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Constructing ergodic exponential maps with dense post-singular orbits

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Abstract. We construct ergodic exponential maps $f_\lambda(z) = \lambda e^z$ such that the forward orbit of the origin is dense in $\mathbb{C}$.

1. *Introduction*

1.1. *Statement of the main result.* There are many known ergodic examples in the exponential family. For instance, any post-singularly finite exponential map is ergodic. More generally, it was proved that if the ω-limit set of the post-singular orbit is a compact repeller, then the exponential map is ergodic [**3**]. For all of these examples the ω-limit sets of the post-singular orbits do not contain the origin. This is because people need the non-recurrence property to obtain some kind of expanding property which then implies the ergodicity. An interesting question is whether there exist ergodic exponential maps with recurrent post-singular orbits. In this paper we prove the following result.

MAIN THEOREM. *Let $f_\lambda(z) = \lambda e^z$. Then there exist parameters λ such that the maps $f_\lambda : \mathbb{C} \to \mathbb{C}$ are ergodic with respect to the Lebesgue measure on the complex plane and, moreover, the forward orbits of the origin are dense in $\mathbb{C}$.*

This work is largely inspired by [**4**] in which Rees established a similar result for rational maps.

THEOREM 1. (Rees [**4**]) *Let $d \geq 2$. Then there exist ergodic rational maps f of degree d such that the critical orbits of f are dense in $\widehat{\mathbb{C}}$.*

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The general idea of Rees' proof can be sketched as follows. Consider a certain one-parameter family of degree d rational maps F_λ. Let A denote the set of those parameters λ such that every critical point of F_λ is strictly pre-periodic. It was proved that the closure of A is a perfect set in $\widehat{\mathbb{C}}$. The result then follows by showing that the desired parameters consist of a residue subset of $\overline{A}$.

Our proof of the Main Theorem uses the same strategy as above. However, owing to the particular property of the exponential maps, the arguments here are quite different from those in [**4**], especially the part by which we obtain the residue property of the ergodicity. We use a rather direct measure argument instead of certain type of transfer operator used by Rees (compare §3 of this paper and [**4**, p. 316]).

1.2. *Notation and terminologies.* We use $\mathbb{C}$ to denote the complex plane. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. For $\lambda \in \mathbb{C}^*$, let $f_\lambda(z) = \lambda e^z$ and

$$P_\lambda = \overline{\{f_\lambda^k(0)\}_{k=0}^\infty}.$$

We call P_λ the post-singular set of f_λ. Let $\mathbb{C}_\lambda = \mathbb{C} \setminus P_\lambda$. Since P_λ contains at least two distinct points 0 and λ, it follows that $\mathbb{C}_\lambda$ is a hyperbolic Riemann surface.

We say that f_λ is *post-singularly finite* if P_λ is a finite set. In this case, the forward orbit of the origin eventually enters into a repelling periodic cycle and the Julia set is the whole complex plane. By [**3**, Theorem 1], f_λ acts ergodically on $\mathbb{C}$. Let

$$\Lambda = \{\lambda \in \mathbb{C}^* \mid f_\lambda \text{ is post-singularly finite}\}.$$

For $r > 0$ and $z \in \mathbb{C}$, we use $B_r(z)$ to denote the Euclidean disk with center z and radius r. For two subsets A and B of the complex plane, we use $d(A, B)$ to denote the distance between A and B with respect to the Euclidean metric. For $X \subset \mathbb{C}$, we use $\operatorname{diam}(X)$ to denote the diameter of X with respect to the Euclidean metric.

Let

$$\Gamma = \{x + iy \mid x \text{ and } y \text{ are both rational numbers}\}.$$

It follows that Γ is a dense and countable subset in $\mathbb{C}$. Let us order the elements in Γ as $\{p_1, p_2, \ldots\}$.

For a holomorphic isomorphism $f : U \to V$, we say the distortion of f on U is bounded by some constant $M \geq 1$ if $|Df(x)/Df(y)| \leq M$ for all $x, y \in U$.

