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Basic dynamical stability results for $\mathbb{P}^n$

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Abstract. We study stability questions for dynamics on $\mathbb{P}^n$ using holomorphic families, which we require to be locally simply parameterized. Given such a holomorphic family $f\colon \mathbb{P}^n \times M \to \mathbb{P}^n$, we define the notion of a 'Fatou section', which is intuitively a choice of one point in $\mathbb{P}^n$ for each map in the family such that: (1) the chosen point depends holomorphically on M, and (2) the points chosen for two different maps behave dynamically 'comparably' under iteration of those respective maps. We prove a weak version of the λ-lemma for dynamical systems on $\mathbb{P}^n$ by showing that the set of all Fatou sections is a compact space. We introduce the notion of a postcritically bounded holomorphic family and show that if a (locally simply parameterized) holomorphic family is postcritically bounded by some open subset of $\mathbb{P}^n$ then: (1) repelling periodic points in the Julia set for one member of the family cannot bifurcate, nor can they leave the Julia set in other members of the family, (2) there is a Fatou section which never leaves the Julia set through each point of the Julia set of any member of the family, and (3) any intersection between the Julia set and the critical set is persistent, in the sense that there is a Fatou section through the point of intersection which never leaves either the critical set or the Julia set. Here, for conciseness, we are using the term 'Julia set' to mean the support of the unique measure of maximal entropy, although we recognize that how this term should be used in higher dimensions is still undecided. We also provide a brief summary of the development of the field in our introduction.

1. *Introduction*

We start with a brief overview of the development of higher-dimensional holomorphic dynamics, with a focus on stability.

Higher-dimensional holomorphic dynamics was preceded by one-dimensional holomorphic dynamics, so we start there. One-dimensional holomorphic dynamics was studied in the first part of the twentieth century by Fatou and Julia, but then lay fairly dormant until computer pictures of Fatou and Julia sets, the basic dynamical sets for one-dimensional holomorphic systems, rekindled interest. Developments that followed led to a great deal of study of these systems.

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Given a holomorphic self-map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, the Fatou set is the set on which the iterates of the map are equicontinuous, and the Julia set is the complement of the Fatou set. The Julia set is the set on which the dynamics are interesting.

Important results in the one-dimensional setting include a flurry of results by Hubbard and Douady [**DH84**], including the proof that the Mandelbrot set is connected (this was also proven by Sibony), the conjecture that the Mandelbrot set is locally connected, and the development of the theory of external rays. An essential contribution was made by Sullivan, who, using techniques from Kleinian groups, showed that no Fatou component is wandering, thus completing the classification of Fatou components [**Sul85**]. Sullivan also proposed a dictionary between rational maps and Kleinian groups relating concepts and results in the two different subjects.

Particularly relevant to our discussion are the results of [**MnSS83**] and [**Lyu83**]. These papers concern J-stability. A rational map is J-stable if it is stable on its Julia set. In [**MnSS83**] it was shown that in any holomorphic family of rational maps the J-stable maps form an open and dense subfamily. An essential tool was the introduction of holomorphic motions. It is an old dynamical notion that a subset S of a dynamical system is dynamically stable if, as we vary the system, we can vary the points in S to obtain a new set S', conjugate under the new system to S under the original system. The idea of a holomorphic motion is that, given a holomorphic family of rational maps parameterized by a complex manifold, each point of S is required to move holomorphically as we move among the members of the family. The fact that complex varieties always intersect positively combined with Kobayashi hyperbolicity of a thrice punctured $\overline{\mathbb{C}}$ immediately gives the important λ-lemma of [**MnSS83**], namely that any holomorphic motion of a set S can be extended uniquely to a holomorphic motion of the closure of S. One is then able to control bifurcation on the entire Julia set over a dense open subset of parameter space. This is because one can control bifurcation of all repelling periodic points on an open set of parameter space near any given parameter point by controlling only a finite number of attracting periodic points (since if a repelling point ceases to be repelling, it must become attracting for nearby parameters). Then one uses the λ-lemma to extend the holomorphic motion of the repelling periodic points to a holomorphic motion on the closure of the repelling periodic points which is the Julia set. This holomorphic motion defines a conjugacy from one map to nearby maps on the Julia set, and hence gives J-stability on these maps. Thus the J-stable maps are open and dense.

Lyubich [**Lyu83**] independently did work which overlapped with Mané, Sad and Sullivan's. One of his results bears particular similarity to the main result in this paper. He showed that, given a holomorphic family of rational maps f_m parameterized by m in some complex manifold M, if one considers the simultaneous self-map of the whole family $\tilde{f} : \overline{\mathbb{C}} \times M \to \overline{\mathbb{C}} \times M$ and if the iterates of $\tilde{f}$ applied to the critical locus form a normal family, then the family is J-stable.

There has been a vast deal more important work done in the one-dimensional holomorphic case, but we move on at this point to the development of higher-dimensional complex dynamics. Many of the tools of one-dimensional complex analysis that proved fruitful in one-dimensional dynamics do not carry over to higher dimensions in a form that remains as useful. Pluripotential theory has provided the most important basic tool

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thus far in higher-dimensional complex dynamics. There is typically more than one set that is in some way or another analogous to the Julia set of one-dimensional dynamics. In fact, analogs of the Julia 'set' in higher-dimensional complex dynamical systems are typically constructed first as a closed positive (p, p) current, rather than as a set. Invariant currents that generalize the Julia set in one way or another in a higher-dimensional system are called Green's currents. Typically it is easiest to construct invariant closed positive $(1, 1)$ currents first, show that they have bounded potentials, and thereby conclude that a generalized wedge product of these currents is well defined, giving invariant closed positive (p, p) currents. For p equal to the complex dimension of the ambient manifold, the current so obtained is an invariant measure, typically with properties analogous to those of the Julia set of a one-dimensional system along with its unique measure of maximal entropy.

Hénon maps had long been studied as dynamical systems on $\mathbb{R}^2$, and their extension to $\mathbb{C}^2$ became a prime example of a higher-dimensional complex dynamical system. A (complex) Hénon map of $\mathbb{C}^2$ is a map of the form $(x, y) \mapsto (p(x) - ay, x)$, where $p(x)$ is a polynomial of degree at least two and a is a non-zero complex number. All Hénon maps are biholomorphisms (i.e. holomorphically invertible). Sets of interest for Hénon maps include: the sets K^+ and K^- of points with bounded forward and backward orbits respectively, the complements $U^\pm$ of each of the sets $K^\pm$, the boundaries $J^\pm \equiv \partial K^\pm$ and the intersection $J \equiv J^+ \cap J^-$. Hubbard and Oberste-Vorth defined plurisubharmonic potentials G_+ and G_- which measure the rate of divergence to infinity at each point under forward and backward iteration respectively. Hubbard and Oberste-Vorth used these functions and holomorphic counterparts of them to give a description of the dynamics of Hénon maps on the sets U^+ and U^-. In [**HOV95**] Hubbard and Oberste-Vorth gave a very detailed topological description of the dynamics of a Hénon map $(p(x) - ay, x)$ when $p(x)$ is hyperbolic and a is sufficiently small. Friedland and Milnor showed [**FM89**] that all automorphisms of $\mathbb{C}^2$ are conjugate either to an elementary automorphism (which has simple dynamics) or to a composition of finitely many Hénon mappings. Previously, potential theory had been successfully used in one-dimensional dynamics by Brolin [**Bro65**]. According to [**BLS93a**], it was Sibony who had the idea of introducing pluripotential theory into $\mathbb{C}^2$ dynamics. He introduced the closed positive $(1, 1)$ currents $\mu^\pm = dd^c G_\pm$ and the invariant measure $\mu = \mu^+ \wedge \mu^-$. In [**BS91**] , Bedford and Smillie showed that the successive preimages (respectively pushforwards) of a smooth algebraic curve converge to a multiple of μ^+ (respectively μ^-). For a Hénon map which is hyperbolic (on J), they showed that the non-wandering set is equal to the union of J and a finite number of hyperbolic periodic points, that the periodic points are dense in J (hence such maps are Axiom A), that J is the unique infinite basic set and that $J = \mathrm{Supp}(\mu)$. Owing to hyperbolicity the sets J^+ and J^- are foliated by Riemann surfaces. Bedford and Smillie also showed that μ^+ and μ^- induce transverse measures on these foliations. In [**BS92**] it was shown that the closure of the saddle orbits contains $\mathrm{Supp}(\mu)$ and that μ is ergodic with one positive and one negative Lyapunov exponent.

Fornaess and Sibony [**FS92**] showed that the functions $G_\pm$ are Hölder continuous and that the basin of a fixed point with one attracting eigenvalue and one indeterminant eigenvalue satisfying the Bryuno condition is either $\mathbb{C}^2$ or the product of $\mathbb{C}$ and the disk. In [**BLS93b**] Bedford *et al* showed that μ is the unique measure of maximal entropy, that

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every saddle orbit lies in Supp(μ) and that for any pair p, q of distinct saddle periodic points Supp(μ) is the closure of the intersections of the unstable manifold of p and the stable manifold of q. They also showed that every periodic saddle point creates homoclinic intersections and μ is Bernoulli. In [**BLS93a**] the same authors showed that μ describes the distribution of periodic points, almost all of which are saddle periodic points and that the positive Lyapunov exponent of μ is a subharmonic function on any holomorphic family of Hénon maps. They also showed that if a member f_c of such a holomorphic family is topologically conjugate to nearby maps in the family then the positive Lyapunov exponent is harmonic. In fact, it was shown that if the Lyapunov exponent is not harmonic at f_c then in the case where f_c is dissipative there a sequence of members of the family f_{c_i} approaching f_c such that f_{c_i} has a sink of period i for i large, and in the case where f_c is volume preserving there is a sequence of members of the family f_{c_i} approaching f_c such that f_{c_i} has a Siegel ball of period i for i large. In [**BS98a**] an integral formula was obtained relating the Lyapunov exponents of μ to the Green's functions on what might be termed a 'critical locus' of the map.

The dichotomy between a connected Julia set and a totally disconnected Julia set is important in one-dimensional complex dynamics. In [**BS98b**] a Hénon map was defined to be unstably connected if the intersection of the unstable manifold of a saddle periodic point and U^+ is simply connected. This notion is shown not to depend on the choice of saddle periodic point. One defines stable connectivity analogously. It was then shown that J is connected if and only if the Hénon map is either stably connected or unstably connected. Also, a dissipative Hénon map is not stably connected. A volume-preserving Hénon map was shown to be stably connected if and only if it is unstably connected. Moreover if a Hénon map is unstably connected then on $J^- \cap U^+$ the map is semiconjugate to a complex solenoid. In [**BS99**] it was shown that for hyperbolic Hénon maps this semiconjugacy is a conjugacy. This same paper showed that for an unstably connected hyperbolic Hénon map all external rays along leaves of $J^- \cap U^+$ land. In [**BS02**] a notion of quasihyperbolic Hénon maps was defined and studied. These maps were proposed as an analog to the semihyperbolic maps of one-dimensional complex dynamics. In another direction on Hénon maps, Lyubich and the author [**LR**] have given a careful description of the 'critical locus' of Hénon maps when a is small and $p(x)$ is hyperbolic with simple critical points.

Buzzard [**Buz99**] showed that structural stability is not dense in the family of polynomial automorphisms of $\mathbb{C}^2$ of fixed degree, and that, for d sufficiently large, hyperbolicity is not dense in the family of polynomial automorphisms of $\mathbb{C}^2$ of degree d. It had been shown by Jonsson [**Jon98**] that, for holomorphic families of automorphisms of Hermitian manifolds, a hyperbolic set K of one member of the family will follow a holomorphic motion as we move to nearby members of the family. The same was also shown to be true for families of endomorphisms provided that K was a repellor. Buzzard and Verma [**BV01**] constructed invariant holomorphic motions for hyperbolic holomorphic families of Hénon maps on $J^+ \cup J^-$ by using the Bers–Royden extension of a holomorphic motion to extend the natural holomorphic motion of J restricted to the one-dimensional complex leaves of the foliation of J^+ and J^- to obtain a holomorphic motion on all of $J^+ \cup J^-$. Essential use is made in this approach of the existence of an invariant foliation by one-dimensional leaves. This allows holomorphic motion to be constructed

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within each leaf using one-dimensional methods after it is shown that the punctures at infinity in the leaves vary holomorphically. The fact that such a foliation does not exist for the maps in our paper necessitates a different approach. The purpose of using the Bers–Royden extension in [**BV01**] is that leaves of the foliation are dense and one needs the canonical nature of this extension to guarantee that a holomorphic motion on one leaf induces a homeomorphism on all of J^+ or J^-. In [**BHI05**] it is shown that among degree d polynomial automorphisms of $\mathbb{C}^2$ the set of Kupka–Smale maps is both residual and of full Lebesgue measure. They also showed that a heteroclinic point of two saddle periodic points is persistent in the sense that it can be continued for almost the entire parameter space.

