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Low-Mach-number asymptotics for two-phase flows of granular materials

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In this paper, we generalize the concept of low-Mach-number approximation to multi-phase flows and apply it to the two-phase flow model of Papalexandris (*J. Fluid Mech.*, vol. 517, 2004, p. 103) for granular materials. In our approach, the governing system of equations is first non-dimensionalized with values that correspond to a reference thermodynamic state of the phase with the smaller speed of sound. By doing so, the Mach number based on this reference state emerges as a perturbation parameter of the equations in hand. Subsequently, we expand each variable in power series of this parameter and apply singular perturbation techniques to derive the low-Mach-number equations. As expected, the resulting equations are considerably simpler than the unperturbed compressible equations. Our methodology is quite general and can be directly applied for the systematic reduction of continuum models for granular materials and for many different types of multi-phase flows. The structure of the low-Mach-number equations for two special cases of particular interest, namely, constant-density flows and the equilibrium limit is also discussed and analysed. The paper concludes with some proposals for experimental validation of the equations.

Key words: complex fluids, granular media, particle/fluid flow

1. Introduction

According to the well-known incompressibility criterion, a flow is said to be incompressible if both the velocity variations and the reference velocity, i.e. the average velocity over space and time, are small with respect to the speed of sound (Gallavotti 2002). In other words, a fluid flow is incompressible if the Mach number remains much smaller than unity. In particular, according to this criterion, compressibility effects are absent from flows in which density variations result from temperature variations only. Such flows are usually referred to as low-Mach-number or dilatational flows and are encountered in numerous technological applications and natural phenomena.

On the other hand, the Mach number appears explicitly in the non-dimensionalized energy balance equation for simple fluids. More specifically, it appears as a multiplying factor in the expression for the work performed by the surface forces that act on the fluid. Therefore, it arises naturally as a perturbation parameter for low-Mach-number flows. This, in turn, suggests the application of singular perturbation techniques to derive a simplified version of the balance equations when the Mach number is small.

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The resulting system is usually referred to as the *low-Mach-number equations*. This singular perturbation analysis has the following advantages:

(i) It identifies terms whose magnitude remains sufficiently small and, therefore, can be neglected. More specifically, it removes all terms associated with compressibility effects. As a consequence, the momentum and energy equations essentially decouple from each other. Thus, the resulting system is much simpler than the unperturbed, compressible one.

(ii) As the Mach number approaches zero, the fluid velocity becomes increasingly smaller than the sound speed and, therefore, the time step required for the numerical integration of the (compressible) equations becomes very small, due to the constraint imposed by the Courant-Friedrichs-Lewy (CFL) condition. Special acceleration techniques such as pre-conditioning, multi-grid and residual smoothing can only partially alleviate this problem. However, the low-Mach-number equations are not subject to such constraints, due to the removal of all terms associated with compressibility effects. Therefore, they can be integrated with much larger time steps, which results in dramatic computational savings.

(iii) In the low-Mach-number regime, the accuracy and efficiency of standard numerical methods for compressible flows are reduced considerably (Volpe 1993), independently of their performance in other regimes, e.g. in the transonic or the supersonic regime. This deterioration is likely to be due to the fact that the zero-Mach-number limit of the unperturbed, compressible equations is singular. On the other hand, the low-Mach-number equations are not singular in the limit as the Mach number goes to zero, because they result from singular perturbation expansion. This, in turn, considerably simplifies the design of time-accurate and efficient algorithms for their numerical treatment.

The price that must be paid for the use of the low-Mach-number approximation is the complete loss of acoustic effects. Indeed, terms that describe these effects are of higher order and, therefore, are neglected from the low-Mach-number equations. Nonetheless, for many flows of interest, this is usually a small price to pay when compared to the computational gain resulting from the low-Mach-number approximation.

In the past, several authors have contributed to the development of algorithms for the low-Mach-number equations of simple fluids. For example, Chenoweth & Paolucci (1986) developed a predictor–corrector type algorithm and studied natural convection in enclosed domains, whereas Herczynski & Kassoy (1991) studied the same problem in the absence of gravity via semi-analytical techniques based on Laplace transform, in one dimension. Further, Majda & Sethian (1985) presented an algorithm for low-Mach-number flows with combustion based on random vortex techniques. Also, for the same type of flows, Cook & Riley (1996) proposed an algorithm based on sixth-order compact differences and a fifth-order compact boundary scheme, and Najm, Wyckoff & Knio (1998) proposed an additive semi-implicit projection method. More recently, Lessani & Papalexandris (2006) designed and implemented an algorithm for low-Mach-number flows with strong temperature gradients, with or without combustion, based on a predictor–corrector time integration scheme and a projection method for the momentum equation; see also Lessani & Papalexandris (2008) for an application of this algorithm to dilatational turbulent channel flows. A detailed list and a comprehensive description of various numerical methods for low-Mach-number flows can be found in the recent article of Knikker (2010). Comparisons of the efficiency and robustness of these methods can also be found in that article.

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Alongside algorithm development and numerical analysis, considerable research efforts have been devoted to the mathematical analysis of the low-Mach-number equations for simple fluids. For example, Klainerman & Majda (1981) developed a general theoretical framework for the study of singular limits of quasi-linear hyperbolic systems. As an application, the authors proved the convergence of strong (classical) solutions of the isentropic compressible Navier–Stokes equations to strong solutions of the incompressible ones, provided that strong solutions for the compressible equations exist and that the initial data are subject to scaling constraints so as to balance the acoustic components, i.e. they are 'nearly incompressible'. In other words, for isentropic flows, the low-Mach-number expansion is convergent. This result has subsequently been generalized to other types of flows. The most general case, i.e. the Navier–Stokes–Fourier equations, has been studied by Alazard (2006) and Feireisl & Novotný (2007). More specifically, Alazard (2006) proved the short-time convergence of the low-Mach-number expansion for classical solutions, albeit for 'nearly incompressible' initial data. Finally, Feireisl & Novotný (2007) proved the large-time convergence of the low-Mach-number expansion for weak solutions, under similar assumptions for the initial data.

In this paper, we present the generalization of low-Mach-number asymptotics to continuum models for multi-phase mixtures and apply it to the two-phase flow model of granular materials proposed by Papalexandris (2004). However, our methodology is quite general and can be applied directly to other models of two-phase granular mixtures and to the governing equations for other types of multi-phase flows.

To the best of our knowledge, low-Mach-number asymptotics have not yet been generalized to multi-phase flow models. This can possibly be attributed to the size and complexity of these models. In fact, the number of equations in these models is much larger than in single-phase flow models. Further, they contain high-order nonlinear differential operators and relaxation terms to describe interactions between phases and phenomena due to the microstructure of the mixture. These additional terms significantly increase the complexity of the equations in hand. Nonetheless, incompressible multi-phase flows with strong temperature gradients are encountered in numerous technological applications and natural phenomena. Therefore, it would be useful to generalize the concept of the low-Mach-number approximation for these flows and benefit from the aforementioned advantages of this approximation.

In particular, low-Mach-number equations are well adapted for the study of irreversible phenomena such as diffusive processes and relaxation mechanisms in the mixtures of interest, by virtue of the decoupling between the energy and momentum balance equations. They are also convenient for the study of the interplay between these phenomena as two-phase granular mixtures are driven to an equilibrium or a stationary state. Further, they are well adapted for the study of a number of phenomena emerging in two-phase flows that are currently not well understood. Examples include the structure of boundary layers, the evolution of interfaces, the formation and evolution of zones of high particle concentrations (compaction zones), and others.

As regards the numerical treatment of the flows of interest, the low-Mach-number expansion results in equations that are much easier to study numerically than the unperturbed compressible ones. This is of particular importance because, up to now, numerical results for incompressible multi-phase flows with temperature gradients exist only for very special cases that are of little practical interest, such as very dilute mixtures in which the volume occupied by the solid particles is negligible.

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This paper is organized as follows. In § 2, we briefly describe the mathematical model in hand. We also give the equation of state of the solid phase and discuss its implications for the modelling of the microstructure of the mixture and for the structure of the solid energy equation. In § 3, we generalize the concept of low-Mach-number asymptotics to two-phase flows, derive the low-Mach-number equations for the constitutive model in hand and discuss some important characteristics of these equations. In § 4, we discuss the structure of the low-Mach-number equations for two interesting special cases, namely, constant-density flows and the equilibrium limit of two-phase granular materials. Finally, § 5 contains some remarks about the experimental validation of the low-Mach-number equations derived herein.

2. The two-phase flow model for granular materials

Let us consider a mixture of an isotropic granular material and a simple fluid occupying a domain $\Omega \subset \mathbb{R}^3$. It is assumed that the mixture is immiscible and saturated, and that there are no phase transformations and chemical reactions. According to the continuum-theory approach, the mixture is assumed to consist of two separate, identifiable continua occupying the same space: the fluid phase and the granular phase, also referred to as the solid phase.

The mass, momentum and energy equations for the fluid phase read (Drew & Passman 1999; Papalexandris 2004)

$$\frac{\mathrm{d}\hat{\rho}_f\phi_f}{\mathrm{d}\hat{t}_f} + \hat{\rho}_f\phi_f\hat{\nabla}\cdot\hat{\boldsymbol{u}}_f = 0, \tag{2.1a}$$

$$\hat{\rho}_f\phi_f\frac{\mathrm{d}\hat{\boldsymbol{u}}_f}{\mathrm{d}\hat{t}_f} + \hat{\nabla}(\hat{p}_f\phi_f) = -\hat{\nabla}\big(\hat{p}^v_f\phi_f\big) - \hat{\nabla}\cdot\big(\phi_f\hat{\boldsymbol{P}}^v_f\big) - \hat{\boldsymbol{f}} - \hat{\rho}_f\,\phi_f\,\hat{\boldsymbol{g}}, \tag{2.1b}$$

$$\hat{\rho}_f\phi_f\frac{\mathrm{d}\,\hat{e}_f}{\mathrm{d}\hat{t}_f} + \hat{p}_f\phi_f\hat{\nabla}\cdot\hat{\boldsymbol{u}}_f = -\hat{p}^v_f\phi_f\hat{\nabla}\cdot\hat{\boldsymbol{u}}_f - \phi_f\hat{\boldsymbol{P}}^v_f : \hat{\boldsymbol{V}}^v_f - \hat{\nabla}\cdot(\phi_f\hat{\boldsymbol{q}}_f) + \hat{\boldsymbol{f}}\cdot(\hat{\boldsymbol{u}}_f - \hat{\boldsymbol{u}}_s) - \hat{E}, \tag{2.1c}$$

respectively, whereas, for the solid phase, they read

$$\frac{\mathrm{d}\hat{\rho}_s\phi_s}{\mathrm{d}\hat{t}_s} + \hat{\rho}_s\phi_s\hat{\nabla}\cdot\hat{\boldsymbol{u}}_s = 0, \tag{2.1d}$$

$$\hat{\rho}_s\phi_s\frac{\mathrm{d}\hat{\boldsymbol{u}}_s}{\mathrm{d}\hat{t}_s} + \hat{\nabla}(\hat{p}_s\phi_s) = -\hat{\nabla}\big(\hat{p}^v_s\phi_s\big) - \hat{\nabla}\cdot\big(\phi_s\hat{\boldsymbol{P}}^v_s\big) + \hat{\boldsymbol{f}} - \hat{\rho}_s\,\phi_s\,\hat{\boldsymbol{g}}, \tag{2.1e}$$

$$\hat{\rho}_s\phi_s\frac{\mathrm{d}\,\hat{e}_s}{\mathrm{d}\hat{t}_s} + \hat{p}_s\phi_s\hat{\nabla}\cdot\hat{\boldsymbol{u}}_s = -\hat{p}^v_s\phi_s\hat{\nabla}\cdot\hat{\boldsymbol{u}}_s - \phi_s\hat{\boldsymbol{P}}^v_s : \hat{\boldsymbol{V}}^v_s - \nabla\cdot(\phi_s\,\hat{\boldsymbol{q}}_s) + \hat{E}, \tag{2.1f}$$

respectively.

