, we furthermore sum over the number of running maxima k, the values of the running maxima $a_{I_1}, \ldots a_{I_k}$, the number of entries d_ℓ between each maxima I_ℓ and $I_{\ell+1}$ (or between I_k and m if $\ell = m$), and all sequences $a \in \mathscr{P}_k$ that admit such running maxima and inter-maxima lengths. Moreover, we use (3.5) to bound $\mathrm{P}\,(E_a)$ from above, now that we know the number of running maxima in a.

We can now take the sum over $a \in \mathscr{P}_k$ into the product, because we can decompose each sequence $a \in \mathscr{P}_k(a_{I_1}, \ldots, a_{I_k}, d_1, \ldots d_k)$ into a concatenation of sequences $a^{(1)} \ldots a^{(k)}$, with $a^{(\ell)} := a_{I_\ell} \cdots a_{I_{\ell+1}-1} \in \mathscr{P}_1(a_{I_\ell}, d_\ell)$ for each $\ell \in [k]$. This yields

$$\sum_{a\in\mathscr{P}_k} \prod_{\ell=1}^{k} \mathrm{P}(\widetilde{E}_{a,\ell}) = \prod_{\ell=1}^{k} \sum_{\substack{a^{(\ell)}:|a^{(\ell)}|=d_\ell+1\\ a^{(\ell)}\ a_{I_\ell}\text{-conservative}}} \mathrm{P}(\widetilde{E}_{a,\ell}).$$

We can then directly apply Lemma 1 to each of the sums in the product. Indeed, as $a_{I_\ell} > a_{I_1} = a_1 > K'$, we can take K' large enough to obtain, for some $\zeta < 1$, the upper bound

$$\prod_{\ell=1}^{k} \sum_{\substack{a^{(\ell)}:|a^{(\ell)}|=d_\ell+1\\ a^{(\ell)}\ a_{I_\ell}\text{-conservative}}} \mathrm{P}(\widetilde{E}_{a,\ell}) \le \prod_{\ell=1}^{k} \zeta^{d_\ell} \mathcal{M}_{\lambda_{a_{I_\ell}}}\big(Y_{a_{I_\ell}}\big)\mathcal{L}_{\lambda_{a_{I_\ell}}}\big(\mathcal{P}_{a_{I_\ell}}\big).$$

We can take out the factors ζ^{d_ℓ} and use the fact that the d_ℓ sum to $m-k$. Using this in (3.6) yields

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{k=1}^{m} \sum_{a_{I_k}>\ldots>a_{I_1}>K'} \zeta^{m-k} \sum_{\substack{(d_j)_{j\in[k]}\in\mathbb{N}_0^k\\ \sum_{\ell=1}^k d_\ell = m-k}} \prod_{\ell=1}^{k} \mathcal{M}_{\lambda_{a_{I_\ell}}}\big(Y_{a_{I_\ell}}\big)\mathcal{L}_{\lambda_{a_{I_\ell}}}\big(\mathcal{P}_{a_{I_\ell}}\big)\\
&\quad\le \sum_{m=1}^{\infty}\sum_{k=1}^{m} \binom{m-1}{k-1} \zeta^{m-k} \sum_{a_{I_1}>K'} \cdots \sum_{a_{I_k}>K'} \prod_{\ell=1}^{k} \mathcal{M}_{\lambda_{a_{I_\ell}}}\big(Y_{a_{I_\ell}}\big)\mathcal{L}_{\lambda_{a_{I_\ell}}}\big(\mathcal{P}_{a_{I_\ell}}\big)\\
&\quad= \sum_{m=1}^{\infty}\sum_{k=1}^{m} \binom{m-1}{k-1} \zeta^{m-k} \Big(\sum_{n>K'} \mathcal{M}_{\lambda_n}(Y_n)\mathcal{L}_{\lambda_n}(\mathcal{P}_n)\Big)^k.
\end{aligned}$$

By Assumption 1(ii), we can bound the innermost sum from above by ζ/M for some $M > 0$ when K' is sufficiently large, so that we obtain the upper bound

$$\sum_{m=1}^{\infty}\sum_{k=1}^{m} \binom{m-1}{k-1} \zeta^m M^{-k} = \frac{\zeta}{M} \sum_{m=1}^{\infty} \Big(\zeta\big(1+\tfrac{1}{M}\big)\Big)^{m-1} < \infty,$$

where the last step follows when M is large enough that $\zeta(1+1/M) < 1$, which holds by choosing K' sufficiently large, as follows from (3.3). □

