# ON $A$-GROUPS

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## Introduction

A sequence of papers by P. Hall ((1)–(4)) on some fundamental properties of soluble groups was followed in 1940 by his account of a new construction theory for such groups (5). We shall have frequent occasion to refer to these papers, and we begin with a brief account of their contents.

In (1) and (2) Hall showed that a certain extension of Sylow's theorem is a characteristic property of soluble groups. This leads to the existence in a soluble group of *Sylow systems* (3), and so to the important and striking properties of the *system normalizers* (4). The basic idea of the construction theory in (5) is the *partially complemented extension* of a specific characteristic subgroup, by a partial complement belonging to a characteristic class of conjugate subgroups. In a soluble group, the relative system normalizers of a characteristic subgroup form such a characteristic class; the vital result is:

*If $H/N$ is any nilpotent factor of the group $G$, $H$ is soluble and $M_G(H)$ is a system normalizer of $H$ relative to $G$, then*

$$G = M_G(H)\,N.$$

In particular, if $G$ is itself soluble, having nilpotent length $n$ and lower nilpotent series $G = L_0 > L_1 > L_2 > \ldots > L_n = 1$, we may take $H = L_{i-1}$, $N = L_i$. Then $G$ is a partially complemented extension of its characteristic subgroup $L_i$, of nilpotent length $n-i$, by $M_G(L_{i-1})$, which is of length $i$ and is one of a characteristic class of conjugate subgroups. If all soluble groups of lengths less than $n$ are known, all soluble groups of length $n$ can thus be constructed from them. (If $n = 1$, $G$ is nilpotent. It is assumed that a construction theory is available for such groups.) The inherent difficulties of all construction theories—those of the *factor systems* and the *isomorphism problem*—are reduced by the partial complement idea, and by the characteristic nature of the components, respectively.

Hall ends his paper (5) with some remarks about *$A$-groups*, which is the name he gives to soluble groups whose Sylow subgroups are all Abelian. The construction theory mentioned above is particularly appropriate for $A$-groups, since then the $M_G(L_{i-1})$ are actually *complements* of $L_i$. This statement depends on a theorem for $A$-groups which he enunciates, but for which no proof is offered; and in the course of his remarks other interesting properties of $A$-groups are suggested, again without proof. The primary object of the present paper is to provide proofs for these results, and this has been done. Other properties (including some generalizations of Hall's theorems) have been incorporated, and the concluding section on soluble groups of cube-free order is substantially new.

We devote §§ 1 and 2 to various definitions and lemmas which are needed later, and

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introduce the theory of $A$-groups in § 3, where a number of elementary properties are given and (where necessary) proved. With these preliminaries completed, we prove in § 4 Hall's conjectured theorem that *the meet of the centre and the first derived group of an A-group is the identity*, and deduce various corollaries of this extremely fruitful result. One consequence has been mentioned above, that *in an A-group* $L_i$ *and* $M_G(L_{i-1})$ *are complements*. In § 5 we establish properties of the normal subgroups of an $A$-group, including Hall's theorem on the form of the characteristic subgroup $U_1$ (the unique maximal normal nilpotent subgroup). The definitions and lemmas of § 6 are used in § 7 to prove the Basis Theorem; our justification for the invention of a number of new terms is that they help to reduce circumlocution in a proof whose concise expression presents more difficulties than its conception.

Any soluble group whose order contains no prime to a power greater than 2 is an $A$-group, and we conclude by investigating the structure of such groups. The main result of § 8 is that the length of a soluble group of cube-free order cannot exceed 3, and that if the order is odd the length cannot exceed 2.

## 1. Definitions and results involving covering and avoidance

We begin by introducing a slight generalization of the concept of *covering and avoidance* (4), and then we deduce some simple but useful properties.

Let $L$ be any subgroup, and $H/K$ any factor, of a finite group $G$. We shall call the number of cosets of $K$ in $KL$, which are also cosets of $K$ in $H$, *the trace of* $L$ *on the factor* $H/K$. This number we write $\mathrm{tr}\,(L; H/K)$, or simply $\mathrm{tr}\,(L)$ if the factor involved can be taken as understood. Clearly $\mathrm{tr}\,(L)$ is a factor of $(H/K)$ and of $(L)$†, and it is readily seen that

$$\mathrm{tr}\,(L; H/K) = (H \wedge L)/(K \wedge L). \tag{1·1}$$

This equation could have been taken as the definition of trace.

The two extreme cases that may arise are: (*a*) $\mathrm{tr}\,(L) = (H/K)$, occurring if and only if $KL \wedge H = H$. We then say that $L$ *covers* $H/K$. (*b*) $\mathrm{tr}\,(L) = 1$, occurring if and only if $KL \wedge H = K$. We then say that $L$ *avoids* $H/K$.

If $\mathrm{tr}\,(L) > 1$, we shall say that $L$ *overlaps* $H/K$; this, of course, includes the special case when $L$ covers $H/K$.

Suppose now that $G = G_0 > G_1 > G_2 > \ldots > G_r = 1$ is any chain of normal subgroups of $G$, and that $L$ is any subgroup of $G$. Then

$$(L) = (L \wedge G_0)/(L \wedge G_r) = \prod_{i=1}^{r} (L \wedge G_{i-1})/(L \wedge G_i) = \prod_{i=1}^{r} \mathrm{tr}\,(L; G_{i-1}/G_i).$$

Thus we have:

(1·2) *The order of* $L$ *is the product of the traces of* $L$ *on the factors* $G_{i-1}/G_i$ *derived from any chain of normal subgroups of* $G$. $L$ *avoids any factor of* $G$ *whose order is prime to* $(L)$. *If* $L$ *covers or avoids each of the factors* $G_{i-1}/G_i$, *then* $(L)$ *is the product of the orders of those factors which* $L$ *covers. Conversely, if* $(L)$ *is the product of the orders of those factors which* $L$ *overlaps, then* $L$ *covers or avoids each of the factors* $G_{i-1}/G_i$.

† Here and subsequently we use the convenient bracket notation to signify the order of a group

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A factor $H/K$ of $G$ may be decomposable into component factors $H_0/H_1$, $H_1/H_2$, ..., $H_{s-1}/H_s$ (where $H_0 = H$, $H_s = K$). It is then clear that

$$\mathrm{tr}\,(L; H/K) = \prod_{i=1}^{s} \mathrm{tr}\,(L; H_{i-1}/H_i).$$

In particular, $L$ covers (avoids) $H/K$ if and only if $L$ covers (avoids) every component factor of $H/K$.

From our original definition of trace, we see that if $L \geqslant M$, then $\mathrm{tr}\,(L) \geqslant \mathrm{tr}\,(M)$ on any factor $H/K$. If $M$ covers $H/K$, so also does $L$; and if $L$ avoids $H/K$, so does $M$. A very simple example, however, shows that in general it is not true that if $P$, $Q$ both cover $H/K$, so also does $P \wedge Q$; nor does it follow that if $P$, $Q$ both avoid $H/K$, so does $P \vee Q$, even if $P \vee Q = PQ$. For let $G$ be the quadratic group (or *Vierergruppe*) consisting of the elements 1, $p$, $q$, $r = pq$, and let $P$, $Q$, $R$ be the subgroups of order 2 generated by $p$, $q$, $r$, respectively. Then $P$, $Q$ both avoid the chief factor $R/1$ and cover the chief factor $G/R$, but $P \vee Q = PQ = G$ covers $R/1$ and $P \wedge Q = 1$ avoids $G/R$.

(1·3) *If the subgroups $L$, $M$ of $G$, each of which covers or avoids any chief factor of $G$, are such that no chief factor is covered by both of them, then $L \wedge M = 1$; and if $L$, $M$ are permutable, their join $LM$ avoids every chief factor avoided by both $L$ and $M$, and covers every other chief factor.*

In fact, if $P$ and $Q$ are *any* subgroups such that no chief factor is overlapped by both of them, then $P \wedge Q = 1$, since $P \wedge Q$ can overlap no chief factor. Secondly, $LM$ covers any chief factor covered either by $L$ or by $M$; and consideration of a particular chief series and the equation $(LM) = (L)(M)$ shows that $LM$ must avoid every other chief factor.

An interesting class of subgroups of a group $G$ consists of all those which either cover or avoid any chief factor of $G$ (i.e. which cover any chief factor overlapped). For any group, every normal subgroup has this property; and an important theorem of Hall (4) is that any system normalizer of a normal soluble subgroup of $G$, relative to $G$, covers or avoids any chief factor of $G$. We shall later mention some subgroups of $A$-groups with this property, but meanwhile we give a general result for soluble groups.