In the above equations, the subscripts 's' and 'f' denote the solid and fluid phases, respectively. Further, $\hat{\rho}_i$, $\hat{p}_i$, $\hat{\boldsymbol{u}}_i$, ϕ_i and $\hat{e}_i$, $i = s, f$, are the density, pressure, velocity vector, volume fraction and internal energy of phase i. Also, in the above equations and throughout the paper, the hat symbol '$\hat{}$' over a variable denotes that the variable is in dimensional form and the symbol '$:$' stands for the component-wise (Hadamard) product of two tensors. Further, the operator $\mathrm{d}/\mathrm{d}\hat{t}_i = (\partial/\partial\,\hat{t}) + \hat{\boldsymbol{u}}_i\cdot\hat{\nabla}$ stands for the derivative along the streamlines of phase i (material derivative).

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The quantities $\hat{p}_i^v$ and $\hat{\boldsymbol{P}}_i^v$ result from the natural decomposition of the stress tensor $\hat{\boldsymbol{P}}$ into a diagonal part and a traceless deviatoric part,

$$\hat{\boldsymbol{P}}_i = -\big(\hat{p}_i + \hat{p}_i^v\big)\,\boldsymbol{I} + \hat{\boldsymbol{P}}_i^v, \quad i = s,\, f, \tag{2.2}$$

where $\hat{p}_i^v$ are the bulk viscous pressures, respectively, and $\boldsymbol{I}$ is the identity matrix. Further, $\hat{\boldsymbol{V}}_i^v$ are the traceless deviatoric parts of the deformation tensors $\hat{\boldsymbol{V}}_i$. In other words, $\hat{\boldsymbol{V}}_i = (1/2)(\hat{\nabla}\hat{\boldsymbol{u}}_i + (\hat{\nabla}\hat{\boldsymbol{u}}_i)^{\mathrm{T}})$ and $\hat{\boldsymbol{V}}_i^v = \hat{\boldsymbol{V}}_i - (1/3)(\hat{\nabla}\cdot\hat{\boldsymbol{u}}_i)\,\boldsymbol{I}$. Finally, $\hat{\boldsymbol{q}}_i$ are the heat conduction fluxes inside each phase, and $\hat{\boldsymbol{f}}$, $\hat{E}$ represent the momentum and heat exchanges between the two phases, respectively.

This system is closed with an evolution equation for the solid volume fraction, state equations for both phases and the saturation condition,

$$\phi_s(\boldsymbol{x}) + \phi_f(\boldsymbol{x}) = 1, \quad \boldsymbol{x} \in \Omega. \tag{2.3}$$

Equations (2.1*a*)–(2.1*f*) must be supplemented with constitutive relations for all dissipative (irreversible) processes in the mixture. Papalexandris (2004) provided such constitutive relations by assuming that a generalized Gibbs equation holds for each phase and by taking advantage of the constraints imposed by the entropy inequality for the mixture. In the present paper, we derive the low-Mach-number approximation of this particular model, which will be, henceforth, referred to as the *two-phase flow model of reference*. For the derivation of this model, the following constitutive assumptions are made:

(i) Each phase of the mixture is endowed with its own set of thermodynamic variables and velocity vector.

(ii) Each phase is in a local (equilibrium) state. Therefore, the entropy of each phase is a state variable, and its differential 1-form is exact and defined by a Gibbs equation.

(iii) The momentum and energy exchanges between the two phases are pure, i.e. their sum vanishes.

(iv) The principle of phase separation holds. This implies that irreversible phenomena that take place 'inside' one phase depend only on thermodynamic variables of this phase.

(v) Microinertia effects of the solid phase can be ignored.

In conclusion, the two phases of the mixture are treated as two separate thermodynamic subsystems that interact with each other. Nonetheless, the entire system, i.e. the mixture, can be open, closed or isolated, depending on the boundary conditions.

Since granular materials possess microstructure, additional variables are needed for the complete description of the thermodynamic state of the solid phase. In the two-phase flow model of reference, these additional variables are the solid volume fraction and its spatial gradient (ϕ_s and $\hat{\nabla}\phi_s$, respectively). The use of ϕ_s as a concentration parameter is standard in multi-phase flow problems. Usually it is defined through averaging over a so-called representative volume. According to its usual interpretation, the mixture consists of two co-existing continua. However, when applied to granular materials, this definition is unsatisfactory from both the mathematical and the physical point of view, because of the microstructure of these materials. In fact, in many granular mixtures, even the naked eye can observe the phase separation and determine if a given point in space is occupied by one phase or the other. For this reason, we consider that it is more appropriate to interpret the volume fraction as the probability of finding one phase or the other in a given measurable set in space and at a given

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time. In other words, the volume fraction is a probability density function defined through the Radon–Nikodym theorem (Goodman & Cowin 1972). This axiomatic definition of the volume fraction is consistent with the measure-theoretic framework of thermodynamics and has important consequences in the mathematical analysis of the governing equations.

Further, it is well known that granular materials support shear at equilibrium. These shear stresses result from non-local effects related to the microstructure of the system and depend on the intergranular and interfacial surface densities. On the other hand, the volume fraction gradient, $\hat{\nabla}\phi_s$, is the natural measure of the interfacial area density (Drew & Passman 1999). Therefore, it is introduced as the additional degree of freedom that is required for the constitutive description of these stresses.

Summing up, the fluid phase is treated as a simple Newtonian fluid; hence, its state-variable vector is $(\hat{\rho_f}, \hat{e}_f)^{\mathrm{T}}$. On the other hand, the state-variable vector for the solid phase is $(\hat{\rho}_s, \hat{e}_s, \phi_s, \hat{\nabla}\phi_s)^{\mathrm{T}}$. Therefore, the standard Gibbs relation holds for the fluid phase,

$$\hat{T}_f \,\mathrm{d}\hat{\eta}_f = \mathrm{d}\hat{e}_f - \frac{\hat{p}_f}{\hat{\rho}_f^2}\,\mathrm{d}\hat{\rho}_f, \tag{2.4}$$

whereas a generalized Gibbs relation holds for the solid phase,

$$\hat{T}_s \,\mathrm{d}\hat{\eta}_s = \mathrm{d}\hat{e}_s - \frac{\hat{p}_s}{\hat{\rho}_s^2}\,\mathrm{d}\hat{\rho}_s - \frac{\hat{\beta}_s}{\hat{\rho}_s\phi_s}\,\mathrm{d}\phi_s - \frac{1}{\hat{\rho}_s\phi_s}\hat{\boldsymbol{h}}_s \cdot \mathrm{d}(\hat{\nabla}\phi_s). \tag{2.5}$$

In these equations, $\hat{\eta}_i$, $i = f, s$, is the entropy of phase i. Further, the quantities $\hat{\beta}_s$ and $\hat{\boldsymbol{h}}_s$ are defined via the granular material's Helmholtz free energy $\hat{\psi}_s$ as follows:

$$\hat{\beta}_s = \hat{\rho}_s\phi_s\,\frac{\partial\hat{\psi}_s}{\partial\phi_s}, \quad \hat{\boldsymbol{h}}_s = 2\hat{\rho}_s\phi_s\,\frac{\partial\hat{\psi}_s}{\partial(\hat{\nabla}\phi_s)}. \tag{2.6}$$

In particular, $\hat{\beta}_s$ is the *configuration pressure* and represents the contact forces that are developed between grains under compaction. Also, the vector $\hat{\boldsymbol{h}}_s$ represents forces due to rearrangements in the distribution of the interfacial area density. Thus, in the context of the present formalism, rearrangements in the interfacial area density are considered to be dispersive, weakly non-local effects. Henceforth, and in analogy with the name adopted for $\hat{\beta}_s$, it will be referred to as the *configuration stress vector*. The presumed isotropy of the granular material implies that the Helmholtz free energy of the solid phase $\hat{\psi}_s$ is an isotropic function. This means that the configuration stress vector $\hat{\boldsymbol{h}}_s$ can be rewritten as

$$\hat{\boldsymbol{h}}_s = \hat{\Gamma}_s\hat{\nabla}\phi_s, \quad \text{where} \quad \hat{\Gamma}_s \equiv 2\hat{\rho}_s\phi_s\,\frac{\partial\hat{\psi}_s}{\partial|\hat{\nabla}\phi_s|^2}. \tag{2.7}$$

Henceforth, the coefficient $\hat{\Gamma}_s$ will be referred to as the *configuration stress coefficient*. In principle, both $\hat{\beta}_s$ and $\hat{\Gamma}_s$ are functions of all state variables of the solid phase.

It is worth mentioning that the Gibbs relation for the solid phase (see (2.5)) implies that the only elastic shear stresses acting on the grains are those related to the microstructure of the granular medium, i.e. the stresses due to the grain distribution and configuration in space. Experience shows that this constitutive assumption holds for many incompressible granular flows of interest under moderate external forcings (see Bagnold 1954 and Savage 1979). Further, it has been adopted in all previous mixture theories for granular media; see, for example, Goodman & Cowin (1972),

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Passman, Nunziato & Bailey (1986), Wang & Hutter (1999), Wang & Hutter (2006) and others. Nonetheless, this constitutive assumption can be relaxed easily. For example, one can model the granular material as an elastic solid by adding to (2.5) the terms that describe the reversible work performed by (elastic) shear stresses.