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We then state three results, proved by Iyer and the author in [11]. First, we introduce, for $L \in \mathbb{N}$,

$$\mathcal{U}_L := \{u \in \mathcal{U}_\infty : u_i \leq L \text{ for all } i \in [|u|]\} \cup \{\varnothing\}.$$

Following [16], we say that elements of $\mathcal{U}_L$ are *L-moderate*. We then have the following result, which tells us that the subtree consisting of L-moderate individuals does not explode, almost surely.

Proposition 2 ([11].) *Let $(\mathscr{T}_t)_{t\geq 0}$ be a (X,W)-CMJ branching process satisfying Condition (iii) of Assumption 1. Then, almost surely, for all $t \in (0, \infty)$, we have $|\mathscr{T}_t \cap \mathcal{U}_L| < \infty$.*

Recall that, for an (X, W)-CMJ branching process $(\mathscr{T}_t)_{t\geq 0}$, we have

$$\tau_\infty = \lim_{k\to\infty} \tau_k = \inf\{t > 0 : |\mathscr{T}_t| = \infty\}.$$

Recall also that we have $\mathcal{T}_\infty = \bigcup_{k=1}^\infty \mathcal{T}_k = \bigcup_{k=1}^\infty \mathscr{T}_{\tau_k}$. We then have the following result.

Lemma 2 ([11].) *Let $(\mathscr{T}_t)_{t\geq 0}$ be a (X,W)-CMJ branching process satisfying Condition (iii) of Assumption 1. Then, almost surely, $\mathcal{T}_\infty = \{u \in \mathcal{U}_\infty : \mathcal{B}(u) < \tau_\infty\} \subseteq \mathscr{T}_{\tau_\infty}$.*

Finally, we relate τ_∞ to the explosion times of all individuals in $\mathcal{U}_\infty$.

Lemma 3 ([11].) *Let $(\mathscr{T}_t)_{t\geq 0}$ be a (X,W)-CMJ branching process that satisfies Assumption 1. Almost surely, $\tau_\infty = \inf_{u\in\mathcal{U}_\infty}\{\mathcal{B}(u) + \mathcal{P}(u)\}$.*

Combining these three lemmas with Proposition 1, we are ready to prove Theorem 1. Although the proof is identical to the proof of Theorem 2.5 in [11], as it uses these three results and a weaker version of Proposition 1, we include it for the article to be self-contained.

Proof of Theorem 1. Fix K' as in Proposition 1. We may view any $w \in \mathcal{U}_\infty$ as a concatenation $w = uv$, where $u \in \mathcal{U}_{K'}$ is K'-moderate, and $v = v_1 \cdots v_k$, where $v_1 > K'$ (here we also allow v to be empty, so that K'-moderate nodes w may also be interpreted as a concatenation). Now, note that (on $\mathcal{B}(u) < \infty$) the birth times $\mathcal{B}(uv) - \mathcal{B}(u) \sim \mathcal{B}(v)$, and thus, by arguments analogous to those appearing in Proposition 1, for any $u \in \mathcal{U}_\infty$ (in particular for $u \in \mathcal{U}_{K'}$),

$$\mathrm{E}\left[\left|\left\{a = a_1 \cdots a_m \in \mathcal{U}_\infty : a_1 > K', ua \text{ explodes before } ua_{|_{m-1}}, ua_{|_{m-2}}, \ldots, u\right\}\right|\right] < \infty. \quad (3.7)$$

Now, since $\tau_\infty < \infty$ almost surely, we infer from Proposition 2 with $L = K'$, that $|\{u \in \mathcal{U}_{K'} : \mathcal{B}(u) \leq \tau_\infty\}| < \infty$ almost surely. Therefore, by (3.7), the set

$$E_{\mathrm{expl}} := \{u \in \mathcal{U}_\infty : \mathcal{B}(u) \leq \tau_\infty, u \text{ explodes before all of its ancestors}\}$$

is finite almost surely. By the definition of E_{expl} and the fact that the infimum of a finite set is attained by (at least) one of its elements, Lemma 3 implies that, almost surely,

$$\exists u^* \in E_{\mathrm{expl}} : \mathcal{B}(u^*) + \mathcal{P}(u^*) = \inf_{v\in E_{\mathrm{expl}}}\{\mathcal{B}(v) + \mathcal{P}(v)\} = \inf_{u\in\mathcal{U}_\infty}\{\mathcal{B}(u) + \mathcal{P}(u)\} = \tau_\infty.$$