Not long after the study of Hénon maps had begun, another direction was taken up. Complex projective n space, denoted $\mathbb{P}^n$, is the n-dimensional analog of the Riemann sphere. Just as on the Riemann sphere, automorphisms of $\mathbb{P}^n$ are dynamically trivial, but there are holomorphic families of endomorphisms of every (algebraic) degree greater than one which are dynamically interesting. These maps are always algebraic in nature, and are the direct analogs of rational self-maps of the Riemann sphere. To avoid trivialities all endomorphisms to which we refer will be implicitly assumed to be neither automorphisms nor constant maps. Hubbard and Papadopol [**HP94**] showed the existence of an invariant closed positive $(1, 1)$ current for endomorphisms of $\mathbb{P}^n$, which they called the Green's current. The Green's current has locally bounded potential, hence one can take a generalized p-fold self-intersection of this current and obtain invariant closed positive (p, p) currents for each p. The invariant closed positive (n, n) current is an invariant measure, typically denoted μ. Hubbard and Papadopol also showed that the Fatou set (defined to be the largest open set on which the iterates form a normal family) is contained in the complement of the support of the Green's current. Ueda [**Ued94**] and Fornaess and Sibony independently showed the converse, hence the Fatou set is exactly the complement of the support of the Green's current. Ueda also showed that Fatou components are Caratheodory hyperbolic, hence also Kobayashi hyperbolic. In [**FS94**], Fornaess and Sibony studied maps satisfying Kobayashi hyperbolicity assumptions and introduced and studied strictly critically finite maps of $\mathbb{P}^2$. Briend and Duval [**BD01**] showed that μ is the unique measure of maximal entropy, and that μ equals the distribution of the successive preimages of all points outside of a small exceptional set. They also showed that this exceptional set is a totally invariant algebraic variety. Such exceptional sets are the higher-dimensional analog of exceptional points in $\overline{\mathbb{C}}$ dynamics, and are typically empty. In [**BD99**] Briend and Duval showed that μ represents the distribution of repelling periodic points. We will make essential use of Briend and Duval's results in this paper. Jonsson and Weickert [**JW00**] showed that endomorphisms of $\mathbb{P}^2$ can have non-algebraic attractors, unlike endomorphisms of $\overline{\mathbb{C}}$. Jonsson and Favre [**FJ03**] showed that, given any closed positive $(1, 1)$ current C in $\mathbb{P}^2$ which does not charge the exceptional set, the successive preimages of C converge to the Green's current.

In [**FS95**] Fornaess and Sibony studied the dynamics of meromorphic self-maps of $\mathbb{P}^2$. This generalizes the study of both holomorphic endomorphisms of $\mathbb{P}^2$ and Hénon maps to a common framework, but creates interesting dilemmas as not every point has a well-defined orbit. That is, meromorphic maps are not *bona fide* maps. Some points do not have a well-defined image. Fornaess and Sibony showed the existence of a Green's current T supported

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exactly on the complement of the Fatou set for a 'generic meromorphic normal' map. They also showed that the measure $\mu = T \wedge T$ is the unique measure of maximal entropy.

Bimeromorphic maps of compact complex surfaces were then studied by Diller and Favre [**DF01**]. Letting $f^*\colon H^{1,1}(X) \to H^{1,1}(X)$ be the induced cohomological map, they define a bimeromorphism $f\colon X \to X$ to be algebraically stable if $(f^n)^* = (f^*)^n$. Not all bimeromorphisms of compact complex surfaces are algebraically stable, but Diller and Favre showed that there is a proper modification of X on which f lifts to an algebraically stable map. Diller and Favre also showed that the growth rate of $\|f^{n*}\|$ can be bounded, grow linearly, grow quadratically, or grow exponentially. They showed that f must be of a very specific type in the first three cases and show that in the last case there is a unique numerically effective (nef) class expanded by f^*. They constructed a current representing this nef class and show that the Fatou set lies outside the support of this current. Bimeromorphic maps have become increasingly important in the study of higher-dimensional complex dynamics. In [**Sib99**], Sibony showed the existence of a Green's current for algebraically stable maps of $\mathbb{P}^k$.

Another direction was pursued by Cantat [**Can01**], who showed that K3 surfaces have automorphisms of positive entropy. He also showed that there is a unique measure of maximal entropy, and that this measure describes the distribution of periodic points. McMullen [**McM02**] showed the existence of (full dimensional) Siegel disks for automorphisms of K3 surfaces. McMullen also showed that there are only countably many such examples, and that each such K3 surface must be non-projective.

Dinh and Sibony [**DS03**] have introduced a notion of polynomial-like maps for higher-dimensional complex dynamics and have shown the existence of a unique measure of maximal entropy that describes the distribution of successive preimages of points for these maps. Dinh and Sibony [**DS05**] have also shown the existence of an invariant closed positive (p, p) current for automorphisms of compact Kähler manifolds under fairly general conditions, as well as for holomorphic endomorphisms of compact Kähler manifolds.

The author of the present paper [**Robb**] has provided a general method to show the existence, convergence, regularity and positivity results for Green's currents in a very general setting.

There are certainly other results that should have been mentioned here. The author apologizes for not giving a broader exposition that could have included more of the important results in this field. Choices about what to include were in many instances determined by what seemed related to the present paper.

In the present paper we investigate basic stability questions for holomorphic families of self-maps of $\mathbb{P}^n$. These families are assumed to have locally a special form, namely they are locally represented by homogeneous polynomials whose coefficients are holomorphic functions of the parameterizing space. We anticipate that it can be shown that any holomorphic family has this property.

Definition. A *holomorphic family* of self-maps of $\mathbb{P}^n$ parameterized by a complex manifold M is given by a holomorphic map $f\colon \mathbb{P}^n \times M \to \mathbb{P}^n$. A *holomorphic family* $f\colon \mathbb{P}^n \times M \to \mathbb{P}^n$ will be said to be *simply parameterized* if f can be written in the form

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$f = (f_0 : \cdots : f_n)$ where each of $f_0, \ldots, f_n$ is of the form $f_i(x, y) = \sum_{|j|=d} a_{ij}(y)x^j$ where $x = (x_0 : \cdots : x_n) \in \mathbb{P}^n$, $y \in M$ and $j = (j_0, \ldots, j_n)$ is a multi-index, and each a_{ij} is a holomorphic function on M. The integer d is required to be the same for each f_i and is called the *degree* of the family.

We use the term 'holomorphic family' because for each $y \in M$ there is a holomorphic self-map $f_y \colon \mathbb{P}^n \to \mathbb{P}^n$ given by $f_y(x) = f(x, y)$. We define the total self-map $\dot{f} \colon \mathbb{P}^n \times M \to \mathbb{P}^n \times M$ by $\dot{f}(x, y) = (f(x, y), y)$. We iterate the family f by iterating each of the members of the family, i.e. $(f^{\circ k})_y = (f_y)^{\circ k}$. Thus the correct definition for iteration of a holomorphic family f is $f^{\circ k} \equiv f \circ \dot{f}^{\circ(k-1)}$ (where since $\dot{f}$ is a self-map of $\mathbb{P}^n \times M$ we understand $\dot{f}^{\circ k-1}$ to be $\dot{f}$ iterated $k-1$ times). Hence as a map $f^{\circ k}$ lands in $\mathbb{P}^n$.

For a holomorphic family to be simply parameterized means that when each of the maps f_y is written as a regular polynomial map from $\mathbb{P}^n$ to itself then the coefficients of the polynomials are holomorphic functions of M. This condition is clearly equivalent to requiring that there be a 'lift' of f to a holomorphic family $F \colon \mathbb{C}^{n+1} \times M \to \mathbb{C}^{n+1}$ of polynomial self-maps of $\mathbb{C}^{n+1}$.

Our principal tool to study basic stability questions on simply parameterized holomorphic families will be something we call a Fatou section (a variation on the notion of a *Fatou map* defined originally in [**Ued97**] and independently in [**Rob00**]).

Definition. A *Fatou section* is a holomorphic section $s \colon M \to \mathbb{P}^n \times M$ of the trivial bundle $\mathbb{P}^n \times M \to M$ such that the sequence of iterates $\{f^{\circ k} \circ s\}_{k \geq 0}$ is a normal family of maps from M to $\mathbb{P}^n$ (recall that $f^{\circ k} \colon \mathbb{P}^n \times M \to \mathbb{P}^n$ is defined to be $f \circ \dot{f}^{\circ(k-1)} \colon \mathbb{P}^n \times M \to \mathbb{P}^n$). We will use $\Gamma_{\text{Fatou}}(f)$ to denote the set of Fatou sections for the holomorphic family f.

If p is a periodic point that does not bifurcate in a holomorphic family of maps then the trace of the point p is an example of a Fatou section. Fatou sections allow us to tell which points behave comparably in different members of the family.

In this paper we will prove a weak analog for $\mathbb{P}^n$ of the λ-lemma of Mané, Sad and Sullivan [**MnSS83**]. The λ-lemma does not hold in higher dimensions because sections of the trivial bundle $\mathbb{P}^n \times D \to D$ have codimension n and can thus be close to each other at a single point without being close everywhere. Our weak λ-lemma states that given a simply parameterized holomorphic family of self-maps $f \colon \mathbb{P}^n \times M \to \mathbb{P}^n$ of degree at least two then the set $\Gamma_{\text{Fatou}}(f)$ of Fatou sections $s \colon M \to \mathbb{P}^n \times M$ is compact.

We will also prove a generalization of a theorem of Ueda [**Ued98**, Theorem 1.1]. Ueda's theorem says that the set of 'lifts' of a holomorphic map $h \colon Z \to \mathbb{P}^n$ through a holomorphic self-map $f \colon \mathbb{P}^n \to \mathbb{P}^n$ is a normal family. We generalize this (in Theorem 3.2) to normal families of sections in order to provide a means of constructing Fatou sections. In the simplest case our generalization states that, if we construct sections of the trivial bundle $\mathbb{P}^n \times M \to M$ by starting with an arbitrary section σ_0 and inductively building a sequence of sections σ_i, where σ_{i+1} is a preimage of σ_i under $\dot{f}$, then the sequence of sections σ_i is a normal family of maps, and any limit of a convergent subsequence of σ_i is a Fatou section.

We digress momentarily to fix our terminology. We recognize that given a holomorphic self-map $f \colon \mathbb{P}^n \to \mathbb{P}^n$ the question of how the term 'Julia set' should be used is not decided

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for $n > 1$. In this paper we use the term 'Julia set' to mean the support of μ, the unique measure of maximal entropy.

The portions of this paper described above largely provide groundwork. Having accomplished this will bring us to the significant portion of the paper. We introduce postcritically bounded families of holomorphic self-maps of $\mathbb{P}^n$. We will say that a holomorphic family $f: \mathbb{P}^n \times M \to \mathbb{P}^n$ is *postcritically bounded* by a set $B \subset \mathbb{P}^n$ if the postcritical set of the map $f_y: \mathbb{P}^n \to \mathbb{P}^n$ is disjoint from B for each $y \in M$. We will show that if $f: \mathbb{P}^n \times M \to \mathbb{P}^n$ is locally simply parameterized and postcritically bounded by an open subset of $\mathbb{P}^n$ then no repelling periodic point in the Julia set can bifurcate and additionally (after reviewing some work of Briend and Duval [**BD99**]) no repelling periodic point in the support of μ for one map in the family can be outside the support of μ for other members of the family. It will also follow that there is a Fatou section which never leaves the Julia set through any point of the Julia set of any member of the family and additionally that if the Julia set intersects the critical set for one member of the family, then this intersection is persistent.

As mentioned previously these results should be compared with the result of Lyubich [**Lyu83**] described earlier. Since there are not any (known) invariant foliations to make use of, and since we do not assume hyperbolicity, the methods of Buzzard and Verma cannot be applied in this setting.

2. *The weak λ-lemma*

Conventions. All complex spaces used in this paper will be assumed to be reduced and to have a countable basis of open sets. We are studying the dynamics of holomorphic self-maps of $\mathbb{P}^n$ and families of such maps. We will *always* implicitly assume such maps to have degree at least two.

Assume that $f: \mathbb{P}^n \times M \to \mathbb{P}^n$ is a simply parameterized holomorphic family. Then we can write $f = (f_0 : \cdots : f_n)$ with $f_i = \sum_{|j|=d} a_{ij}(y)x^j$ for each $i \in 0, \ldots, n$, where $a_{ij} \in \mathcal{O}(M)$, $x_0, \ldots, x_n$ are coordinates on $\mathbb{C}^{n+1}$, $y \in M$ and d is a fixed integer (the degree of the maps). Since f is a map to $\mathbb{P}^n$ then $f_0, \ldots, f_n$ do not all vanish simultaneously at any point. It is clear that there is a corresponding holomorphic family of self-maps of $\mathbb{C}^{n+1}$, namely $F: \mathbb{C}^{n+1} \times U \to \mathbb{C}^{n+1}$ given by $F = (F_0, \ldots, F_n)$ with $F_i = \sum_{|j|=d} a_j(y)x^j$ for each $i \in 0, \ldots, n$. Of course then $F_0, \ldots, F_n$ do not all vanish simultaneously at any point of $(\mathbb{C}^{n+1} \setminus \{0\}) \times U$. Clearly $\rho \circ F = f \circ (\rho \times \mathrm{id}_U)$ where $\rho: \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the usual projection, and id_U is the identity map of U. Since $F: \mathbb{C}^{n+1} \times M \to \mathbb{C}^{n+1}$ is a holomorphic family of self-maps of $\mathbb{C}^{n+1}$ then one defines $\dot{F}: \mathbb{C}^{n+1} \times M \to \mathbb{C}^{n+1} \times M$ by $\dot{F}(x, y) = (F(x, y), y)$ and $F^{\circ n} \equiv F \circ \dot{F}^{\circ(n-1)}$ just as with f.

Definition. We define the total Green's function G for a simply parameterized holomorphic family $f: \mathbb{P}^n \times M \to \mathbb{P}^n$ to be a map $G: \mathbb{C}^{n+1} \times M \to \mathbb{R} \cup \{-\infty\}$ given by

$$G(x, y) = \lim_{n\to\infty} \frac{\log \|F^{\circ n}(x, y)\|}{d^n},$$

where $F: \mathbb{C}^{n+1} \times M \to \mathbb{C}^{n+1}$ is a lift of f.

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Of course the Green's function depends upon the lift F of f, but it is easy to see that F is unique up to multiplication by a member h of $\mathcal{O}^*(M)$ and that if we replace F by hF then the Green's function G is changed to $G + \log|h|$. Hence G is unique up to addition of a pluriharmonic function of M. The important thing is the relationship between F and G so we will always assume when using both F and G that the Green's function G is the one computed from the given F.

For convenience we note that the sections of the trivial bundle $\mathbb{C}^{n+1} \times M \to M$ are an $\mathcal{O}(M)$ module via the following definition.

Definition. Given $\alpha \in \mathcal{O}(M)$ and σ a section of the trivial bundle $\mathbb{C}^{n+1} \times M \to M$ we define the product $\alpha\sigma = (\alpha \cdot s, \mathrm{id}_M)$ (i.e. $(\alpha\sigma)(y) = (\alpha(y)s(y), y)$) where s is the $\mathbb{C}^{n+1}$ coordinate function of σ, i.e. $\sigma(y) = (s(y), y)$.