The balance equations of the mixture (i.e. (2.1*a*)–(2.1*f*)) and the Gibbs equations (i.e. (2.4) and (2.5)) can be used to derive an expression for the (non-negative) entropy production in the mixture as a sum of products between thermodynamic fluxes J_i and thermodynamic forces X_k (de Groot & Mazur 1984; Lebon, Jou & Casas-Vázquez 2008). By assuming linear relations between conjugate fluxes and forces ($i = k$), and by ignoring couplings between non-conjugate ones ($i \neq k$), the following constitutive expressions are derived for the thermodynamic fluxes:

$$\hat{E} = \hat{h}\left(\frac{1}{\hat{T}_s} - \frac{1}{\hat{T}_f}\right), \tag{2.8a}$$

$$\hat{p}_f^{\,v} = -\hat{\zeta}_f \hat{\boldsymbol{\nabla}} \cdot \hat{\boldsymbol{u}}_f, \tag{2.8b}$$

$$\hat{p}_s^{\,v} = \frac{\hat{\Gamma}_s}{3\,\phi_s}|\hat{\boldsymbol{\nabla}}\phi_s|^2 - \hat{\zeta}_s\,\hat{\boldsymbol{\nabla}} \cdot \hat{\boldsymbol{u}}_s, \tag{2.8c}$$

$$\hat{\boldsymbol{q}}_f = -\hat{\kappa}_f \hat{\boldsymbol{\nabla}} \hat{T}_f, \tag{2.8d}$$

$$\hat{\boldsymbol{q}}_s = -\hat{\kappa}_s \hat{\boldsymbol{\nabla}} \hat{T}_s, \tag{2.8e}$$

$$\hat{\boldsymbol{f}} = \hat{p}_f \hat{\boldsymbol{\nabla}}\phi_s + \hat{\delta}\,(\hat{\boldsymbol{u}}_f - \hat{\boldsymbol{u}}_s), \tag{2.8f}$$

$$\hat{\boldsymbol{P}}_f^{\,v} = -\hat{\mu}_f \hat{\boldsymbol{V}}_f^{\,v}, \tag{2.8g}$$

$$\hat{\boldsymbol{P}}_s^{\,v} = \frac{\hat{\Gamma}_s}{\phi_s}\hat{\boldsymbol{\Phi}}_s^{\,v} - \hat{\mu}_s \hat{\boldsymbol{V}}_s^{\,v}, \tag{2.8h}$$

$$\frac{\mathrm{d}\phi_s}{\mathrm{d}\,\hat{t}_s} = \hat{\mu}_c^{-1}\left(\frac{\hat{p}_s - \hat{\beta}_s}{\hat{T}_s} - \frac{\hat{p}_f}{\hat{T}_f} + \hat{\boldsymbol{\nabla}} \cdot \left(\frac{\hat{\Gamma}_s}{\hat{T}_s}\hat{\boldsymbol{\nabla}}\phi_s\right)\right). \tag{2.8i}$$

In the above equations, $\hat{\zeta}_i$ is the bulk viscosity, $\hat{\mu}_i$ is the shear viscosity and $\hat{k}_i$ is the thermal conductivity of phase i. Further, $\hat{h}$ is the interfacial heat transfer coefficient, $\hat{\delta}$ is the interfacial drag coefficient and $\hat{\mu}_c$ is the dynamic compaction coefficient. Also, in the above equations, $\hat{\boldsymbol{\Phi}}_s^{\,v}$ is defined as the traceless deviatoric part of $\hat{\boldsymbol{\nabla}}\phi_s \otimes \hat{\boldsymbol{\nabla}}\phi_s$,

$$\hat{\boldsymbol{\nabla}}\phi_i \otimes \hat{\boldsymbol{\nabla}}\phi_i = \tfrac{1}{3}|\hat{\boldsymbol{\nabla}}\phi_i|^2 \boldsymbol{I} + \hat{\boldsymbol{\Phi}}_s^{\,v}, \tag{2.9}$$

where the symbol '$\otimes$' denotes the tensor product of two vectors. The compaction equation (2.8*i*) describes the evolution of the solid volume fraction and takes the form of a nonlinear reaction–diffusion equation along the streamlines of the granular material. In contrast to other models that have been proposed for two-phase flows of granular materials, the two-phase flow model of reference predicts that the evolution of ϕ_s depends on the phasial temperatures, and hence, on the thermal non-equilibrium between the two phases. This implies that the evolution of ϕ_s is not reversible, even in the absence of configuration stresses, for as long as the two phases are not in thermal equilibrium with each other. Consequently, the solid entropy is not conserved along streamlines of the solid phase. As we will see below, these properties have important

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implications for the structure of the low-Mach-number equations for the mixtures of interest.

2.1. *Equations of state*

The dynamics of the thermodynamic state of the mixtures of interest are complex, even in low-Mach-number flows, because of the microstructure of granular materials and because of the large size of the mixture's thermodynamic state vector. For this reason, it is useful to discuss the implications of the equations of state for each phase in the governing system and the derivation of the low-Mach-number approximation.

At this point, it is important to make the distinction between *solid material* and *solid phase*. By the term 'solid material', we denote the material that the solid particles (grains) consist of. In this paper, and for the sake of simplicity, this material is treated as a stiffened gas and, therefore, its thermodynamic state is defined by two independent variables. By contrast, by the term 'solid phase' or 'granular material' we denote the ensemble of the grains, whose thermodynamic state vector consists of the thermodynamic vector of the solid material plus the solid volume fraction and its spatial gradient. On the other hand, such distinction is not necessary for the fluid phase because we assume that the fluid material is a simple Newtonian fluid. As such, this material does not have any microstructure and it does exhibit scale segregation. Therefore, the thermodynamic state vectors for both fluid material and fluid phase are the same.

For the sake of simplicity, we assume that the fluid phase obeys the ideal-gas equation of state,

$$\hat{p}_f = \hat{R}_f \hat{\rho}_f \hat{T}_f, \tag{2.10}$$

where $\hat{R}_f$ is the fluid constant, i.e. $\hat{R}_f = \hat{c}_{p,f} - \hat{c}_{v,f}$, with $\hat{c}_{p,f}$ and $\hat{c}_{v,f}$ being the specific heats of the fluid under constant pressure and constant volume, respectively.

As regards the equation of state for the solid phase, we first observe that a given thermodynamic potential of this phase can be decomposed into two parts: one part that is independent of the additional variables ϕ_s and $\hat{\nabla}\phi_s$, and another part that is a function of them. Therefore, the first part relates to the thermodynamics of the solid material, i.e. the material that the grains consist of. It is a function of the solid density and temperature only and is referred to as the *pure part* of the thermodynamic potential. The remaining second part is a function of ϕ_s and $\hat{\nabla}\phi_s$ and possibly other thermodynamic variables, and is referred to as the *configuration part* of the thermodynamic potential.

Therefore, the following decompositions are introduced for the thermodynamic potentials of internal energy, entropy and Helmholtz free energy:

$$\hat{e}_s = \hat{e}_s^p(\hat{\rho}_s, \hat{T}_s) + \hat{e}_s^c(\hat{\rho}_s, \hat{T}_s, \phi_s, \hat{\nabla}\phi_s), \tag{2.11}$$

$$\hat{\eta}_s = \hat{\eta}_s^p(\hat{\rho}_s, \hat{T}_s) + \hat{\eta}_s^c(\hat{\rho}_s, \hat{T}_s, \phi_s, \hat{\nabla}\phi_s), \tag{2.12}$$

$$\hat{\psi}_s = \hat{\psi}_s^p(\hat{\rho}_s, \hat{T}_s) + \hat{\psi}_s^c(\hat{\rho}_s, \hat{T}_s, \phi_s, \hat{\nabla}\phi_s). \tag{2.13}$$

The Helmholtz free energies $\hat{\psi}_s^p$ and $\hat{\psi}_s$ are defined as follows:

$$\hat{\psi}_s^p = \hat{e}_s^p - \hat{T}_s \hat{\eta}_s^p, \tag{2.14}$$

$$\hat{\psi}_s = \hat{e}_s - \hat{T}_s \hat{\eta}_s. \tag{2.15}$$

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From (2.14) and (2.15), and by virtue of the decompositions (2.11)–(2.13), we deduce that

$$\hat{\psi}_s^c = \hat{e}_s^c - \hat{T}_s\,\hat{\eta}_s^c. \tag{2.16}$$

Further, from the thermodynamics of the solid material, the following Gibbs relation holds:

$$\hat{T}_s\,\mathrm{d}\hat{\eta}_s^p = \mathrm{d}\hat{e}_s^p - \frac{\hat{p}_s}{\hat{\rho}_s^2}\,\mathrm{d}\hat{\rho}_s, \tag{2.17}$$

where, as mentioned earlier, it is assumed that elastic effects are negligible at length scales *smaller* than the grain diameter. Then, by subtracting (2.17) from (2.5), we obtain the Gibbs relation for the configuration part of the entropy of the granular material,

$$\hat{T}_s\,\mathrm{d}\hat{\eta}_s^c = \mathrm{d}\hat{e}_s^c + \frac{\hat{\beta}_s}{\hat{\rho}_s\phi_s}\,\mathrm{d}\phi_s + \frac{1}{\hat{\rho}_s\phi_s}\hat{\boldsymbol{h}}_s\cdot\mathrm{d}(\hat{\nabla}\phi_s). \tag{2.18}$$

Upon differentiation of (2.16) and introduction of the result in (2.18), we arrive at

$$\mathrm{d}\hat{\psi}_s^c = -\hat{\eta}_s^c\,\mathrm{d}\hat{T}_s - \frac{\hat{\beta}_s}{\hat{\rho}_s\phi_s}\,\mathrm{d}\phi_s - \frac{1}{\hat{\rho}_s\phi_s}\hat{\boldsymbol{h}}_s\cdot\mathrm{d}(\hat{\nabla}\phi_s), \tag{2.19}$$

from which we finally deduce that

$$\hat{\eta}_s^c = -\left(\frac{\partial\hat{\psi}_s^c}{\partial\hat{T}_s}\right)_{\hat{\rho}_s,\phi_s,\hat{\nabla}\phi_s}. \tag{2.20}$$

This relation implies that if $\hat{\psi}_s^c$ does not depend on temperature $\hat{T}_s$, then the configuration entropy $\hat{\eta}_s^c$ is zero. Evidently, in the general case where the configuration free energy is a function of the temperature, then the configuration entropy is non-zero.

In this paper, for the sake of simplicity, we assume that the solid material obeys the stiffened-gas equation of state,

$$\hat{p}_s = -\gamma_s\hat{P}_\infty + (\gamma_s - 1)\hat{\rho}_s\hat{e}_s^p, \tag{2.21}$$

where $\hat{P}_\infty$ is a constant parameter, usually referred to as the 'correction pressure', and γ_s is a non-dimensional constant. In fact, it can be shown that γ_s equals the heat capacity ratio, $\gamma_s = \hat{c}_{p,s}/\hat{c}_{v,s}$, with $\hat{c}_{p,s}$ and $\hat{c}_{v,s}$ being the specific heats under constant pressure and constant volume, respectively, of the solid material (LeMetayer, Massoni & Saurel 2004). Further, from definition (2.21), one can readily obtain the thermal equation of state of a stiffened gas,

$$\hat{p}_s = \hat{R}_s\hat{\rho}_s\hat{T}_s - \hat{P}_\infty, \tag{2.22}$$

where $\hat{R}_s$ is defined as the difference between the specific heats, $\hat{R}_s = \hat{c}_{p,s} - \hat{c}_{v,s}$. Usually, in the stiffened-gas approximation, the specific heats are considered constant. However, the above relations can be extended to the more general case where the specific heats are functions of the temperature.

Further, after some simple algebraic manipulations involving definition (2.21) and standard thermodynamic relations, one can obtain the following canonical forms of

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the internal energy and entropy of a stiffened gas:

$$\hat{e}_s^p = \frac{\hat{P}_\infty}{\hat{\rho}_s} + \hat{c}_{v,s}\hat{T}_s, \tag{2.23}$$

and

$$\hat{\eta}_s^p = \hat{c}_{v,s}\,\log\left(\left(\frac{\hat{T}_s}{T_0}\right)\left(\frac{\hat{\rho}_s}{\rho_0}\right)^{1-\gamma_s}\right) + \eta_0. \tag{2.24}$$

In the above equation, T_0, ρ_0 and η_0 are the temperature, density and entropy of the solid material in a reference state, respectively.