This implies that u^* has infinite degree in $\mathscr{T}_{\tau_\infty}$. By Condition (iii) of Assumption 1, we have $\sum_{i=n+1}^\infty X(u^*i) > 0$ almost surely, so that $\mathcal{B}(u^*i) < \tau_\infty$ for each $i \in \mathbb{N}$, almost surely. Therefore, by Lemma 2, u^* has infinite degree in $\mathcal{T}_\infty$ as well, which yields the desired result. □

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3.2. Proof of Theorem 3

Proof of Theorem 3. The desired result follows by verifying the conditions on λ_n and the vertex-weights in Corollary 1. We start by constructing a sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that there exists $N \in \mathbb{N}$ for which we have $\lambda_n < f(i, w)$, for all $i \geq n \geq N$. We recall that there exist $\epsilon > 0$, $n_0 \in \mathbb{N}$, and a sequence $(k_n)_{n\in\mathbb{N}}$ such that $\mathrm{P}(g(W) > k_n) \leq n^{-(1+\epsilon)}$, for all $n \geq n_0$. Then, let $\delta \in (0, \epsilon)$, as in the statement of Theorem 3, recall μ_n from (2.2), and set

$$\lambda_n := \delta \log(n)\mu_n^{-1}. \tag{3.8}$$

It is clear that μ_n is decreasing in n and tends to zero by (2.5), so that λ_n is increasing and diverges. Then, by Assumption 2, we fix $\eta > 0$ small and take N large so that, for all $n \geq N$,

$$\mu_n^{-1} = \left(\sum_{j=n+1}^{\infty} \frac{1}{g(w^*)s(j)}\right)^{-1} \leq g(w^*)\left(\sum_{j=n+1}^{\infty} \frac{1}{\eta j^p}\right)^{-1} = (\eta g(w^*)(p-1) + o(1))n^{p-1}. \tag{3.9}$$

At the same time, again for N large and all $n \geq N$, we have $s(n) \geq \eta^{-1}n^{\beta}$, so that

$$\begin{aligned}\lambda_n = \delta \log(n)\mu_n^{-1} &\leq (\eta\delta g(w^*)(p-1) + o(1)) \log(n)n^{p-1} \\ &\leq (\eta^2\delta g(w^*)(p-1) + o(1))\frac{\log(n)}{n^{(1+\beta)-p}}s(n).\end{aligned}$$

The final condition of Assumption 2 then implies that

$$\lambda_n \leq (\eta^2\delta C g(w^*)(p-1) + o(1)) \inf_{i\geq n} s(i),$$

for all $n \geq N$ and some large $N \in \mathbb{N}$. As we can choose η arbitrarily small, we can make the constant on the right-hand side arbitrarily small, so that

$$\lambda_n = o(\inf_{i\geq n} s(i)),$$

as desired.

We then prove (2.3). By the choice of f and assuming (without loss of generality) that $g(w^*) = 1$, we have that (2.3) is equivalent to

$$\sum_{n=N}^{\infty}\prod_{i=n}^{\infty}\left(\frac{s(i)}{s(i)-\lambda_n}\right)\mathrm{E}\left[\prod_{i=0}^{n-1}\frac{g(W)s(i)}{g(W)s(i)+\lambda_n}\right] < \infty. \tag{3.10}$$