The measure used throughout the paper is the usual Lebesgue measure on the complex plane and is denoted by $m(\cdot)$.

There have been many research articles on the dynamics of exponential family. For readers who would like to read more related material, we refer them to [**1**] and [**2**].

2. *Some preliminary lemmas*

LEMMA 2.1. *Let $\lambda \in \Lambda$. Then any inverse branch of f_λ contracts the hyperbolic metric in $\mathbb{C}_\lambda$.*

Proof. Let $Q_\lambda = f_\lambda^{-1}(P_\lambda)$. Then Q_λ is a discrete subset of $\mathbb{C}$. Note that the map

$$f_\lambda : \mathbb{C} \setminus Q_\lambda \to \mathbb{C}_\lambda$$

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is a holomorphic covering map and is thus locally isometric. The lemma then follows from the fact that P_λ is a proper subset of Q_λ. □

Recall that $\Gamma = \{p_1, p_2, \ldots\}$.

LEMMA 2.2. *For any $\lambda_0 \in \Lambda$, $\epsilon > 0$ and integer $l \geq 1$, there exist a $\lambda \in B_\epsilon(\lambda_0) \cap \Lambda$ and an integer $n_0 \geq 0$ such that*

$$f_\lambda^{n_0}(0) \in B_{1/l}(p_l).$$

Proof. For $k \geq 0$, let

$$\lambda_k = f_{\lambda_0}^{k+1}(0).$$

By the definition of Λ, there exist two integers $0 \leq m < m'$ such that $\lambda_m = \lambda_{m'}$. Let $p = m' - m$. For $n \geq 0$, define a sequence of maps

$$F_n : \mathbb{C}^* \to \mathbb{C}^*$$

by $F_n(\lambda) = f_\lambda^{n+1}(0)$. In particular, we have

$$F_{m+(k+1)p}(\lambda) = (f_\lambda^p \circ F_{m+kp})(\lambda). \tag{1}$$

Let

$$t = |(Df_{\lambda_0}^p)(\lambda_m)|.$$

Note that λ_m is a repelling periodic point of f_{λ_0} with period p. It follows that $|t| > 1$. From (1) and direct calculations, we have

$$DF_{m+(k+1)p}(\lambda_0) = C + tDF_{m+kp}(\lambda_0) \tag{2}$$

where C is some constant independent of k. Take a large $K > 1$. From (2) and the fact that $|t| > 1$, we can take $k_0 \geq 1$ large enough such that

$$K_0 = |DF_{m+k_0p}(\lambda_0)| > 2K.$$

Let us fix the k_0. Then, when $0 < r \ll 1$ is small, the map

$$F_{m+k_0p} : B_r(\lambda_0) \to F_{m+k_0p}(B_r(\lambda_0))$$

is like a linear map. In particular, by taking $r > 0$ small, we have

$$B_{Kr}(\lambda_m) \subset F_{m+k_0p}(B_r(\lambda_0)). \tag{3}$$

Now let $t_0 = (1 + |t|)/2$. It follows that $1 < t_0 < |t|$. Write

$$f_\lambda(z) = (\lambda/\lambda_0) f_{\lambda_0}(z).$$

When $r > 0$ is small, λ/λ_0 is close to 1 and f_λ^p is close to $f_{\lambda_0}^p$. This, together with (3) and the fact that K is large, implies that for $l \geq 0$, the set

$$F_{m+(k_0+l)p}(B_r(\lambda_0)) = f_\lambda^{lp}(F_{m+k_0p}(B_r(\lambda_0)))$$

contains a Euclidean disk with center λ_m and radius $t_0^l Kr$ provided that $t_0^l Kr$ is not too large (note that the error caused by regarding f_λ^p as $f_{\lambda_0}^p$ is $O(r)$ and can be neglected when

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K is large). This means that there is a $\delta > 0$ independent of r such that for any $r > 0$, there is an $l_0 \geq 0$ such that

$$B_\delta(\lambda_m) \subset F_{m+(k_0+l_0)p}(B_r(\lambda_0)), \tag{4}$$

and, moreover, there is an open set $X \subset B_r(\lambda_0)$ such that

$$F_{m+(k_0+l_0)p} : X \to B_\delta(\lambda_m) \tag{5}$$

is a homeomorphism.