(1·4) *If the subgroup $H$ of the soluble group $G$ covers or avoids any chief factor of $G$, so also does any $S$-subgroup† $J$ of $H$.*

Let $(H) = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, $(J) = p_1^{\alpha_1} \dots p_s^{\alpha_s}$, where $s < r$. If $G = G_0 > G_1 > \dots > G_m = 1$ is any chief series of $G$, the order of each factor $G_{i-1}/G_i$ is a power of a prime. We have

$$\prod_{i=1}^{m} \mathrm{tr}\,(J; G_{i-1}/G_i) = p_1^{\alpha_1} \dots p_s^{\alpha_s}.$$

Now any factor $G_{i-1}/G_i$ which is overlapped by $J$ is covered by $H$, and its order must be a power of one of the primes $p_1, \dots, p_s$. The product of the orders of all factors of the series overlapped by $J$ must therefore divide $(H)$ and involve only the primes $p_1, \dots, p_s$ —i.e. it must divide $p_1^{\alpha_1} \dots p_s^{\alpha_s} = (J)$. Hence $J$ covers every factor $G_{i-1}/G_i$ that it overlaps, and this statement holds also for any other chief series of $G$.

† I.e. a subgroup whose order is prime to its index in $G$. See (1).

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## 2. A LEMMA ON ABELIAN $p$-GROUPS

As a further weapon to assist in the proofs of our main theorems we now give a lemma on Abelian $p$-groups with operators. We shall call a group with operators, whose operators form the set $\Omega$, an $\Omega$-*group*. An $\Omega$-*subgroup* of such a group is a subgroup which is itself an $\Omega$-group, with respect to the same set $\Omega$ of operators.

LEMMA 2·1. *Let $G$ be an Abelian $\Omega$-group of order a power of a prime $p$, where $\Omega$ is a subgroup of the group of all automorphisms of $G$, and is of order $q$, prime to $p$. Then any $\Omega$-subgroup $G_1$ of $G$ which is complemented in $G$ has a complement which is an $\Omega$-subgroup of $G$.*

*Proof.* Let the greatest possible order of an element of $G$ be $p^\alpha$. We can choose integers $r$, $s$ such that

$$rp^\alpha + sq = 1.$$

Then if $x \in G$,

$$x = x^{rp^\alpha + sq} = x^{sq}.$$

Suppose now that $G = G_1 \times G_2$, where $G_1$ is an $\Omega$-subgroup of $G$, and $G_2$ is a subgroup of $G$. Then if $x \in G$, $x = x_1 x_2$ uniquely, with $x_1 \in G_1$, $x_2 \in G_2$. Consider the set $G_3$ consisting of the elements

$$x_3 = \left\{\prod_{\theta \in \Omega} (x\theta)_2 \theta^{-1}\right\}^s$$

for all $x \in G$. $G_3$ is a subgroup of $G$, since if $x, y \in G$,

$$x_3 y_3 = (xy)_3.$$

Let $\phi$ be any element of $\Omega$. Then

$$x_3\phi = \left\{\prod_{\theta \in \Omega} (x\theta)_2 \theta^{-1}\phi\right\}^s.$$

Write $y = x\phi$ and $\psi = \phi^{-1}\theta$. For fixed $\phi$, $\psi$ runs through all the elements of $\Omega$ as $\theta$ does; and

$$x\theta = x\phi\psi = y\psi.$$

Hence

$$x_3\phi = \left\{\prod_{\psi \in \Omega} (y\psi)_2 \psi^{-1}\right\}^s = y_3 \in G_3.$$

Thus $G_3$ is an $\Omega$-subgroup of $G$.

For any $x \in G$, write $x = (xx_3^{-1})\, x_3$. Then

$$xx_3^{-1} = x^{q^s} x_3^{-1} = \left\{\prod_{\theta \in \Omega} (x\theta)\, \theta^{-1}\right\}^s \left\{\prod_{\theta \in \Omega} (x\theta)_2 \theta^{-1}\right\}^{-s}$$

$$= \left[\prod_{\theta \in \Omega} \{(x\theta)\, (x\theta)_2^{-1}\} \theta^{-1}\right]^s = \left\{\prod_{\theta \in \Omega} (x\theta)_1 \theta^{-1}\right\}^s \in G_1,$$

since $G_1$ is an $\Omega$-group. Hence $G = G_1 G_3$. Since $G = G_1 \times G_2$, it follows that $(G_3) \geqslant (G_2)$. If $x \in G$, $x_1 \in G_1$, $(x_1\theta) \in G_1$ for all $\theta$, and so $(x_1\theta)_2 = 1$, $(x_1)_3 = 1$. Hence

$$x_3 = (x_1 x_2)_3 = (x_1)_3 (x_2)_3 = (x_2)_3.$$

Therefore $(G_3) \leqslant (G_2)$, and so $(G_3) = (G_2)$, $G = G_1 \times G_3$, completing the proof of our lemma.

The above argument has been adapted from a proof of the following well-known result:

If $M$ is an $\Omega \times G$ module, and $G$ is a group of finite order $g$, where the operator $1/g$ (in the obvious sense) belongs to $\Omega$, then $M$ is completely reducible qua $\Omega \times G$ module if it is so qua $\Omega$ module.

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## 3. Elementary theory of $A$-groups

DEFINITION. An *$A$-group* is a soluble group, all of whose Sylow subgroups are Abelian.

*Examples.* Any group whose Sylow subgroups are all cyclic, and in particular any group of square-free order, is an $A$-group. Again, any soluble group of cube-free order is an $A$-group.

*Preliminary results*

(3·1) *Any subgroup of an $A$-group is itself an $A$-group.*

For any subgroup of a soluble group is soluble, and any one of its Sylow subgroups is contained in some Sylow subgroup of the whole group.

(3·2) *Any homomorphic image of an $A$-group is itself an $A$-group.*

For any homomorphic image of a soluble group is soluble, and any Sylow subgroup of the image group is the image of some Sylow subgroup of the original group.

(3·3) *A nilpotent $A$-group must be Abelian, since it is the direct product of its Sylow subgroups.* (Of course, any Abelian group is both nilpotent and an $A$-group.)

(3·4) *The lower nilpotent series of an $A$-group coincides with the derived series. In particular, $L_1$ coincides with the first derived group (or commutator subgroup) $G'$.*

For $L_1$ is the (unique) minimal normal subgroup with nilpotent quotient group, i.e. (in an $A$-group) with Abelian quotient group. Hence $L_1 = G'$, and by repetition of the argument we see that $L_i$ is the $i$th derived group.

Thus the *nilpotent length* of an $A$-group $G$ coincides with the '*Stufenzahl*', the length of the derived series. [See H. Fitting, (6).] We shall call this number the *length* of $G$.

(3·5) *If $M$ is any absolute system normalizer of the $A$-group $G$, then $G = MG'$.*

This is an immediate consequence of (3·4) and the general result $G = L_i M_G(L_{i-1})$ for any soluble group, quoted in the introduction.

(3·6) *If $G' \leqslant Z_1$, the centre of the $A$-group $G$, then $G$ is Abelian. In fact, for any group $G$, $G' \leqslant Z_1$ implies that $G$ is soluble and nilpotent; for then $H_3 = [G, G'] = 1$, so that $L_1 = 1$, $G$ is nilpotent.*

(3·7) *Any system normalizer $M$ of an $A$-group $G$ is Abelian.*

For by a property of soluble groups (4), $M$ is nilpotent.

(3·8) *The hypercentral $Z_\infty$ of an $A$-group $G$ coincides with the centre $Z_1$.*

For any group $G$, $[Z_2, G'] = 1$, and for any soluble group, $Z_2 \leqslant Z_\infty \leqslant M$, where $M$ is any system normalizer of $G$ (4). If $G$ is an $A$-group, $M$ is Abelian, and so $[Z_2, M] = 1$. Thus $[Z_2, G] = [Z_2, MG'] = 1$, $Z_2 \leqslant Z_1$. Hence $Z_\infty = Z_1$.

(3·9) *Any system normalizer $M$ of an $A$-group $G$ is contained in the centre of its normalizer, since in any soluble group a system normalizer is contained in the hypercentral of its normalizer* (4). *Thus the normalizer of $M$ coincides with $M^*$, the centralizer of $M$ in $G$.*

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## 4. The fundamental theorem on $A$-groups, and some consequences of it

We are now in a position to prove the important theorem on $A$-groups, which was stated without proof by P. Hall (5), and which leads to many interesting properties.

Theorem 4·1. *The meet of the centre $Z_1$ and the derived group $G'$ of an $A$-group $G$ is the identity.*

*Proof.* The theorem is trivially true if $G$ is of order $p^\alpha$ (where $p$ is any prime, $\alpha$ is any positive integer), since $G$ is then Abelian and $G' = 1$. We proceed by induction, making the assumption that the theorem holds for any $A$-group of order less than that of $G$.

Suppose the theorem to be false for $G$. Then $Z_1 \wedge G' = H$, a normal subgroup of $G$ different from the identity.