By using that $1 - x \leq \mathrm{e}^{-x}$ for all $x \in \mathbb{R}$, we can bound each term in the sum from above by

$$\begin{aligned}&\exp\left(\lambda_n\sum_{i=n}^{\infty}\frac{1}{s(i)-\lambda_n} - \lambda_n\sum_{i=0}^{n-1}\frac{1}{k_n s(i)+\lambda_n}\right) + \exp\left(\lambda_n\sum_{i=n}^{\infty}\frac{1}{s(i)-\lambda_n}\right)\mathrm{P}(g(W) > k_n) \\ &\leq \exp\left(\lambda_n\sum_{i=n}^{\infty}\frac{1}{s(i)-\lambda_n} - \frac{\lambda_n}{k_n}\sum_{i=0}^{n-1}\frac{1}{s(i)+\lambda_n/k_n}\right) + \exp\left(\lambda_n\sum_{i=n}^{\infty}\frac{1}{s(i)-\lambda_n}\right)\frac{1}{n^{1+\epsilon}},\end{aligned} \tag{3.11}$$

where we take $N \geq n_0$ and use the fact that $\mathrm{P}(g(W) > k_n) \leq n^{-(1+\epsilon)}$ for all $n \geq N$ to arrive at the upper bound. Using $\lambda_n = o(\inf_{i\geq n} s(i))$ and $g(w^*) = 1$,

$$\sum_{i=n}^{\infty}\frac{1}{s(i)-\lambda_n} = (1 + o(1))\mu_n.$$

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Then, by the choice of λ_n in (3.8), we can write (3.11) as

$$\exp\left((\delta + o(1))\log(n) - \frac{\lambda_n}{k_n}\sum_{i=0}^{n-1}\frac{1}{s(i)+\lambda_n/k_n}\right) + n^{-(1+\epsilon-\delta+o(1))}. \tag{3.12}$$

Since $\delta < \epsilon$, the second term is summable in n. It thus remains to bound the first term.

We recall that $p \in (1, 1+\beta)$ and that $k_n \geq g(w^*) = 1$ for all n. As a result, for some constant $C_\beta > 0$ and using (3.9),

$$\left(\frac{\lambda_n}{k_n}\right)^{1/\beta} \leq \lambda_n^{1/\beta} \leq C_\beta (\log n)^{1/\beta} n^{(p-1)/\beta} = o(n).$$

We then observe that λ_n/k_n tends to infinity with n. Indeed, as $\sup_{x\geq 0}\mu_x$ is finite (where we define μ_x for non-integer x by linear interpolation), for all n large,

$$\frac{\lambda_n}{k_n} = \delta\frac{\log n}{\mu_n k_n} \geq \frac{\mu_{(\delta\log(n)/(\mu_n k_n))^{1/\beta}}}{\mu_n k_n},$$

and the right-hand side tends to infinity with n by the condition in (2.8). Further, Assumption 2 yields that $s(i) \geq Mi^\beta$ for any M large and all $i \geq I = I(M) \in \mathbb{N}$. Combined, we obtain, for all n large, that

$$s(i) \geq Mi^\beta \geq M\frac{\lambda_n}{k_n}, \quad \text{for all } i \geq \left(\frac{\lambda_n}{k_n}\right)^{1/\beta}.$$

As a result,

$$\sum_{i=0}^{n-1}\frac{1}{s(i)+\lambda_n/k_n} \geq \sum_{i=(\lambda_n/k_n)^{1/\beta}}^{n-1}\frac{1}{s(i)+\lambda_n/k_n} = \frac{\mu_{(\lambda_n/k_n)^{1/\beta}} - \mu_n}{1+1/M}.$$

Since $\lambda_n = \delta\log(n)\mu_n^{-1}$, we use this in the first term in (3.12) to obtain

$$\exp\left(\left(\delta\frac{2M+1}{M+1} + o(1)\right)\log(n) - \delta\frac{M}{M+1}\log(n)\frac{\mu_{(\lambda_n/k_n)^{1/\beta}}}{\mu_n k_n}\right).$$

By the assumption in (2.6), it follows that we can bound the entire term from above by n^{-C} for any $C > 0$, since the ratio $\mu_{(\lambda_n/k_n)^{1/\beta}}/(\mu_n k_n)$ can be bounded from below by a sufficiently large constant for all n large by (2.8). This yields (3.10) and concludes the proof. □

4. Examples

In this section, we discuss the examples that Theorem 4 deals with. The proof comes down to verifying the conditions in Theorem 3. We recall that we set $g(w) = w + 1$. In each case, we asymptotically determine μ_n, provide k_n, and show that the required assumptions are satisfied.

Proof of Theorem 4.

Infinite star. To prove there exists a unique infinite star almost surely, we verify the conditions in Theorem 3.