Finally, upon introduction of (2.23) and (2.24) in (2.14), we arrive at the canonical form of the Helmholtz free energy of a stiffened gas,

$$\hat{\psi}_s^p = \frac{\hat{P}_\infty}{\hat{\rho}_s} + \hat{c}_{v,s}\hat{T}_s\left(1 - \log\left(\left(\frac{\hat{T}_s}{T_0}\right)\left(\frac{\hat{\rho}_s}{\rho_0}\right)^{1-\gamma_s}\right)\right) - \hat{T}_s\eta_0. \tag{2.25}$$

As regards the configuration part of the Helmholtz free energy of the solid phase, the simplest choice would be to consider a quadratic function of ϕ_s and $\hat{\nabla}\phi_s$, i.e.

$$\hat{\psi}_s^c = \tfrac{1}{2}\hat{k}_1\phi_s^2 + \tfrac{1}{2}\hat{k}_2|\hat{\nabla}\phi_s|^2, \tag{2.26}$$

where $\hat{k}_1$ and $\hat{k}_2$ are positive, material-dependent constants. The weighted (mass) integral of this function is a special case of the well-known Ginzburg–Landau functional that is often used in statistical physics.

According to (2.26), the configuration Helmholtz free energy is assumed to be independent of the temperature. As mentioned above, by virtue of (2.20), this implies that the configuration entropy is zero, $\hat{\eta}_s^c = 0$. Consequently, relation (2.16) yields that $\hat{e}_s^c = \hat{\psi}_s^c$. Also, by employing relation (2.6), we arrive at the following expressions for the configuration pressure and configuration stress coefficient:

$$\hat{\beta}_s = \hat{k}_1\hat{\rho}_s\phi_s^2, \quad \hat{\Gamma}_s = \hat{k}_2\hat{\rho}_s\phi_s. \tag{2.27}$$

It is straightforward to verify that these expressions satisfy the convexity requirements for stability of thermodynamic equilibria and that the second-order differential operator on the right-hand side of the compaction equation (2.8*i*) is elliptic.

Evidently, one can consider other, more elaborate and more accurate expressions for the pure and the configuration part of the Helmholtz free energy of the solid phase; see, for example, Passman *et al.* (1986), Powers (2004) and references therein. As regards the configuration free energy, detailed experimental measurements in a variety of physical conditions currently do not exist in the literature. For this reason, in the past, researchers in the field of granular flows considered various ad hoc or empirical expressions for the configuration pressure and stress vector. Nonetheless, for the purpose of deriving low-Mach-number asymptotics for two-phase granular mixtures, the above expressions for the free energies of the solid phase are deemed to be adequate. More important, as it will be evident in the next section, our derivation is not limited to the above expressions but can be directly applied with more general equations of state.

It is worth mentioning that by virtue of the continuity equation (i.e. (2.1*a*)) and the ideal-gas equation of state (i.e. (2.10)), the energy equation of the fluid phase

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(see (2.1c)) can be written in terms of the fluid temperature as follows:

$$\hat{\rho}_f \phi_f \hat{c}_{p,f} \frac{\mathrm{d}\hat{T}_f}{\mathrm{d}\hat{t}_f} - \frac{\mathrm{d}\,\hat{p}_f \phi_f}{\mathrm{d}\hat{t}_f} = -\hat{p}_f^v \phi_f \hat{\nabla}\cdot\hat{\boldsymbol{u}}_f - \phi_f \hat{\boldsymbol{P}}_f^v : \hat{\boldsymbol{V}}_f^v - \hat{\nabla}\cdot(\phi_f \hat{\boldsymbol{q}}_f) + \hat{\boldsymbol{f}}\cdot(\hat{\boldsymbol{u}}_f - \hat{\boldsymbol{u}}_s) - \hat{E}. \quad (2.28)$$

Similarly, the solid energy equation (2.1f) can be written in terms of the solid temperature as follows:

$$\hat{\rho}_s \phi_s \left(\hat{c}_{p,s} \frac{\mathrm{d}\hat{T}_s}{\mathrm{d}\hat{t}_s} + \frac{\mathrm{d}\,\hat{\psi}_s^c}{\mathrm{d}\hat{t}_s} \right) - \frac{\mathrm{d}\,\hat{p}_s \phi_s}{\mathrm{d}\hat{t}_s} = -\hat{p}_s^v \phi_s \hat{\nabla}\cdot\hat{\boldsymbol{u}}_s - \phi_s \hat{\boldsymbol{P}}_s^v : \hat{\boldsymbol{V}}_s^v - \hat{\nabla}\cdot(\phi_s \hat{\boldsymbol{q}}_s) + \hat{E}. \quad (2.29)$$

3. Derivation of the low-Mach-number equations

Thus far, all equations have been given in dimensional form. In this section, we first perform the non-dimensionalization of the balance equations, which is a necessary step before the derivation of their low-Mach-number approximation. In what follows, the subscript 'r' is used to denote the (dimensional) reference values with respect to which the non-dimensionalization is performed.

First, velocity vectors, length vectors and time are non-dimensionalized with respect to the characteristic dimension of the flow domain L_r, and the reference velocity of the mixture u_r, i.e. the average velocity over space and time, as follows:

$$\boldsymbol{u}_i = \frac{\hat{\boldsymbol{u}}_i}{u_r}, \quad \boldsymbol{x} = \frac{\hat{\boldsymbol{x}}}{L_r}, \quad t = \hat{t}_r \frac{u_r}{L_r}. \quad (3.1)$$

Next, the non-dimensionalization of the thermodynamic variables, their derivatives and the various transport and relaxation coefficients is performed with respect to a reference state of the material with the smallest speed of sound. In almost all cases of interest, it is the fluid material that has the smallest speed of sound. Therefore, and without loss of generality, the non-dimensionalization is performed with respect to values that correspond to a *reference state of the fluid phase*. For example, if the flow domain is open, then the thermodynamic state of the fluid in the open boundary can be chosen as the reference state. By virtue of the ideal-gas equation of state for the fluid phase (see (2.10)), the following relation holds between the thermodynamic variables in the reference state:

$$p_r = \hat{R}_f \rho_r T_r. \quad (3.2)$$

Accordingly, the non-dimensionalization of the thermodynamic variables reads

$$p_i = \frac{\hat{p}_i}{p_r}, \quad \rho_i = \frac{\hat{\rho}_i}{\rho_r}, \quad T_i = \frac{\hat{T}_i}{T_r}, \quad i = s, f, \quad (3.3)$$

and

$$c_{v,i} = \frac{\hat{c}_{v,i}}{c_{v,r}}, \quad c_{p,i} = \frac{\hat{c}_{p,i}}{c_{p,r}}, \quad i = s, f, \quad (3.4)$$

with $c_{v,r} = \hat{c}_{v,f}(T_r)$ and $c_{p,r} = \hat{c}_{p,f}(T_r)$.

Further, the non-dimensionalization of the transport coefficients is performed with respect to the values of the transport coefficients of the fluid at the reference

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thermodynamic state, i.e.

$$\mu_i = \frac{\hat{\mu}_i}{\mu_r}, \quad \kappa_i = \frac{\hat{\kappa}_i}{\kappa_r}, \quad \zeta_i = \frac{\hat{\zeta}_i}{\zeta_r}, \quad i = s, f. \tag{3.5}$$

The Reynolds number Re, the Prandtl number Pr and the reference Mach number $M_{f,r}$ of the flow are defined as follows:

$$Re = \frac{\rho_r u_r L_r}{\mu_r}, \quad Pr = \frac{\mu_r c_{p,r}}{\kappa_r}, \quad M_{f,r} = \frac{u_r}{c_{f,r}}. \tag{3.6}$$

In the above, $c_{f,r}$ is the reference speed of sound of the fluid, defined by

$$c_{f,r} = \sqrt{\gamma_{f,r} \hat{R}_f T_r}, \tag{3.7}$$

with $\gamma_{f,r}$ being the value of the heat-capacity ratio of the fluid at the reference state, $\gamma_{f,r} = c_{p,r}/c_{v,r}$. Next, the interfacial drag coefficient $\hat{\delta}$, the interfacial heat transfer coefficient $\hat{h}$ and the dynamic compaction coefficient $\hat{\mu}_c$ are non-dimensionalized as follows:

$$\delta = \hat{\delta} \frac{L_r}{\rho_r u_r}, \quad h = \hat{h} \frac{L_r}{\rho_r u_r c_{p,r} T_r^2}, \quad \mu_c = \hat{\mu}_c \frac{T_r}{\rho_r u_r L_r}. \tag{3.8}$$

As regards the non-dimensionalization of $\hat{\beta}_s$ and $\hat{\Gamma}_s$, we remark the following. Experience with incompressible flows of granular materials shows that the configuration pressure and configuration stresses are much smaller than the reference pressure of the carrier fluid p_r (see Bagnold 1954, Passman *et al.* 1986 and Papalexandris 2004). In particular, it is expected that $\hat{\beta}_s$ and $\hat{\Gamma}_s$ scale with $\rho_r u_r^2$, which is also much smaller than p_r in the low-Mach-number regime. On the other hand, it is convenient to perform the non-dimensionalization in a way that the non-dimensional variables are of order unity. For this reason, we chose to non-dimensionalize $\hat{\beta}_s$ and $\hat{\Gamma}_s$ with respect to $\rho_r u_r^2$ as follows:

$$\beta_s = \frac{\hat{\beta}_s}{\rho_r u_r^2}, \quad \Gamma_s = \frac{\hat{\Gamma}_s}{\rho_r u_r^2 L_r^2}. \tag{3.9}$$

By performing the above non-dimensionalizations and by introducing the constitutive laws (see (2.8*a*)–(2.8*h*)) to the balance equations (see (2.1*a*)–(2.1*f*)), we arrive at the non-dimensional form of the two-phase flow model of reference. In particular, the balance equations of the fluid phase read

$$\frac{\mathrm{d}\rho_f \phi_f}{\mathrm{d}t_f} + \rho_f \phi_f \nabla \cdot \boldsymbol{u}_f = 0, \tag{3.10a}$$

$$\rho_f \phi_f \frac{\mathrm{d}\boldsymbol{u}_f}{\mathrm{d}t_f} + \frac{1}{\epsilon} \nabla (p_f \phi_f) = \frac{\xi}{Re} \nabla (\phi_f \zeta_f \nabla \cdot \boldsymbol{u}_f) + \frac{1}{Re} \nabla \cdot \left(\phi_f \mu_f \boldsymbol{V}_f^v \right) - \frac{1}{\epsilon} p_f \nabla \phi_s - \delta (\boldsymbol{u}_f - \boldsymbol{u}_s) - \rho_f \phi_f \boldsymbol{g}, \tag{3.10b}$$

$$\rho_f \phi_f c_{p,f} \frac{\mathrm{d}T_f}{\mathrm{d}t_f} - \frac{\gamma_{f,r} - 1}{\gamma_{f,r}} \frac{\mathrm{d}p_f \phi_f}{\mathrm{d}t_f} = \epsilon \frac{\xi}{Re} \zeta_f \phi_f (\nabla \cdot \boldsymbol{u}_f)^2 + \epsilon \frac{1}{Re} \phi_f \mu_f \boldsymbol{V}_f^v : \boldsymbol{V}_f^v + \frac{\gamma_{f,r} - 1}{\gamma_{f,r}} p_f (\boldsymbol{u}_f - \boldsymbol{u}_s) \cdot \nabla \phi_s + \epsilon \delta (\boldsymbol{u}_f - \boldsymbol{u}_s)^2 + \frac{1}{Pr\, Re} \nabla \cdot (\kappa_f \phi_f \nabla T_f) - h \left(\frac{1}{T_s} - \frac{1}{T_f} \right). \tag{3.10c}$$

