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Non-intersecting splitting σ-algebras in a non-Bernoulli transformation

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This paper is dedicated to the memory of Dan Rudolph.

Abstract. Given a measure-preserving transformation T on a Lebesgue σ-algebra, a complete T-invariant sub-σ-algebra is said to split if there is another complete T-invariant sub-σ-algebra on which T is Bernoulli which is completely independent of the given sub-σ-algebra and such that the two sub-σ-algebras together generate the entire σ-algebra. It is easily shown that two splitting sub-σ-algebras with nothing in common imply T to be K. Here it is shown that T does not have to be Bernoulli by exhibiting two such non-intersecting σ-algebras for the T, T^{-1} transformation, negatively answering a question posed by Thouvenot in 1975.

1. *Introduction*

Notation 1. Throughout, any transformation T that we consider is an ergodic transformation of a Lebesgue space Ω endowed with σ-algebra A and measure μ. T, Ω, A and μ are assumed unless we say otherwise.

Definition 1. The Pinsker algebra is the largest T-invariant sub-σ-algebra S of A such that T has entropy 0 on S.

Definition 2. T is called a K transformation if its Pinsker algebra is trivial (i.e. consists only of the sets whose measure is either 0 or 1). This is equivalent to saying that T has no non-trivial 0 entropy factors (The trivial 0 entropy factor is the unique transformation T' which acts on the unique Lebesgue space consisting of only one point.)

Definition 3. A T-invariant sub-σ-algebra B of A is said to split if there is another T-invariant sub-σ-algebra C of A such that T on C is Bernoulli, B is independent of C, and together B and C generate A.

If a sub-σ-algebra splits then it contains the Pinsker algebra. Hence if two sub-σ-algebras both split, their intersection must contain the Pinsker algebra, and thus if that intersection is trivial the process must be K. In [**1**] it was shown that there are uncountably

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many transformations that are K and not Bernoulli. In [**2**] it was shown that a particular transformation called the T, T^{-1} transformation was one of them.

As a result of the second sentence of the above paragraph, Thouvenot posed the following question.

Question 1. If there are two complete T-invariant sub-σ-algebras of a given σ-algebra which both split but whose intersection is trivial, does it follow that the transformation is Bernoulli?

Actually this is a weakening of the question he asked in 1975 [**3**, Question A].

Question A. If there are two complete T-invariant sub-σ-algebras A_1 and A_2 of a given σ-algebra A such that both A_1 and A_2 split in A, does it follow that the intersection of A_1 and A_2 splits in A_1?

The reason why a positive answer to Question 1 follows from a positive answer to Question A is that if the trivial σ-algebra splits in A_1 and A_1 splits in A then T on A_1 is Bernoulli and hence T on A is Bernoulli. Thouvenot mentioned Question A again in [**4**] where he proved that if T_1 has the weak Pinsker property (it is not necessary for the reader to know what that means in this paper), T_B is Bernoulli of finite entropy and $T_1 \times T_B$ is isomorphic to $T_2 \times T_B$, then T_1 is isomorphic to T_2.

The reason why he still wanted to resolve Question A was that the above result would be rendered easy for any T_1 and T_2 if $T_1 \times T_B$ had finite entropy and the statement of Question A were valid.

Proof. From the preconditions it would follow that T_1 has the same entropy as T_2. Let Ω_1, Ω_2, Ω_3, Ω_4, and Ω_5 respectively be the spaces that T_1 acts on, T_2 acts on, T_B acts on, $\Omega_1 \times \Omega_3$, and $\Omega_2 \times \Omega_3$. If we look at the isomorphism from $T_1 \times T_B$ to $T_2 \times T_B$ (from Ω_4 to Ω_5) let A_2 be the embedding of the σ-algebra generated by T_2 in Ω_2 into Ω_5. Let A_1 be the isomorphic image of (the embedding of σ-algebra generated by T_1 into Ω_4) into Ω_5. Let T be $T_2 \times T_B$ (which acts on Ω_5). Let A be the entire σ-algebra generated by T in Ω_5. We have that A_1, A_2, and A are all on Ω_5, and under T both A_1 and A_2 split in A, and hence a positive result to Question A would give that their intersection splits in both A_1 and A_2 under T. The restriction of T to A_1 gives a transformation isomorphic to T_1 which can be written as the cross of T restricted to that intersection with a Bernoulli, and the restriction of T to A_2 gives a transformation isomorphic to T_2 which can be written as the cross of T restricted to that intersection with another Bernoulli. Since T_1 has the same entropy as T_2 the two Bernoullis have the same entropy and hence are isomorphic, making T_1 isomorphic to T_2. □

The purpose of this paper is to answer Question 1 negatively (thereby answering Question A negatively) by showing that we can do this with the T, T^{-1} transformation. But it would be unfair to give this paper complete credit for answering Question 1. The two T-invariant sub-σ-algebras we will give will be called B and C (to be defined later). The creation of B and the fact that it split was established by Thouvenot. He never actually wrote this in a paper but he made it general knowledge. The creation of C and the fact that it split was established by Hoffman in [**5**]. The reason why Question 1 has until now been

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too hard to solve is that Hoffman's result was not yet known. After Hoffman established that C splits it was clear that if someone could show that B and C had trivial intersection Question 1 would be resolved. Hoffman himself never had the opportunity of resolving this issue because he was not aware of the question. Our contribution here is to establish that B and C have trivial intersection, thereby completing the resolution of Question 1 negatively. Actually the σ-algebra B that we use here is not precisely the σ-algebra that Thouvenot proved to split but just a small modification of that σ-algebra, and the proof that it splits is identical. Here we will state the modified Thouvenot σ-algebra B, show that B and C have trivial intersection, and then give the proof (essentially Thouvenot's) that B splits.