It is easy to see that $G(\alpha(y)x, y) = G(x, y) + \log|\alpha(y)|$ for $\alpha \in \mathcal{O}^*(M)$. We will often need to refer to the zero set of the Green's function G so we will define $\mathcal{Z} = \{(x, y) \in \mathbb{C}^{n+1} \times M \mid G(x, y) = 0\}$. We have the usual result for such Green's functions.

THEOREM 2.1. *The Green's function G is continuous and plurisubharmonic to $\mathbb{R} \cup \{-\infty\}$. Also $G(x, y) = -\infty$ if and only if $x = 0$, for $(x, y) \in \mathbb{C}^{n+1} \times M$.*

We omit the necessarily slight modification of the proof as it appears in [**For96**].

Observation. Suppose we are given an open $U \Subset M$. Let S be the unit sphere in $\mathbb{C}^{n+1}$. Since G is continuous and finite valued on $S \times \overline{U}$, then $|G|$ must be bounded by some value $B > 0$ on $S \times \overline{U}$. It follows that if $G(x, y) = 0$ where $x \in \mathbb{C}^{n+1}$ and $y \in \overline{U}$ then

$$B > \left| G\left(\frac{x}{\|x\|}, y\right) \right| = |G(x, y) - \log\|x\|| = |\log\|x\||,$$

so $e^{-B} < |x| < e^B$.

The following lemma follows easily from the observation by noting that the hypothesis guarantees that $\mathfrak{V}$ is locally a normal family, and therefore $\mathfrak{V}$ is a normal family.

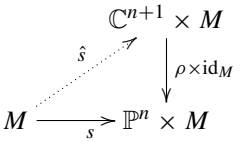
LEMMA 2.1. *If $f\colon \mathbb{P}^n \to \mathbb{P}^n$ is simply parameterized then any collection $\mathfrak{V}$ of holomorphic sections of the trivial bundle $\mathbb{C}^n \times M \to \mathbb{C}^n$ which all land in the zero set of the Green's function has compact closure (in the space of all holomorphic sections of the trivial bundle).*

We will now prove a theorem for Fatou sections in simply parameterized families. The lemma is motivated by work of Hubbard and Papadopol [**HP94**], Fornaess and Sibony [**For96**] and Ueda [**Ued94**]. Our method of proof is essentially borrowed from their work.

THEOREM 2.2. *Given a simply parameterized holomorphic family $f\colon \mathbb{P}^n \times M \to \mathbb{P}^n$ and given that M is simply connected, a holomorphic section $s\colon M \to \mathbb{P}^n \times M$ of the trivial bundle is a Fatou section if and only if $s\colon M \to \mathbb{P}^n \times M$ has a holomorphic lift*

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$\hat{s}\colon M \to \mathbb{C}^{n+1} \times M$ to a holomorphic section of the trivial bundle $\mathbb{C}^{n+1} \times M \to M$ in the sense that the diagram



commutes and such that $\hat{s}\colon M \to \mathbb{C}^{n+1} \times M$ lands in $\mathcal{Z}$, the zero set of the Green's function $G\colon \mathbb{C}^{n+1} \times M \to \mathbb{R} \cup \{-\infty\}$.

Proof. If such a lift $\hat{s}\colon M \to \mathbb{C}^{n+1} \times M$ exists then each of the maps $F^{\circ n} \circ \hat{s}$, $n \geq 0$, lands in $\mathcal{Z}$. From Lemma 2.1 we see that the family of holomorphic sections $\{F^{\circ n} \circ \hat{s}\}_{n \geq 0}$ has compact closure (in the space of all holomorphic sections of $\mathbb{C}^{n+1} \times M \to M$). It follows that every subsequence of $\{F^{\circ n_k} \circ s\}$ has a convergent subsequence. Thus every subsequence of $\rho \circ \{F^{\circ n_k} \circ s\} = \{f^{\circ n_k} \circ s\}$ has a convergent subsequence so f is a Fatou section.

Conversely assume there is a convergent subsequence $f^{\circ n_i} \circ s$ to some map $h\colon M \to \mathbb{P}^n$. Each point $y \in M$ has an open neighborhood $U \Subset M$ such that $h(U)$ has compact closure in some affine coordinate plane of $\mathbb{P}^n$. Without loss of generality we can assume that this is the affine coordinate plane of points $(z_0 : \cdots : z_{n-1} : 1)$. Then for all sufficiently large i the image $f^{\circ n_i} \circ s(U)$ lands in a bounded subset, say the ball of radius N about the origin of this affine coordinate patch. By replacing the sequence $f^{\circ n_i}$ with a subsequence if necessary we can assume that $f^{\circ n_i} \circ s(U)$ lands in the ball of radius N about the origin of this affine coordinate patch for all $i \geq 1$. We can also assume that $s\colon U \to \mathbb{P}^n \times M$ lifts to a map $\hat{s}\colon U \to \mathbb{C}^{n+1} \setminus \{0\} \times M$ (by shrinking U if necessary so that the $\mathbb{P}^n$ component of $s|_U$ lands in a coordinate patch and then using the coordinate functions for s in this coordinate patch to construct the lift).

We now consider the holomorphic map $\hat{s}\colon U \to \mathbb{C}^{n+1} \times M$. The sequence of maps $F^{\circ n_i} \circ \hat{s}$ satisfies $\rho \circ F^{\circ n_i} \circ \hat{s} = f^{\circ n_i} \circ s$. If we write any map $F^{\circ n_i} \circ \hat{s}$ for $i \geq 0$ in coordinates in $\mathbb{C}^{n+1}$ then our map is $z \mapsto (A_{0i}, A_{1i}, \ldots, A_{ni})$ where each of the functions A_{ji} is a holomorphic function from U to $\mathbb{C}$. The important thing is that if we 'projectivize' this map by composing with $\rho\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ to get the map $f^{\circ n_i} \circ s$ into $\mathbb{P}^n$ then we can rewrite it as $(A_{0i}/A_{ni} : \cdots : A_{n-1i}/A_{ni} : 1)$ and the first n coordinates give the map $f^{\circ n_i} \circ s$ in the coordinates of our affine plane. Note that A_{ni} never vanishes on U for large $i \geq 1$ because by assumption $f^{\circ n_i} \circ s(U)$ lands in the affine coordinate chart of points of the form $(z_0 : \cdots : z_{n-1} : 1)$. Since we noted that for $i \geq 1$ the image $f^{\circ n_i} \circ s(U)$ lands in the ball of radius N about the origin of this coordinate patch we conclude that each of the ratios A_{ji}/A_{ni} must remain uniformly bounded (by N) for $i \geq 1$. Also $(A_{0i}/A_{ni}, \ldots, A_{n-1i}/A_{ni})$ must converge uniformly to the function h written in the coordinates on our affine coordinate chart. Thus if we let $\lambda_i(z) = 1/A_{ni}(z)$ for $z \in U$ and for each $i \geq 1$ then the sequence of maps $\lambda_i(z) \cdot (F^{\circ n_i} \circ \hat{s}(z))$ converges uniformly on U to a lift $\hat{h}$ of h to $\mathbb{C}^{n+1}$.

Now the limit

$$\lim_{i \to \infty} \left| \frac{\log |\lambda_i(z)|}{d^{n_i}} + \frac{\log \|F^{\circ n_i} \circ \hat{s}\|}{d^{n_i}} \right| = \lim_{i \to \infty} \left| \frac{\log \|\lambda_i(z) \cdot (F^{\circ n_i} \circ \hat{s})\|}{d^{n_i}} \right|, \qquad (1)$$

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goes to zero uniformly on U since $\lambda_i \cdot (F^{\circ n_i} \circ s)$ is uniformly bounded on U for $i \geq 1$. We can thus conclude that

$$\lim_{i\to\infty} \frac{\log|\lambda_i|}{d^{n_i}} = -\lim_{i\to\infty} \frac{\|F^{\circ n_i} \circ \hat{s}\|}{d^{n_i}} = -G(\hat{s}(z)), \tag{2}$$

and that the convergence is uniform on U (since by formula (1) we see that the difference between the two sides of the first equality in formula (2) goes to zero uniformly on U, and it is clear in the proof of Theorem 2.1 that the two sides of the second equality in (2) converge uniformly over any $U \Subset M$).

We can assume that U is simply connected (by shrinking U if necessary). Then $\log \lambda_i$ can be taken to be a holomorphic function since λ_i never vanishes. We consider our lift $\hat{s}\colon U \to \mathbb{C}^{n+1} \times M$. Note that given an open neighborhood $U_0 \Subset U$ of z_0 then $\hat{s}(U_0) \Subset (\mathbb{C}^{n+1} \setminus \{0\}) \times M$. Thus $G(\hat{s}(U_0)) \Subset \mathbb{R}$ and is therefore bounded. It follows from the uniform convergence over U of equation (2) that the functions $\log|\lambda_i|/d^{n_i}$ must be uniformly bounded on U_0 for all $i \geq 0$, and hence that regardless of how we choose to lift λ_i by log that $\operatorname{Re}\log\lambda_i = \log|\lambda_i|$ is uniformly bounded on U_0 for all large i by some value B.

Now U_0 was chosen to be an open neighborhood of the point we started with, z_0. We can assume without loss of generality that each of our functions $\log\lambda_i$ satisfies $0 \leq \operatorname{Im}\log\lambda_i(z_0) < 2\pi i$ and this uniquely determines the functions $\log\lambda_i$. Then the sequence of functions $\log\lambda_i$ must be a normal family on U_0, since each member lands in the taut region $W \equiv \{z \in \mathbb{C} \mid -2B < \operatorname{Re} z < 2B\}$ and additionally each member maps the point z_0 into a compact region of W given by $\{z \in \mathbb{C} \mid -B < \operatorname{Re} z < B, 0 \leq \operatorname{Im} z < 2\pi\}$ so this sequence cannot be compactly divergent. Thus this sequence must have a convergent subsequence to some holomorphic function $k\colon U \to \mathbb{C}$ with $\operatorname{Re} k = \lim_{i\to\infty} \operatorname{Re}\log|\lambda_i|/d^{n_i} = -G(\tilde{s}(z))$. Then the holomorphic function $e^k \cdot \tilde{s}\colon U_0 \to \mathbb{C}^{n+1}$ satisfies $G(e^k \cdot \hat{s}) = G(\hat{s}) + \log|e^k| = G(\hat{s}) + \operatorname{Re} k = 0$ and thus is a lift of the holomorphic map $s\colon M \to \mathbb{P}^n$ on the open set U_0 about the point z to the zero set $\mathcal{Z}$ of G.

We have shown that each point $z_0 \in Z$ has a neighborhood U_0 such that there is a holomorphic lift $e^k \cdot \hat{s}\colon U_0 \to \mathbb{C}^{n+1}$ which lands in $\mathcal{Z}$, the zero set of G. The proof will be complete if these lifts patch together to give a lift on all of M.

Now M is a simply connected complex manifold. Using standard methods from Riemannian geometry we can choose a locally finite cover $\{B_i\}_{i\in\Lambda}$ of M by convex open balls in some fixed Riemannian metric on M, which each lie in some neighborhood U_0 as constructed above, and such that any non-empty intersection of these balls is contractible.

Then there is a holomorphic lift $\hat{s}_i\colon B_i \to \mathbb{C}^{n+1} \times B_i$ of s on B_i and $\hat{s}_i$ lands in the zero set $\mathcal{Z}$ of the Green's function for each $i \in \Lambda$. For each $i \in \Lambda$ let $t_i\colon B_i \to \mathbb{C}^{n+1}$ denote the $\mathbb{C}^{n+1}$ component of $\hat{s}_i\colon B_i \to \mathbb{C}^{n+1} \times B_i$. We note that if two balls B_i and B_j overlap then t_i and t_j are constant multiples of each other on $B_i \cap B_j$ by a constant that lies on S^1, the unit circle in $\mathbb{C}$ (i.e. $\hat{s}_i$ and $\hat{s}_j$ have a well-defined holomorphic ratio $\tau\colon B_1 \cap B_2 \to \mathbb{C}^*$ and since $\hat{s}_i$ and $\hat{s}_j$ each land in the zero set of the Green's function then $|\tau| = 1$ and therefore τ is a constant in S^1). For each pair of overlapping balls B_i, B_j, let ω_{ij} be the constant such that $t_i = \omega_{ji} t_j$ on $B_i \cap B_j$. Then we have a Cech cochain $c_1 \in \check{C}^1(\{B_i\}; S^1)$ by assigning ω_{ij} to $B_i \cap B_j$. Since $\omega_{ij}\omega_{jk}\omega_{ki} = 1$ then c_1 is a cocycle. Since non-empty intersections of these balls are always contractible then

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$\check{H}^1(\{B_i\};\, S^1) = \check{H}^1(M;\, S^1)$. Now we know that $H_1(M) = 0$ since M is simply connected. Using the isomorphism between singular and (constant sheaf) Cech cohomology on locally contractible spaces, combined with the exact sequence $0 \to \mathrm{Ext}(H_0(C),\, S^1) \to H^1(C,\, S^1) \to \hom(H_1(C),\, S^1) \to 0$ (applied to the singular chain complex C over $\mathbb{Z}$) one concludes that $\check{H}^1(M;\, S^1) = 0$ (see e.g. Spanier [**Spa66**, pp. 243, 334, 340]). Then choose $c_0 \in \check{C}^0(\{B_i\};\, S^1)$ such that $\delta c_0 = c_1$. For each $i \in \Lambda$ let $\alpha_i = c_0(B_i)$. Then $\delta c_0(B_i \cap B_j) = \alpha_i/\alpha_j = \omega_{ij}$ and so $t_i = \alpha_j/\alpha_i t_j$ and we see that $\alpha_i t_i = \alpha_j t_j$ for all $i,\, j \in \Lambda$, so letting $\hat{s}\colon M \to \mathbb{C}^{n+1} \times M$ be given by $\hat{s}(y) = (\alpha_i t_i(y),\, y)$ on B_i we have a holomorphic lift of s on all of M. Moreover $G(\hat{s}(y)) = G(\alpha_i t_i(y),\, y) = G(t_i(y),\, y) = G(\hat{s}_i(y)) = 0$ for any $y \in B_i$ so $\hat{s}$ lands in the zero set of the Green's function. □

Since being a Fatou section is a local property, and since we proved one direction under the mere assumption that there was some convergent subsequence of forward iterates then (exactly as in Ueda [**Ued94**]) we have the following as a corollary of the proof.