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The balance equations for the solid phase read

$$\frac{\mathrm{d}\rho_s\phi_s}{\mathrm{d}t_s} + \rho_s\phi_s\nabla\cdot\boldsymbol{u}_s = 0, \tag{3.10d}$$

$$\begin{aligned}\rho_s\phi_s\frac{\mathrm{d}\boldsymbol{u}_s}{\mathrm{d}t_s} + \frac{1}{\epsilon}\nabla(p_s\phi_s) = {}& \frac{\xi}{Re}\nabla(\phi_s\zeta_s\nabla\cdot\boldsymbol{u}_s) + \frac{1}{Re}\nabla\cdot(\phi_s\mu_s\boldsymbol{V}_s^v)\\ & -\nabla\cdot(\Gamma_s\nabla\phi_s\otimes\nabla\phi_s) + \frac{1}{\epsilon}p_f\nabla\phi_s\\ & +\delta(\boldsymbol{u}_f-\boldsymbol{u}_s) - \rho_s\phi_s\boldsymbol{g},\end{aligned} \tag{3.10e}$$

$$\begin{aligned}\rho_s\phi_s c_{p,s}\frac{\mathrm{d}T_s}{\mathrm{d}t_s} - \frac{\gamma_{f,r}-1}{\gamma_{f,r}}\frac{\mathrm{d}p_s\phi_s}{\mathrm{d}t_s} + \epsilon\rho_s\phi_s\frac{\mathrm{d}\psi_s^c}{\mathrm{d}t_s} = {}& -\epsilon\left(\frac{\Gamma_s}{3}|\nabla\phi_s|^2 - \frac{\xi}{Re}\phi_s\zeta_s\nabla\cdot\boldsymbol{u}_s\right)\nabla\cdot\boldsymbol{u}_s\\ & -\epsilon\left(\Gamma_s\boldsymbol{\Phi}_s^v - \frac{1}{Re}\phi_s\mu_s\boldsymbol{V}_s^v\right):\boldsymbol{V}_s^v\\ & +\frac{1}{PrRe}\nabla\cdot(\kappa_s\phi_s\nabla T_s)\\ & +h\left(\frac{1}{T_s}-\frac{1}{T_f}\right).\end{aligned} \tag{3.10f}$$

Also, the compaction equation for the solid volume fraction (i.e. (2.8*i*)) in non-dimensional form is written as

$$\frac{\mathrm{d}\phi_s}{\mathrm{d}t_s} = \mu_c^{-1}\left(\frac{1}{\epsilon}\left(\frac{p_s}{T_s}-\frac{p_f}{T_f}\right) - \frac{\beta_s}{T_s} + \nabla\cdot\left(\frac{\Gamma_s}{T_s}\nabla\phi_s\right)\right). \tag{3.10g}$$

The parameter ϵ that appears in the above equations is defined as

$$\epsilon = \gamma_{f,r}\,M_{f,r}^2, \tag{3.11}$$

whereas the parameter ξ is the ratio of the reference bulk viscosity to the reference shear viscosity, i.e. $\xi = \zeta_r/\mu_r$.

Finally, the non-dimensionalized state equation for the fluid phase (i.e. (2.10)) reads

$$p_f = \rho_f\,T_f, \tag{3.12}$$

whereas the non-dimensionalized state equation for the solid material (i.e. (2.22)) reads

$$p_s = -P_\infty + R_s\rho_s\,T_s, \tag{3.13}$$

with $P_\infty = \hat{P}_\infty/p_r$ and $R_s = \hat{R}_s/\hat{R}_f$.

Each phase of the mixtures of interest has its own speed of sound and, therefore, its own Mach number. Evidently, a flow of the mixtures of interest is a low-Mach-number flow if the reference velocity is much smaller than the speeds of sound of both phases. Consequently, it is the smaller of the phasial speeds of sound that determines whether the flow of interest is in the low-Mach-number regime or not. We recall that herein, for the sake of simplicity and without loss of generality, we have assumed that the smaller speed of sound corresponds to the fluid phase.

Next, we assume that the reference Mach number of the fluid phase is sufficiently small so that $\epsilon \ll 1$. Thus, the Mach numbers of both phases are much smaller than unity and, therefore, the flow of each phase is incompressible, in accordance with the incompressibility criterion that was mentioned in § 1 (Gallavotti 2002). In turn, this implies that the governing system of equations (see (3.10*a*)–(3.13)) is singularly

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perturbed with respect to the parameter ϵ, i.e. with respect to the reference Mach number $M_{f,r}$. As such, it is subject to singular perturbation analysis. If, by contrast, the reference speed of sound of one phase is not much larger than the reference velocity, then ϵ is not small anymore. In this case, ϵ is no longer a perturbation parameter and the governing system is not subject to the proposed perturbation analysis.

Next, each of the unknown variables of the system in hand is expanded into a perturbation series of ϵ, i.e. we assume expansions of the form

$$\omega = \omega^0 + \epsilon\,\omega^1 + O(\epsilon^2), \tag{3.14}$$

whereas by the term 'regular series' we mean series of non-negative powers. Moreover, similar expansions are considered for products, such as $p_s\phi_s$, and for functions, such as $\boldsymbol{V}_s^v$.

Subsequently, we substitute the expansions of type (3.14) into (3.10*a*)–(3.13) and into the saturation relation (2.3). We also multiply (3.10*b*), (3.10*c*), (3.10*e*) and (3.10*f*) by the perturbation parameter ϵ, so that their leading-order terms are of order one, $O(1)$. Finally, we group terms of same order. The results are the following.

3.1. *Saturation relation*

At order $O(1)$, the saturation relation (2.3) reads

$$\phi_s^0 + \phi_f^0 = 1. \tag{3.15}$$

Since ϕ_s and ϕ_f are positive, it follows that ϕ_s^0 and ϕ_f^0 are also positive, for otherwise the overall perturbation series could be negative, which is a contradiction. Thus, the first-order terms of the perturbation expansion of the volume fractions ϕ_s^0 and ϕ_f^0 are both probability densities and convex coefficients, in analogy with the unperturbed variables ϕ_s and ϕ_f.

However, these properties do not hold for the higher order terms. Indeed, at order $O(\epsilon)$, the saturation relation (2.3) takes the form

$$\phi_s^1 + \phi_f^1 = 0, \tag{3.16}$$

which obviously implies that one of the second-order terms, ϕ_s^1 and ϕ_f^1, must have negative sign.

3.2. *Continuity equations*

At order $O(1)$, the continuity equations for the solid phase (see (3.10*d*)) and the fluid phase (see (3.10*a*)), respectively, read

$$\frac{\mathrm{d}\rho_s^0\phi_s^0}{\mathrm{d}t_s} + \rho_s^0\phi_s^0\nabla\cdot\boldsymbol{u}_s^0 = 0, \tag{3.17}$$

$$\frac{\mathrm{d}\rho_f^0\phi_f^0}{\mathrm{d}t_f} + \rho_f^0\phi_f^0\nabla\cdot\boldsymbol{u}_f^0 = 0. \tag{3.18}$$

We observe that the structure of these equations is the same as the structure of the unperturbed continuity equations, just as in low-Mach-number asymptotics for single-phase flows. This is to be expected because all terms of the continuity equations are of the same order.

3.3. *Momentum equations*

At order $O(1)$, the momentum equation of the fluid phase (i.e. (3.10*b*)) reads

$$\nabla\big(p_f^0\phi_f^0\big) + p_f^0\nabla\phi_s^0 = 0. \tag{3.19}$$

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By virtue of (3.15), this equation yields

$$\nabla p_f^0 = 0. \tag{3.20}$$

In other words, p_f^0 is a function of time only,

$$p_f^0 = p_f^0(t), \tag{3.21}$$

and is determined by the boundary conditions. In particular, if the domain is open, then p_f^0 is constant and equal to the open boundary pressure. If the domain is closed, then p_f^0 may vary in time.

At order $O(\epsilon)$, (3.10*b*) reads

$$\begin{aligned}\rho_f^0\phi_f^0\frac{\mathrm{d}\boldsymbol{u}_f^0}{\mathrm{d}t_f} + \nabla\big(p_f^1\phi_f^0 + p_f^0\phi_f^1\big) \;&= \frac{\xi}{Re}\nabla\big(\phi_f^0\,\zeta_f\,\nabla\cdot\boldsymbol{u}_f^0\big) + \frac{1}{Re}\nabla\cdot\big(\phi_f^0\mu_f\,\boldsymbol{V}_f^{v,0}\big)\\ &\quad - \big(p_f^1\nabla\phi_s^0 + p_f^0\nabla\phi_s^1\big) - \delta\big(\boldsymbol{u}_f^0 - \boldsymbol{u}_s^0\big)\\ &\quad - \rho_f^0\phi_f^0\boldsymbol{g}.\end{aligned} \tag{3.22}$$

Combining (3.22) with (3.15) and (3.20), we conclude that

$$\begin{aligned}\rho_f^0\phi_f^0\frac{\mathrm{d}\boldsymbol{u}_f^0}{\mathrm{d}t_f} + \phi_f^0\nabla p_f^1 = \frac{\xi}{Re}\nabla\big(\phi_f^0\,\zeta_f\,\nabla\cdot\boldsymbol{u}_f^0\big) + \frac{1}{Re}\nabla\cdot\big(\phi_f^0\mu_f\,\boldsymbol{V}_f^{v,0}\big)\\ - \delta\big(\boldsymbol{u}_f^0 - \boldsymbol{u}_s^0\big) - \rho_f^0\phi_f^0\boldsymbol{g}.\end{aligned} \tag{3.23}$$

At order $O(1)$, the momentum equation of the solid phase (i.e. (3.10*e*)) reads

$$\nabla\big(p_s^0\phi_s^0\big) + p_f^0\nabla\phi_f^0 = 0. \tag{3.24}$$

Combining this equation with (3.15) and (3.20), we readily obtain that

$$p_s^0 = p_f^0(t). \tag{3.25}$$

In other words, the first-order pressures p_i^0 are equal and functions of time only, and are determined by the boundary conditions. This implies that in low-Mach-number flows, the two phases are, to the first-order, in mechanical equilibrium with each other.