Let $N$ ($\neq 1$) be any normal subgroup of $G$, and $\theta$ be any homomorphism of $G$ whose kernel is $N$. By (3·2), $G\theta$ is an $A$-group. The derived group of $G\theta$ is $G'\theta$, and $Z_1\theta$ is contained in the centre of $G\theta$. Since $(G\theta) < (G)$, by the induction hypothesis the centre and the derived group of $G\theta$ meet in the identity of $G\theta$. Thus we have

$$H\theta \leqslant G'\theta \wedge Z_1\theta = N\theta,$$

and $N\theta$ is the identity of $G\theta$. Hence $H \leqslant N$, and so $H$ is the unique minimal normal subgroup of $G$.

Any subgroup of $Z_1$ is normal in $G$, so that $H$ is the unique minimal subgroup of the Abelian group $Z_1$. Hence $Z_1$ is cyclic and of order $p^\beta$, for some prime $p$, and $H$ is of order $p$.

Since $G$ is soluble, $(G') < (G)$. Suppose $G'' \neq 1$; since $G''$ is a normal subgroup of $G$, $H \leqslant G''$. But $H = G' \wedge Z_1$ is also contained in the centre of $G'$, so the meet of $G''$ and the centre of $G'$ contains $H$, a contradiction of the induction hypothesis. Thus $G'' = 1$ and $G'$ is Abelian. Then every Sylow subgroup of $G'$ is a characteristic subgroup of $G'$, and so also of $G$. Hence every Sylow subgroup of $G'$ contains $H$, which is of order $p$. It follows that $G$ is of order $p^\gamma$, for some integer $\gamma$.

Further, $G'$ must be cyclic. For suppose that this is not so. Then $H$ is a proper subgroup of $L$, the characteristic subgroup of $G'$ consisting of all elements of $G'$ of order $p$ (together with the identity). $L$ is of type $(p, p, \ldots, p)$, and we can write $L = H \times J$, where $J \neq 1$. $L$ is a normal subgroup of $G$; hence every inner automorphism of $G$ gives rise to an automorphism of $L$, which leaves every element of $H$ unchanged (since $H \leqslant Z_1$). $L$ is contained in every Sylow $p$-subgroup of $G$, all of which are Abelian. Therefore every element of $L$ permutes with every element of $G$ whose order is a power of $p$. The group $\Omega$ of all automorphisms of $L$, generated by transformations by elements of $G$, is thus of order prime to $p$. We see that $L$ is an Abelian $\Omega$-group of order a power of $p$, $H$ is an $\Omega$-subgroup of $L$, complemented in $L$, and $\Omega$ is a subgroup of the group of automorphisms of $L$, and is of order prime to $p$. Hence, by Lemma 2·1, $H$ has as complement in $L$ an $\Omega$-subgroup $K$. $K \neq 1$, is normal in $G$, and is such that $H \wedge K = 1$. But this is a contradiction of the unique minimal property of $H$, and we are forced to conclude that $G'$ is cyclic.

Let $G' = \{x\}$, of order $p^\gamma$. Then $H = \{x^{p^{\gamma-1}}\}$. We certainly have $\gamma > 1$; for if $\gamma = 1$, $G' = H \leqslant Z_1$, and $G$ is Abelian, by (3·6), contrary to hypothesis.

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Let $y$ be any element of $G$ of order $r$ (say), prime to $p$. Then $y^{-1}xy = x^{\lambda}$ for some $\lambda$. Since $x^{p^{\gamma-1}} \in Z_1$,

$$x^{p^{\gamma-1}} = y^{-1}x^{p^{\gamma-1}}y = x^{\lambda p^{\gamma-1}}.$$

Hence

$$\lambda p^{\gamma-1} \equiv p^{\gamma-1} \pmod{p^{\gamma}}, \quad \lambda \equiv 1 \pmod{p}.$$

We also have $x^{\lambda^r} = y^{-r}xy^r = x$, since $y^r = 1$. Hence

$$\lambda^r \equiv 1 \pmod{p^{\gamma}},$$

i.e.

$$(\lambda - 1)(\lambda^{r-1} + \lambda^{r-2} + \ldots + \lambda + 1) \equiv 0 \pmod{p^{\gamma}}.$$

Since $\lambda \equiv 1 \pmod{p}$, each of $\lambda, \lambda^2, \ldots, \lambda^{r-1} \equiv 1 \pmod{p}$, and hence

$$(\lambda^{r-1} + \lambda^{r-2} + \ldots + \lambda + 1) \equiv r \pmod{p} \not\equiv 0 \pmod{p},$$

since $r$ is prime to $p$. Thus

$$(\lambda - 1) \equiv 0 \pmod{p^{\gamma}}, \quad y^{-1}xy = x.$$

Hence every $y \in G$ whose order is prime to $p$ permutes with $x$, as does every $z \in G$ which is of order a power of $p$. But any element $w \in G$ can be expressed as $yz$, and hence *every* element of $G$ permutes with $x$. Thus $x \in Z_1$, $G' \leqslant Z_1$, $G$ is Abelian, by (3·6). But this is a contradiction of our initial hypothesis that our theorem does not hold in $G$.

Therefore the theorem is true of $G$, and so, by induction, is true of any $A$-group.

*Corollaries of Theorem* 4·1

(4·2) *If $K$ is the derived group of the $A$-group $H$, a subgroup of $G$, and $Q/R$ is any factor of $G$ which is central† for $H$, then $K$ avoids $Q/R$.* (In this result $G$ need not be an $A$-group, nor yet even a soluble group.)

Let the natural homomorphism $g \to Rg$ of $G$ onto $G/R$ be $\theta$. Then $H\theta$ is contained in the centralizer of $Q\theta$ in $G\theta$. Hence $H\theta \wedge Q\theta$ is contained in the centre of $H\theta$. But $H\theta$ is an $A$-group, with derived group $K\theta$. Applying Theorem 4·1 we have that $H\theta \wedge Q\theta \wedge K\theta$ consists of the identity of $H\theta$. Therefore $Q \wedge KR = R$, i.e. $K$ avoids $Q/R$.

(4·3) *In particular, the derived group $G'$ of an $A$-group $G$ avoids every central factor of $G$.*

(4·4) *If $K$ is the derived group of the $A$-group $H$, a normal subgroup of $G$, then $K$ is complementary in $G$ to $M = M_G(H)$, any system normalizer of $H$ relative to $G$.* (Here again, $G$ need not even be soluble.)

Since $K$ is a characteristic subgroup of $H$, $K$ is a normal subgroup of $G$. $H/K$ is thus an Abelian factor of $G$, and so by the general result quoted in the introduction, $G = KM$.

By a theorem of Hall (4), $M$ covers every chief factor of $G$ central for $H$, and avoids every chief factor eccentric for $H$. By (4·2), $K$ avoids every chief factor central for $H$ and, being normal in $G$, must cover or avoid any chief factor. Applying (1·3), we have that $K \wedge M = 1$, and so $K$ is complementary to $M$ in $G$.

(4·5) *If $G$ is an $A$-group, any system normalizer of $L_{i-1}$ relative to $G$ is a complement of $L_i$. In particular, any absolute system normalizer of $G$ is complementary to the derived group $G' = L_1$ of $G$.*

(4·6) *If the $A$-group $H$ is a normal subgroup of $G$, $M = M_G(H)$ is any system normalizer of $H$ relative to $G$, and $K$ is the derived group of $H$, then a chief factor of $G$ is (a) covered*

† $Q/R$ is *central for* $H$ if $H$ is contained in the centralizer of $Q/R$ in $G$, i.e. if $HR/R$ is contained in the centralizer of $Q/R$ in $G/R$. $Q/R$ is a *central* factor of $G$ if the centralizer of $Q/R$ in $G$ is $G$. If a chief factor is not central, it is called *eccentric*. See (4).

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*by $M$ and avoided by $K$, if it is central for $H$, or (b) avoided by $M$ and covered by $K$, if it is eccentric for $H$.*

We have only to show that $K$ covers every chief factor of $G$ eccentric for $H$. But this follows from our argument at (4·4) above and (1·3), since any chief factor, eccentric for $H$, avoided by $K$ would also be avoided by $KM = G$.

(4·7) *In particular, the derived group $G'$ of an A-group $G$ covers every eccentric chief factor of $G$.*

(4·8) *The total number of distinct Sylow systems of an A-group is equal to the order of the derived group.*

For this number equals the index of any system normalizer.