COROLLARY 2.1. *Given a locally simply parameterized holomorphic family $f\colon \mathbb{P}^n \times M \to \mathbb{P}^n$ then a section $s\colon M \to \mathbb{P}^n \times M$ is a Fatou section if there exists at least one convergent subsequence of forward iterates of $\{f^{\circ k} \circ s\}_{k \geq 0}$.*

As in the work of Fornaess and Sibony, Hubbard and Papadopol and Ueda cited previously and using the same methods we also have the following equivalent characterization of a Fatou section.

THEOREM 2.3. *Given a simply parameterized holomorphic family $f\colon \mathbb{P}^n \times M \to \mathbb{P}^n$ and given that M is simply connected then a holomorphic section $s\colon M \to \mathbb{P}^n \times M$ is a Fatou section if and only if there is a holomorphic lift $\hat{s}\colon M \to (\mathbb{C}^{n+1} \setminus \{0\}) \times M$ such that $G \circ \hat{s}$ is pluriharmonic. If one such holomorphic lift of s to a section $\hat{s}$ of $(\mathbb{C}^{n+1} \setminus \{0\}) \times M \to M$ has this property then all such holomorphic lifts of s do.*

As essentially the same theorem (although for Fatou maps, not Fatou sections), with the same proof, appears in [**Rob00**] we omit the proof.

We can now prove our weak λ-lemma.

THEOREM 2.4. (Weak λ-lemma) *Given a locally simply parameterized holomorphic family then the set $\Gamma_{\mathrm{Fatou}}(f)$ of Fatou sections $s\colon M \to \mathbb{P}^n \times M$ is compact. In other words the following three things are true.*

(i) *The collection $\Gamma_{\mathrm{Fatou}}(f)$ is a normal family of maps from M to $\mathbb{P}^n \times M$.*
(ii) *The collection $\Gamma_{\mathrm{Fatou}}(f)$ has no compactly divergent subsequence.*
(iii) *Any limit of maps in $\Gamma_{\mathrm{Fatou}}(f)$ is a map in $\Gamma_{\mathrm{Fatou}}(f)$.*

Proof. It follows immediately from Theorem 2.2 and Lemma 2.1 that for a simply parameterized holomorphic family of maps $f\colon \mathbb{P}^n \times M \to \mathbb{P}^n$ the set of Fatou sections is compact. Hence chose an open cover $\underline{U}$ of M such that f is simply parameterized over each $U \in \underline{U}$. The cover $\underline{U}$ can be assumed to be countable and it can be assumed that each member of the cover is simply connected (by starting with a dense open set S in M and then constructing $\underline{U}$ by choosing a simply connected open set U on which f is simply parameterized about each point of S).

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We then have an obvious map $\chi\colon \Gamma_{\text{Fatou}}(f) \to \prod_{U\in\underline{U}} \Gamma_{\text{Fatou}}(f|_U)$. This map is clearly continuous and injective. Since the cover $\underline{U}$ was taken to be countable then $\prod_{U\in\underline{U}} \Gamma_{\text{Fatou}}(f|_U)$ is metrizable (since each $\Gamma_{\text{Fatou}}(f|_U)$ is metrizable as follows from the fact that the set of all holomorphic maps from one complex space into another is metrizable). Then given any sequence of members σ_i in $\Gamma_{\text{Fatou}}(f)$ it follows from compactness of the product that the sequence $\chi(\sigma_i)$ has a convergent subsequence $\chi(\sigma_{k_i})$. However, the limit $\tau = \prod_{U\in\underline{U}} \tau_U$ of this subsequence must be a well-defined section on all of M since for any intersecting pair $U_1, U_2 \in \underline{U}$ we see that $\tau_{U_1}|_{U_1\cap U_2} = \tau_{U_2}|_{U_1\cap U_2}$ because τ_{U_1} and τ_{U_2} must agree on any relatively compact open subset of $U_1 \cap U_2$ (since τ_{U_1} and τ_{U_2} are each the limit of the sequence σ_{k_i} which converges on both U_1 and U_2). Hence τ is simply given by the restriction to each $U \in \underline{U}$ of a well-defined section ν over M. Let $\underline{\underline{U}}$ be the cover of M obtained by replacing each member $U \in \underline{U}$ with a countable collection of relatively compact open sets which cover U. Then σ_{k_i} is uniformly convergent on each member of $\underline{\underline{U}}$. Thus if K is any compact subset of M then choose a finite subcover of $\underline{\underline{U}}$ which covers K and it follows that σ_{k_i} converges uniformly on K. Hence σ_{k_i} converges locally uniformly to ν. Since the forward iterates of τ under $\dot{f}$ are locally a normal family (i.e. on each $U \in \underline{U}$) then ν is a Fatou section as being a normal family is a local property. Thus $\Gamma_{\text{Fatou}}(f)$ is sequentially compact and is therefore also compact (since it is metrizable). □

3. *Generalizing a theorem of Ueda to normal families of sections*

Our central result in this section will be a generalization of a theorem of Ueda. We will need to develop some basic facts about the map $\dot{f}$ and about finding preimages of sections of the trivial bundle $\mathbb{P}^n \times M \to M$ under $\dot{f}$. After doing so we will state the original theorem of Ueda in question, state our generalization of it, and then prove our generalization.

We will need to know the following.

LEMMA 3.1. *Each of* $\dot{f}\colon \mathbb{P}^n \times M \to \mathbb{P}^n \times M$ *and* $\dot{F}\colon \mathbb{C}^{n+1} \times M \to \mathbb{C}^{n+1} \times M$ *is proper, open and finite and each is a local biholomorphism off of its respective critical set.*

Proof. It is known that any non-constant holomorphic map $f\colon \mathbb{P}^n \to \mathbb{P}^n$ is finite and surjective, hence this is strictly an exercise using the fact that $F(\beta x, y) = \beta^d F(x, y)$, $(\rho \times \mathrm{id}_M) \circ \dot{F} = \dot{f} \circ (\rho \times \mathrm{id}_M)$ and the open mapping theorem (see e.g. [**GR84**, p. 107]). □

We include here a lemma which we will use in a couple of proofs. It is a nice tool in our setting for showing that a given map is actually a complex cover.

LEMMA 3.2. *Given complex spaces* X, Y *and a holomorphic map* $f\colon X \to Y$ *which is proper and a local biholomorphism and given that* Y *is connected then* f *is also a finite-to-one complex covering map from* X *to* Y.

Proof. To see this, note that given any point $y \in Y$ the preimages of y under a local biholomorphism must form a discrete set. Since the map f is proper this discrete set must be compact, and therefore it is finite. Let $a_1, \ldots, a_k$ be the preimages of y and for each $i \in 1, \ldots, k$ choose an open neighborhood $U_i \subset X$ about a_i such that $f\colon U_i \to Y$

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is a biholomorphism onto its image. The intersection $\bigcap_{i=1}^{k} f(U_i)$ of the images $f(U_i)$ is then an open neighborhood of y and each U_i contains a unique open neighborhood of a_i mapped biholomorphically to $\bigcap_{i=1}^{k} f(U_i)$. Now define $n\colon Y \to \mathbb{Z}$ by setting $n(y)$ to be the number of elements in $f^{-1}(y)$. Then from the above we see that if $n(y) = k$ then there is an open neighborhood U of y on which $n \geq k$. Thus for any integer $k \geq 0$ the set $P_k \equiv \{y \in Y \mid n(y) \geq k\}$ is open. However if $y' \in \partial P_k$ then $y' \notin P_k$ so y' has fewer than k preimages. Choose V an open neighborhood of y' such that each of the preimages $b_1, \ldots, b_\ell$ of y' has an open neighborhood $U_1', \ldots, U_\ell'$ which is mapped biholomorphically to V (note that $\ell < k$). Choose W to be a compact neighborhood of y contained in V. Then $L = f^{-1}(V) \setminus (U_1' \cup U_2' \cdots \cup U_\ell')$ is a compact set since it is a compact set minus a collection of open sets. Moreover $V \cap P_k \subset f(L)$ because every point of $V \cap P_k$ has a preimage in L (since it has at least one preimage not in any U_i'). However $y' \notin f(L)$ since y' has no preimage in L. This contradicts the compactness of L since it contains $V \cap P_k$ but not y' which lies in the closure of P_k. Hence our assumption that $y' \in \partial P_k$ must have been impossible so $\partial P_k = \emptyset$. Thus each P_k is either empty or all of Y, since Y is connected. It follows that every point of Y has the same number of preimages under f. From this we see that if $a_1, \ldots, a_k$ are preimages of a point $y \in Y$ and if U is an open neighborhood of y for which there are corresponding open neighborhoods $U_1, \ldots, U_k$ of $a_1, \ldots, a_k$ such that each neighborhood U_i is mapped biholomorphically to U, then in fact $f^{-1}(U) = \bigcup_{i=1}^{k} U_i$ since this is all the preimages of any point in U. Thus $f\colon X \to Y$ is a finite-to-one complex covering map. □

We next show that if a section $s\colon M \to \mathbb{P}^n \times M$ is disjoint from the image of the critical set then the section s pulls back through $\dot{f}$ to a disjoint union of the images of a finite number of sections.

LEMMA 3.3. *Assume that M is a simply connected complex manifold. Let $s\colon M \to \mathbb{P}^n \times M$ be a section of the trivial bundle $\mathbb{P}^n \times M \to M$. If the image $s(M)$ is disjoint from the image of the critical set of $\dot{f}$ then any connected component of $\dot{f}^{-1}(s(M))$ is the image of a holomorphic section $r\colon M \to \mathbb{P}^n \times M$ such that $\dot{f} \circ r = s$.*

Proof. Assume that the image $s(M)$ is disjoint from the image of the critical set of $\dot{f}$. Let V be any connected component of $\dot{f}^{-1}(s(M))$. Then the restriction $\dot{f}|_V$ of $\dot{f}$ to V is a proper local biholomorphism and hence a complex covering map by Lemma 3.2.

Since $s(M)$ is biholomorphic to M and since M is simply connected then $\dot{f}|_V\colon V \to s(M)$ is a biholomorphism. Letting $t\colon s(M) \to V$ be the inverse biholomorphism then $t \circ s\colon M \to \mathbb{P}^n \times M$ gives the desired section. □

Remark. Note that given any two holomorphic sections s, t of $\mathbb{P}^n \times M \to M$ such that the image of t is disjoint from the critical set of $\dot{f}$ and such that $\dot{f} \circ s = t$ then s must be one of the sections described in Lemma 3.3.

COROLLARY 3.1. *Assume $t\colon M \to \mathbb{P}^n \times M$ is a holomorphic section of the trivial bundle $\mathbb{P}^n \times M \to M$. Assume further that the image $t(M)$ of t is disjoint from the postcritical set of $\dot{f}$. Given any integer $n \geq 0$ and any point $(x', y') \in \mathbb{P}^n \times M$ such that $\dot{f}^{\circ n}(x', y') \in t(M)$ there is a unique holomorphic section $s\colon M \to \mathbb{P}^n \times M$ such that $s(y') = (x', y')$ (i.e. x' is the point in $\mathbb{P}^n$ over y') and $\dot{f}^{\circ n} \circ s = t$.*

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Proof. The existence of the section s follows from Lemma 3.3 applied to the section t and the map $f^{\circ n}$ by taking s to be the component of $f^{\circ -n}(t(M))$ containing (x', y'). The uniqueness of s holds since any other such section s' would have both to contain the point (x', y') and to satisfy $f^{\circ n} \circ s' = t$ and so $s'(M)$ would lie in $f^{\circ -n}(t(M))$ and contain the point (x', y') contrary to $s(M)$ being the unique component of $f^{\circ -n}(t(M))$ containing (x', y'). □

The following lemma is sufficiently clear that we omit its proof.

LEMMA 3.4. *The critical set of* $\dot{F}$ *is precisely the union of* $\{0\} \times M$ *and the preimage of the critical set of* $\dot{f}$ *under the projection* $\rho \times \mathrm{id}_M : \mathbb{C}^{n+1} \setminus \{0\} \times M \to \mathbb{P}^n \times M$.

We will need to be able to take sections of the trivial bundle $\mathbb{P}^n \times M \to M$ and 'lift' these to sections of the trivial bundle $\mathbb{C}^{n+1} \times M \to M$. We will need to be able to do this over certain open subsets of M as well. That this can be done is shown in the following lemma.

LEMMA 3.5. *Let* U *be any contractible Stein open subset of* M. *For any holomorphic section* $\sigma : U \to \mathbb{P}^n \times U$ *of the trivial bundle* $\mathbb{P}^n \times U \to U$ *there is a lift of* σ *to a holomorphic section* $\hat{\sigma} : U \to \mathbb{C}^{n+1} \times U$ *of the trivial bundle* $\mathbb{C}^{n+1} \times U \to U$ *which avoids the zero section and satisfies* $(\rho \times \mathrm{id}_U) \circ \hat{\sigma} = \sigma$.

Proof. Write $\sigma : U \to \mathbb{P}^n \times U$ in $\mathbb{P}^n$ and U coordinates as $\sigma(y) = (s(y), y)$ for some holomorphic $s : U \to \mathbb{P}^n$.

Let $t : L \to \mathbb{P}^n$ be the tautological line bundle over $\mathbb{P}^n$. Then there is a canonical holomorphic map from $b : L \to \mathbb{C}^{n+1}$ which blows down the zero section of L to the origin of $\mathbb{C}^{n+1}$ and which is compatible with t in the sense that $\rho \circ b = t$.