At order $O(\epsilon)$, the momentum equation of the solid phase (3.10*e*) reads

$$\begin{aligned}\rho_s^0\phi_s^0\frac{\mathrm{d}\boldsymbol{u}_s^0}{\mathrm{d}t_s} + \nabla\big(p_s^1\phi_s^0 + p_s^0\phi_s^1\big) = \frac{\xi}{Re}\nabla\big(\phi_s^0\zeta_s\nabla\cdot\boldsymbol{u}_s^0\big) + \frac{1}{Re}\nabla\cdot\big(\phi_s^0\mu_s\boldsymbol{V}_s^{v,0}\big)\\ - \nabla\cdot\big(\Gamma_s^0\nabla\phi_s^0\otimes\nabla\phi_s^0\big) + \big(p_f^1\nabla\phi_s^0 + p_f^0\nabla\phi_s^1\big)\\ + \delta(\boldsymbol{u}_f^0 - \boldsymbol{u}_s^0) - \rho_s^0\phi_s^0\boldsymbol{g}.\end{aligned} \tag{3.26}$$

By virtue of (3.25), (3.26) takes the form

$$\begin{aligned}\rho_s^0\phi_s^0\frac{\mathrm{d}\boldsymbol{u}_s^0}{\mathrm{d}t_s} + \nabla\big(p_s^1\phi_s^0\big) = \frac{\xi}{Re}\nabla\big(\phi_s^0\,\zeta_s\,\nabla\cdot\boldsymbol{u}_s^0\big) + \frac{1}{Re}\nabla\cdot\big(\phi_s^0\,\mu_s\,\boldsymbol{V}_s^{v,0}\big)\\ - \nabla\cdot\big(\Gamma_s^0\nabla\phi_s^0\otimes\nabla\phi_s^0\big) + \;p_f^1\nabla\phi_s^0 + \delta\big(\boldsymbol{u}_f^0 - \boldsymbol{u}_s^0\big)\\ - \rho_s^0\phi_s^0\boldsymbol{g}.\end{aligned} \tag{3.27}$$

The presence of p_f^1 in the last equation is due to the constitutive relation (2.8*f*) for the momentum exchange between the two phases, $\boldsymbol{f}$. According to this constitutive relation, one component of $\boldsymbol{f}$ is the non-conservative product of the fluid pressure and the gradient of the solid volume fraction, $\hat{p}_f\hat{\nabla}\phi_s$.

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3.4. *Equations of state*

At order $O(1)$, the equation of state of the fluid phase (see (3.12)) reads

$$p_f^0 = \rho_f^0 T_f^0. \tag{3.28}$$

Hence, since p_f^0 is constant in space, spatial variations in the first-order term of the fluid density ρ_f^0 result only from spatial variations of T_f^0. We also observe that the structure of (3.28) is the same as the structure of the unperturbed equation of state, (3.12).

Further, at order $O(\epsilon)$, the equation of state of the fluid phase (i.e. (3.12)) yields

$$p_f^1 = \rho_f^1 T_f^0 + \rho_f^0 T_f^1. \tag{3.29}$$

At order $O(1)$, the equation of state of the solid material (3.13) reads

$$p_s^0 = -P_\infty + R_s \rho_s^0 T_s^0. \tag{3.30}$$

Thus, by virtue of (3.25), we observe that spatial variations in the first-order term of the solid density ρ_s^0 are caused only by spatial variations of T_s^0, in analogy with the fluid phase. Also, the structure of (3.30) is the same as the structure of the unperturbed solid equation of state (i.e. (3.13)).

Further, at order $O(\epsilon)$, the equation of state of the solid phase (see (3.13)) yields

$$p_s^1 = R_s\left(\rho_s^1 T_s^0 + \rho_s^0 T_s^1\right). \tag{3.31}$$

According to typical terminology for single-phase flows, the first-order terms of the phasial pressures, $p_i^0,\ i = f, s$, are referred to as 'thermodynamic pressures' because they appear in the state equations of their corresponding phases (i.e. (3.28) and (3.30)). Similarly, the second order of the phasial pressures, $p_i^1,\ i = f, s$, are referred to as 'dynamic pressures' because their gradients act as forcing terms in the momentum equations (i.e. (3.23) and (3.27)).

3.5. *Compaction equation*

At order $O(1)$, the compaction equation (3.10*g*) reads

$$\frac{p_s^0}{T_s^0} - \frac{p_f^0}{T_f^0} = 0, \tag{3.32}$$

which, by virtue of (3.25), yields

$$T_f^0 = T_s^0. \tag{3.33}$$

In other words, in low-Mach-number flows and to the first-order approximation, the temperatures of the two phases are equal and, therefore, there is no heat exchange between the two phases. Nonetheless, the first-order temperature T_f^0 can exhibit strong spatial and temporal variations, similarly to single-phase low-Mach-number flows.

At order $O(\epsilon)$, (3.10*g*) reads

$$\frac{\mathrm{d}\phi_s^0}{\mathrm{d}t_s} = \mu_c^{-1}\left(\frac{p_s^1 T_s^0 - p_s^0 T_s^1}{\left(T_s^0\right)^2} - \frac{p_f^1 T_f^0 - p_f^0 T_f^1}{\left(T_f^0\right)^2} - \frac{\beta_s^0}{T_s^0} + \nabla\cdot\left(\frac{\Gamma_s^0}{T_s^0}\nabla\phi_s^0\right)\right). \tag{3.34}$$

Upon substitution of (3.25) and (3.33), this equation is written as follows:

$$\frac{\mathrm{d}\phi_s^0}{\mathrm{d}t_s} = \mu_c^{-1}\left(\frac{p_s^1 - p_f^1}{T_s^0} - \frac{\left(T_s^1 - T_f^1\right)p_s^0}{\left(T_s^0\right)^2} - \frac{\beta_s^0}{T_s^0} + \nabla\cdot\left(\frac{\Gamma_s^0}{T_s^0}\nabla\phi_s^0\right)\right). \tag{3.35}$$

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We observe that second-order differences between the pressures and temperatures of the two phases act as forcing terms in the evolution of the solid volume fraction, even though *to the first order* the phasial pressures and temperatures are equal. In other words, second-order mechanical and thermal non-equilibrium between the two phases plays a role critical to the (irreversible) low-Mach-number dynamics of the system, even though *to the first order* the two phases are in thermal and mechanical equilibrium with each other. This is analogous to the situation in the momentum equations, for both single- and two-phase flows. In that case, the gradients of the second-order terms of the pressures act as forcing terms in the low-Mach-number momentum equations, even though the first-order terms of the pressures are spatially uniform.

3.6. *Energy equations*

At order $O(1)$, and by virtue of (3.33), the energy equation of the fluid phase (i.e. (3.10*c*)) reads

$$\rho_f^0\phi_f^0 c_{p,f}^0\frac{\mathrm{d}T_f^0}{\mathrm{d}t_f}-\frac{\gamma_{f,r}-1}{\gamma_{f,r}}\frac{\mathrm{d}\,p_f^0\phi_f^0}{\mathrm{d}t_f}=\frac{\gamma_{f,r}-1}{\gamma_{f,r}}p_f^0\left(\boldsymbol{u}_f^0-\boldsymbol{u}_s^0\right)\cdot\nabla\phi_s^0+\frac{1}{PrRe}\nabla\cdot\left(\kappa_f^0\phi_f^0\nabla T_f^0\right).\tag{3.36}$$

Similarly, the energy equation for the solid phase (i.e. (3.10*f*)) at order $O(1)$ reads

$$\rho_s^0\phi_s^0 c_{p,s}^0\frac{\mathrm{d}T_s^0}{\mathrm{d}t_s}-\frac{\gamma_{f,r}-1}{\gamma_{f,r}}\frac{\mathrm{d}p_s^0\phi_s^0}{\mathrm{d}t_s}=\frac{1}{PrRe}\nabla\cdot\left(\kappa_s^0\phi_s^0\nabla T_s^0\right).\tag{3.37}$$

3.7. *The low-Mach-number equations and their properties*

In analogy with single-phase flows, we define the two-phase low-Mach-number equations as the set of equations that govern the evolution of the first-order terms of the low-Mach-number expansion. Under this definition, we arrive at the following system which corresponds to the low-Mach-number equations for two-phase granular materials:

$$\phi_s^0+\phi_f^0=1,\quad p_s^0=p_f^0=p_f^0(t),\quad T_f^0=T_s^0,\tag{3.38a}$$

$$p_f^0=\rho_f^0\,T_f^0,\quad p_s^0=-P_\infty+R_s\,\rho_s^0\,T_s^0,\tag{3.38b}$$

$$\frac{\mathrm{d}\rho_f^0\phi_f^0}{\mathrm{d}t_f}+\rho_f^0\phi_f^0\nabla\cdot\boldsymbol{u}_f^0=0,\tag{3.38c}$$

$$\frac{\mathrm{d}\rho_s^0\phi_s^0}{\mathrm{d}t_s}+\rho_s^0\phi_s^0\nabla\cdot\boldsymbol{u}_s^0=0,\tag{3.38d}$$

$$\rho_f^0\phi_f^0\frac{\mathrm{d}\boldsymbol{u}_f^0}{\mathrm{d}t_f}+\phi_f^0\nabla p_f^1=\frac{\xi}{Re}\nabla\left(\phi_f^0\zeta_f\nabla\cdot\boldsymbol{u}_f^0\right)+\frac{1}{Re}\nabla\cdot\left(\phi_f^0\mu_f\boldsymbol{V}_f^{v,0}\right)-\delta\left(\boldsymbol{u}_f^0-\boldsymbol{u}_s^0\right)-\rho_f^0\phi_f^0\boldsymbol{g},\tag{3.38e}$$

$$\rho_s^0\phi_s^0\frac{\mathrm{d}\boldsymbol{u}_s^0}{\mathrm{d}t_s}+\nabla\left(p_s^1\phi_s^0\right)=\frac{\xi}{Re}\nabla\left(\phi_s^0\zeta_s\nabla\cdot\boldsymbol{u}_s^0\right)+\frac{1}{Re}\nabla\cdot\left(\phi_s^0\mu_s\boldsymbol{V}_s^{v,0}\right)-\nabla\cdot\left(\Gamma_s^0\nabla\phi_s^0\otimes\nabla\phi_s^0\right)+p_f^1\nabla\phi_s^0+\delta\left(\boldsymbol{u}_f^0-\boldsymbol{u}_s^0\right)-\rho_s^0\phi_s^0\boldsymbol{g},\tag{3.38f}$$

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$$\rho_f^0\phi_f^0 c_{p,f}^0\frac{\mathrm{d}T_f^0}{\mathrm{d}t_f}-\frac{\gamma_{f,r}-1}{\gamma_{f,r}}\frac{\mathrm{d}p_f^0\phi_f^0}{\mathrm{d}t_f}=\frac{\gamma_{f,r}-1}{\gamma_{f,r}}p_f^0\left(\boldsymbol{u}_f^0-\boldsymbol{u}_s^0\right)\cdot\nabla\phi_s^0 + \frac{1}{Pr\,Re}\nabla\cdot\left(\kappa_f^0\phi_f^0\nabla T_f^0\right), \tag{3.38g}$$

$$\rho_s^0\phi_s^0 c_{p,s}^0\frac{\mathrm{d}T_s^0}{\mathrm{d}t_s}-\frac{\gamma_{f,r}-1}{\gamma_{f,r}}\frac{\mathrm{d}p_s^0\phi_s^0}{\mathrm{d}t_s}=+\frac{1}{Pr\,Re}\nabla\cdot\left(\kappa_s^0\phi_s^0\nabla T_s^0\right), \tag{3.38h}$$

$$\frac{\mathrm{d}\phi_s^0}{\mathrm{d}t_s}=\mu_c^{-1}\left(\frac{p_s^1-p_f^1}{T_s^0}-\frac{\left(T_s^1-T_f^1\right)p_s^0}{\left(T_s^0\right)^2}-\frac{\beta_s^0}{T_s^0}+\nabla\cdot\left(\frac{\Gamma_s^0}{T_s^0}\nabla\phi_s^0\right)\right). \tag{3.38i}$$

The low-Mach-number equations (i.e. (3.38*c*)–(3.38*i*)) constitute a system of seven partial differential equations that is closed by the algebraic equations (3.38*a*) and (3.38*b*). The seven independent unknown variables are

$$\phi_s^0,\ \boldsymbol{u}_s^0,\ \boldsymbol{u}_f^0,\ p_s^1,\ p_f^1,\ T_s^0,\ \Delta T^1=T_s^1-T_f^1. \tag{3.39}$$

The above system has the following interesting properties:

(i) The momentum equations (3.38*e*) and (3.38*f*) are decoupled from their corresponding energy equations (3.38*g*) and (3.38*h*), respectively. As mentioned in § 1, this decoupling constitutes a major simplification over the unperturbed, compressible system (3.10*a*)–(3.13).