Case (i). It is clear that Assumption 2 is satisfied with $\beta = 1$ and any $p \in (1, 2)$. By switching from summation to integration:

$$\mu_n = \sum_{i=n}^{\infty}\frac{1}{s(i)} = \int_n^\infty \frac{1+o(1)}{x(\log x)^\sigma}\,\mathrm{d}x = \int_{\log n}^\infty \frac{1+o(1)}{y^\sigma}\,\mathrm{d}y = \big((\sigma-1)^{-1} + o(1)\big)(\log n)^{-(\sigma-1)}. \tag{4.1}$$

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We set $k_n := ((1+\epsilon)\log n)^{1/\kappa}$ with $\epsilon > 0$, and take $\delta \in (0, \epsilon)$. There exists $C_1 > 0$ such that

$$\mathrm{P}\left(g(W) > k_n\right) \leq C_1 \mathrm{e}^{-k_n^{\kappa}} = C_1 n^{-(1+\epsilon)}, \tag{4.2}$$

which is summable. Then there exists $C_2 > 0$ such that

$$\frac{\mu_{\delta \log(n)/(\mu_n k_n)}}{\mu_n k_n} = (C_2 + o(1)) \frac{(\log n)^{\sigma-1}(\log\log n)^{-(\sigma-1)}}{(\log n)^{1/\kappa}} = (\log n)^{(\sigma-1)-1/\kappa-o(1)}.$$

When $(\sigma-1)\kappa > 1$, this quantity diverges with n, so that the condition in Theorem 3 holds.

Case (ii). It is clear that Assumption 2 is satisfied with $\beta = 1$ and any $p \in (1, 2)$. By switching from summation to integration and substituting $y = (\log\log x)^{\nu}$:

$$\mu_n = \sum_{i=n}^{\infty} \frac{1}{s(i)} = \int_n^{\infty} \frac{1+o(1)}{x \log x \exp\left((\log\log x)^{\nu}\right)} \mathrm{d}x = (1+o(1)) \int_{(\log\log n)^{\nu}}^{\infty} y^{1-1/\nu} \mathrm{e}^{-y} \mathrm{d}y. \tag{4.3}$$

By using the incomplete gamma function, we arrive at $\mu_n = \exp\left(-(1+o(1))(\log\log n)^{\nu}\right)$. We set $k_n := \exp\left((\log((1+\epsilon)\log n))^{1/\gamma}\right)$ with $\epsilon > 0$, and take $\delta \in (0, \epsilon)$. There exists $C_1 > 0$ such that

$$\mathrm{P}\left(g(W) > k_n\right) \leq C_1 \exp\left(-\mathrm{e}^{(\log k_n)^{\gamma}}\right) = C_1 n^{-(1+\epsilon)}, \tag{4.4}$$

which is summable. Then

$$\frac{\mu_{\delta \log(n)/(\mu_n k_n)}}{\mu_n k_n} = \exp\left(\left[(\log\log n)^{\nu} - (\log\log n)^{1/\gamma} - \left(1 - \tfrac{1}{\gamma}\right)^{\nu} (\log\log\log n)^{\nu}\right](1+o(1))\right).$$

When $\nu\gamma > 1$, this quantity diverges with n, so that the condition in Theorem 3 holds.

Case (iii). It is clear that Assumption 2 is satisfied with $\beta = 1$ and any $p \in (1, 2)$. By switching from summation to integration and with similar steps as in (4.1):

$$\mu_n = \sum_{i=n}^{\infty} \frac{1}{s(i)} = \int_n^{\infty} \frac{1+o(1)}{x \log(x)(\log\log x)^{\sigma}} \mathrm{d}x = \left((\sigma-1)^{-1} + o(1)\right)(\log\log n)^{-(\sigma-1)}. \tag{4.5}$$

We set $k_n := (\log((1+\epsilon)\log n))^{1/\kappa}$ with $\epsilon > 0$, and take $\delta \in (0, \epsilon)$. There exists $C_1 > 0$ such that

$$\mathrm{P}\left(g(W) > k_n\right) \leq C_1 \exp\left(-\mathrm{e}^{k_n^{\kappa}}\right) = C_1 n^{-(1+\epsilon)}, \tag{4.6}$$

which is summable. Then there exists $C_2 > 0$ such that

$$\frac{\mu_{\delta \log(n)/(\mu_n k_n)}}{\mu_n k_n} = (C_2 + o(1)) \frac{(\log\log n)^{\sigma-1}(\log\log\log n)^{-(\sigma-1)}}{(\log\log n)^{1/\kappa}} = (\log\log n)^{(\sigma-1)-1/\kappa-o(1)}.$$

When $(\sigma-1)\kappa > 1$, this quantity diverges with n, so that the condition in Theorem 3 holds.