Now let s^*L be the pullback of L to a line bundle over U. Thus we have the following commutative diagram of holomorphic maps.

$$\begin{array}{ccc} s^*L & \xrightarrow[i]{} & L \\ {\scriptstyle s^*t}\downarrow & & \downarrow{\scriptstyle t} \\ U & \xrightarrow[s]{} & \mathbb{P}^n \end{array}$$

A contractible Stein space has no non-trivial line bundles (or even non-trivial vector bundles; see e.g. the translation of Leiterer [**Laĭ86**] in [**Khe90**, p. 72]). Thus U has no non-trivial line bundles and s^*L is the trivial bundle.

Hence there is a non-vanishing section $r : U \to s^*L$. Now from the above diagram of maps we have $t \circ i = s \circ s^*t$. Precomposing both sides of this equation with r we obtain $t \circ i \circ r = s$. Then using the fact that $t = \rho \circ b$ we get $\rho \circ b \circ i \circ r = s$. Using the facts that r avoids the zero section of s^*L, that $i : s^*L \to L$ is a line bundle morphism, and that $b : L \to \mathbb{C}^{n+1}$ sends only the zero section of L to the origin of $\mathbb{C}^{n+1}$ it follows that $b \circ i \circ r : U \to \mathbb{C}^{n+1}$ avoids the origin. Now letting $\hat{\sigma} : U \to \mathbb{C}^{n+1} \times U$ be given by $\hat{\sigma}(y) = (b(i(r(y))), y)$ we see that $\hat{\sigma}$ avoids the zero section of $\mathbb{C}^{n+1} \times U \to U$ (since $b \circ i \circ r$ is never the origin) and $(\rho \times \mathrm{id}_U) \circ \hat{\sigma}(y) = (\rho(b(i(r(y)))), y) = (s(y), y) = \sigma(y)$. □

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We are now in a position to generalize Ueda's theorem. Ueda proved the following in [**Ued98**, Theorem 1.1].

THEOREM 3.1. (Ueda's theorem) *Let $h\colon \mathbb{P}^n \to \mathbb{P}^n$ be a holomorphic map of degree $d \geq 2$ and $g_0\colon Z \to \mathbb{P}^n$ a holomorphic map from a reduced complex space Z. Let $\{g_\nu\}$ be a family of lifts of g_0 by iterates of h (that is $\{g_\nu\}$ is a family of maps from Z to $\mathbb{P}^n$ such that for each index ν there is an integer i such that $h^{\circ i} \circ g_\nu = g_0$). Then $\{g_\nu\}$ is a normal family of maps from Z into $\mathbb{P}^n$.*

We will prove a sort of enhancement of this for families of holomorphic (not necessarily Fatou) sections. We will only need to use it in the case where the collection $\mathfrak{S}_0$ is a single section, but the proof is not particularly more difficult for this more general case.

The basic idea is the same as Ueda's original argument.

THEOREM 3.2. (Ueda's theorem for sections) *Let $f\colon \mathbb{P}^n \times M \to \mathbb{P}^n$ be a locally simply parameterized holomorphic family. Assume that $\mathfrak{S}_0$ is a normal family of holomorphic sections of the trivial bundle $\mathbb{P}^n \times M \to M$. Assume further that we are given a sequence of families $\{\mathfrak{S}_{-n}\}_{n\geq 0}$ of holomorphic sections of the trivial bundle $\mathbb{P}^n \times M \to M$ such that $\dot{f} \circ \mathfrak{S}_{-(n+1)} \subset \mathfrak{S}_{-n}$ for each $n \geq 0$ (i.e. for each n any section $s_{-(n+1)} \in \mathfrak{S}_{-(n+1)}$ is a preimage of some section $s_{-n} \in \mathfrak{S}_{-n}$ under $\dot{f}$). Then the union $\mathfrak{T} = \bigcup_{n=0}^{\infty} \mathfrak{S}_{-n}$ is a normal family of maps. The family $\mathfrak{T}$ then also has compact closure, since there are obviously no compactly divergent subsequences. What is more, let s_{-n_k} be any convergent sequence of maps in $\mathfrak{T}$ such that $s_{-n_k} \in \mathfrak{S}_{-n_k}$ for each $k \geq 0$ and such that the sequence $\{n_k\}_{k\geq 0}$ diverges to infinity. Then $\sigma = \lim_{k\to\infty} s_{-n_k}$ is a Fatou section.*

The proof is long and will occupy the rest of this section. We begin with three lemmas.

The first lemma says that if a family of maps from M into $\mathbb{P}^n$ has compact closure then we can break M up into small pieces which can be lifted to uniformly bounded maps into $\mathbb{C}^{n+1}$.

Given any two positive numbers $r < R$ let $S_{r,R}$ denote $\{x \in \mathbb{C}^{n+1} \mid r < \|x\| < R\}$. In other words $S_{r,R}$ is the hollow spherical shell in $\mathbb{C}^{n+1}$ whose inner radius is r and whose outer radius is R.

LEMMA 3.6. *Let $\mathfrak{S}$ be a collection of holomorphic maps from M to $\mathbb{P}^n$, and let $\underline{V}$ be an open cover of M. Assume that the family of maps S has compact closure (i.e. is a normal family with no divergent subsequences). Then for any two positive numbers $R > r > 0$ there is an open cover $\underline{U}$ of M and, associated to each $U \in \underline{U}$, a collection $\hat{\mathfrak{S}}_U$ of holomorphic maps from U to $\mathbb{C}^{n+1}$ such that the image of any $\hat{s} \in \hat{\mathfrak{S}}_U$ lies in $S_{r,R}$ and such that for every $s \in \mathfrak{S}$ and every $U \in \underline{U}$ there is some $\hat{s} \in \hat{\mathfrak{S}}_U$ such that $s|_U = \rho \circ \hat{s}$, i.e. the restriction of s to U is the projection of $\hat{s}$ down to $\mathbb{P}^n$. Every one of the open sets $U \in \underline{U}$ can be taken to be biholomorphic to a polydisk and to be relatively compact in some member of $\underline{V}$.*

Proof. Cover $\mathbb{P}^n$ with open sets V_i such that for each V_i there is a holomorphic lift $c_i\colon V_i \to \mathbb{C}^{n+1}$ such that $\rho \circ c_i = \mathrm{id}_{V_i}$ and such that $c_i(V_i) \subset S_{r,R}$. This can be done using lifts to hyperplanes in $\mathbb{C}^{n+1}$. Fix ϵ to be smaller than half the Lebesgue number δ of the cover V_i of $\mathbb{P}^n$ (see e.g. [**Mun75**, p. 179] for a review of Lebesgue numbers). Since $\mathfrak{S}$ has

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compact closure then it is equicontinuous by Ascoli's theorem (see e.g. [**Mun75**, pp. 276, 290]). Hence given any $y \in M$ let U_y be a neighborhood of y such that for all $y' \in U_y$ and all $s \in \mathfrak{S}$ we have $d(s(y'), s(y)) < \epsilon$ and such that U_y is an open polydisk about y relatively compact in some member of $\underline{V}$. Thus for each $y \in M$, $s \in \mathfrak{S}$, there is some V_i containing $s(U_y)$ and hence the map $\hat{s}_y \equiv c_i \circ s|_{U_y}$ satisfies $\rho \circ \hat{s}_y = s|_{U_y}$ and the image of $\hat{s}_y$ lies in $S_{r,R}$. Thus $\{U_y\}_{y \in M}$ gives the desired cover of M and letting $\hat{\mathfrak{S}}_U \equiv \{\hat{s}_y, s \in \mathfrak{S}\}$ completes the proof. □

The second lemma we will need for our generalization of Ueda's theorem says that given a holomorphic map from a complex space to $\mathbb{P}^n$ and given two lifts of that map to a map into $\mathbb{C}^{n+1}$ then the two lifts are just holomorphic rescalings of one another.

LEMMA 3.7. *If Z is a connected complex space and $g\colon Z \to \mathbb{P}^n$ is a holomorphic map and if h_0, h_1 are two holomorphic lifts of g to $\mathbb{C}^{n+1} \setminus \{0\}$ (i.e. two holomorphic maps such that $\rho \circ h_0 = \rho \circ h_1 = g$ where $\rho\colon \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^n$ is the standard projection) then there is a holomorphic function $\tau\colon Z \to \mathbb{C}^*$ such that $h_0 = \tau \cdot h_1$.*

Proof. This is a standard enough construction so we only sketch the proof. It can be found in [**Rob00**]. To do it one writes the maps h_0 and h_1 in coordinates as $h_0 = (h_{00}, \ldots, h_{0n})$ and $h_1 = (h_{10}, \ldots, h_{1n})$. Then, since $\rho \circ h_0 = \rho \circ h_1$, for each pair i, j we have

$$\frac{h_{0i}}{h_{1i}} = \frac{h_{0j}}{h_{1j}},$$

at each point where these are both defined. This ratio gives τ. □

The third lemma we will need for our generalization of Ueda's theorem says that, given a section τ of the trivial bundle $\mathbb{P}^n \times M \to M$ and given a lift of τ to a section $\hat{\tau}$ of the trivial bundle $\mathbb{C}^{n+1} \times M \to M$, if there is a section σ of $\mathbb{P}^n \times M \to M$ which is mapped to τ by $\dot{f}$, then there is also a section $\hat{\tau}$ which is mapped to $\hat{\sigma}$ by $\dot{F}$. Of course F only exists if f is simply parameterized, so if f is only locally simply parameterized then we need to use a smaller subset $U \subset M$ on which f is simply parameterized.

LEMMA 3.8. *Assume that $f\colon \mathbb{P}^n \times M \to \mathbb{P}^n$ is locally simply parameterized, and that U is a contractible Stein open subset of M on which f is simply parameterized. Given $\sigma, \tau\colon U \to \mathbb{P}^n \times U$ both sections of $\mathbb{P}^n \times U \to U$ such that $\dot{f} \circ \sigma = \tau$ and given a lift $\hat{\tau}\colon U \to \mathbb{C}^{n+1} \times U$ of τ (meaning that $(\rho \times \mathrm{id}_U) \circ \hat{\tau} = \tau$) then there is a lift $\hat{\sigma}\colon U \to \mathbb{C}^{n+1} \times U$ of σ (meaning $(\rho \times \mathrm{id}_U) \circ \hat{\sigma} = \sigma$) such that $\dot{F} \circ \hat{\sigma} = \hat{\tau}$.*

Proof. Let $\hat{\sigma}_0\colon U \to \mathbb{C}^{n+1} \times U$ be *some* lift (by Lemma 3.5) of σ (meaning $(\rho \times \mathrm{id}_U) \circ \hat{\sigma}_0 = \sigma$). We first note that there is some $k \in \mathcal{O}^*(U)$ such that $\dot{F}(\hat{\sigma}_0) = k\hat{\tau}$. This is because writing $\hat{\sigma}_0(y) = (\hat{s}_0(y), y)$ and $\hat{\tau}(y) = (\hat{t}(y), y)$ it follows from our assumption $\dot{f} \circ \sigma = \tau$ that $\rho \circ F \circ \hat{\sigma}_0 = \rho \circ \hat{t}$ (to see this use $\dot{f} \circ \sigma = \tau$ implies $\dot{f} \circ (\rho \times \mathrm{id}_U) \circ \hat{\sigma}_0 = (\rho \times \mathrm{id}_U) \circ \hat{\tau}$ implies $f \circ (\rho \times \mathrm{id}_U) \circ \hat{\sigma}_0 = \rho \circ \hat{t}$ implies $\rho \circ F \circ \hat{\sigma}_0 = \rho \circ \hat{t}$). Both $F \circ \hat{\sigma}_0$ and $\hat{t}$ are maps from U to $\mathbb{C}^{n+1}$ and since they both have the same projection to $\mathbb{P}^n$ then $F \circ \hat{\sigma}_0 = k \cdot \hat{t}$ for some $k \in \mathcal{O}^*(U)$ by Lemma 3.7. Then $\dot{F} \circ \hat{\sigma}_0 = k\hat{\tau}$ by the definition of our multiplication of members of $\mathcal{O}(U)$ (such as k) and sections of $\mathbb{C}^{n+1} \times U \to U$ (such as $\hat{\tau}$).

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Now we recall that for any $\gamma \in \mathcal{O}^*(U)$ we have $\dot{F}(\gamma\hat{\sigma}_0) = \gamma^d \dot{F}(\hat{\sigma}_0)$. Since $k \in \mathcal{O}^*(U)$ and U is simply connected (since U is contractible) then there exists a holomorphic function $h \in \mathcal{O}(U)$ such that $e^h = k$. Letting $\hat{\sigma} = e^{-(1/d)h}\hat{\sigma}_0$ we then have $\dot{F}(\hat{\sigma}) = e^{-h}\dot{F}(\hat{\sigma}_0) = k^{-1}k\hat{\tau} = \hat{\tau}$, which is what we wanted to show. □

We now prove our generalization of Ueda's theorem.

Proof of Theorem 3.2. (Ueda's theorem for sections) To prove this we will apply Lemma 3.6 to conclude that the family of sections $\mathfrak{S}_0$ lifts to a normal family of sections of $\mathbb{C}^{n+1} \times M \to M$ if we are willing to break M up into small pieces by using an appropriate open cover $\underline{U}$. Each of these small pieces will be a subset of M biholomorphic to the unit ball; thus each will be a contractible Stein manifold.

We make a trivial observation that will make the proof easier to follow.

Observation. Given complex manifolds W, U, a family $\mathfrak{S}'$ of holomorphic sections of the trivial bundle $W \times U \to U$ has a natural corresponding family S' of holomorphic maps from W to U given by replacing a section $\sigma(y) = (s(x),\ y)$ (written in terms of its W and U coordinate functions) with the function $s(x)$ and vice versa. The family of holomorphic sections $\mathfrak{S}'$ has compact closure if and only if the family of holomorphic maps S' has compact closure.