To be more precise, Thouvenot starts with a set to make his construction. He insists that the set be a stretch of the path (we will later explain what 'path' means). However, we will show that he could have let it be a stretch of both the path and the scenery (we will later explain what 'scenery' means). We use exactly the Thouvenot construction to get B, except that we use a set that depends on more than just the path. For that reason we include in the last section a proof (essentially Thouvenot's) that B splits.

Thus we are faced in this paper with the following tasks.

(1) Define the T, T^{-1} transformation. (Although this has already been done in [**2**], we do it again here to make this paper self-contained.)
(2) Define B.
(3) Define C.
(4) Show that B and C have trivial intersection.
(5) Use Thouvenot's proof to show that B splits.

2. *Dedication to Dan Rudolph*

Everyone understands that Dan was the best ergodic theorist of our generation in the world, but that is not the purpose of this dedication. Its purpose is to indicate the kind of person Dan was as a person. While we shared a house together in Berkeley with about three other people he did most the work of organizing and supporting the house without seeming to in any way resent doing so. When I got sick and was afraid I would not be able to handle a job, he and Ken Berg encouraged me to come to Maryland anyway and took on my workload for a semester when I was hospitalized. While he was chairman of the department and a father and a thesis adviser he still found time to sit in on my ergodic theory course as a service to me (he obviously did not need to learn ergodic theory from me). One of my criteria for judging a person is whether he tends to say bad things about others. Dan never did. He was always positive in his impressions of others, always looking for the good in them.

3. *General theory*

Since this paper makes heavy use of the concept of fibers and since we fear that many readers are uncomfortable with fiber arguments, we wish to give a discussion about fiber arguments and why they are valid. We will usually omit proofs of theorems in this section as we are only discussing theory that the reader should already be familiar with. When we do give proofs in this section they will be only sparse proofs as the reader should be able to fill in the details himself. As throughout this paper, all transformations T are on a Lebesgue space and T acts on a σ-algebra A.

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In this paper all transformations considered have finite entropy and all such transformations have a finite generator.

Definition 4. Let T be a transformation, Q be a partition and ω be a point in the space. The Q, T name of ω is the sequence $\ldots a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ such that a_i is the element of Q which $T^i(\omega)$ is in.

Notation 2. A_1 and A_2 are complete sub-σ-algebras of A.

Notation 3. Δ means symmetric difference.

Definition 5. The literal σ-algebra generated by Q, T is the smallest σ-algebra containing all sets in the partitions $\bigvee_{-n}^{n}(T^i(Q))$ for all n. The σ-algebra generated by Q, T is all sets of the form $S_1 \Delta Z$ where S_1 is in the literal σ-algebra generated by Q, T and Z has measure 0.

THEOREM 3.1. *Let T_1 and T_2 be two measure-preserving transformations acting on the same space and let Q_1 and Q_2 be two partitions. Suppose that the σ-algebras generated by Q_1, T_1 and Q_2, T_2 both equal A_1. Then there exists a set Z of measure* 0 *such that if ω_1 and ω_2 are not in Z then they have the same Q_1, T_1 name if and only if they have the same Q_2, T_2 name.*

Proof. Let $S \in A_1$. You can approximate S arbitrarily well with a finite union set in the partition $\bigvee_{-n}^{n}(T_1^i(Q_1))$ for some n. It follows that after removing a set of measure 0, the T_1, Q_1 name of ω_1 determines whether or not you are in S. By letting S be a set in $\bigvee_{-n}^{n}(T_2^i(Q_2))$, we get that after removing a set of measure 0 the T_1, Q_1 name of ω_2 determines the T_2, Q_2 name of ω_2 from $-n$ to n and hence (after removing a set of measure 0) you can get it to determine the entire T_2, Q_2 name of ω_2. Argue symmetrically to get the converse. □

Comment 1. As a matter of fact there is no reason to restrict to finite entropy because the above theorem is still true if one or both of those partitions are countably infinite. In the proof you still need to get an increasing sequence of finite partitions which generate A_1 but this can be done by truncating your countable generator to make it finite at any finite stage.

The above theorem makes the following definition well defined up to measure 0, that is, it does not depend on what transformation T we use or what generator Q we use.

Definition 6. Let Q, T generate A_1. Then ω_1 and ω_2 are said to be in the same fiber of A over A_1 if they have the same Q, T name. Being in the same fiber of A over A_1 is an equivalence relation and a fiber of A over A_1 is an equivalence class for that equivalence relation. We can just refer to 'fiber of A_1' or 'same fiber of A_1' if A is understood.

THEOREM 3.2. *Let S be a set in the literal σ-algebra generated by Q, T. Then if ω_1 and ω_2 have the same Q, T name, either they are both in S or neither of them is in S.*

Proof. Just show that it is true for all sets in the appropriate partitions and that the collection of sets for which it is true is a σ-algebra. □

From this we are led to the following theorem.