THEOREM 4·9. *Suppose that the group $G$ has as a normal subgroup the A-group $H$, and that $M$ is any system normalizer of $H$ relative to $G$. Then any normal subgroup $L$ of $M$ covers or avoids any given chief factor of $G$.*

*Proof.* The derived group $K$ of $H$ is a normal subgroup of $G$, and $G = MK$. Since $L$ is normal in $M$, $LK$ is normal in $G$ and hence covers or avoids any given chief factor of $G$. Let $G = G_0 > G_1 > \ldots > G_r = 1$ be any chief series of $G$. If $G_{i-1}/G_i$ is eccentric for $H$, it is covered by $K$ and avoided by $M$ (and hence avoided by $L \leqslant M$). If $G_{i-1}/G_i$ is central for $H$, it is avoided by $K$ and covered by $M$; if it is overlapped by $L$ it must be overlapped (and therefore covered) by $LK$. Thus $LK$ covers any chief factor which is either covered by $K$ or overlapped by $L$, and these two classes of chief factors are exclusive. Now $(L)(K) = (LK)$, which is not less than the product of the orders of the chief factors covered by $K$, times the product of the orders of the chief factors overlapped by $L$. The former of these products is equal to $(K)$; the latter is not less than $(L)$, and therefore is equal to $(L)$. Hence $(L)$ equals the product of the orders of the chief factors overlapped by $L$, and so by (1·2) $L$ covers or avoids any factor $G_{i-1}/G_i$ of the chosen chief series of $G$, and therefore *any* chief factor of $G$.

## 5. NORMAL SUBGROUPS OF AN $A$-GROUP

THEOREM 5·1. *Let $H$ be any normal subgroup of the A-group $G$, $M$ be any absolute system normalizer of $G$, and $G'$ be the derived group of $G$. Then if $H \wedge M = J$ and $H \wedge G' = L$, $J$ is a complement of $L$ in $H$. If $H$ is Abelian, $H = J \times L$.*

*Proof.* Since $H$, $G'$ are both normal in $G$, so also is $L$. Hence $JL$ is a subgroup of $G$, and clearly $JL \leqslant H$.

Consider a chief series of $G$ through $H$; $M$ and $G'$ both 'cover or avoid' any chief factor. If $P \leqslant H$, we have

$$\mathrm{tr}(R \wedge H; P/Q) = (R \wedge H \wedge P)/(R \wedge H \wedge Q) = (R \wedge P)/(R \wedge Q) = \mathrm{tr}(R; P/Q)$$

for any $R$. Thus $J$ covers any chief factor 'below' $H$ which is covered by $M$, and avoids any chief factor either 'above' $H$ or avoided by $M$; and $L$ bears the same relationship to $H$ and $G'$. Thus

$(H)$ is the product of the orders of all chief factors below $H$,
$(J)$ is the product of the orders of all central chief factors below $H$,
$(L)$ is the product of the orders of all eccentric chief factors below $H$

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(where 'chief factor' implies 'chief factor from the chosen chief series through $H$'). Hence $(H) = (J)(L) = (JL)$, since $J \wedge L = 1$.

Therefore $H = JL$, and $J$ is a complement of $L$ in $H$. It follows at once that if $H$ is Abelian, $H = J \times L$.

COROLLARY 5·2. $$H = H_0 H_1 \dots H_{n-1},$$

*where* $H_i = H \wedge M(L_i)$. [$L_0 = G$, $L_n = 1$, *and* $M(L_i)$ *is any absolute system normalizer of* $L_i$.]

THEOREM 5·3. *If* $A$ *is any normal Abelian subgroup of an* $A$*-group* $G$, *then*

$$A = A_0 \times A_1 \times \dots \times A_{n-1},$$

*where* $A_i = A \wedge Z_1(L_i)$ *and is hence itself a normal Abelian subgroup of* $G$.

*Proof.* Any Abelian group is the direct product of its Sylow subgroups, which are characteristic subgroups. Hence a normal Abelian subgroup of $G$ is the direct product of normal Abelian subgroups of $G$, each of which is of order a power of some prime. There is no loss of generality, therefore, if we suppose $A$ to be of order $p^\alpha$, $p$ being a prime. Since $A$ is normal, it is contained in every Sylow $p$-subgroup of $G$. Let $S$ be any Sylow $p$-subgroup, $M$ be any system normalizer of $G$, and let $S'$ be a Sylow $p$-complement which is normalized by every element of $M$. Write

$$A \wedge M = J, \quad A \wedge G' = L.$$

Then, by Theorem 5·1, $A = J \times L$; and $L$ is a normal subgroup of $G$. Let $j$ be any element of $J$, and $g$ be any element of $G$. Then $g^{-1}jg \in A$; say $g^{-1}jg = j'l'$, where $j' \in J$, $l' \in L$. Since $S$, $S'$ are complements in $G$, we may write $g = ss'$, where $s \in S$, $s' \in S'$. Then, since $j$ permutes with $s$ ($J \leqslant A \leqslant S$),

$$g^{-1}jg = s'^{-1}s^{-1}jss' = s'^{-1}js' = j \,.\, j^{-1}s'^{-1}js' = j'l'.$$

But $j^{-1}s'^{-1}js' \in G'$, and since the expression of any element of $G$ as the product of an element of $M$ and an element of $G'$ is unique, we must have $j = j'$ and $j^{-1}s'^{-1}js' = l'$.

Now $j \in M$ and hence normalizes $S'$, so that $j^{-1}s'^{-1}j = \bar{s} \in S'$. Therefore $l' = \bar{s}s' \in S'$, $l' \in L \wedge S' \leqslant S \wedge S' = 1$, so that $l' = 1$. Thus $g^{-1}jg = j$ for every $g \in G$, $j \in J$. Therefore $J \leqslant Z_1(G) \leqslant M$ for every system normalizer $M$. Clearly $J = Z_1(G) \wedge A = A_0$ (say), a normal subgroup of $G$. By a repetition of the same argument we have $L = A_1 \times R$, where $A_1 = L \wedge Z_1(G') = A \wedge Z_1(L_1)$ and $R = L \wedge G'' = A \wedge L_2$. Thus $A = A_0 \times A_1 \times R$, and by further repetitions of this argument we eventually obtain the result

$$A = A_0 \times A_1 \times \dots \times A_{n-1}$$

where $A_i = A \wedge Z_1(L_i)$, a normal subgroup of $G$.

An immediate consequence of this result is

THEOREM 5·4†. *The term* $U_1$ *of the upper nilpotent series of an* $A$*-group* $G$ (*which is the unique maximal normal Abelian subgroup of* $G$) *is given by*

$$U_1 = Z_1(G) \times Z_1(L_1) \times Z_1(L_2) \times \dots \times Z_1(L_{n-1}).$$

For each component $Z_1(L_i)$ is an Abelian normal subgroup of $G$, and the expression on the right is in fact a direct product, by Theorem 4·1. Hence this expression is

† This theorem was stated (without proof) by Hall (5).

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certainly a normal Abelian subgroup of $G$, and by Theorem 5·3 contains any other such subgroup $A$ of $G$; it is therefore $U_1$.

The fact that any $A$-group contains a unique maximal normal Abelian subgroup suggests a simple extension for $A$-groups of a well-known elementary theorem, which states that if two normal subgroups $N$, $P$ of a group $G$ meet in the identity only, then every element of $N$ permutes with every element of $P$, i.e. that $N \leqslant P^*$, the centralizer of $P$, and $P \leqslant N^*$. Conversely, it is clear that a *necessary* condition for $N \leqslant P^*$ is that the meet $L = N \wedge P$ should be contained in the centres of both $N$ and $P$. But in general this is not a *sufficient* condition; for if it were, it would follow that the join of any two normal Abelian subgroups of a group is itself Abelian, implying the existence of a unique maximal normal Abelian subgroup of the group. Such a maximal subgroup does not in general exist; but this objection is not valid for an $A$-group. In fact, we have

THEOREM 5·5. *Suppose that $N$, $P$ are two normal subgroups of an $A$-group $G$. Then the necessary and sufficient condition that $N \leqslant P^*$ is that the meet $L = N \wedge P$ should be contained in the centres of both $N$ and $P$.*

*Proof.* We have only to establish that the condition is sufficient. Any element of $L$ is contained in the centre $Z_1(NP)$ of $NP$, since it permutes with every element of $N$ and of $P$. Since $N$, $P$ are both normal, $[N, P] \leqslant L \leqslant Z_1(NP)$. But $[N, P] \leqslant L_1(NP)$, and so $[N, P] \leqslant L_1(NP) \wedge Z_1(NP) = 1$, by Theorem 4·1. Hence $N \leqslant P^*$

## 6. SOME FURTHER DEFINITIONS AND LEMMAS ON ABELIAN $p$-GROUPS

We now come to certain properties of Abelian $p$-groups with operators which are required in the proof of the Basis Theorem (which we give in the next section). In order to make our arguments lucid, and to avoid circumlocution, we shall first introduce some terms and notations which are found to be useful.

Let $G$ be an Abelian $p$-group. We say that the element $x$ of $G$ is *of exponent $\lambda$ in $G$* if there exists $y \in G$ such that $x = y^{p^\lambda}$, but there exists no $z \in G$ such that $x = z^{p^{\lambda+1}}$. For example, every element of a basis of $G$ is of exponent 0 in $G$.