Case (iv). It is clear that the first part of Assumption 2 is satisfied for any $\beta < \alpha$ and any $p \in (1, 1+\beta)$. Since, for large n, the smallest value $s(i)$ among all $i \geq n$ is attained at $i = \lceil\sqrt{n}\rceil^2$, with value $s(i) = \lceil\sqrt{n}\rceil^{2\alpha} = (1+o(1))n^{\alpha}$, it follows that the second part of Assumption 2 is also satisfied, since $\alpha \in (\frac{1}{2}, 1]$ and we can thus choose p close enough to 1 and β close enough to α so that $1 + \beta - p > \frac{1}{2} > 1 - \alpha$. Hence, for all n large,

$$s(n) \leq (n+1)(\log(n+2))^{\sigma} \leq \frac{n^{1+\beta-p}}{\log n} n^{\alpha} = (1+o(1)) \frac{n^{1+\beta-p}}{\log n} \inf_{i \geq n} s(i).$$

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The last part of Assumption (2) is thus satisfied with $C > 1$ and N sufficiently large. Then

$$\mu_n = \sum_{i=n}^{\infty} \frac{1}{s(i)} = \sum_{\substack{i=n \\ \sqrt{i}\in\mathbb{N}}}^{\infty} \frac{1}{i^{\alpha}} + \sum_{\substack{i=n \\ \sqrt{i}\notin\mathbb{N}}}^{\infty} \frac{1}{(i+1)(\log(i+2))^{\sigma}}. \tag{4.7}$$

As $i^{\alpha} \le (i+1)(\log(i+2))^{\sigma}$ for all $i \ge n$ when n is large, since $\alpha \le 1$, we can use (4.1) to obtain the lower bound:

$$\mu_n \ge \big((\sigma-1)^{-1} + o(1)\big)(\log n)^{-(\sigma-1)}.$$

For an upper bound, we include all $i \ge n$ in the second sum in (4.7) and again use (4.1). The first sum on the right-hand side of (4.7) is then bounded from above by

$$\sum_{\substack{i=n \\ \sqrt{i}\in\mathbb{N}}} \frac{1}{i^{\alpha}} = \sum_{i=\lceil\sqrt{n}\rceil}^{\infty} i^{-2\alpha} = \frac{1+o(1)}{2\alpha-1} n^{1/2-\alpha}.$$

As a result, the upper bound matches the lower bound, regardless of the value of $\alpha \in (\frac{1}{2}, 1]$, to obtain

$$\mu_n = ((\sigma-1)^{-1} + o(1))(\log n)^{-(\sigma-1)}. \tag{4.8}$$

The remainder of the computations then follow the same approach as Case (i).

Infinite path. To prove that there exists a unique infinite path, we use [11, Lemma 6.4] to verify the conditions in Theorem 2. That is, if, for any $w \ge 0$,

$$\limsup_{n\to\infty} \frac{1}{\mu_n^w} \sum_{i=0}^{\infty} \frac{1}{f(i,k_n)} < 1,$$

where k_n is a sequence such that $\mathrm{P}(g(W) > k_n)$ is not summable, then T_{∞} contains a unique infinite path and no node of infinite degree, almost surely. As $f(i,x) = g(x)s(i) = (x+1)s(i)$, it follows from (2.5) that it suffices to show that $\mu_n k_n$ diverges.

Case (i). With μ_n as in (4.1) and $k_n := ((1-\epsilon)\log n)^{1/\kappa}$, it follows that $\mathrm{P}(g(W) > k_n)$ is not summable when the inequality in (4.2) is reversed (and using a constant $C_1' < C_1$). We then conclude that $\mu_n k_n$ diverges when $(\sigma-1)\kappa < 1$.