Breaking up M so bounded lifts exist for $\mathfrak{S}_0$. We let S_0 be the family of holomorphic maps from M to $\mathbb{P}^n$ corresponding to the family of sections $\mathfrak{S}_0$ (i.e. S_0 is just the set of the $\mathbb{P}^n$ coordinate functions of the members of $\mathfrak{S}_0$). Choose some positive numbers $R > r > 0$ and fix them. Choose a cover $\underline{V}$ of M such that f is simply parameterized on each $V \in \underline{V}$. Then by Lemma 3.6 there is a cover $\underline{U}$ of M by open sets U and associated to each U is a family of holomorphic maps $\hat{S}_U$ from U to $\mathbb{C}^{n+1}$ such that the image of each member of $\hat{S}_U$ lies in $S_{r,R} = \{x \in \mathbb{C}^{n+1} \mid r < \|x\| < R\}$ and such that for each $s \in S_0$ and each $U \in \underline{U}$ there is some $\hat{s} \in \hat{S}_U$ such that $s|_U = \rho \circ \hat{s}$. Moreover each member of the cover $\underline{U}$ is relatively compact in M and biholomorphic to an open polydisk. For each $U \in \underline{U}$ we let $\hat{\mathfrak{S}}_U$ be the set of sections of the trivial bundle $\mathbb{C}^{n+1} \times U \to U$ corresponding to the set of holomorphic maps $\hat{S}_U$, each of which map U to $\mathbb{C}^{n+1}$.

Inductively build lifts for $\mathfrak{S}_{-n}$. We will now fix an arbitrary member U of $\underline{U}$ and fix V a member of underlineV such that $U \Subset V$. Let $F_V : \mathbb{C}^{n+1} \times V \to V$ be a lift of f on V and let $G_V : \mathbb{C}^{n+1} \times V \to \mathbb{R} \cup \{-\infty\}$ be the Green's function computed from F_V. For most of the rest of this proof we will only deal with the restrictions of the families $\mathfrak{S}_{-n}$ to U. To keep the notation from being too cumbersome we let $\mathfrak{R}_{-n} \equiv \mathfrak{S}_{-n}|_U$ for each $n \geq 0$. This puts us back in the setting of the original hypothesis of the theorem we are proving with the families $\mathfrak{S}_{-n}$ replaced by $\mathfrak{R}_{-n}$ and M replaced by U, for $\mathfrak{R}_0$ is a normal family of holomorphic sections of the trivial bundle $\mathbb{P}^n \times U \to U$ and $\dot{f} \circ \mathfrak{R}_{-(n+1)} \subset \mathfrak{R}_{-n}$ for each $n \geq 0$. We let $\widehat{\mathfrak{R}}_0 \equiv \hat{\mathfrak{S}}_U$ as defined above. Then $(\rho \times \mathrm{id}_M) \circ \widehat{\mathfrak{R}}_0 = \mathfrak{R}_0$ by the construction of $\hat{\mathfrak{S}}_U = \widehat{\mathfrak{R}}_0$. We now inductively build families of sections $\widehat{\mathfrak{R}}_{-n}$ for every $n \geq 0$ as follows. Given $\widehat{\mathfrak{R}}_{-n}$ such that $(\rho \times \mathrm{id}_M) \circ \widehat{\mathfrak{R}}_{-n} = \mathfrak{R}_{-n}$ then for every $r_{-(n+1)} \in \mathfrak{R}_{-(n+1)}$ we do the following. Temporarily let $r_{-n} = \dot{f} \circ r_{-(n+1)}$. Then there exists $\hat{r}_{-n} \in \widehat{\mathfrak{R}}_{-n}$ such that $(\rho \times \mathrm{id}_U) \circ \hat{r}_{-n} = r_{-n}$ by the inductive hypothesis. Now applying Lemma 3.8 we conclude that there exists some holomorphic section

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$\hat{r}_{-(n+1)}\colon U \to \mathbb{C}^{n+1} \times U$ such that $\dot{F} \circ \hat{r}_{-(n+1)} = \hat{r}_{-n}$ and $(\rho \times \mathrm{id}_U) \circ \hat{r}_{-(n+1)} = r_{-(n+1)}$. Given this procedure we let $\widehat{\mathfrak{R}}_{-(n+1)}$ be the union of all such $\hat{r}_{-(n+1)}$ obtained from any $r_{-(n+1)} \in \mathfrak{R}_{-(n+1)}$. Then $(\rho \times \mathrm{id}_U) \circ \widehat{\mathfrak{R}}_{-(n+1)} = \mathfrak{R}_{-(n+1)}$ and $\dot{F}_V \circ \widehat{\mathfrak{R}}_{-(n+1)} \subset \widehat{\mathfrak{R}}_{-n}$ by construction. We have thus obtained a sequence $\widehat{\mathfrak{R}}_{-n}$ of sections of the trivial bundle $\mathbb{C}^{n+1} \times U \to U$ which lifts the sequence $\mathfrak{R}_{-n}$ of sections of the trivial bundle $\mathbb{P}^n \times U \to U$ in the sense that for each $n \geq 0$ we have the following commutative diagram.

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\dot{F}_V} & \widehat{\mathfrak{R}}_{-(n+1)} & \xrightarrow{\dot{F}_V} & \widehat{\mathfrak{R}}_{-n} & \xrightarrow{\dot{F}_V} & \cdots \\
 & & \downarrow{\scriptstyle \rho\times\mathrm{id}_U} & & \downarrow{\scriptstyle \rho\times\mathrm{id}_U} & & \\
\cdots & \xrightarrow[\dot{f}]{} & \mathfrak{R}_{-(n+1)} & \xrightarrow[\dot{f}]{} & \mathfrak{R}_{-n} & \xrightarrow[\dot{f}]{} & \cdots
\end{array}$$

Lifts uniformly bounded; compactness conclusion on each piece U of M. For each $\hat{r}_0 \in \widehat{\mathfrak{R}}_0$ the image of $\hat{r}_0\colon U \to \mathbb{C}^{n+1} \times U$ lies in $S_{r,R} \times U \Subset (\mathbb{C}^{n+1} \setminus \{0\}) \times V$ by construction. Since $S_{r,R} \times U \Subset (\mathbb{C}^{n+1} \setminus \{0\}) \times V$ then $G_V(S_{r,R} \times U) \Subset \mathbb{R}$ and so there is some $C > 0$ such that $-C < G_V(S_{r,R} \times U) < C$. Hence for any $\hat{r}_0 \in \widehat{\mathfrak{R}}_0$ and any $y \in U$ we see that $-C < G_V(\hat{r}_0(y)) < C$.

Now given any $n \geq 0$ and any $\hat{r}_{-n} \in \widehat{\mathfrak{R}}_{-n}$ we let $\hat{r}_0 = \dot{F}_V^{\circ n} \circ \hat{r}_{-n} \in \widehat{\mathfrak{R}}_0$. Then we have the relationship $G_V(\hat{r}_0) = G_V(\dot{F}^{\circ n} \circ \hat{r}_{-n}) = d^n \cdot G_V(\hat{r}_{-n})$. Rewriting this we see that $G_V(\hat{r}_{-n}) = 1/d^n G_V(\hat{r}_0)$ so $-C/d^n < G_V(\hat{r}_{-n}(y)) < C/d^n$ for any $y \in U$.

Now for any $n \geq 0$ and any $y \in U$ we have

$$\begin{aligned}
G_V(e^{-G_V(\hat{r}_{-n}(y))} \cdot \hat{r}_{-n}(y)) &= G_V(\hat{r}_{-n}(y)) + \log|e^{-G_V(\hat{r}_{-n}(y))}| \\
&= G_V(\hat{r}_{-n}(y)) - G_V(\hat{r}_{-n}(y)) \\
&= 0,
\end{aligned}$$

and so the section $e^{-G_V \circ \hat{r}_{-n}} \cdot \hat{r}_{-n}$ lands in the zero set $\mathcal{Z}$ of the Green's function G_V. Just as in the proof of Lemma 2.1 it follows easily from the compactness of $\overline{U}$ and the continuity of $G_V\colon (\mathbb{C}^{n+1} \setminus \{0\}) \times V \to \mathbb{R}$ that there is some constant $B > 0$ such that if (x, y) is in the zero set of G_V and $y \in U$ then $e^{-B} < \|x\| < e^B$. Writing $\hat{r}_{-n}\colon U \to \mathbb{C}^{n+1} \times U$ as $\hat{r}_{-n} = (t_{-n}(y), y)$ where t_{-n} is the $\mathbb{C}^{n+1}$ component of $\hat{r}_{-n}$ we see for each $y \in U$ that

$$e^{-G_V(\hat{r}_{-n}(y))} \cdot \hat{r}_{-n}(y) = (e^{-G_V(\hat{r}_{-n}(y))} \cdot t_{-n}(y), y),$$

in $\mathbb{C}^{n+1} \times U$ coordinates and thus conclude that $e^{-B} < \|e^{-G_V(\hat{r}_{-n}(y))} \cdot t_{-n}(y)\| < e^B$. Using the fact that $-C \leq -C/d^n < G_V(\hat{r}_{-n}(y)) < C/d^n \leq C$ we can simplify this to $e^{-B-C} < \|t_{-n}(y)\| < e^{B+C}$ for each $y \in U$. The values B and C did not depend upon n, nor upon which $\hat{r}_{-n} \in \widehat{\mathfrak{R}}_{-n}$ was used and thus we see that if we let $\hat{R}_{-n}$ denote the set of all $\mathbb{C}^{n+1}$ components of members of $\widehat{\mathfrak{R}}_{-n}$ then for each member $t_{-n} \in \hat{R}_{-n}$ we have $e^{-B-C} < \|t_{-n}(y)\| < e^{B+C}$ for all $y \in U$. Put differently this means that if we let $K = e^{B+C}$ then for any $n \geq 0$ and any $\hat{r}_{-n} \in \widehat{\mathfrak{R}}_{-n}$ the image of $\hat{r}_{-n}$ lies in $S_{K^{-1},K} \times U$. If we let $\hat{\mathfrak{T}}_U \equiv \bigcup_{n \geq 0} \widehat{\mathfrak{R}}_{-n}$ and let $\hat{T}_U$ denote the set of all $\mathbb{C}^{n+1}$ components of members of $\hat{\mathfrak{T}}_U$ then every member of $\hat{T}_U$ is a holomorphic map from U to $\mathbb{C}^{n+1}$ whose image lies in $S_{K^{-1},K}$. Thus $\hat{T}_U$ has compact closure since it is a normal family of maps into $\mathbb{C}^{n+1}$ and no sequence of members of $\hat{T}_U$ could be compactly divergent (into $\mathbb{C}^{n+1}$). Hence also $\hat{\mathfrak{T}}_U$ has compact closure. Since every sequence of members of $\hat{T}_U$ has a convergent

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subsequence to some map h whose image lands in $\overline{S_{K^{-1},K}} \times U$ then any sequence of maps in $\bigcup_{n\geq 0} \mathfrak{R}_{-n} = \bigcup_{n\geq 0}(\mathfrak{S}_{-n}|_U)$ has a convergent subsequence to $(\rho \times \mathrm{id}_U) \circ h$. *Hence* $\bigcup_{n\geq 0}(\mathfrak{S}_{-n}|_U)$ *has compact closure.*

Fatou section conclusion of each piece U of M. Now assume that $\{s_{-n_k}\}_{k\geq 0}$ is a convergent sequence of holomorphic sections of the trivial bundle $\mathbb{P}^n \times U \to U$ to some section $\sigma : U \to \mathbb{P}^n \times U$. Assume further that $s_{-n_k} \in \mathfrak{R}_{-n_k}$ for each $k \geq 0$ and that the sequence $\{n_k\}_{k\geq 0}$ diverges to infinity. Then for each $k \geq 0$ let $\hat{s}_{-n_k}$ be a member of $\widehat{\mathfrak{R}}_{-n_k}$ such that $(\rho \times \mathrm{id}_U) \circ \hat{s}_{-n_k} = s_{-n_k}$. Since $\hat{\mathfrak{T}}_U = \bigcup_{n\geq 0} \widehat{\mathfrak{R}}_{-n}$ has compact closure then by replacing $\{n_k\}_{k\geq 0}$ with a subsequence if necessary we can assume that $\hat{s}_{-n_k}$ converges to a section $\hat{\sigma} : U \to \mathbb{C}^{n+1} \times U$. It follows that $(\rho \times \mathrm{id}_U) \circ \hat{\sigma} = \sigma$ (since $\rho \times \mathrm{id}_U$ is continuous). We then compute $|G_V \circ \hat{\sigma}| = \lim_{k\to\infty} |G_V \circ \hat{s}_{-n_k}| \leq \lim_{k\to\infty} C/d^{n_k} = 0$ so $\hat{\sigma}$ lands in the zero set of the Green's function G_V. Thus $\sigma : U \to \mathbb{P}^n \times U$ is a Fatou section over U by Theorem 2.2.

Putting the pieces back together. Lastly we take our results on an arbitrary member U of $\underline{U}$ and extend them to results over M. Since the restriction of the family of holomorphic sections $\mathfrak{T} = \bigcup_{n\geq 0} \mathfrak{S}_{-n}$ to each member U of the open cover $\underline{U}$ is a normal family then $\mathfrak{T}$ is a normal family of maps (since being a normal family is a local property). If we consider the family of maps T from M to $\mathbb{P}^n$ comprising the $\mathbb{P}^n$ coordinate functions of each member of $\mathfrak{T}$ we see that no subsequence can be compactly divergent. Hence T must have compact closure. Since T has compact closure then $\mathfrak{T}$ has compact closure.

If $\{s_{-n_k}\}_{k\geq 0}$ is a convergent sequence of holomorphic sections of the trivial bundle $\mathbb{P}^n \times M \to M$ to some section $\sigma : M \to \mathbb{P}^n \times M$ and if $s_{-n_k} \in \mathfrak{S}_{-n_k}$ for each $k \geq 0$ and the sequence $\{n_k\}_{k\geq 0}$ diverges to infinity then the sequence of maps $\{f^{\circ n} \circ \sigma|_U\}_{n\geq 0}$ is a normal family of maps from U to $\mathbb{P}^n \times U$ for each $U \in \underline{U}$ since $\sigma|_U$ is a Fatou section over U. Since $\{f^{\circ n} \circ \sigma\}_{n\geq 0}$ is a normal family locally then it must be a normal family (since being a normal family is a local property). Thus σ must be a Fatou section. □

4. *Postcritically bounded families on $\mathbb{P}^n$*

Conventions. Given a holomorphic family $f : \mathbb{P}^n \times M \to \mathbb{P}^n$, for each $y \in M$ we will let μ_y be the unique measure of maximal entropy. We will let $\mathrm{Crit}(f_y)$ denote the critical set of f_y, $\mathrm{PC}(f_y)$ denote the postcritical set of f_y and $E(f_y)$ denote the exceptional set of f_y.