THEOREM 3.3. *Let S be a set in A_1. There exists a set Z of measure* 0 *dependent on what set S we use and on what transformation and generator we used to define fibers, such that*

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if ω_1 and ω_2 are in same fiber of A over A_1 and neither ω_1 nor ω_2 is in Z, then either they are both in S or neither of them is.

Definition 7. We say that A_2 is a two-point extension of A_1 if there is a set Z of measure 0 such that outside of Z, each fiber of A over A_1 is a union of two fibers of A over A_2.

4. *Definitions of the T, T^{-1} transformation, B and C*

Definition 8. Let $\ldots a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ (called the scenery) be independent tosses of a fair coin and let $\ldots b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots$ (called the path) also be independent tosses of a fair coin. The path and scenery are chosen independently of each other. The a_i take on the values h (heads) or t (tails) and the b_i each take on the values L (left) or R (right), but the distribution of both processes is the same (1/2, 1/2 product measure). The T, T^{-1} transformation is a stationary process on a four-letter alphabet $\{(h, L), (t, L), (h, R), (t, R)\}$. To generate a word in the T, T^{-1} transformation we take a random walk on $\ldots a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ using the $\ldots b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots$ to tell you how to walk.

Example 1. Suppose that the terms $b_{-2}, b_{-1}, b_0, b_1, b_2$ take on the values L, L, L, L, R and $a_{-2}, a_{-1}, a_0, a_1, a_2$ take on the values h, h, t, t, t. Here is how we start to generate a word $\ldots c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots$ in the T, T^{-1} transformation . Since $b_0 = L$ and $a_0 = t$, $c_0 = (t, L)$. Now since $b_0 = L$, we walk to the left on the scenery. On the path the 0 coordinate always goes to the right (i.e. the sequence $\ldots b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots$ always shifts to the left). Since $b_1 = L$ and $a_{-1} = h$, $c_1 = (L, h)$. Now since the first coordinate of c_1 is L, we walk to the left again on the scenery. Since $b_2 = R$ and $a_{-2} = h$, $c_2 = (h, R)$. We also go backwards in time to get c_{-1}, c_{-2}, etc. Since $b_{-1} = L$ we just walked left and since we start at 0 we must have been at position 1 at time -1, and so since $a_1 = t$ and $b_{-1} = L$, $c_{-1} = (L, t)$.

Definition 9. In the above definition, $\ldots a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ is called the scenery and $\ldots b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots$ is called the path. $\ldots X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots$ is the T, T^{-1} process where each $X(i)$ takes on the value

$$c_i \in P := \{(h, L), (h, R), (t, L), (t, R)\}$$

where c_i is as in Example 1. If you use the T, T^{-1} process as a measure on doubly infinite words in alphabet P, then taking a doubly infinite word and shifting it to the right is the T, T^{-1} transformation.

Comment 2. There is an easier way to define the T, T^{-1} transformation which explains why it is called the T, T^{-1} transformation. Let S be the transformation on doubly infinite sequences with independent and identically distributed (i.i.d.) 1/2, 1/2 probability (in this case the $\ldots b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots$ process) which shifts a word to the left, and T be an independent transformation on doubly infinite sequences with i.i.d. 1/2, 1/2 probability (in this case the $\ldots a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ process) which shifts a word to the left. Partition the space into sets; Left $= (b_0 = L)$ and Right $= (b_0 = R)$. The T, T^{-1} transformation is the transformation on the product space which takes (ω_1, ω_2) to $(S(\omega_1), T(\omega_2))$ if ω_1 is in right and $(S(\omega_1), T^{-1}(\omega_2))$ if ω_1 is in left and then the T, T^{-1} process is

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the $(T, T^{-1}$ transformation, $P)$ process where P is as defined above, namely the four-set partition determined by whether a_0 is heads or tails and whether you are in the right or left.

Definition 10. Since the T, T^{-1} process is defined to be a random walk on a random scenery we now explicitly define $n(i)$ to be the desired random walk,

$$f(i) := \begin{cases} 1 & \text{if } b_i = R, \\ -1 & \text{if } b_i = L; \end{cases} \tag{1}$$

$$n(i) := \begin{cases} \sum_{j=0}^{i-1} f(j) & \text{if } i \text{ is non-negative,} \\ -\sum_{j=i}^{-1} f(j) & \text{if } i \text{ is negative.} \end{cases} \tag{2}$$

Note that $n(0) = 0$.

Definition 11. For each i, $X_i[1]$ and $X_i[2]$ are the first and second coordinates of X_i respectively.

Comment 3. $X_i[1] = a_{n(i)}$, $X_i[2] = b_i$.

Definition 12. In the T, T^{-1} process, $\ldots a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ is called the scenery at time 0 and $\ldots b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots$ is called the path at time 0. Furthermore,

$$\ldots A(-1), A(0), A(1), A(2), \ldots$$

is called the scenery at time i if $A(k) = a_{n(i)+k}$, and the process

$$\ldots B(-1), B(0), B(1), B(2), \ldots$$

is called the path at time i if $B(k) = b_{i+k}$. Finally, $\ldots X_{-1}[1], X_0[1], X_1[1] \ldots$ is called the scenery process (which is distinct from the scenery) at time 0, but there is no distinction between the path process $(\ldots X_{-1}[2], X_0[2], X_1[2] \ldots)$ and the path at time 0.