A *homocyclic* group $H$ is an Abelian group of type $(p^\alpha, p^\alpha, \ldots, p^\alpha)$ and order $p^{\alpha\beta}$. Any basis consists of $\beta$ elements, each of order $p^\alpha$; we shall call $\alpha$ the *power* of the group and $\beta$ the *span*. Thus a homocyclic group of power 1 is elementary Abelian. If $H$ is homocyclic of power $\alpha$ and span $\beta$, and $K$ is a homocyclic subgroup of power $\alpha$ and span $\gamma < \beta$, then $K$ is complemented in $H$.

For an Abelian $p$-group $G$, we shall use a notation of Burnside, denoting by $G_{(\mu)}$ the characteristic subgroup consisting of all elements whose orders divide $p^\mu$, and by $G^{(\nu)}$ that consisting of all elements with exponent in $G$ at least $\nu$. We shall also write $G_{(\mu)} \wedge G^{(\nu)} = K_{(\mu,\nu)}$, another characteristic subgroup. Note that if $H$ is homocyclic, of power $\alpha$, and $0 \leqslant \mu \leqslant \alpha$, then $H_{(\mu)} = H^{(\alpha-\mu)}$.

Suppose that $G$ is an Abelian $p$-group, and $\Omega$ is a subgroup of the group of all automorphisms of $G$. Then $G$ and all characteristic subgroups of $G$ are $\Omega$-groups. If $G_{(1)}$ is an indecomposable $\Omega$-group, so also is $G$. For suppose that $G$ can be decomposed into a direct product $G_1 \times G_2 \times \ldots \times G_r$ of $\Omega$-subgroups $G_i$. Then

$$G_{(1)} = (G_1)_{(1)} \times (G_2)_{(1)} \times \ldots \times (G_r)_{(1)},$$



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and each $(G_i)_{(1)}$ is an $\Omega$-group, contradicting the indecomposability of $G_{(1)}$. In particular, if $G_{(1)}$ is irreducible (qua $\Omega$-group), then $G$ is indecomposable.

LEMMA 6·1. *Let $G$ be an Abelian $\Omega$-group of order a power of the prime $p$, where $\Omega$ is a subgroup of the group of all automorphisms of $G$ and is of order prime to $p$. Let a basis of $G$ contain $\beta_i$ elements of order $p^{\alpha_i}$ for each $i = 1, 2, \ldots, s$, where $\alpha_1 > \alpha_2 > \ldots > \alpha_s \geqslant 1$. Then $G$ can be expressed as the direct product of $r\,(\geqslant s)$ indecomposable homocyclic $\Omega$-groups $H_j$. If $H_j$ is of power $\gamma_j$ and span $\delta_j$, then $\gamma_j = \alpha_i$ for some $i$, and*

$$\sum_{(j;\,\gamma_j = \alpha_i)} \delta_j = \beta_i \quad (i = 1, 2, \ldots, s).$$

*Proof.* It is clearly sufficient to show that $G$ can be expressed as the direct product of $s$ homocyclic $\Omega$-groups $J_i$ $(i = 1, 2, \ldots, s)$, where $J_i$ is of power $\alpha_i$ and span $\beta_i$; the subsequent expression as a product of *indecomposable* homocyclic $\Omega$-groups $H_j$ requires no further proof. If $G$ is itself homocyclic, the result is obviously true; we therefore argue by induction, assuming that any $\Omega$-group whose order (a power of $p$) is less than that of $G$ can be expressed as a direct product of homocyclic $\Omega$-groups.

$G_{(1)}$ is an $\Omega$-subgroup of $G$ of elementary Abelian type, in which every subgroup is complemented. $K_{(1,\alpha_s)}$ is an $\Omega$-subgroup of $G_{(1)}$, and therefore has a complement in $G_{(1)}$. By Lemma 2·1, $K_{(1\,\alpha_s)}$ has a complement in $G_{(1)}$ which is itself an $\Omega$-group. Let such a complement be $N$. Any basis of $N$ consists of $\beta_s$ elements, each of order $p$ and exponent $\alpha_s - 1$ in $G$.

The $\Omega$-group $G/N$ is of order less than that of $G$, and a basis consists of $\beta_k$ elements of order $p^{\alpha_k}$, for each $k = 1, 2, \ldots, s-1$, together with $\beta_s$ elements of order $p^{\alpha_s - 1}$. By the induction hypothesis, we can write $G/N = Q/N \times R/N$, where $Q/N$ is homocyclic, of power $\alpha_s - 1$ and span $\beta_s$, and $R/N$ has $\beta_k$ basis elements of order $p^{\alpha_k}$, for each $k = 1, 2, \ldots, s-1$. $Q$, $R$ are $\Omega$-subgroups of $G$, such that $Q \wedge R = N$; $Q$ is homocyclic, of power $\alpha_s$ and span $\beta_s$, while $R$ has $\beta_k$ basis elements of order $p^{\alpha_k}$, for each $k = 1, 2, \ldots, s-1$, and $\beta_s$ basis elements of order $p$. Let $T_0$ be a subgroup of $R$ having $\beta_k$ basis elements of order $p^{\alpha_k}$ for each $k = 1, 2, \ldots, s-1$. Then $T_0 \wedge N = 1$, since every element of $(T_0)_{(1)}$ is of exponent at least $\alpha_{s-1} - 1 > \alpha_s - 1$ in $G$. Hence $R = T_0 \times N$, and so, by Lemma 2·1, $R = T \times N$, where $T$ is an $\Omega$-group. Now $T \wedge Q \leqslant R \wedge Q = N$, and hence $T \wedge Q = 1$. Considerations of order show that $G = Q \times T$, and $Q$ is a homocyclic $\Omega$-group, $T$ is an $\Omega$-group of order less than that of $G$. The induction hypothesis shows that $T$ may be expressed as the direct product of homocyclic $\Omega$-groups, from which it follows that $G$ may be so expressed. The truth of our lemma is thus established by induction.

COROLLARY 6·2. *If $H$ is an indecomposable homocyclic $\Omega$-group of order a power of the prime $p$, where $\Omega$ is a subgroup of the group of automorphisms of $H$ and is of order prime to $p$, then $H_{(1)}$ and each $H_{(i)}/H_{(i-1)}$ (for $i = 2, \ldots, \alpha$, where $\alpha$ is the power of $H$) is an irreducible $\Omega$-group.*

For suppose that $H_{(1)}$ has a proper $\Omega$-subgroup $N$, of span $\beta_0$, where $\beta_0 < \beta$, the span of $H$. Then $H/N$ is an $\Omega$-group having $\beta - \beta_0$ basis elements of order $p^{\alpha}$, and $\beta_0$ basis elements of order $p^{\alpha-1}$. By Lemma 6·1,

$$H/N = Q/N \times R/N,$$

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where $Q/N$, $R/N$ are homocyclic $\Omega$-groups of powers $\alpha-1$, $\alpha$ and spans $\beta_0$, $\beta-\beta_0$ respectively. Then $R$ is an $\Omega$-group having $\beta-\beta_0$ basis elements of order $p^\alpha$ and $\beta_0$ basis elements of order $p$, and so by Lemma 6·1 contains a homocyclic $\Omega$-group of power $\alpha$ and span $\beta-\beta_0$. This proper homocyclic $\Omega$-subgroup of $H$ is complemented in $H$, contrary to the hypothesis that $H$ is indecomposable. Hence $H_{(1)}$ is irreducible, qua $\Omega$-group.

If $H_{(i)}/H_{(i-1)}$ were not irreducible, $H$ would contain a proper non-homocyclic $\Omega$-subgroup, and this would imply that $H_{(1)}$ had a proper $\Omega$-subgroup, which we have already shown to be impossible.

COROLLARY 6·3. *$G$, as defined in Lemma* 6·1, *contains a sequence of $\Omega$-subgroups* $G = R_0 > R_1 > \ldots > R_{n-1} > R_n = 1 \left(n = \sum_{j=1}^{r} \gamma_j\right)$, *where each factor $R_{i-1}/R_i$ is an irreducible (and therefore elementary Abelian) $\Omega$-group which is covered by just one of the $H_j$ and avoided by all the rest, $H_j$ covering precisely $\gamma_j$ of the factors (each of order $p^{\beta_j}$) and avoiding all the rest.*

For we may take

$$\begin{aligned}
R_1 &= (H_1)_{(\gamma_1-1)} \times H_2 \times \ldots \times H_r,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
R_{\gamma_1-1} &= (H_1)_{(1)} \times H_2 \times \ldots \times H_r,\\
R_{\gamma_1} &= H_2 \times \ldots \times H_r,\\
R_{\gamma_1+1} &= (H_2)_{(\gamma_2-1)} \times \ldots \times H_r,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
R_{n-1} &= (H_r)_{(1)},
\end{aligned}$$

and our result then follows from Lemma 6·1 and Corollary 6·2.