In this section we will be studying a certain type of holomorphic family of self-maps of $\mathbb{P}^n$.

Definition. We will say that a holomorphic family $f : \mathbb{P}^n \times M \to \mathbb{P}^n$ is *postcritically bounded* by a collection $\mathfrak{S}$ of sections of $\mathbb{P}^n \times M \to \mathbb{P}^n$ if $s(y) \notin \mathrm{PC}(f_y)$ for every $y \in M$ and every $s \in \mathfrak{S}$. We specialize this definition in the following two ways:

- (Constant sections) We say that f is postcritically bounded by a *set* $B \subset \mathbb{P}^n$ if the postcritical set of the map $f_y : \mathbb{P}^n \to \mathbb{P}^n$ is disjoint from B for each $y \in M$. We will say the set B bounds the postcritical set.
- (Single section) We will say that a holomorphic family $f : \mathbb{P}^n \times M \to \mathbb{P}^n$ is postcritically bounded by a *section* $s : M \to \mathbb{P}^n \times M$ (not necessarily Fatou) if for every $y \in M$ one has $s(y) \notin \mathrm{PC}(f_y)$. We will say the section s bounds the postcritical set.

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Note that if a section s bounds the postcritical set of a family f then $s(y)$ never belongs to the exceptional set of f_y since the exceptional set belongs to the critical set of f_y as noted in [**BD01**].

We will now apply our weak λ-lemma and our generalized Ueda's theorem to show some basic stability results for $\mathbb{P}^n$.

THEOREM 4.1. *Assume that* $f\colon \mathbb{P}^n \times M \to \mathbb{P}^n$ *is a locally simply parameterized holomorphic family of self-maps of* $\mathbb{P}^n$ *which is postcritically bounded by some section and assume that* M *is simply connected. It follows that for every* $y \in M$ *and every* $x \in \mathrm{Supp}(\mu_y)$ *there is a Fatou section* $s\colon M \to \mathbb{P}^n \times M$ *passing through the point* $(x, y) \in \mathbb{P}^n \times M$.

Proof. Let $\sigma_0(y) = (s(y), y)$ be a section which bounds the postcritical set of the family. Now fix a point $y \in M$ and $x \in \mathrm{Supp}(\mu_y)$. It follows from [**BD01**, Theorem 1] that there is a sequence $x_k \in f^{-k}(s(y))$ such that $\lim_{k\to\infty} x_k = x$. Then using Corollary 3.1 let σ_k denote the unique section such that $\dot{f}^{\circ k} \circ \sigma_k = \sigma_0$ and $\sigma_k(y) = (x_k, y)$. Then by Theorem 3.2 the sequence σ_k has a convergent sequence to a Fatou section σ. Clearly $\sigma(y) = (x, y)$ and we are done. □

We now prove a lemma that is fairly intuitive. If a section σ through a periodic point is repelling for f_{y_0} at some point $x_0 \in \mathbb{P}^n$ then another Fatou section cannot intersect σ at (x_0, y_0) because it will be repelled from the image of σ at (x_0, y_0) under iteration (so the iterates cannot form a normal family).

LEMMA 4.1. *Given a holomorphic family* $f\colon \mathbb{P}^n \times M \to \mathbb{P}^n$, *assume that* $\sigma\colon M \to \mathbb{P}^n \times M$ *is the section of a periodic point of period* m, *meaning* $\dot{f}^{\circ m} \circ \sigma = \sigma$ *(so* σ *is necessarily a Fatou section). If* $y_0 \in M$ *and the periodic point* $\sigma(y_0)$ *of* f_{y_0} *is repelling then* $\sigma(M)$ *cannot intersect any other Fatou section at* $\sigma(y_0)$.

Proof. Write $\sigma(y) = (s(y), y)$, $s\colon M \to \mathbb{P}^n$ and let $x_0 = s(y_0)$. Let $U \subset \mathbb{P}^n$ be an affine coordinate chart (which we identify with $\mathbb{C}^2$) chosen so that x_0 corresponds to the origin and let $V \subset M$ be a coordinate chart of M (which we identify with the corresponding open subset of $\mathbb{C}^m$) about y_0 chosen so that y_0 corresponds to the origin of $\mathbb{C}^m$ (so $\sigma(y_0) = (x_0, y_0) \in U \times V$). Choose V_0 an open neighborhood of y_0 in V such that $\sigma(V_0) \Subset U \times V$. We note that $(u, v) \mapsto (u - s(v), v)\colon \mathbb{C}^n \times V_0 \to \mathbb{C}^n \times V_0$ gives well-defined new coordinates on $U \times V_0 = \mathbb{C}^n \times V_0$. In this new coordinate system σ is given by $\sigma(y) = (0, y)$ for $y \in V_0$. We replace the former coordinate system on $U \times V_0$ with this new one. Note that $(0, 0)$ still corresponds to the point $(x_0, y_0) \in \mathbb{P}^n \times M$.

By replacing f with an iterate if necessary we can assume that $\dot{f} \circ \sigma = \sigma$, and hence that x_0 is a fixed point of f_{y_0}. Then for each $v \in V_0$ we see that $\dot{f}(0, v) = \dot{f}(\sigma(v)) = \sigma(v) = (0, v)$. Hence $f(0, v) = 0 \in U = \mathbb{C}^n$ for $v \in V_0$. It follows that $f(u, v)$ can be written locally about $(0, 0) \in U \times V_0$ as

$$f(u, v) = u_1 g_1(u, v) + u_2 g_2(u, v) + \cdots + u_n g_n(u, v),$$

where $g_1, \ldots, g_n$ are holomorphic and $\mathbb{C}^n$ valued in a neighborhood of $(0, 0) \in U \times V_0$.

Now assume that there is some other Fatou section $\tau_0\colon M \to \mathbb{P}^n \times M$ and that $\tau_0(y_0) = \sigma(y_0)$. We are interested in the germs of the maps $\dot{f} \circ \tau_0$ about y_0. Hence we let $\tau_k = \dot{f}^{\circ k} \circ \tau_0$ for each integer $k \geq 0$ and we write the germ $\tau_{k y_0}$ of each τ_k at y_0 in terms

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of our coordinates on $U \times V_0$ as $\tau_{ky_0}(y) = (t_k(y), y)$ for each integer $k \geq 0$ where $t_k(y)$ is a (multivariable) power series about y_0. We let $t_{k1}, \ldots, t_{kn}$ be the coordinate functions of the power series t_k.

From the definition of τ_k we have $\dot{f}(\tau_k) = \tau_{k+1}$ and hence

$$\dot{f}(t_k(y), y) = (f(t_k(y), y), y) = \left(\sum_{j=1}^{n} t_{kj}(y) \cdot g_j(t_k(y), y), y\right),$$

for $g_1, \ldots, g_n$ (as above) defined in a neighborhood of $s(y_0)$ (whose coordinates are $(0, 0)$ in the coordinate patch $U \times V_0$). It follows that

$$t_{k+1}(y) = \sum_{j=1}^{n} t_{kj}(y) \cdot g_j(t_k(y), y), \tag{3}$$

for each integer $k \geq 0$. For each t_k let $\mathrm{Ord}(t_k)$ denote the smallest degree (in $v_1, \ldots, v_m$) of a non-zero term of t_k. We can see that $\mathrm{Ord}(t_0) \geq 1$ by writing the statement $\tau_0(y_0) = \sigma(y_0)$, in our coordinates, to obtain $\tau_0(0) = (t_0(0), 0) = (0, 0)$ and so $t_0(0) = 0$. What is more, from the equation (3) above we see that $\mathrm{Ord}(t_{k+1}) \geq \min_{j=1,\ldots,n} \mathrm{Ord}(t_{kj}) = \mathrm{Ord}(t_k)$.

We let $\ell = \mathrm{Ord}(t_0)$. From the above discussion it follows that $\mathrm{Ord}(t_k) \geq \ell$ for each integer $k \geq 0$. For each integer $k \geq 0$ we let $\check{t}_k$ denote the degree ℓ part of t_k and for each of the coordinate functions t_{kj} of t_k we let $\check{t}_{kj}$ be the degree ℓ part of t_{kj}. Now take the degree ℓ part of each side of equation (3). This yields $\check{t}_{k+1} = \sum_{j=1}^{n} \check{t}_{kj} \cdot g_j(0, 0)$. What is more, for each $j \in 1, \ldots, n$ we can differentiate the formula

$$f(u, v) = u_1 g_1(u, v) + u_2 g_2(u, v) + \cdots + u_n g_n(u, v),$$

with respect to u_j and then evaluate at $(0, 0)$. Doing so we see that

$$\left.\frac{\partial f}{\partial u_j}\right|_{(0,0)} = g_j(0, 0),$$

for each $j \in 1, \ldots, n$. Thus

$$\check{t}_{k+1}(0) = \sum_{j=1}^{n} \check{t}_{kj}(0) \cdot \left.\frac{\partial f}{\partial u_j}\right|_{(0,0)}.$$

Since the matrix whose jth column is given by $\partial f/\partial u_j$ is just $D_{x_0} f_{y_0}$ written in the coordinates of U, (where $x_0 \in \mathbb{P}^n$ is the repelling fixed point of f_{y_0} given by $\sigma(y_0)$, i.e. $\sigma(y_0) = (x_0, y_0) \in \mathbb{P}^n \times M$) then we see that $\check{t}_{k+1} = (D_{x_0} f_{y_0})\check{t}_k$ for each integer $k \geq 0$. Hence inductively we see that $\check{t}_k = (D_{x_0} f_{y_0})^k \check{t}_0$ for each integer $k \geq 0$. Note that since x_0 is a *repelling* fixed point of f_{y_0} then $(D_{x_0} f_{y_0})^k$ is non-singular. Hence for all $k \geq 0$ we see that $\check{t}_k \neq 0$ since $\check{t}_0 \neq 0$ (by our choice of ℓ). Since τ_0 is a Fatou section then the sequence of forward iterates $\tau_k = \dot{f}^{\circ k} \circ \tau_0$ must have compact closure. It follows that the sequence of germs $(t_k(y), y)$ at y_0 of the sequence $\tau_k(y)$ must have compact closure and hence that the sequence of power series $t_k(y)$ must have compact closure. Specifically we see that the sequence of homogeneous polynomial maps $\check{t}_k(y)$ must have compact closure.

Since f_{y_0} has x_0 as a repelling periodic point by assumption then decomposing $\check{t}_k(y)$ into components in different eigenspaces of $D_{x_0} f_{y_0}$ we see that it is impossible for the sequence $\check{t}_k(y)$ to have compact closure unless all of these components are zero. Hence we obtain the contradiction $\check{t}_k(y) = 0$ for all $k \geq 0$. This completes the proof. □

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LEMMA 4.2. *Given a locally simply parameterized family* $f\colon \mathbb{P}^n \times M \to \mathbb{P}^n$ *which is postcritically bounded by some section, let* x *be a repelling periodic point in the Julia set of some* f_y. *Then there is a unique Fatou section* $s\colon M \to \mathbb{P}^n \times M$ *of period* m *passing through* (x, y).

Proof. The uniqueness follows immediately from Lemma 4.1. We need to show the existence.

We assume that x_0 is a repelling periodic point of period m for f_{y_0} and that x_0 lies in $\mathrm{Supp}(\mu_{y_0})$. By Theorem 4.1 there is a Fatou section $\sigma\colon M \to \mathbb{P}^n \times M$ such that $\sigma(y_0) = (x_0, y_0)$. Now let $\mathcal{V} \subset \mathbb{P}^n \times M$ be the analytic variety given by

$$\mathcal{V} = \{(x, y) \in \mathbb{P}^n \times M \mid f^{\circ m}(x, y) = x\}.$$

Then by hypothesis $(x_0, y_0) \in \mathcal{V}$. Locally $\mathcal{V}$ will be the graph of a map from a neighborhood V of y_0 in M to $\mathbb{P}^n$ as long as $D_x(f^{\circ m}(x, y) - x) = D_x f^{\circ m}(x, y) - I_m$ is non-singular (where by D_x we mean the derivative at (x_0, y_0) with respect to the variables $x \in \mathbb{P}^n$ and where I_m is the $m \times m$ identity matrix), as follows from the implicit function theorem.

Since $D_{x_0}(f^{\circ m})(x_0, y_0) = D_{x_0}(f_{y_0}^{\circ m})(x_0, y_0)$ and since x_0 is a repelling fixed point of $f_{y_0}^{\circ m}$ then $D_{x_0}(f_{y_0}^{\circ m})$ does not have the number one as an eigenvalue. Hence there is a neighborhood $V \subset M$ of y_0 such that $\mathcal{V}$ is the graph of a map $t\colon V \to \mathbb{P}^n$ which satisfies $t(y_0) = x_0$. But then the map $\tau\colon V \to \mathbb{P}^n \times V$ given by $\tau(y) = (t(y), y)$ is a section of the trivial bundle $\mathbb{P}^n \times V \to V$ and τ satisfies $\dot{f}^{\circ m} \circ \tau = \tau$ because τ lands in $\mathcal{V}$. Thus τ is a periodic Fatou section over V which contains (x_0, y_0) in its image. Now by Lemma 4.1 it follows that $\tau = \sigma|_V$. But then σ must be periodic of period m, meaning $\dot{f}^{\circ m} \circ \sigma = \sigma$, because it is periodic with period no more than m on an open set and has period no less than m at y_0. □

We now include the following lemma which will help to shorten a couple of proofs by avoiding repeating the same argument.

LEMMA 4.3. *Given a sequence* g_i *of holomorphic maps* $g_i\colon X \to Y$, *where* X, Y *are complex spaces, assume that every subsequence of* g_i *has a convergent subsequence and that every convergent subsequence of* g_i *has the same limit* $h\colon X \to Y$, *then the sequence of maps* g_i *converges to* h.