Comment 4. (Intuitive idea of the T, T^{-1} process) You should envisage the T, T^{-1} process to be a random walk on a random scenery where L means that you walk left, R means you walk right and the heads and tails are the scenery.

Definition 13. The σ-algebra generated by just the knowledge of the $X_i[1]$ where i runs over all the integers is called the scenery σ-algebra.

Comment 5. By now the reader might be getting confused about the various partitions introduced here. Throughout the remainder of the text P always means $P := ((h, L), (h, R), (t, L), (t, R))$. P is a refinement of two partitions. One is the path partition (R, L) and the other is the scenery partition (t, h). Whenever we want to refer to those two partitions we will explicitly refer to them as the (R, L) partition and the (t, h) partition. If T is the T, T^{-1} transformation, the T, P process is the T, T^{-1} process, the $T, (R, L)$ process is the path process and the $T, (t, h)$ process is the scenery process. The most confusing concept is the scenery at time 0, often referred to simply as the scenery.

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The trouble is that there is no partition Q whatsoever such that the scenery at time 0 is the T, Q process. You start with the scenery at time 0 to create an output of the T, T^{-1} process, but you cannot get the scenery at time 0 as a standard factor of the T, T^{-1} process.

Comment 6. If we just provide you with an ω it is clear what we mean by the scenery process for ω. It is just the first coordinates of the T, P name of ω where T is the T, T^{-1} transformation, that is, it is the T, (h, t) process. However, there is no *a priori* reason why you should know the scenery at time 0 for ω if all you know about ω is its T, P name. But in fact, the T, P name (or just the past T, P name or just the future T, P name) above does determine the scenery at time 0 because the random walk is recurrent, so if you know the whole T, P name then you can watch the random walk cover the entire scenery and thus you can observe it telling you exactly what that scenery is. Thus the following definition makes sense.

Definition 14. The scenery at time 0 for ω is the scenery at time 0 determined by the T, P name of ω where T is the T, T^{-1} transformation and $P = \{(h, L), (h, R), (t, L), (t, R)\}$. This is distinct from the scenery process at time 0 which is just the first coordinates of that T, P name, that is,

$$\ldots X_{-2}(\omega)[1], X_{-1}(\omega)[1], X_0(\omega)[1], X_1(\omega)[1], X_2(\omega)[1], \ldots.$$

Comment 7. While the scenery process and the scenery at time 0 are distinct, they both have the same zeroth coordinate. The zeroth coordinate for the output of the scenery process of ω at time 0 is $X_0[1]$, but here we used ω to define

$$\ldots, X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots.$$

If we similarly took $T^{100}(\omega)$ and used that to define

$$\ldots, X'_{-2}, X'_{-1}, X'_0, X'_1, X'_2, \ldots$$

then $X'_0 = X_{100}$ and hence $X'_0[1] = X_{100}[1]$. Thus the scenery process is the sequence of zeroth coordinates of the scenery processes of

$$\ldots T^{-2}(\omega), T^{-1}(\omega), T^0(\omega), T^1(\omega), T^2(\omega), \ldots$$

and this implies (fixing ω) that the scenery process can be thought of as the zeroth coordinates of the sceneries at times $\ldots -2, -1, 0, 1, 2 \ldots$.

Comment 8. The purpose of [**5**] by Hoffman was to establish that the scenery process is not loosely Bernoulli, a fact that is quite significant but irrelevant to this paper. However, in the process of proving that, Hoffman established the σ-algebra C which we are about to describe. He starts by quoting a result of Matzinger [**6**] which states that if T is the T, T^{-1} transformation, there is a Z of measure 0 so that if $\omega_1 \notin Z$ and $\omega_2 \notin Z$, and ω_1 and ω_2 are in the same fiber of A over the scenery σ-algebra (i.e. if $T^i(\omega_1)$ and $T^i(\omega_2)$ are in the same element of the two-set partition (h, t) for all $i \in \mathbb{Z}$) then either the scenery at time 0 of ω_1 is an even translate of the scenery at time 0 of ω_2 or the scenery at time 0 of ω_1 is an even translate of the inverse of the scenery at time 0 of ω_2 (i.e. if $\ldots a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ is the scenery of ω_1 at time 0 and $a'_{-2}, a'_{-1}, a'_0, a'_1, a'_2$ is the scenery of ω_2 at time 0 then there exists an even integer k such that either $a'_i = a_{i+k}$ for all $i \in \mathbb{Z}$ or $a'_i = a_{k-i}$ for all $i \in \mathbb{Z}$). Hoffman then got a two-point extension of the scenery σ-algebra such that, outside

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of a set of measure 0, they are in the same fiber of A over that two-point extension if and only if ω_1 and ω_2 are in the same fiber of A over the scenery σ-algebra and there exists an even integer k such that $a'_i = a_{i+k}$ for all i. He then went on to show that the two-point extension he defined splits.

Definition 15. C is the two-point extension of the scenery process that Hoffman defined above. In other words, C is a σ-algebra with the following properties.

(1) There is a set Z of measure 0 such that if $\omega_1 \notin Z$ and $\omega_2 \notin Z$, letting the scenery at time 0 for ω_1 be $\ldots a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ and the scenery at time 0 for ω_2 be $a'_{-2}, a'_{-1}, a'_0, a'_1, a'_2, \omega_1$ and ω_2 are in the same fiber of A over C if and only if they have the same name in the scenery process and there is an even k such that $a'_i = a_{i+k}$ for all i.