LEMMA 6·4. *Any normal Abelian subgroup $A$ of an $A$-group $H$ is expressible as a direct product $A = A_1 \times A_2 \times \ldots \times A_r$, where the $A_i$ are all normal in $H$ and homocyclic, and are such that no $A_i$ can be further decomposed into a direct product of normal homocyclic subgroups. If $A_i$ is of order $p_i^{\alpha_i\beta_i}$, of power $\alpha_i$ and span $\beta_i$, then there exists a chief series of $H$ through $A$ such that each $A_i$ covers $\alpha_i$ factors (each of order $p_i^{\beta_i}$) and avoids all the rest, each factor below $A$ being covered by one and only one $A_i$.*

$A$ may be expressed as the direct product of its Sylow subgroups, which are characteristic subgroups of $A$ and hence normal subgroups of $H$. Hence we may confine our attention to normal $p$-subgroups of $H$, which are automatically Abelian since $H$ is an $A$-group. Let $\Omega$ be the group of all automorphisms of the normal $p$-subgroup $A$, generated by transformations by elements of $H$. Then $\Omega$ is of order prime to $p$, and any $\Omega$-subgroup of $A$ is a normal subgroup of $H$. We now apply Lemma 6·1 and its corollaries, and Lemma 6·4 follows by translating those results into the new notation. For example, a factor of $A$ which is an irreducible $\Omega$-group is a chief factor of $H$.



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## 7. THE BASIS THEOREM

THEOREM 7·1†. *Any $A$-group $G$ contains a system of subgroups $G_1, G_2, \ldots, G_\rho$ with the following properties:*

(i) *Of any pair $G_i$, $G_j$, at least one is contained in the normalizer of the other. Thus the $G_i$ are all permutable.*

(ii) *The product of the $G_i$'s is $G$, and the product of their orders is the order of $G$. Every element of $G$ is uniquely expressible in the form $\xi_1 \xi_2 \ldots \xi_\rho$, with $\xi_i$ in $G_i$; and the product of any selection of the $G_i$'s gives a subgroup complementary in $G$ to the product of the remainder.*

(iii) *Each $G_i$ is Abelian, of prime-power order $p_i^{\alpha_i \beta_i}$ and type $(p_i^{\alpha_i}, p_i^{\alpha_i}, \ldots, p_i^{\alpha_i})$. There exists a particular chief series of $G$ with the property that each of its factors is covered by just one of the $G_i$'s and avoided by all the rest, $G_i$ covering $\alpha_i$ distinct factors, each of order $p_i^{\beta_i}$. The product of all those $G_i$'s of order a power of a given prime $p$ is a Sylow $p$-subgroup of $G$. The complete set of permutable Sylow subgroups so formed defines a Sylow system $\mathfrak{S}$ of $G$, every one of whose members is obtainable as a product of $G_i$'s.*

(iv) *The $G_i$'s can be divided into $n$ exclusive sets, where $n$ is the length of $G$. If we write for the products of the $G_i$'s in the various sets $M_0, M_1, \ldots, M_{n-1}$, then $M_k$ is a system normalizer of $L_k$—viz., the normalizer of the Sylow system $\mathfrak{S} \wedge L_k$ of $L_k$. The product $M_k M_{k+1} \ldots M_{n-1}$ gives $L_k$ and, for $j < k$, $M_j M_{j+1} \ldots M_k = M_{L_j}(L_k)$, the normalizer relative to $L_j$ of the Sylow system $\mathfrak{S} \wedge L_k$ of $L_k$.*

(v) *Any product of $G_i$'s (and, in particular, any $G_i$ itself) covers or avoids any given chief factor of $G$.*

*Proof.* Let $\mathfrak{S}_0$ be any Sylow system of $G$. Since each term $L_k$ of the derived series is normal in $G$, $\mathfrak{S}_k = \mathfrak{S}_0 \wedge L_k$ is a Sylow system of $L_k$. Write $M_k = M_{L_k}(L_k)$, the absolute system normalizer of the Sylow system $\mathfrak{S}_k$ of $L_k$. Repeated application of (4·5) shows that $G = M_0 M_1 \ldots M_{n-1}$, and that the product of the orders of the $M_k$'s is the order of $G$.

We show first that the normalizer $M_{L_j}(L_k)$ of the Sylow system $\mathfrak{S}_k$ of $L_k$, relative to $L_j (j < k)$, is $M_j M_{j+1} \ldots M_k$. If $l \leqslant k$, each element of $M_l$ normalizes the Sylow system $\mathfrak{S}_l$ of $L_l$, and hence normalizes the system $\mathfrak{S}_k = \mathfrak{S}_l \wedge L_k$ of $L_k$, since $L_k$ is normal in $G$. Thus the set $M_j M_{j+1} \ldots M_k$ (not yet proved to be a group) is contained in $M_{L_j}(L_k)$. But this set and $M_{L_j}(L_k)$ are both complements of $L_{k+1}$ in $L_j$, and hence

$$M_j M_{j+1} \ldots M_k = M_{L_j}(L_k).$$

In particular,

$$M_0 M_1 \ldots M_k = M_G(L_k).$$

Now $H_k = M_0 M_1 \ldots M_k \cong G/L_{k+1}$, and in this isomorphism $M_k \cong L_k/L_{k+1}$. Thus $M_k$ is a normal Abelian subgroup of the $A$-group $H_k$, for each $k = 0, 1, \ldots, n-1$. Applying Lemma 6·4, with $H_k = H$, $M_k = A$, we have $M_k = A_{k1} \times A_{k2} \times \ldots \times A_{kr_k}$, where each $A_{kj}$ is a normal homocyclic subgroup of $H_k$, of order $p_{kj}^{\alpha_{kj}\beta_{kj}}$ (say) and power $\alpha_{kj}$, span $\beta_{kj}$. Also there exists a chief series of $H_k$ through $M_k$ such that, of the chief factors below $M_k$, each is covered by one and only one of the $A_{kj}$'s and avoided by all the rest, each

† This is another of the theorems conjectured by Hall(5). We have rearranged the clauses, and introduced a generalization into (v).

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$A_{kj}$ covering $\alpha_{kj}$ distinct factors, each of order $p_{kj}^{\beta_{kj}}$, and avoiding all the rest. Let the part of this chief series below $M_k$ be

$$M_k = M_{k0} > M_{k1} > \ldots > M_{ks_k} = 1 \quad \left(s_k = \sum_{j=1}^{r_k} \alpha_{kj}\right).$$

Then each factor

$$L_{k+1}M_{kl}/L_{k+1}M_{k(l+1)} \quad (l = 0, 1, \ldots, s_k - 1)$$

is a chief factor of $G$.

The totality of subgroups $A_{kj}$ $(k = 0, 1, \ldots, n-1;\ j = 1, 2, \ldots, r_k)$ forms the required set of subgroups $G_1, G_2, \ldots, G_\rho$ of $G$, where $\rho = \sum_{k=0}^{n-1} r_k$. We proceed to establish the properties claimed for the $G_i$'s:

(i) Let any pair of the $G_i$'s be $A_{k_1j_1}$, $A_{k_2j_2}$, and for definiteness suppose that $k_1 \leqslant k_2$. Then $A_{k_1j_1}$ is contained in $H_{k_2}$, of which group $A_{k_2j_2}$ is a normal subgroup, and so $A_{k_1j_1}$ is contained in the normalizer of $A_{k_2j_2}$.

(ii) The product of all the $A_{kj}$'s with given $k$ is $M_k$, and the product of their orders is the order of $M_k$. Hence the product of *all* the $A_{kj}$'s is $M_0M_1 \ldots M_{n-1} = G$, and the product of their orders is the product of the orders of the $M_k$'s, i.e. the order of $G$. This implies that any $g \in G$ is uniquely expressible as $g = \xi_1\xi_2 \ldots \xi_\rho$ with $\xi_i \in G_i$.

By (i), any product of a selection of the $G_i$'s is a subgroup of $G$. Let such a product be $U$, and let $V$ be the product of the remainder. Then $UV = G$ and $(U)(V) = (G)$, and so $U$ and $V$ are complementary in $G$.

(iii) and (iv) We have already seen that each $A_{kj}$ is homocyclic. A particular chief series of $G$ may be constructed as a refinement of the derived series

$$G = L_0 > L_1 > \ldots > L_k > L_{k+1} > \ldots > L_n = 1$$

by inserting between $L_k$ and $L_{k+1}$, for each $k$, the normal subgroups

$$L_k = L_{k+1}M_k = L_{k+1}M_{k0} > L_{k+1}M_{k1} > \ldots > L_{k+1}M_{ks_k} = L_{k+1}.$$

As we have seen, each factor $L_{k+1}M_{kl}/L_{k+1}M_{k(l+1)}$ is a chief factor of $G$, and it is covered by $A_{kj}$ if and only if the chief factor $M_{kl}/M_{k(l+1)}$ of $H_k$ is covered by $A_{kj}$. (It is, of course, avoided by each $A_{k'j'}$ for which $k' \neq k$). In this chief series of $G$, every chief factor is covered by just one of the $A_{kj}$'s and is avoided by all the rest, and each $A_{kj}$ covers precisely $\alpha_{kj}$ factors, each of order $p_{kj}^{\beta_{kj}}$, and avoids all the rest.