Now for $L \geq 1$, let

$$S_L = \{x + iy \mid L < x < L + 1, 0 < y < 2\pi\}.$$

Take L large enough and fix it so that S_L does not intersect P_{λ_0} (note that the hyperbolic diameter of S_L in $\mathbb{C}_{\lambda_0}$ is small provided that L is large) and, moreover, there exists a non-zero integer q satisfying

$$B_1(2q\pi i) \subset f_{\lambda_0}(S_L).$$

Note that $\{f_{\lambda_0}^{-k}(S_L)\}_{k=1}^{\infty}$ is dense in $\mathbb{C}$ in the sense that for any point z and $\eta > 0$, there is some integer $k \geq 1$ such that one of the components of $f_{\lambda_0}^{-k}(S_L)$ intersects $B_\eta(z)$. Since the hyperbolic diameter of S_L in $\mathbb{C}_{\lambda_0}$ is small by taking L large, and since the inverse branch of f_{λ_0} contracts the hyperbolic metric in $\mathbb{C}_{\lambda_0}$ by Lemma 2.1, there is an integer $u \geq 1$ and a topological disk $Q_L \subset B_{1/2l}(p_l)$ such that $f_{\lambda_0}^u : Q_L \to S_L$ is a homeomorphism. For the same reason, there is an integer v and a topological disk $W_L \subset B_{\delta/2}(\lambda_m)$ such that $f_{\lambda_0}^v : W_L \to Q_L$ is a homeomorphism.

Now we see that $f_{\lambda_0}^{u+v+2}(W_L)$ contains a small disk $B_R(\lambda_0)$ for some $R > 0$. Pull back $B_R(\lambda_0)$ along the orbit $f_{\lambda_0}^i(W_L)$, $0 \leq i \leq u + v + 2$. Then we obtain a topological disk $Y_L \subset W_L$ such that

$$f_{\lambda_0}^{u+v+2} : Y_L \to B_R(\lambda_0)$$

is a homeomorphism. So for $0 < r < R$ small, there is a topological disk $U \subset B_r(\lambda_0)$ such that

$$F_{m+(k_0+l_0)p} : U \to Y_L$$

is a homeomorphism. We claim that for any $r < s < R$, provided that $r > 0$ is small, the curve

$$F_{m+(k_0+l_0)p+u+v+2}(\lambda); \quad \lambda \in \partial U,$$

turns around $B_s(\lambda_0)$ exactly once when λ runs through ∂U once. In fact, when $r > 0$ is small, f_λ is close to f_{λ_0} and, thus,

$$F_{m+(k_0+l_0)p+u+v+2}(\partial U) = (f_\lambda^{u+v+2} \circ F_{m+(k_0+l_0)p})(\partial U)$$

is close to

$$(f_{\lambda_0}^{u+v+2} \circ F_{m+(k_0+l_0)p})(\partial U) = f_{\lambda_0}^{u+v+2}(\partial Y_L) = \partial B_R(\lambda_0).$$

The claim thus follows.

This implies that there is a topological disk $V \subset U \subset B_r(\lambda_0)$ such that

$$F_{m+(k_0+l_0)p+u+v+2} : V \to B_s(\lambda_0)$$

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is a holomorphic isomorphism. Since $V \subset B_r(\lambda_0) \subset B_s(\lambda_0)$, by a standard argument, it follows that there exists a unique $\lambda \in V$ such that

$$F_{m+(k_0+l_0)p+u+v+2}(\lambda) = \lambda.$$

Let $T = m + (k_0 + l_0)p + u + v + 3$. It follows that $f_\lambda^T(0) = \lambda$. This completes the proof of the lemma. □

A point x is called an *escaping point* of f_λ if $f_\lambda^k(x) \to \infty$ as $k \to \infty$. For $\lambda \in \mathbb{C}^*$, let E_λ denote the set of all of the escaping points of f_λ.