(2) C splits.

Comment 9. Note that we are not giving an explicit definition of C in this paper, because we do not need it. All we need to know about C is (1) and (2) above.

Definition 16. We put an equivalence relation, to be called equivalence relation 1, on the set of all outputs of the T, T^{-1} process by saying that ω_1 and ω_2 are equivalent-1 if their outputs in the T, T^{-1} process have the same first coordinates, that is, if they have the same name in the scenery process.

THEOREM 4.1. *Let S be a set in C. Then there exists a set Z of measure* 0 *such that for any ω_1 and ω_2 if:*

(1) *neither ω_1 nor ω_2 are in Z;*

(2) *ω_1 and ω_2 are equivalent-1;*

(3) *the outputs of ω_1 and ω_2 in the T, T^{-1} process only differ on finitely many coordinates.*

Then either both ω_1 and ω_2 are in S or neither of them is.

Proof. (3) implies that there is an N such that the outputs of ω_1 and ω_2 are identical for all $i \leq N$, and that in turn implies that they have the same scenery at time N (generally one would want to think of N as negative, but it could be positive). Since the scenery at time 0 is a translate of the scenery at time N, the sceneries at time 0 for the two of them are translates of each other. In fact they are even translates of each other because to get to the first scenery at 0 from the scenery process at time N you have to take $|N|$ steps of size 1 or -1, and to get to the second scenery at time 0 from the scenery at time N you also have to take $|N|$ such steps. The result follows by (1) of Definition 15 and Theorem 3.3. □

Definition 17. Let $d < e$ be integers. Then $X_d, X_{d+1}, \ldots, X_e$ sees only a finite subset of the scenery $\ldots a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$, that is, there exist a j and a k such that the walk from time d to time e only covers $a_j, a_{j+1}, a_{j+2}, \ldots, a_k$. More precisely, j and k are the maximum and minimum of $\{n(i) : d \leq i \leq e\}$ respectively, where $n(i)$ is as in Definition 10. $a_j, a_{j+1}, a_{j+2}, \ldots a_k$ is called the block of scenery seen from time d to time e.

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Comment 10. The 'block of scenery seen from time d to time e' is not the same as the scenery seen from time d to time e in the scenery process. The former is a concept of scenery in space and the latter is a concept of scenery in time.

Definition 18. Let $d < e$ be integers. Then $b_d, b_{d+1}, b_{d+2}, \ldots b_{e-1}$ is a portion of the path taking on values in $\{L, R\}$. Let f be as in Definition 10. Then

$$\text{fo} := \max_{-1 \le k \le e-d-1} \sum_{i=0}^{k} f(b_{d+i})$$

is called the forwards distance you walk from time d to time e, and

$$\text{ba} := \min_{-1 \le k \le e-d-1} \sum_{i=0}^{k} f(b_{d+i})$$

is called the backwards distance you walk from time d to time e. The net distance you walk from time d to time e is $\sum_{i=0}^{e-d-1} f(b_{d+i})$. Here $\sum_{i=0}^{-1}$ (of anything) always takes on value 0.

Comment 11. Let $a_j, a_{j+1}, a_{j+2}, a_k$ be the block of scenery seen from time d to time e. Then $k - j = \text{fo} - \text{ba} + 1$.

We now describe the σ-algebra B (a specific modification of Thouvenot's σ-algebra).

Definition 19. Let S be the set of all ω whose first 11 outputs in the T, T^{-1} process are

$$(h, L), (h, L), (h, L), (h, R), (h, R)(h, L), (h, R), (h, L), (h, L), (h, L), (t, L).$$

Let P' be the partition of Ω which contains the complement of S and the following partition of S. We let ω_1 in S and ω_2 in S be in the same piece of that partition if and only if the following three conditions are satisfied:

$$\text{Let } m, n \text{ be the least positive integers such that } T^{-m}(\omega_1) \in S \text{ and } T^{-n}(\omega_2) \in S. \text{ Then } m = n. \tag{3}$$

$$\text{The net distance, forwards distance, and backwards distance from time } -m \text{ until time 0 are the same for } \omega_1 \text{ and } \omega_2. \tag{4}$$

$$\text{The block of scenery seen from time } -m \text{ to time 0 is the same for } \omega_1 \text{ and } \omega_2. \tag{5}$$

Definition 20. BB is the literal σ-algebra generated by $\{T^i P' : i \in \mathbb{Z}\}$ and B is the σ-algebra generated by $\{T^i P' : i \in \mathbb{Z}\}$, that is, $S' \in B$ if it can be written as $SS'\Delta Z$ where $SS' \in BB$ and Z has measure 0.

Comment 12. (Intuitive idea of B) If you are in S, by equation (3), P' tells you how long it has been since the previous time you were in S. Then by equation (5), P' tells you the block of scenery you saw since the last time you were in S; and by equation (4), P' tells you where you were in that scenery the last time you were in S. Now look at the last time you were in S: P' told you when you were in S the previous time before that, what block of scenery you had seen between that time and the time before that and where you were in the block of scenery before that. Piecing this all together, you can deduce from the

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entire T, P' process (or just your past T, P' process or just the future T, P' process) your complete scenery because the random walk is recurrent. If you are not in S you cannot know exactly where you are in that scenery, but since you did know the scenery the last time you were in S and since the scenery at any time is just a translate of the scenery at any other time you know the scenery up to a translate. Furthermore, you know how long it has been since the last time you were in S which is an upper bound of how big that translate is. Since a fiber of the σ-algebra generated by T, P' tells you the future as well as the past, you also know what the scenery will be the next time you enter S and an upper bound for how much of a translate your scenery is from that scenery.