The remaining clauses of (iii) follow at once from (i) and (ii), and from the fact that each $G_i$ is of prime-power order. Moreover, the Sylow system $\mathfrak{S}$ of $G$ thus obtained is precisely the system $\mathfrak{S}_0$ from which we started. This is a consequence of the well-known theorem that the normalizer of a Sylow $p$-subgroup $S$ of $G$ contains no elements of order a power of $p$ other than the elements of $S$ itself. Hence the Sylow $p$-subgroup of $M_k$ is contained in the Sylow $p$-subgroup of $L_k$ which belongs to the Sylow system $\mathfrak{S}_0 \wedge L_k$ of $L_k$, and is therefore contained in the Sylow $p$-subgroup of $G$ which belongs to $\mathfrak{S}_0$. Since this is true for each $k$, the Sylow $p$-subgroup of $G$ contained in the system $\mathfrak{S}$ coincides with that contained in the system $\mathfrak{S}_0$, for each $p$, and so $\mathfrak{S}$ and $\mathfrak{S}_0$ are identical. From the definitions of the $M_k$, and the properties established at the beginning of this proof, we see at once that (iv) is true.

(v) Let $V = G_{\lambda_1}G_{\lambda_2} \ldots G_{\lambda_\sigma}$ be any product of $G_i$'s. We may split the $G_{\lambda_j}$'s into $n$ sets,

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each set consisting of $G_i$'s from the same $M_k$. Thus $V = V_0V_1 \dots V_{n-1}$ (say), where $V_k \leqslant M_k$ is a product of $G_i$'s. (Of course, some of the $V_k$'s may consist only of 1.)

Now $V_k$ is a normal subgroup of $H_k$, which is the relative system normalizer $M_G(L_k)$. By Theorem 4·9, $V_k$ covers or avoids any given chief factor of $G$; any chief factor it covers must also be covered by $M_G(L_k)$ and by $L_k$, and is therefore ($a$) central for $L_k$ (and hence also for $L_{k+1}, L_{k+2}, \dots, L_{n-1}$), and ($b$) eccentric for $L_{k-1}$ (and hence also for $L_{k-2}, \dots, L_1, G$), by (4·6).

It is clear, therefore, that no chief factor covered by $V_k$ can be covered by any other component $V_l$ ($l \neq k$); we thus have, by repeated applications of (1·3), that the product $V$ of the $V_k$ must also cover or avoid any given chief factor of $G$.

This completes our proof of the Basis Theorem.

## 8. Soluble groups of cube-free order

Any group whose order contains no prime to a power greater than 2 is soluble, provided that it contains no subgroup which is a tetrahedral group†. Clearly any soluble group of cube-free order is an $A$-group, since any Sylow subgroup, being of order a prime or the square of a prime, is Abelian. In this section we shall investigate the possible lengths of 'cube-free' soluble groups. An $A$-group of length 1 is Abelian, while one of length 2 is metabelian. It is convenient to speak of a group consisting of the identity alone as being of length 0.

If $G = G_0 > G_1 > \dots > G_m = 1$ is any series of normal subgroups of the $A$-group $G$ such that $G_{i-1}/G_i$ is Abelian, for each $i = 1, 2, \dots, m$, then it is easily seen that $m \geqslant n$, the length of $G$, and that $G_i \geqslant L_i$ for each $i = 0, 1, \dots, m$ (where we write

$$L_n = L_{n+1} = \dots = L_m = 1,$$

if $m > n$).

Lemma 8·1. *If $K$ is a subgroup of the centre of the $A$-group $G$, then $G$ and $G/K$ have the same length.*

(We exclude the trivial case when $G$ is Abelian and $K = G$.)

For if the length of $G$ is $n$, $K$ does not contain $L_{n-1}$, and hence $G/K$ is of length $n$.

If $G$ is any $A$-group of length $n$, $L_{n-1}$ is a characteristic Abelian subgroup of $G$, so that all its Sylow subgroups are characteristic in $G$. Let the Sylow $p$-subgroup of $L_{n-1}$ be $S$, and denote by $S^*$ the centralizer of $S$ in $G$. $S^*$ certainly contains $L_{n-1}$, and also every Sylow $p$-subgroup of $G$. Every inner automorphism of $G$ gives rise to an automorphism of $S$; the subgroup of elements of $G$ corresponding to the identical automorphism of $S$ is $S^*$. A homomorphism is thus set up between $G$ and a subgroup of the group of all automorphisms of $S$. The kernel of this homomorphism is $S^*$, so that the order of the image must be prime to $p$. The image group is isomorphic to $G/S^*$, and is therefore of the same length.

† See Burnside, *The theory of groups*, 1st ed. (Cambridge, 1897), p. 359. In the 2nd ed. (Cambridge, 1911), p. 328, he gives the more restrictive condition that the cube-free order should not be divisible by 12.

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LEMMA 8·2. *If $G$ is an $A$-group of length $n$, and $S$ is any Sylow subgroup of $L_{n-1}$, then $G/S^*$ is of length $n-1$.*

*Proof.* Since $S^* \geqslant L_{n-1}$, the length of $G/S^*$ cannot exceed $n-1$. It can only be less than $n-1$ if $L_{n-2} \leqslant S^*$. But this would imply that $S$ is contained in the centre of $L_{n-2}$, which is impossible by Theorem 4·1, since $S$ is contained in the derived group $L_{n-1}$ of $L_{n-2}$. Hence $G/S^*$ is of length precisely $n-1$.

Before applying this result to the consideration of cube-free groups, we use it to prove a general property of $A$-groups, mentioned by Hall (5). This is

THEOREM 8·3. *The length of an $A$-group cannot exceed the number of different primes which divide its order.*

*Proof.* Let the length of the $A$-group $G$, whose order contains $r$ different primes, be $n$. We use an induction argument to show that $n \leqslant r$. This is trivially true if $r = 1$; we assume the theorem true for any $A$-group whose order contains less than $r$ distinct primes.

Let $S$ be the Sylow $p$-subgroup of $L_{n-1}$. We have seen that $G/S^*$ is of order prime to $p$; so its order cannot contain more than $r-1$ different primes. By the induction hypothesis, the length $\lambda$ of $G/S^*$ cannot exceed $r-1$. But, by Lemma 8·2, $\lambda = n-1$. Hence $n \leqslant r$, and our induction argument is completed.

Now suppose that $G$ is an $A$-group of cube-free order, of length $n$. Let $S$ be the Sylow $p$-subgroup of $L_{n-1}$. We have seen that $G/S^*$ is isomorphic to a subgroup of the group $\Omega$ of automorphisms of $S$, and is of length $n-1$. We are thus led to investigate the possible lengths of subgroups of $\Omega$ which are themselves $A$-groups of cube-free order, and whose orders are factors of the order of $G$ and prime to $p$. There are two possibilities:

(1) $S$ is cyclic. Then the automorphism group $\Omega$ is cyclic, and so $G/S^*$ is of length 0 or 1.

(2) $S$ is Abelian, of order $p^2$ and type $(p, p)$. Then $\Omega$ is isomorphic to the group of all homogeneous linear substitutions of two variables (mod $p$), whose determinants are not congruent to 0 (mod $p$)†. This group is itself isomorphic to the group $\Gamma$ of all $(2 \times 2)$ matrices whose elements are integers (mod $p$) and whose determinants are not congruent to 0 (mod $p$). $\Gamma$ is of order $(p^2-1)(p^2-p) = p(p-1)^2(p+1)$.

If $p = 2$, $\Gamma$ is of order 6, so that the order of $G/S^*$ must be 1 or 3; the length of $G/S^*$ must be 0 or 1.

If $p = 3$, $\Gamma$ is of order $3 \cdot 2^4$, so that the order of $G/S^*$ must be 1, 2 or 4, and the length of $G/S^*$ is again 0 or 1.

In the following work, therefore, we assume that $p \geqslant 5$.

$\Gamma$ contains a normal subgroup $N$ of index 2, consisting of all elements of $\Gamma$ whose determinants are quadratic residues of $p$. The centre $C$ of $\Gamma$ is cyclic of order $p-1$, and consists of all the 'scalar matrices'

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad (\lambda = 1, 2, \ldots, p-1).$$

† See, for example, Speiser, *Die Theorie der Gruppen von endlicher Ordnung*, 3rd ed. (Berlin, 1937; New York, 1945), Satz 112, p. 128.