LEMMA 2.3. *Let $\lambda \in \Lambda$. Then E_λ is a zero measure set.*

Proof. Assume that E_λ has positive Lebesgue measure. Take a large $K > 1$. For $n \geq 1$, let

$$E_\lambda^n = \{z \in E_\lambda \mid \Re(f_\lambda^k(z)) > K \text{ for all } k \geq n\}.$$

It follows that

$$E_\lambda = \bigcup_{n=1}^{\infty} E_\lambda^n.$$

In particular, E_λ^n has positive Lebesgue measure for all n large enough. Take such an n. Let x be a Lebesgue point of E_λ^n. For $m > n$, let S_m denote the square with center $f_\lambda^m(x)$ and side length 2π. Then the hyperbolic diameter of S_m in $\mathbb{C}_\lambda$ approaches zero as $m \to \infty$. Let W_m denote the pull back of S_m by f_λ^{-m} along the orbit of x. Since the inverse branch of f_λ contracts the hyperbolic metric in $\mathbb{C}_\lambda$ by Lemma 2.1, the hyperbolic diameter, and thus the Euclidean diameter of W_m approaches zero also. Note that a definite part of S_m is mapped by f_λ^2 to a small neighborhood of the origin. However, on the other hand, by Koebe's distortion theorem, the distortion of the map f_λ^{-m} on S_m is bounded. This implies that there is a sequence $r_k \to 0$ such that $B_{r_k}(x)$ contains a definite part which does not belong to E_λ^n. It follows that x is not a Lebesgue point of E_λ^n. This is a contradiction. The proof of the lemma is completed. □

LEMMA 2.4. *Let $\lambda \in \Lambda$ and $V \subset \mathbb{C}$ be an open subset. Then the forward orbit of almost every $z \in \mathbb{C}$ intersects V.*

Proof. Assume that it were not true. Let

$$E = \mathbb{C} \setminus \bigcup_{k=0}^{\infty} f_\lambda^{-n}(V).$$

It follows from the assumption that E has positive Lebesgue measure. Take a Lebesgue point of E, say x. By Lemma 2.3, we may assume that x is not an escaping point. So there exist an $0 < R < \infty$, a $\delta > 0$ and a sequence $l_k \to \infty$ such that

$$d(P_\lambda, f_\lambda^{l_k}(x)) > \delta$$

and

$$|f_\lambda^{l_k}(x)| < R.$$

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Now we can pull back the disk $B_{\delta/2}(f_\lambda^{l_k}(x))$ along the orbit $\{f_\lambda^i(x)\}_{0\le i\le l_k}$ and obtain a small neighborhood of x, say V_{l_k}. Note that as $l_k \to \infty$, $\operatorname{diam}(V_{l_k}) \to 0$ (this is because when restricted to a compact subset of $\mathbb{C}_\lambda$, the inverse branches of f_λ strictly contract the hyperbolic metric in $\mathbb{C}_\lambda$). Since the distortion of $f_\lambda^{-l_k}$ on $B_{\delta/2}(f_\lambda^{l_k}(x))$ is uniformly bounded by Koebe's distortion theorem, it follows that there is a uniform $\delta_0 > 0$ such that

$$m(V_{l_k}) \ge \delta_0 \operatorname{diam}^2(V_{l_k}).$$

This implies that

$$m(V_{l_k} \cap E)/m(V_{l_k}) \to 1.$$

By taking a subsequence we may assume that $f_\lambda^{l_k}(x) \to y$. Since E is forward invariant and since the distortion of $f_\lambda^{-l_k}$ on $B_{\delta/2}(f_\lambda^{l_k}(x))$ is uniformly bounded, it follows that $B_{\delta/4}(y) \subset E$ up to a set of zero Lebesgue measure. Note that

$$\mathbb{C}^* \subset \bigcup_{k=1}^{\infty} f_\lambda^k(B_{\delta/4}(y)).$$

It follows that $\mathbb{C} = E$ up to a set of zero Lebesgue measure. This is impossible since $E \cap V = \emptyset$. The lemma follows. □