(ii) The first-order (thermodynamic) pressures p_s^0 and p_f^0 are equal and functions of time only. As such, their values are determined by the boundary conditions. In particular, in open domains, the thermodynamic pressures are constant.

(iii) The second-order difference between the phasial temperatures, ΔT^1, appears as a forcing term in the compaction equation (3.38*i*). This implies that (3.38*i*) is not a convenient rate equation for the first-order term of the solid volume fraction. Instead, it is more convenient to consider it as an algebraic equation for ΔT^1. However, in low-Mach-number flows, we are typically interested in values only up to $O(1)$, and ΔT^1 does not appear in any other equation of the governing system. Therefore, we conclude that (3.38*i*), when viewed as an equation for ΔT^1, essentially becomes redundant and does not need to be solved. In conclusion, for all practical purposes, the low-Mach-number approximation consists of six equations with six unknowns. This can have important numerical implications because it means that the nonlinear and possibly stiff right-hand side of (3.38*i*) does not need to be integrated numerically.

(iv) Since the first-order phasial temperatures are equal, i.e. $T_f^0=T_s^0$, one needs to solve only one of the energy equations (i.e. (3.38*g*) and (3.38*h*)) for T_s^0. Then, the other energy equation can play the role of a rate equation for ϕ_s^0. In other words, one can integrate the other energy equation for the computation of ϕ_s^0. In fact, from the numerical point of view, linear combinations of the two energy equations (addition, subtraction, etc.) might be more advantageous for the computation of T_s^0 and ϕ_s^0.

(v) The viscous forces do enter the low-Mach-number momentum equations (3.38*e*) and (3.38*f*). However, the work performed by these forces is of order $O(\epsilon)$ with respect to variations of the enthalpies of the two phases. Therefore, this work does not appear in the low-Mach-number energy equations, just as in the low-Mach-number approximation of single-phase flows.

(vi) The divergence of the tensor $(\Gamma_s^0\nabla\phi_s^0\otimes\nabla\phi_s^0)$, which models additional stresses associated with the microstructure of the solid phase, appears in the

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low-Mach-number solid momentum equation (3.38*f*). Moreover, this tensor does not vanish at equilibrium, which is expected because granular materials support shear stresses at equilibrium. However, the work of these stresses is also of $O(\epsilon)$ with respect to variations of the enthalpy of the solid phase. Therefore, this work does not appear in the low-Mach-number solid energy equation (3.38*g*).

(vii) The configuration Helmholtz free energy of the solid phase $\psi^{c,0}$ is of order $O(\epsilon)$, with respect to the solid's pure-phase enthalpy, $c^0_{p,s}T^0_s$. Therefore, its material derivative is neglected from the solid energy equation (3.38*g*).

As regards property (*c*), we observe that the dependence of the evolution of ϕ^0_s on ΔT^1 is a consequence of the following: (i) the two phases are, in general, in non-equilibrium with each other and (ii) the evolution of the volume fraction is not a fully reversible precess. As such, the forcing term of the compaction equation (2.8*i*) consists, in general, of a reversible and an irreversible part. Consistency with the second thermodynamic law requires that the irreversible part be a function of the difference between the temperatures of the two phases (i.e. (2.8*i*)). This requirement is also reflected in the low-Mach-number approximation (3.38*i*) of (2.8*i*), albeit in a subtle way, i.e. in the appearance of the second-order term of the difference of the phasial temperatures as a forcing term of (3.38*i*).

According to some models for two-phase granular mixtures, such as the one of Baer & Nunziato (1986) and its variations, the phasial temperatures do not appear on the right-hand side of the compaction equation. In other words, according to those models, the evolution of the volume fraction does not depend on the thermal non-equilibrium between the two phases. This implies that in those models, and in absence of configuration stress vector, the volume fraction evolution is a purely reversible process. This, however, is an additional constitutive *assumption* of questionable validity and not a constraint imposed by the second thermodynamic law or by any other thermodynamic argument. In any case, since the compaction equation in those models is independent of the phasial temperatures, it is expected that the low-Mach-number approximation of those models would allow first-order differences between the phasial temperatures, i.e. $T^0_s \neq T^0_f$, in general.

The situation is also different if one makes a different constitutive assumption, namely that the two phases are in thermal equilibrium with each other, i.e. $T_s = T_f$, as is done in some other models; see, for example, Svendsen & Hutter (1995), Wang & Hutter (1999) and Passman *et al.* (1986). Evidently, in this case, the forcing terms of the compaction equation are independent of the temperature and, furthermore, there is only one energy equation. Nonetheless, because of this additional assumption, the predictions of those models for the evolution of the mixture towards an equilibrium or stationary state are different from the evolution predicted by the two-phase flow model of reference.

4. Special cases: constant-density flows and the equilibrium limit

In this section, we first examine the form of the low-Mach-number equations in the special case of constant-density flows, i.e. when the phasial densities ρ_s and ρ_f remain constant. The study of these equations has practical interest because such flows are encountered in various technological applications. It also has theoretical interest because these equations are the equivalent of the incompressible Navier–Stokes equations to two-phase granular mixtures.

In our study, we focus on open domains and on closed domains under constant pressure, which are the cases that are most often encountered in practice. In other

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words, we have that

$$p_f^0 = p_s^0 = \text{constant}. \tag{4.1}$$

Further, since the densities are constant, the first-order densities, ρ_s^0 and ρ_f^0, are also constant. In turn, by virtue of the first-order approximation of the equations of state (i.e. (3.38*b*)), the last equation implies that

$$T_f^0 = T_s^0 = \text{constant}, \tag{4.2}$$

which means that, to the first order, the flow is isothermal. We, however, note that T_s^1 and T_f^1 are generally not constant and not equal to each other.

From the above equations, it follows that the energy equation for the solid phase (i.e. (3.38*h*)) reduces to

$$\frac{\mathrm{d}\phi_s^0}{\mathrm{d}t_s} = 0. \tag{4.3}$$

This relation means that ϕ_s^0 is conserved along the streamlines of the solid phase. This, however, does not imply that in constant-density flows, the solid volume fraction becomes a passive scalar that is advected with the flow. Indeed, as is shown below, the configuration stress vector still appears in the momentum equation of the solid phase, (i.e. (4.7)). Therefore, ϕ_s^0 remains strongly coupled with both $\boldsymbol{u}_s^0$ and p_s^1, even in the case of constant-density flows.

Substituting (4.3) into the solid continuity equation (3.38*d*), we deduce that the solid velocity is divergence-free, i.e.

$$\nabla \cdot \boldsymbol{u}_s^0 = 0. \tag{4.4}$$

This, in turn, implies that the solid continuity equation (3.38*c*) reduces to a kinematic constraint for the solid velocity, just as in the incompressible Navier–Stokes equations for single-phase flows.

Further, as regards the velocity of the fluid phase $\boldsymbol{u}_f^0$, by combining the energy equation (3.38*g*) and the continuity equation (3.38*c*), we arrive at

$$\nabla \cdot \left(\left(\boldsymbol{u}_s^0 - \boldsymbol{u}_f^0 \right) \phi_f \right) = 0. \tag{4.5}$$

In other words, the velocity of the fluid phase is not divergence-free but is subject to a different kinematic constraint. The fact that only the velocity of the fluid phase is not divergence-free is due to the momentum exchange between the two phases in the form of hydrodynamic drag exerted on the solid particles by the carrier fluid (see (2.8*f*)).

Subsequently, by substituting (3.38*a*), (3.38*b*) and (4.3) into the compaction equation (3.38*i*), we arrive at

$$\left(T_s^1 - T_f^1 \right) \frac{p_s^0}{T_s^0} = p_s^1 - p_f^1 - \beta_s^0 + \nabla \cdot \left(\Gamma_s^0 \nabla \phi_s^0 \right). \tag{4.6}$$

We observe that the compaction equation has been reduced to an elliptic partial differential equation with respect to ϕ_s^0. Further, the second-order terms of the phasial pressure and temperature differences still appear as forcing terms in the compaction equation (4.6) even though the phasial densities are constant and the flow is isothermal to the first order. We conclude that even in constant-density flows, the distribution of the solid volume fraction depends on the second-order thermal non-equilibrium between the two phases. In any case, since we are interested only in first-order terms, ϕ_s^0 can be computed via (4.3), and (4.6) becomes redundant.

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Next, by introducing the divergence-free condition (4.4) in the solid momentum equation (3.38*f*), we obtain the following expression:

$$\rho_s^0\phi_s^0\frac{\mathrm{d}\boldsymbol{u}_s^0}{\mathrm{d}t_s}+\nabla\big(p_s^1\phi_s^0\big)=\frac{1}{Re}\nabla\cdot\big(\phi_s^0\,\mu_s\,\boldsymbol{V}_s^{v,0}\big)-\nabla\cdot\big(\Gamma_s^0\nabla\phi_s^0\otimes\nabla\phi_s^0\big)+p_f^1\nabla\phi_s^0+\delta\big(\boldsymbol{u}_f^0-\boldsymbol{u}_s^0\big)-\rho_s^0\phi_s^0\boldsymbol{g}. \tag{4.7}$$

Finally, we note that the low-Mach-number momentum equation for the fluid phase (i.e. (3.38*e*)) remains the same.