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$C$ is contained in $N$, and the quotient group $N/C$ is a simple group of order

$$\tfrac{1}{2}p(p-1)(p+1).$$

Suppose now that $G/S^*$ is isomorphic to the subgroup $H$ of $\Gamma$. Write $H \wedge N = M$, $H \wedge C = K$. Then $H \geqslant M \geqslant K \geqslant 1$ is a chain of normal subgroups of $H$; $M$ is of index 1 or 2 in $H$, and $K$ is contained in the centre of $H$. The quotient group $M/K$ is isomorphic to the subgroup $MC/C$ of $N/C$. We therefore consider the soluble subgroups of that simple group, whose orders are prime to $p$ and are factors of the order of $G$.

A complete enumeration of the subgroups of the simple group $N/C$ is given by Burnside†. These subgroups are

($a$) Cyclic subgroups of order $p$ and subgroups containing them normally, of orders dividing $\frac{1}{2}p(p-1)$.

($b$) Cyclic subgroups of various orders $q$, prime to $p$. $q$ must be a factor of $\frac{1}{2}(p-1)$ or $\frac{1}{2}(p+1)$.

($c$) Dihedral subgroups of various orders $2q$, with $q$ as in ($b$). These include 'quadratic groups' of order 4.

($d$) Tetrahedral subgroups (order 12).

($e$) Octahedral subgroups (order 24), if $\frac{1}{2}p(p^2-1)$ is divisible by 24.

($f$) Icosahedral subgroups (order 60), provided that $\frac{1}{2}p(p^2-1)$ is divisible by 60. As is very well known, such a group, though of cube-free order, is *not* soluble.

It is seen at once that the only types we need consider are ($b$), ($c$) and ($d$). In particular, if the order of $G$ is *odd*, we are left merely with subgroups of type ($b$), which are Abelian. In this case we must also have $M = H$, so that $H/K$—and hence, by Lemma 8·1, $H$—is Abelian.

We now show that in any case the length of $H$ cannot exceed 2. If $M/K$ is of type ($b$) or ($c$), it contains a characteristic cyclic subgroup $Q/K$ of order $q$, which must be normal in $H/K$ and of index 1, 2 or 4. Hence $H/Q$ is Abelian, and the length of $H$ is at most 2. On the other hand, if $M/K$ is of type ($d$), we must have $H = M$, since otherwise the order of $H$ would be divisible by $2^3$. $H$ is therefore of length 2.

Thus, in every case, the length of $H$ (and therefore of $G/S^*$) cannot exceed 2, while if $G$ is of odd order, the length of $G/S^*$ cannot exceed 1. With the help of Lemma 8·2 we have now proved our main theorem of this section:

THEOREM 8·4. *The length of a soluble group of cube-free order cannot exceed* 3. *Any group whose order is odd and cube-free is Abelian or metabelian.*

Another result whose truth is apparent from the argument we have given is as follows:

THEOREM 8·5. *If an $A$-group $G$ of cube-free order is of length* 3, *then any Sylow subgroup of $L_2$ is elementary Abelian, of order $p^2$ and type $(p, p)$, where $p$ is some prime not less than* 5.

Thus an $A$-group of cube-free order cannot be of length 3 unless its order is divisible by the square of a prime not less than 5. Incidentally, Theorem 8·5 shows that for a cube-free soluble group of length 3, $L_2$ is an $S$-subgroup.

An example will show that we cannot expect any simple improvement of Theorems 8·4 and 8·5; for example, the order of a soluble group of cube-free order need not be

† Burnside, *Theory of groups*, 2nd ed. (Cambridge, 1911), pp. 434–48.

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divisible by 4, nor by 6, in order that its length may be 3. We consider the group $G$ of order $2 \cdot 7 \cdot 13^2$ defined by

$$A^2 = B^7 = C^{13} = D^{13} = 1,$$

$$A^{-1}BA = B^{-1}, \quad A^{-1}CA = C^{-1}D^{-1}, \quad B^{-1}CB = C^2D^3, \quad B^{-1}DB = C^3D^5$$

(where $D$ permutes with $A$ and $C$). It is easily verified that $L_1 = \{B, C, D\}$, $L_2 = \{C, D\}$, $L_3 = 1$; $G$ is of length 3.

It has been pointed out by Hall (4) that system normalizers do not possess a property that the normalizers of $S$-subgroups have, of being their own normalizers. The example he gives is that of an octahedral group, which is soluble but is not an $A$-group. It might be conjectured that in an $A$-group the system normalizers *are* their own normalizers; but that this is not so is proved by the example of a group of order $2 \cdot 7 \cdot 13^2$ given above. A particular Sylow system is defined by the permutable Sylow subgroups $\{A\}$, $\{B\}$ and $\{C, D\}$, and $\{A\}$ clearly normalizes each of them. As any system normalizer is of order 2, $\{A\}$ must in fact be the normalizer of the system. But $\{A, D\}$ is an Abelian group of order $2 \cdot 13$ containing $\{A\}$, and is therefore contained in the normalizer of $\{A\}$.

We have seen in (3·9) that the normalizer of a system normalizer $M$ of an $A$-group $G$ is its centralizer $M^*$ in $G$. Since $M^* \geqslant M$, $M^* = M \times L$, where $L = M^* \wedge L_1$. If $G$ is metabelian, $L$ must be contained in $Z_1$, and hence $L = 1$. Thus for a *metabelian* $A$-group, the system normalizers *are* their own normalizers. By consideration of $G/L_2$ we see that, in the general case, $M^* = M \times L$, where $L = M^* \wedge L_1 = M^* \wedge L_2$. (This result incidentally shows that in our example above $\{A, D\}$ is precisely the normalizer of $\{A\}$, since $C$ is certainly not contained in it.)

Another speculation, which may be of some interest, is this: *In an A-group, is any complement of $L_1$ a system normalizer?*

We shall give a counter-example to prove that in general this is untrue, but first we show that in certain particular cases it is true.

(i) Suppose that $G$ is metabelian. Let $M = M_1 \times M_2 \times \ldots \times M_r$ be any complement of $L_1$, where $M_i$ is the Sylow $p_i$-subgroup of $M$. (For some $i$, $M_i$ may consist of the identity alone.) Similarly, let $L_1 = N_1 \times N_2 \times \ldots \times N_r$. The Sylow $p_i$-subgroup $N_i$ of $L_1$ is normal in $G$, so that $M_iN_i$ is a subgroup of $G$; in fact, it is a Sylow $p_i$-subgroup $S_i$. Further, these Sylow subgroups form a Sylow system; for each $M_i$ is permutable with each $N_j$, and with each $M_k$, so that $S_iS_j = S_jS_i$. Finally, each $S_i$ is normalized by every element of $M$, which is therefore a system normalizer of $G$.

(ii) Suppose that $L_1$ is an $S$-subgroup of $G$. Then any complement $M$ of $L_1$ is also an $S$-subgroup, and all such subgroups are conjugate. $M$ is therefore conjugate to a system normalizer, and so must be one itself.

(iii) Suppose that $L_2$ is an $S$-subgroup of $G$. Then the argument used in (ii) above shows that all the complements of $L_2$ in $G$ are system normalizers of $L_1$ relative to $G$. Denote by $m_1$, $m_2$ the indices of $L_1$, $L_2$ in $G$.

If $M$ is any complement of $L_1$ in $G$, $M$ is of order $m_1$, which is a factor of $m_2$. Hence $M$ is a subgroup of some $S$-subgroup $K$ of $G$, or order $m_2$, i.e. of some $M_G(L_1)$. Then $L_1 \wedge K = L$ is an absolute system normalizer of $L_1$, and is also the derived subgroup of $K$. $M$, being a complement of $L_1$ in $G$, is clearly a complement of $L$ in $K$. But

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$K \cong G/L_2$ is metabelian, so by (i) $M$ is an absolute system normalizer of $K = M_G(L_1)$. By a theorem of Hall (4), $M$ is an absolute system normalizer of $G$.

We note that all $A$-groups of cube-free order are included, either under (i) or (iii), or of course by being Abelian (for which groups the property is trivial). We may therefore state

THEOREM 8·6. *If $G$ is an $A$-group of cube-free order, every complement of $L_1$ is an absolute system normalizer, and every complement of $L_2$ is a relative system normalizer $M_G(L_1)$.*

We finally give the counter-example for the general $A$-group. This is obtained by taking the direct product of the group of order $2 \cdot 7 \cdot 13^2$ already specified, with a cyclic group of order 13, generated by $E$ (say). The derived group of this direct product is again $\{B, C, D\}$, and the centre is $\{E\}$. Since both the elements $D$, $E$ permute with each other and with $A$, the elements $A$ and $DE$ generate a cyclic subgroup $\{A, DE\}$ of order $2 \cdot 13$. This meets the derived group in the identity only, and is therefore a complement in $G$; but it is certainly not a system normalizer, since it does not contain the centre $\{E\}$ of $G$.

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