3. *Proof of the Main Theorem*

Now let us prove the Main Theorem. We inductively define a sequence $\{t_k\}_{k=1}^{\infty} \subset \Lambda$ and a sequence of positive numbers $\{\epsilon_k\}_{k=0}^{\infty}$ as follows. By Lemma 2.2, one can take an element λ in Λ such that

$$f_\lambda^{m_1}(0) \in B_1(p_1)$$

where $m_1 \ge 1$ is some integer. Set $t_1 = \lambda$ and $\epsilon_0 = |t_1|/2$.

Now assume that $t_k \in \Lambda$ and ϵ_{k-1} are both defined for some $k \ge 1$ such that $t_k \in B_{\epsilon_0}(t_1)$ and

$$f_{t_k}^{m_k}(0) \in B_{1/k}(p_k)$$

where $m_k \ge 1$ is some integer. Let $r_k > 0$ be small such that

$$m\left(\bigcup_{z\in P_{t_k}} B_{kr_k}(z)\right) < 1/2^k. \tag{6}$$

For given $k \ge 1$, one can find finitely many, say n_k points $x_i^k \in B_1(0)$, $1 \le i \le n_k$, such that:

(1) $d(x_i^k, P_{t_k}) > kr_k$ for all $1 \le i \le n_k$; and

(2) for any point $x \in B_1(0)$ with $d(x, P_{t_k}) > kr_k$, there is some $1 \le j \le n_k$ such that

$$m(B_{r_k}(x_j^k) \setminus B_{r_k}(x)) < \pi r_k^2/2. \tag{7}$$

Now, for every $1 \le i \le n_k$, by Lemma 2.4, the forward orbit of almost every point in $B_1(0)$ under f_{t_k} will intersect $B_{r_k}(x_i^k)$. So one can take N_k large enough such that for every $1 \le i \le n_k$, the following inequality holds,

$$m\left(B_1(0) \setminus \bigcup_{0\le l\le N_k} f_{t_k}^{-l}(B_{r_k}(x_i^k))\right) < 1/2^{k+1}.$$

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Since $d(x_i^k, P_{t_k}) > kr_k$ holds for every $1 \le i \le n_k$, it follows from Koebe's distortion theorem that there is some universal $M > 1$ such that for all $k \ge 2$, the distortion of $f_{t_k}^{-l}$ is bounded by M on every $B_{r_k}(x_i^k)$.

In addition, since $d(x_i^k, P_{t_k}) > kr_k$ holds for every $1 \le i \le n_k$, the hyperbolic diameter of $B_{r_k}(x_i^k)$, with respect to the hyperbolic metric in $\mathbb{C}_{t_k}$, approaches zero uniformly as $k \to \infty$. By Lemma 2.1, it follows that as $k \to \infty$, the hyperbolic diameter of every component of $f_{t_k}^{-l}(B_{r_k}(x_i^k))$, approaches zero uniformly also. Since $t_k \in B_{\epsilon_0}(t_1)$ is bounded away from the origin and the infinity and since $\{0, t_k\} \subset P_{t_k}$, it follows that there is a sequence $d_k \to 0$ such that if a component of $f_{t_k}^{-l}(B_{r_k}(x_i^k))$ intersects $B_1(0)$, then its Euclidean diameter is less than d_k.