Summing up, the equations for constant-density flows of two-phase granular mixtures are written as follows:

$$\nabla\cdot\big((\boldsymbol{u}_s^0-\boldsymbol{u}_f^0)\phi_s^0\big)=0, \tag{4.8a}$$

$$\nabla\cdot\boldsymbol{u}_s^0=0, \tag{4.8b}$$

$$\frac{\mathrm{d}\phi_s^0}{\mathrm{d}t_s}=0, \tag{4.8c}$$

$$\rho_f^0\phi_f^0\frac{\mathrm{d}\boldsymbol{u}_f^0}{\mathrm{d}t_f}+\phi_f^0\nabla p_f^1=\frac{\xi}{Re}\nabla\big(\phi_f^0\,\zeta_f\,\nabla\cdot\boldsymbol{u}_f^0\big)+\frac{1}{Re}\nabla\cdot\big(\phi_f^0\mu_f\,\boldsymbol{V}_f^{v,0}\big)-\delta\big(\boldsymbol{u}_f^0-\boldsymbol{u}_s^0\big)-\rho_f^0\phi_f^0\boldsymbol{g}, \tag{4.8d}$$

$$\rho_s^0\phi_s^0\frac{\mathrm{d}\boldsymbol{u}_s^0}{\mathrm{d}t_s}+\nabla\big(p_s^1\phi_s^0\big)=\frac{1}{Re}\nabla\cdot\big(\phi_s^0\,\mu_s\,\boldsymbol{V}_s^{v,0}\big)-\nabla\cdot\big(\Gamma_s^0\nabla\phi_s^0\otimes\nabla\phi_s^0\big)+p_f^1\nabla\phi_s^0+\delta\big(\boldsymbol{u}_f^0-\boldsymbol{u}_s^0\big)-\rho_s^0\phi_s^0\boldsymbol{g}. \tag{4.8e}$$

It is worth mentioning that by virtue of the divergence-free condition for the velocity of the solid phase, the term containing the solid bulk viscous pressure has disappeared from the solid momentum equation (4.8*e*). However, this is not the case for the fluid momentum equation (4.8*d*).

As expected, the equations that govern constant-density flows (i.e. (4.8*a*)–(4.8*e*)) are much simpler than the general low-Mach-number equations (i.e. (3.38*a*)–(3.38*i*)). For example, the continuity and momentum equations for the solid phase resemble the incompressible Navier–Stokes equations for simple fluids. This suggests that it could be possible to generalize the projection methods that are typically used for the Navier–Stokes equations and apply them for the numerical treatment of the balance equations for the solid phase (i.e. (4.8*a*), (4.8*c*) and (4.8*e*)). However, for the balance equations of the fluid phase (i.e. (4.8*b*) and (4.8*d*)), these methods are likely to require certain modifications, because of the different kinematic constraint that holds for the fluid phase (see (4.8*a*)).

4.1. *The equilibrium limit*

At the equilibrium limit, all terms that describe dissipative effects vanish and the temperatures of the two phases become equal and constant. Also, the phasial velocities become zero everywhere. In this case, the low-Mach-number equations (3.38*a*)–(3.38*i*)

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reduce to

$$p_s^0 = p_f^0 = \text{constant}, \quad \phi_s^0 + \phi_f^0 = 1, \tag{4.9a}$$

$$p_f^0 = \rho_f^0 T_f^0, \quad p_s^0 = -P_\infty + R_s \rho_s^0 T_s^0, \tag{4.9b}$$

$$\phi_f^0 \nabla p_f^1 = -\rho_f^0 \phi_f^0 \boldsymbol{g}, \tag{4.9c}$$

$$\nabla\left(p_s^1 \phi_s^0\right) = p_f^1 \nabla \phi_s^0 - \nabla \cdot \left(\Gamma_s^0 \nabla \phi_s^0 \otimes \nabla \phi_s^0\right) - \rho_s^0 \phi_s^0 \boldsymbol{g}, \tag{4.9d}$$

$$p_s^1 - p_f^1 = \beta_s^0 - \nabla \cdot \left(\Gamma_s^0 \nabla \phi_s^0\right). \tag{4.9e}$$

Since the phasial temperatures are equal and constant, their first-order approximations T_s^0 and T_f^0 are also equal and constant. Therefore, by virtue of (4.9*a*) and the state equations (4.9*b*), the first-order phasial densities ρ_s^0 and ρ_f^0 are also constant.

Further, we observe that (4.9*c*) can be integrated analytically for the computation of the fluid dynamic pressure p_f^1. Consequently, the governing system reduces to (4.9*d*) and (4.9*e*) with respect to the two remaining unknown variables, ϕ_s^0 and p_s^1. An algorithm for the numerical treatment of this system has recently been proposed by Varsakelis & Papalexandris (2010).

On the other hand, it is interesting to note that the same set of equations, (4.9*a*)–(4.9*e*), is recovered by a different procedure (Varsakelis & Papalexandris 2010), which is summarized as follows:

(i) Consider the unperturbed, compressible model of reference and set all dissipative processes equal to zero, i.e. take its equilibrium limit.

(ii) Non-dimensionalize the resulting equations.

(iii) Employ standard hydrostatic-type arguments to show the smallness of the parameter

$$\epsilon' = \frac{|\boldsymbol{g}|\, L_r}{p_r/\rho_r}, \tag{4.10}$$

which appears in the non-dimensionalized equations.

(iv) Perturb singularly the non-dimensionalized equations with respect to ϵ'.

In other words, the operations of taking the equilibrium limit and perturbing singularly the two-phase flow model of reference commute. This should come as no surprise because, in the presence of a gravitational field, the term $u_r = \sqrt{|\boldsymbol{g}|\, L_r}$ plays the role of a reference velocity. Therefore, if one considers low-Mach-number flows, this reference velocity is necessarily much smaller than $\sqrt{p_r/\rho_r}$, which is proportional to the reference speed of sound. We conclude that the smallness of ϵ' is included in the low-Mach-number hypothesis. In particular, at equilibrium, the parameter ϵ' controls the incompressibility of the mixture, which explains the observed commutation.

5. Remarks on the validation of the low-Mach-number equations

As is the case for any new theory, both the two-phase flow model of reference and its low-Mach-number approximation presented herein should be validated experimentally. In the past, constant-density, isothermal granular flows have been studied experimentally by various authors. On the other hand, and to the best of our knowledge, there have been no experimental studies of granular flows in the presence of temperature gradients. Nonetheless, some of the isothermal experiments can be

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extended directly to conditions with temperature gradients. Below we briefly describe two of them, namely shear and gravity-driven flow experiments.

Shear-flow experiments can be performed in viscometric shearing apparatus in planar or cylindrical configurations. In a planar shear flow, a mixture of a granular medium and a fluid fills the space between parallel horizontal plates. Shear flow is caused by forcing the upper plate to move parallel to the lower one with a given velocity. On the other hand, cylindrical shear (Couette) flow consists of a flow between two concentric and counter-rotating cylindrical drums; see Bagnold (1954), Savage & McKeown (1983) and references therein. Extension of these experimental configurations to flows with temperature gradients can be achieved by maintaining the two plates (or the drums) at different temperatures, thereby introducing a heat flux across the flow domain.

As regards gravity-driven flow experiments, there are two well-known variations that can be implemented directly. The first one is the free-surface flow of a granular material down an inclined plane; see Savage (1979), Savage & Lun (1988) and references therein. Extension of this experiment to non-isothermal conditions can be achieved by keeping the inclined plane at a temperature different than the ambient one. The second variation is the vertical channel flow, in which a granular material flows due to gravity between two parallel, vertical planes (Savage 1979). Extension of this set-up to flows with temperature gradients can be achieved by maintaining the two planes at different temperatures.

In the past, various authors have obtained numerical solutions for the above problems by employing different granular flow models; see, for example, Passman *et al.* (1986), Wang & Hutter (1999), Wang & Hutter (2006) and references therein. All earlier numerical studies, however, were restricted to isothermal, steady-state, planar flows, for which the flow models reduce to a set of ordinary differential equations. Nonetheless, it is important to numerically simulate isothermal transient three-dimensional flows, so as to study the process of mechanical relaxation between the granular and fluid phases, in addition to low-Mach-number flows with temperature gradients.

Despite the conceptual simplicity of the above experiments, the reader should be warned that validation of granular flow models is by no means an easy task. For example, one difficulty arises from the complexity of the mathematical models, which does not allow the derivation of analytical or even semi-analytical solutions. Thus, the mathematical models have to be discretized and solved numerically. However, the design of efficient and robust methods for these equations is challenging because they contain unconventional and highly nonlinear differential and relaxation terms. Further, during numerical simulations, particular care has to be taken in the implementation of the boundary conditions. In fact, the issue of appropriate boundary conditions for granular materials is still open. For studies and discussion on this issue, the reader is referred to Jenkins & Richmann (1986), Richmann & Chou (1988), Wang & Hutter (2006) and references therein.

Further, another challenging issue is the large number of physical parameters appearing in two-phase flow models for granular materials. For some of these parameters, such as interfacial drag, there are empirical correlations which are considered to be sufficiently accurate. However, there are some transport coefficients of the granular phase for which generally accepted correlations are not yet available. Examples include the shear and bulk viscosity coefficients of the granular phase ($\hat{\mu}_s$ and $\hat{\zeta}_s$, respectively), the dynamic compaction coefficient $\hat{\mu}_c$ and others.

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Furthermore, currently there are no sufficiently accurate estimates for the coefficients appearing in the expression for the configuration part of the Helmholtz free energy, $\hat{k}_1$ and $\hat{k}_2$ (i.e. (2.26)). To this extent, it would be helpful if one could measure the normal and shear configuration stresses that are developed in a granular pile at equilibrium. Then, in principle, one could use these measurements to deduce values for the coefficients $\hat{k}_1$ and $\hat{k}_2$. Alternatively, one could perform numerical simulations of the piles that are formed from discharging a granular material on a horizontal surface and determine the values of those coefficients so that the numerical predictions for the angle of repose of the piles match the experimentally measured angles.

Finally, another point that requires attention is to sufficiently approximate in experiments the numerous constitutive assumptions upon which the flow models are based. For example, the flow model of reference and its low-Mach-number approximation treat the granular phase as a continuum. There are cases where this assumption is no longer valid, e.g. mixtures with very low concentrations of granular matter or with a small number of grains. Consequently, experiments designed for the validation of the models in hand should include large numbers of particles. Further, the grains used in the experiments should be cohesionless, of approximately the same size, without humidity and free of electrostatic forces.

6. Conclusions

In this paper, we have presented a generalization of low-Mach-number asymptotics to multi-phase flows and applied it to derive the low-Mach-number approximation of the mathematical model of Papalexandris (2004) for two-phase flows of granular mixtures. Our methodology amounts to non-dimensionalizing the governing system with respect to a reference thermodynamic state of the phase with the smallest speed of sound and expanding all variables in power series of the Mach number in this reference state. This methodology is quite general and can be directly applied for the systematic reduction of continuum models for granular materials and for many different types of multi-phase flows. As in the case of single-phase flows, the multi-phase low-Mach-number equations are considerably simpler than the unperturbed, compressible models. In particular, the momentum equations of the two phases are essentially decoupled from their corresponding energy equations.

The low-Mach-number approximation, when applied to the above mathematical model for two-phase granular mixtures, requires that the phasial temperatures be equal to the first order. Nonetheless, the second-order term of the temperature difference appears as a forcing term in the rate equation for the volume fraction (compaction equation). Thus, this equation becomes essentially redundant and the solid volume fraction is computed from one of the two energy equations.

In the special but important cases of constant-density flows and of equilibrium limit, the governing equations are further simplified. In particular, in constant-density flows, the velocity of the solid phase is divergence-free, and the solid continuity and momentum equations resemble the classical incompressible Navier–Stokes equations. Also, the volume fraction is constant along the streamlines of the solid phase. On the other hand, the velocity of the fluid phase is not divergence-free but subject to a different kinematic constraint.

It is expected that the low-Mach-number equations that have been derived herein are easier to study than the unperturbed compressible model, both from the numerical and the theoretical point of view. The design and implementation of numerical

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methods for these equations and the mathematical analysis of the low-Mach-number singular limit are currently being pursued.

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