Notation 4. There is a certain ambiguity in our notation. Until now S was a general S. Henceforth S and P' are as in Definition 19.

Definition 21. Let ω_1 and ω_2 be two elements of Ω. We say that ω_1 and ω_2 are equivalent-2 if $T^i(\omega_1)$ and $T^i(\omega_2)$ are in the same atom of P' for all integers i, that is, if ω_1 and ω_2 are in the same atom of $T^i(P')$ for all integers i.

Equivalence-2 is equivalent to being in the same fiber of B (up to measure 0). The following is immediate from Theorem 3.3.

THEOREM 4.2. *Let $S_1 \in B$. Then there exists a Z of measure* 0 *such that if:*
(1) *neither ω_1 nor ω_2 is in Z; and*
(2) *ω_1 and ω_2 are equivalent-2,*
then either both ω_1 and ω_2 are in S_1 or neither of them is.

5. *Proof that B and C have trivial intersection*

Definition 22. Henceforth T is the T, T^{-1} transformation,

$$P := \{(h, L), (h, R), (t, L), (t, R)\},$$

and, for $\omega \in \Omega$, $\ldots (a_{-2}, b_{-2}), (a_{-1}, b_{-1}), (a_0, b_0), (a_1, b_1), (a_2, b_2) \ldots$, called the output of ω for T, where (a_i, b_i) is defined to be the element of P containing $T^i(\omega)$.

Comment 13. Since P is a generator of T, the output of ω completely determines ω.

Definition 23. We suppose the existence of a set SS such that $SS \in B \cap C$.

LEMMA 5.1. *Let Z be any set of measure* 0 *and*

$$ZZ = \{\omega : \exists \omega_1 \in Z \text{ whose output (in the } T, T^{-1} \text{ process)} \\ \text{differs from that of } \omega \text{ in only finitely many coordinates}\}.$$

Then ZZ has measure 0.

Proof. Let Ω be the set of paths of the T, T^{-1} process with the appropriate measure on it. Fix a finite set of integers F. We now define a measure-preserving isomorphism S_F from Ω to itself. Let Le be the least element of F. An output of the T, T^{-1} process is the scenery process and the path process, but in fact if you just know the output of the T, T^{-1} process before a given time and just the path process from that time onward then you know the entire T, T^{-1} process output because the output before a given time determines the scenery

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at that time and therefore the path from then onward will determine scenery process from then onward. We define a transformation S_F from the space of T, T^{-1} process outputs to itself by letting $S_F(\omega)$ be the unique output of the T, T^{-1} process which is identical to that of ω before Le and whose path is identical to that of ω from Le onward except the opposite of the path of ω (that is, switch L to R or R to L) on those times in F. S_F can easily be seen to be measure-preserving so $S_F(Z)$ has measure 0. It is immediate from the definition of ZZ that

$$ZZ \subset \bigcup_{\text{all finite sets } F} (S_F(Z)). \qquad \square$$

LEMMA 5.2. *There exists ZZ of measure 0 such that if the outputs of ω_1 and ω_2 in the T, T^{-1} process differ on only finitely many coordinates and $\omega_1 \notin ZZ$, then $\omega_1 \in SS$ if and only if $\omega_2 \in SS$.*

Proof. By Theorems 4.1 and 4.2 there exists a set Z of measure 0 such that, for any two points which are equivalent-1 or equivalent-2, whose output in the T, T^{-1} process differs on only finitely many terms, neither of which are in Z, either both are in SS or neither of them is. Let

$$ZZ = \{\omega : \exists \omega_{11} \in Z \text{ whose output differs from } \omega \text{ in only finitely many coordinates}\}$$

so that, by Lemma 5.1, ZZ has measure 0. Note that, for any ω_{11}, ω_{12}, if $\omega \notin ZZ$ and ω_{11} and ω_{12} differ from ω in only finitely many coordinates, then $\omega_{11} \notin Z$ and $\omega_{12} \notin Z$. Hence the proof will be complete when we can show that for any outputs ω_1 and ω_2 of the T, T^{-1} process which differ on only finitely many coordinates, there exist ω_3 and ω_4 such that:

(1) any two elements of $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ differ on only finitely many coordinates;
(2) ω_1 and ω_3 are equivalent-1;
(3) ω_2 and ω_4 are equivalent-1;
(4) ω_3 and ω_4 are equivalent-2.

Select M such that ω_1 is the same as ω_2 outside of $[-M, M]$. Note that for any $a < -M$ and $b > M$, ω_1 and ω_2 see identically the same scenery at both a and b because for a they see exactly the same present and past and the present and past of the T, T^{-1} process determines the scenery; for b they see exactly the same present and future which also determines the scenery. Using the recurrence of random walks and the fact that the ω_1 walk is identical to the ω_2 walk before $-M$, select $N_1 < -M$ so that both the ω_1 walk and ω_2 walk from N_1 to M reach their lowest point at time N_1. Select $N_2 > M$ so that both the ω_1 walk and ω_2 walk from N_1 to N_2 reach their highest point at time N_2. We also want to make sure that:

(5) at neither time N_1 nor N_2 are we in the middle of a stretch of the form

$$(h, L), (h, L), (h, L), (h, R), (h, R), (h, L), (h, R), (h, L), (h, L), (h, L), (t, L).$$

It is intuitively obvious that we should be able to insist on (5), but one explicit way to do so is to realize that if we drop condition (5) there are infinitely many values n we could have used for N_1, so, with probability 1, one such n has 24 successive Rs in a row preceding it and then just let $N_1 = n - 12$. Argue symmetrically to choose N_2.