For m_k, N_k, x_i^k, $1 \le i \le n_k$ above, one can take $0 < \epsilon_k < \epsilon_{k-1}/4$ small enough such that for all $\lambda \in B_{\epsilon_k}(t_k)$,

$$f_\lambda^{m_k}(0) \in B_{1/k}(p_k),$$

and

$$m\left(B_1(0) \setminus \bigcup_{0 \le l \le N_k} f_\lambda^{-l}(B_{r_k}(x_i^k))\right) < 1/2^k \tag{8}$$

hold and, for all $k \ge 2$, the distortion of f_λ^{-l}, $0 \le l \le N_k$, is bounded by $2M$ on every $B_{r_k}(x_i^k)$ and, moreover, for any $0 \le l \le N_k$, if a component of $f_\lambda^{-l}(B_{r_k}(x_i^k))$ intersects $B_1(0)$, then the Euclidean diameter of the component is less than $2d_k$.

By Lemma 2.2, one can take a $t_{k+1} \in B_{\epsilon_k/2}(t_k)$ such that

$$f_{t_{k+1}}^{m_{k+1}}(0) \in B_{1/k+1}(p_{k+1}),$$

for some integer $m_{k+1} \ge 1$. In particular, $t_{k+1} \in B_{\epsilon_0}(t_1)$. In this way we obtain a sequence $\{t_k\}_{k=1}^\infty \subset \Lambda$ and a sequence of positive numbers $\{\epsilon_k\}_{k=0}^\infty$ such that $t_{k+1} \in B_{\epsilon_k/2}(t_k)$ and $0 < \epsilon_{k+1} < \epsilon_k/4$ hold for all $k \ge 0$. It is clear that t_k converges to some parameter in $B_{\epsilon_0}(t_1)$. Suppose that $t_k \to \lambda$. Let us prove that λ is a desired parameter.

By the construction it is not difficult to see that the post-singular orbit of f_λ is dense in $\mathbb{C}$. This implies that the Julia set of f_λ is the whole plane. It suffices to prove that f_λ is ergodic. Suppose that it were not true. Then we have a backward invariant subset E with positive Lebesgue measure such that $\mathbb{C} \setminus E$ also has positive Lebesgue measure. Since the Julia set of f_λ is the whole plane, it follows that $\{f_\lambda^k\}_{k=1}^\infty$ is not a normal family in $B_1(0)$. As f_λ omits 0 and ∞, we have

$$\mathbb{C}^* \subset \bigcup_{k=1}^\infty f_\lambda^k(B_1(0)).$$

This implies that $B_1(0) \cap E$ and $B_1(0) \setminus E$ both have positive Lebesgue measure.

By (6), one can take a Lebesgue point x of $B_1(0) \cap E$ such that

$$x \notin \bigcup_{z \in P_{t_k}} B_{kr_k}(z)$$

holds for all k large enough. For k large, let $x_{j_k}^k$ be the point among x_i^k, $1 \le i \le n_k$, such that (7) holds.

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By (8), one can take a Lebesgue point y of $B_1(0) \setminus E$ such that

$$y \in \bigcup_{0 \le l \le N_k} f_{t_k}^{-l}(B_{r_k}(x_{j_k}^k))$$

holds for all k large enough. Then there exist an $0 \le l \le N_k$ and a component of $f_{t_k}^{-l}(B_{r_k}(x_{j_k}^k))$ which contains y. Let us denote the component by Ω_k. As we mentioned before, the distortion of f_λ^{-l}, $0 \le l \le N_k$, is bounded on $B_{r_k}(x_{j_k}^k)$ by $2M$ for some universal $M > 1$. This, together with (7) and the fact that x is a Lebesgue point of E, implies that there is some uniform $\delta_0 > 0$ such that

$$m(\Omega_k \cap E) > \delta_0 \operatorname{diam}^2(\Omega_k)$$

holds for all k large enough. Since $\operatorname{diam}(\Omega_k) \le 2d_k \to 0$ as $k \to \infty$, it follows that y is not a Lebesgue point of $B_1(0) \setminus E$. This is a contradiction.

The proof of the Main Theorem is complete.

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