Case (ii). With μ_n as in (4.3) and $k_n := \exp((\log(1-\epsilon)\log n))^{1/\gamma})$, it follows that $\mathrm{P}(g(W) > k_n)$ is not summable when the inequality in (4.4) is reversed (and using a constant $C_1' < C_1$). We then conclude that $\mu_n k_n$ diverges when $\nu\gamma < 1$.

Case (iii). With μ_n as in (4.5) and $k_n := (\log((1-\epsilon)\log n))^{1/\kappa}$, it follows that $\mathrm{P}(g(W) > k_n)$ is not summable when the inequality in (4.6) is reversed (and using a constant $C_1' < C_1$). We then conclude that $\mu_n k_n$ diverges when $(\sigma-1)\kappa < 1$.

Case (iv). With μ_n as in (4.8), we can follow the same steps as in Case (i) to arrive at the desired result. □

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References

[1] Amini, O., Devroye, L., Griffiths, S. and Olver, N. (2013). On explosions in heavy-tailed branching random walks. *Ann. Probab.* **41**, 1864–1899.
[2] Athreya, K. B. and Karlin, S. (1968). Embedding of urn schemes into continuous time Markov branching processes and related limit theorems. *Ann. Math. Stat.* **39**, 1801–1817.
[3] Athreya, K. B. and Ney, P. E. (1972). *Branching Processes* (Die Grundlehren der mathematischen Wissenschaften 196). Springer, New York.
[4] Crump, K. S. and Mode, C. J. (1968). A general age-dependent branching process. I. *J. Math. Anal. Appl.* **24**, 494–508.
[5] Crump, K. S. and Mode, C. J. (1968). A general age-dependent branching process. II. *J. Math. Anal. Appl.* **25**, 8–17.
[6] Grey, D. (1974). Explosiveness of age-dependent branching processes. *Z. Wahrsch. Verw. Gebiete* **28**, 129–137.
[7] Grishechkin, S. (1987). On the regularity of branching processes with several types of particles. *Theory Probab. Appl.* **31**, 233–243.
[8] Iyer, T. (2023). Degree distributions in recursive trees with fitnesses. *Adv. Appl. Probab.* **55**, 407–443.
[9] Iyer, T. (2024). On a sufficient condition for explosion in CMJ branching processes and applications to recursive trees. *Electron. Commun. Probab.* **29**, 1–12.
[10] Iyer, T. (2024). Persistent hubs in CMJ branching processes with independent increments and preferential attachment trees. *arXiv preprint arXiv:2410.24170*.
[11] Iyer, T. and Lodewijks, B. (2023). On the structure of genealogical trees associated with explosive Crump–Mode–Jagers branching processes. *arXiv preprint arXiv:2311.14664*.
[12] Jagers, P. (1969). A general stochastic model for population development. *Scand. Actuarial J.* **1969**, 84–103.
[13] Jagers, P. (1975). *Branching Processes with Biological Applications* (Wiley Series in Probability and Mathematical Statistics—Applied Probability and Statistics). Wiley-Interscience, London.
[14] Komjáthy, J. (2016). Explosive Crump–Mode–Jagers branching processes. *arXiv preprint arXiv:1602.01657*.
[15] Last, G. and Penrose, M. (2018). *Lectures on the Poisson Process* (Institute of Mathematical Statistics Textbooks **7**). Cambridge University Press.
[16] Oliveira, R. and Spencer, J. (2005). Connectivity transitions in networks with super-linear preferential attachment. *Internet Math.* **2**, 121–163.
[17] Sagitov, S. (2017). Tail generating functions for extendable branching processes. *Stochastic Processes Appl.* **127**, 1649–1675.
[18] Sagitov, S. and Lindo, A. (2016). A special family of Galton–Watson processes with explosions. In *Branching Processes and Their Applications*, eds I. del Puerto, Springer, Cham, pp. 237–254.
[19] Sevast'yanov, B. A. (1967). On the regularity of branching processes. *Mathematical Notes of the Academy of Sciences of the USSR* **1**, 34–40.
[20] Sevast'yanov, B. A. (1970). Necessary condition for the regularity of branching processes. *Mathematical Notes of the Academy of Sciences of the USSR* **7**, 234–238.
[21] Vatutin, V. (1987). Sufficient conditions for regularity of Bellman–Harris branching processes. *Theory Probab. Appl.* **31**, 50–57.

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