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It follows that for any $a \leq N_1$ and any $b \geq N_2$ the backwards, forwards and net distances from a to b are the same for both ω_1 and ω_2. Alter ω_1 to get ω_3 by changing each

$$(h, L), (h, L), (h, L), (h, R), (h, R), (h, L), (h, R), (h, L), (h, L), (h, L), (t, L)$$

between N_1 and N_2 to

$$(h, L), (h, L), (h, L), (h, R), (h, L), (h, R), (h, R), (h, L), (h, L), (h, L), (t, L),$$

and ω_3 is a legitimate output of the T, T^{-1} process. The reason why we added (t, L) at the end of

$$(h, L), (h, L), (h, L), (h, R), (h, R), (h, L), (h, R), (h, L), (h, L), (h, L), (t, L)$$

when we defined S was to guarantee that no such change would create a new instance of S. ω_1 and ω_3 are equivalent-1. Do the same to ω_2 to get a ω_4 such that ω_2 is equivalent-1 to ω_4. If $a \leq N_1 < N_2 \leq b$ none of these changes alter the forwards, backwards, or net distances from a to b. ω_3 and ω_4 have no instance of S between N_1 and N_2 so they are equivalent-2 because they see S at the same times, the forwards, backwards, and net distances between the previous times less than N_1 that they saw S and the next time greater than N_2 that they see S and the sceneries they see at both those times are the same. □

Comment 14. In principle we are already done. Lemma 5.2 establishes that SS is in the double tailfield of the T, T^{-1} process which is trivial so SS is trivial. However, we prefer to give a more self-contained proof that does not refer to the double taifield of the T, T^{-1} process.

LEMMA 5.3. *Select an $N > 0$ and let*

$$G = (a_{-N}, b_{-N}), \ldots, (a_{-1}, b_{-1}), (a_0, b_0), (a_1, b_1), (a_N, b_N)$$

and

$$H = (a'_{-N}, b'_{-N}), \ldots, (a'_{-1}, b'_{-1}), (a'_0, b'_0), (a'_1, b'_1), (a'_N, b'_N)$$

be two words in the T, T^{-1} process from $-N$ to N. Let μ and ν be the conditioned measures of the T, T^{-1} process conditional on G and H, respectively. Then there is a coupling of μ and ν (which means a measure on the product space where μ and ν are the marginals; for example, a joining is a special case of a coupling) such that, with probability 1*, a pair of paths in the coupled process differs only on a finite set.*

Proof. Pick a doubly infinite output, $p = \ldots (a_{-1}, b_{-1}), (a_0, b_0), (a_1, b_1), \ldots$ in accordance with measure μ. Consider the scenery sce at time 0 determined by p. In that scenery look at the w word from $-N$ to N (w is the actual scenery from $-N$ to N of size $2N + 1$, not just the scenery you get to see in G which is usually of order about $\sqrt{N}$ in size). Let w' be the part of the scenery that H tells you, which is typically of order $\sqrt{N}$ in size, part of the past scenery and part of the future scenery. Extend w' randomly to a arbitrary scenery word w_1 from $-N$ to N where we let every possible extension of w' have equal probability of being the w_1 that we choose. Now in sce, let M be the least integer such that $M > N$, $M + N$ is an even integer, and w_1 is the word from M to $M + 2N$. We

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now define a doubly infinite output q in the T, T^{-1} process. One way to pick an output is to tell you the scenery at time 0 and the path at time 0 separately and then let them generate q. We will let the scenery of q at time 0 be the scenery of p at time 0 translated to the right by $M + N$ so that the scenery at time 0 for q from $-N$ to N is precisely w_1. We let the path from time $-N$ to time N be $b'_{-N}, \ldots b'_{-2}, b'_{-1}, b'_0, b'_1, b'_2, \ldots b'_N$. Since w_1 is an extension of w' this forces the output in the T, T^{-1} process for q from time $-N$ to time N to be precisely H. To get the path from time $N + 1$ onward, just run it independently of the path of p until

$$\text{the amount of } R\text{s} - \text{the amount of } L\text{s in } p$$

exceeds

$$\text{the amount of } R\text{s} - \text{the amount of } L\text{s in } q$$

by $M + N$. This will eventually happen with probability 1 because the difference between two random walks is a recurrent random walk on the even integers. When that happens p and q will see the exact same scenery, so from then on couple them to be exactly the same. Using a symmetric argument, run the path of q at times $-(N + 1), -(N + 2), \ldots$ to be independent of the path of p until they see the same scenery and then let them be identical from then on. We claim that our random selection of (p, q) is the desired coupling. It is a measure on the product space of the T, T^{-1} process with itself where, with probability 1, p and q differ on only finitely many terms and p has been chosen in accordance with μ. We need only show that q ends up being chosen in accordance with measure ν. The proof will be complete if we can choose another coupling of p and q in which we first select q in accordance with measure ν and then choose p from q in such a way that in the end the two couplings turn out to be identical. Here we get p from q in exactly the same way that we got q from p except that, instead of picking M to be the least time bigger than N with the appropriate property, we let $-M$ be the largest time below $-N$ with the appropriate property and we produce the coupling in the same way (but translate forward by $M + N$ instead of backward). We think that it is clear that we can get exactly the same coupling that way, that is, that the measures which chose p, q are identical in these two couplings.

It should be intuitively obvious that these two couplings are identical, but, to see it more rigorously, first note that in both cases the measure on the pair ω, ω_1 is the same where ω is the μ word of length $2N + 1$ and ω_1 is the ν word of length $2N + 1$. The probability law on $M + N$ in the first coupling is identical to that of the $-(M + N)$ in the second (and furthermore is independent of the pair ω, ω_1) and, for any k, the distribution of everything else (sceneries and paths) is identical given $\omega, \omega_1, M + N = k$ for the first coupling and $-(M + N) = k$ for the second coupling. □

THEOREM 5.4. *B and C have trivial intersection.*

Proof. Lemmas 5.2 and 5.3 establish that there is a Z of measure 0 such that, for μ and ν as in Lemma 5.3, when p, q is picked as in the coupling of μ and ν given by that lemma and $p \notin Z$, $p \in SS$ if and only if $q \in SS$. It follows that the μ measure of SS is the same as the ν measure of SS and hence that (regarding G and H of Lemma 5.3) the measure of SS given G is the same as the conditional measure of SS given H. Since those are any two words from $-N$ to N for any N, it follows that SS is independent of any cylinder set.

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Since SS can be arbitrarily well approximated by a cylinder set it follows that the measure of SS is either 0 or 1. □

6. *Proof that B splits*

THEOREM 6.1. *B splits.*

Proof. In [**7**], Thouvenot established a condition for determining whether or not a factor splits. The Thouvenot condition (slightly rephrased) is the following generalization of the very weak Bernoulli condition (actually it is a slightly rephrased form of the relative very weak Bernoulli condition which in [**7**] was shown to imply the relatively finitely determined condition which in [**3**] was shown to imply splitting).

Let T be a transformation, let Q_1 be a partition which generates T, and let Q_2 be another partition. Suppose that, for every $\epsilon > 0$, for all sufficiently large n, there is a coupling of the $((Q_1 \vee Q_2), T)$ process with itself so that, for (ω_1, ω_2) in the coupled process:

(1) the Q_2 name of ω_1 is the same as the Q_2 name of ω_2;

(2) the past Q_1 name of ω_1 conditioned on the complete Q_2 name is independent of the past Q_1 name of ω_2 conditioned on the complete Q_2 name;

(3) for $\epsilon > 0$ and all sufficiently large n, the expected mean Hamming distance between

$$\text{the } Q_1 \text{ name from time 0 to time } n \text{ of } \omega_1$$

and

$$\text{the } Q_1 \text{ name from time 0 to time } n \text{ of } \omega_2$$

is less than ϵ.

Then it follows that the Q_2, T process splits, which means that there is another partition B such that the B, T process is independent of the Q_2, T process, the B, T process is Bernoulli, and $Q_2 \vee B$ generates T.

Hence we only have to establish (1)–(3), when Q_2 is the P' of Definition 19 and Q_1 is the standard generator P of the T, T^{-1} process, namely

$$P := \{(h, L), (h, R), (t, L), (t, R)\}.$$

We now start to produce the required coupling. We can just assume (1) and (2) by simply picking a P' name in accordance with the measure on P' names and then couple the past P names independently given that P' name.

Now we have started the coupling. We have coupled the complete P and P' pasts and the P' name of the future. What is left is to couple the complete future P names (given the pasts and the P' futures) so that they are close in mean Hamming distance from time 0 to time n for sufficiently large n. In fact we will do something stronger: we will get the future P names to completely agree eventually. Before continuing the coupling, let us see what we already know.

(a) We know the P' names of the past.

(b) We know that the P' names of the past are the same.

(c) We know the future P' names.

(d) We know that the future P' names are the same.

(e) We know the past P names.

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By (a) and Comment 12, the first positive time either process lands in S they will know their entire scenery; and by (b)–(d), the first time the processes land in S will be the same and the scenery they will both know at that time will be the same. Suppose that the first positive time they land in S will be time 30. The T, T^{-1} transformation has been defined by starting with the path and the scenery at time 0 and using them to generate an element of the T, T^{-1} process. Hence, from time 30 onward, since you know the scenery at time 30, the future (after time 30) path will determine the P and P' name, but, given the scenery, the future path will determine the future P' name *independent of the past P' name and past P name*. Keep in mind that when one parameter determines another then knowledge about the latter parameter tells you information about the former parameter and nothing else; for example, if your name determines whether or not you will be wealthy, then knowing that you will be wealthy determines information about your name and it does not determine anything else. So given the scenery, the future P' name gives information about the future path after time 30 and about nothing else, and the information it gives will be the same for both ω_1 and ω_2 because they see the same scenery. Hence, we can continue the coupling by coupling independently until the first time the two processes reach S (in our example until time 30) and then couple the two paths to be the same after that (which causes the P names to be the same after that). □

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