Proof. Fix a metric d_Y on Y. The sequence g_i converges to $h\colon X \to Y$ if for every compact $K \subset X$ and every $\epsilon > 0$ there exists M such that if $m > M$ then $d_Y(g_m(x), h(x)) < \epsilon$ for every $x \in K$. Thus if g_i does not converge to h then there exists a compact subset $K \subset X$ and an $\epsilon > 0$ such that for every M there is some $m > M$ and some $x \in K$ with $d_Y(g_m(x), h(x)) \geq \epsilon$. Thus there is a subsequence g_{m_k} with points $x_k \in K$ such that $d_Y(g_{m_k}(x_k), h(x_k)) \geq \epsilon$. By hypothesis g_{m_k} has a convergent subsequence and this subsequence necessarily converges to h, but this is a contradiction since no member of g_{m_k} is within ϵ of h on all of K. □

LEMMA 4.4. *Assume that a locally simply parameterized holomorphic family* $f\colon \mathbb{P}^n \times M \to \mathbb{P}^n$ *is postcritically bounded by some section* σ. *Assume further that we are given*

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a periodic repelling point x_1 of the Julia set of some f_{y_1}. Let τ be the unique Fatou section through x_1 (i.e. $\tau(y_1) = (x_1, y_1)$). Then whenever $\tau(y_2)$ is hyperbolic (meaning that if $\tau(y_2)$ has period m then $D_{\tau(y_2)}(f_{y_2}^{\circ m})$ has no eigenvalues which are zero or of unit length) it follows that $\sigma(y_2)$ lies in the global unstable manifold of $\tau(y_2)$.

Proof. To prove this we will use histories. Recall that a history of a point q under a self-map f_y is a sequence of points $\{x_{-k}\}_{k\geq 0}$ such that $f_y(x_{-(k+1)}) = x_{-k}$ and $x_0 = q$.

Write $\sigma(y) = (s(y), y)$ for $y \in M$. Since $s(y_1) \notin \mathrm{PC}(f_{y_1})$ then $s(y_1) \notin E(f_{y_1})$. Thus we can find preimages of $s(y_1)$ arbitrarily close to x_1. It follows that there is a history $\{w_{-k}\}_{k\geq 0}$ of $s(y_1)$ with $\lim_{k\to\infty} w_{-k} = x_1$. For each $k \geq 0$ let $\sigma_k(y)$ be the unique section through (w_{-k}, y_1) with $\dot{f}^{\circ k} \circ \sigma_k = \sigma$. From Theorem 3.2 and Lemma 4.3 it is easy to see that the sequence of sections σ_k converges to $\tau(y)$, the unique Fatou section with $\tau(y_1) = (x_1, y_1)$.

But then $\sigma_k(y_2)$ is a history (since $f_{y_2}(\sigma_{k+1}(y_2)) = \sigma_k(y_2)$) and this history converges to $t(y_2)$ which is a hyperbolic periodic point for f_{y_2}. Since $\sigma_0(y_2) = \sigma(y_2)$ then $\sigma(y_2)$ has a history which converges to the hyperbolic periodic point $\tau(y_2)$ and therefore $\sigma(y_2)$ must lie in the global unstable manifold of $\tau(y_2)$. □

We now prove our non-bifurcation theorem for repelling periodic points in the Julia set. We will later show that under the same conditions a repelling periodic point cannot leave the Julia set.

THEOREM 4.2. *Assume that $f\colon \mathbb{P}^n \times M \to \mathbb{P}^n$ is a locally simply parameterized holomorphic family which is postcritically bounded by some set B. Assume further that B is a set which cannot be contained in the global unstable set of any (non-repelling) hyperbolic periodic point, e.g. B could be an open set (by the Baire category theorem). Then no repelling periodic point in the Julia set of any member of the family bifurcates (meaning no such point intersects any other periodic point, nor changes from being a repelling periodic point).*

Proof. For every $b \in B$ let σ_b be the constant section through b. Assume that τ is the unique section through some repelling periodic point x_1 in the Julia set of some f_{y_1}.

We will show that:

(†) if the repelling periodic point x_1 intersected any other periodic point (even one not locally representable as a section at the point of intersection), or if the repelling periodic point ceased to be repelling for some other y, then there would be some $y_2 \in M$ such that $\tau(y_2)$ is a (non-repelling) hyperbolic periodic point for f_{y_2}.

Given this, it follows by Lemma 4.4 that b lies in the unstable set of $\tau(y_2)$ for each $b \in B$. But this will be a contradiction by our choice of B.

We proceed to prove the statement labeled (†) above. Without loss of generality we may assume that x_1 is a repelling fixed point by replacing f with an iterate if necessary. We first note that the section of fixed points τ cannot intersect some other periodic point without changing at the point of intersection from repelling to some other sort of fixed point. This is a standard dynamical fact which arises by considering the set

$$\mathcal{V}_p \equiv \{(x, y) \in \mathbb{P}^n \times M \mid f^{\circ p}(x, y) = x\},$$

of points in $\mathbb{P}^n \times M$ of period p for each $p \geq 1$. If τ intersects another periodic point, say of period p_1, at some point (x_2, y_2) then $\mathcal{V}_p$ must be singular at (x_2, y_2), for if it is smooth

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at (x_2, y_2) then it must equal the image of τ there. Using local coordinates one concludes (since $\mathcal{V}_p$ is not smooth) from the implicit function theorem that $\det(D_x(f^{\circ p_1}(x, y) - x)|_{(x_2,y_2)}) = 0$ and rewriting this obtains $\det((D_{x_2} f_{y_2})^{p_1} - I_n) = 0$ where I_n is the $n \times n$ identity matrix. Hence $(D_{x_2} f_{y_2})^{p_1}$ has one as an eigenvalue and so $D_{x_2} f_{y_2}$ has a p_1th root of unity as an eigenvalue. Thus the fixed point $\tau(y_2) = (x_2, y_2)$ is not a repelling fixed point if τ intersects another fixed point at y_2. We conclude that if τ intersects some other periodic point then $\tau(y)$ cannot be a repelling fixed point for all $y \in Y$.

Writing τ as $\tau(y) = (t(y), y)$ for some holomorphic $t\colon M \to \mathbb{P}^n$ we see that $D_{x_1} f_{y_1}$ has n eigenvalues all of norm greater than one. If we write the map f in the coordinates of any local coordinate chart on $\mathbb{P}^n \times M$ then the coefficients of $D_{t(y)} f_y$ must vary holomorphically in y since t is holomorphic and $D_{t(y)} f_y$ is just a matrix whose entries are the derivatives of holomorphic coordinate functions of f and t. It follows (and not just locally) that the coefficients of the characteristic polynomial of $D_{t(y)} f_y$ vary holomorphically with y. It then follows, although we omit the proof, that if there is some point $y_2 \in M$ where the fixed point $\tau(y_2)$ is not repelling then there is some point $y_3 \in M$ where $\tau(y_3)$ is a fixed point with no eigenvalues of norm one, no zero eigenvalues and at least one eigenvalue of norm less than one. From a complex analytic point of view this is obvious. Hence $\tau(y_3)$ is a (non-repelling) hyperbolic fixed point. □

5. *Work of Briend and Duval*

We will now consider what is known about repelling periodic points in the support of μ.

The following is a result of Briend and Duval (see [**BD99**, p. 144], in French).

THEOREM 5.1. (Briend and Duval) *The repelling periodic points are equidistributed in the support on μ. In other words, the sequence of measures*

$$\mu_j = \frac{1}{d^{nj}} \sum_{x \in \mathcal{PR}_j} \delta_x,$$

converges weakly to μ, where $\mathcal{PR}_j$ is the set of repelling periodic points of period exactly j. In particular, one obtains the existence of such periodic points and their density in the support of μ.

The above statement guarantees the existence of repelling periodic points, relates the measure of μ to an average over the repelling periodic points, and shows that most periodic points are repelling (since there are at most $d^{(n+1)j} - 1/d^j - 1 \approx d^{nj}$ points of period exactly j but we only sum over the repelling ones; yet the sum converges to a probability measure μ). However, the above statement does not guarantee that there are any repelling periodic points which actually lie in $\mathrm{Supp}(\mu)$ since it only guarantees that the set of repelling periodic points are dense in $\mathrm{Supp}(\mu)$ in the topology of the ambient space, not in the subspace topology on $\mathrm{Supp}(\mu)$.

However, the authors essentially prove a stronger statement (as I am sure they recognized given the nature of the proof). Thanks go to Charles Favre and Nessim Sibony who were able to quickly point out to the author (of this paper) that the stronger statement in Corollary 5.1 is essentially contained in the original proof. We report the necessary details here.

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The proof is carried out in [**BD99**] by using the polynomial lift $f\colon \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ of a holomorphic map $\tilde{f}\colon \mathbb{P}^k \to \mathbb{P}^k$. The authors note that f extends canonically to a self-map of $\mathbb{P}^{k+1}$. They show that it is sufficient to prove their theorem for such a map $f\colon \mathbb{P}^{k+1} \to \mathbb{P}^{k+1}$. Having observed this they change notation, letting k represent the number formerly represented by $k+1$. Hence they are dealing with a special map $f\colon \mathbb{P}^k \to \mathbb{P}^k$. From this point on I will cease to use their notation as I discuss their paper; instead I will use notation consistent with the rest of this paper (i.e. $\mathbb{P}^n$, not $\mathbb{P}^k$).

In the paper [**BD99**] the authors define a measure

$$\nu_j = \frac{1}{d^{(n+1)j}} \sum_{x \in \mathcal{FR}_j} \delta_x,$$

where $\mathcal{FR}_j$ is the set of repelling periodic points of F of period dividing j (see [**BD99**, p. 153]). At exactly the same point in the paper one could introduce $\nu'_j = (1/d^{kn}) \sum_{x \in \mathcal{FR}'_j} \delta_x$ where $\mathcal{FR}'_j$ is the set of repelling periodic points of F of period dividing n which also lie in $\mathrm{Supp}(\mu)$. There is a string of inequalities on the following page [**BD99**, p. 154]. The last of these inequalities involves ν_j. The inequality is:

$$\frac{1}{d^{(n+1)j}} \mu(B(x, r+\epsilon)) \cdot \sharp\{\text{good components of } F^{-j}(B(x, r+\epsilon)) \Subset B(x, r+\epsilon)\} \leq \nu_j(B(x, r+\epsilon))\mu(B(x, r+\epsilon)). \tag{4}$$

In this setting r and ϵ are small, x is a point in $\mathbb{C}^{n+1}$ and $B(x, r+\epsilon)$ is the ball of radius $r+\epsilon$ about x and $\mu(B(x, r+\epsilon)) > 0$. [The actual assumption is not that $\mu(B(x, r+\epsilon)) > 0$ but that $\mu_\delta(B(x, r+\epsilon)) > 0$. However μ is equal to the pushforward of a measure $\hat{\mu}$ on a natural extension $\hat{Y}$ and μ_δ is the pushforward of the restriction of $\hat{\mu}$ to a subspace of $\hat{Y}$; hence $\mu_\delta \leq \mu$ and so $\mu(B(x, r+\epsilon)) > 0$.] A 'good' component of $F^{-j}(B(x, r+\epsilon))$ is one that is sufficiently small and is mapped biholomorphically to $B(x, r+\epsilon)$ by $F^{\circ j}$. The reason for the inequality is that there can be at most $d^{(n+1)j}$ good components of $F^{-j}(B(x, r+\epsilon))$ (since a point has at most $d^{(n+1)j}$ preimages under $F^{\circ j}$) and because each good component U of $F^{-j}(B(x, r+\epsilon))$ contains a repelling fixed point of F of period dividing n (because the inverse biholomorphism g from $B(x, r+\epsilon)$ to the good component U is a strict contraction in the Kobayashi metric and thus has an attracting fixed point $p \in U$). However since $\mathrm{Supp}(\mu)$ is totally invariant under F then if $\mu(B(x, r+\epsilon)) > 0$ it follows that $\mu(g^{\circ \ell}(B(x, r+\epsilon))) > 0$ for all $\ell \geq 0$ (g being a local inverse of $F^{\circ j}$). It follows that the attracting fixed point p of g, which is a repelling fixed point for $F^{\circ j}$, lies in $\mathrm{Supp}(\mu)$. Thus the inequality (4) can be replaced with

$$\frac{1}{d^{(n+1)j}} \mu(B(x, r+\epsilon)) \cdot \sharp\{\text{good components of } F^{-j}(B(x, r+\epsilon)) \Subset B(x, r+\epsilon)\} \leq \nu'_j(B(x, r+\epsilon))\mu(B(x, r+\epsilon)). \tag{5}$$

The remainder of the argument (only a paragraph) follows verbatim with ν'_j replacing ν_j and ν', a limit of the sequence of measures ν'_j, replacing ν. One takes both ends of the string of inequalities on [**BD99**, p. 154] (the last inequality of which was given above) and this gives

$$(1-\epsilon)\mu_\delta(B(x, r))\mu(M(x, r)) \leq \nu'_j(B(x, r+\epsilon))\mu(B(x, r+\epsilon)),$$

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where μ_δ is another measure used in the paper. Here δ is a small positive number and μ_δ converges to μ as δ approaches zero. One concludes that

$$(1-\epsilon)\mu_\delta(B(x,r))\mu(M(x,r)) \le \nu'(B(x,r+\epsilon))\mu(B(x,r+\epsilon)).$$

By varying ϵ one concludes that ν' dominates μ_δ. Then $\nu' \ge \mu$ by letting δ go to zero. One observes that μ is a probability measure and ν has mass at most one, and thus $\mu = \nu'$.

The paper then shows that μ_j converges weakly to μ using the fact that ν_j is close to μ_j (for this last fact it is necessary to use ν_j and not ν'_j).

We thus conclude the following, which, while never stated in [**BD99**], is essentially one of their results.

COROLLARY 5.1. (Briend and Duval) *The sequence of measures* ν'_j *converges weakly to* μ. *Thus for large* j *most repelling periodic points of period dividing* j *are contained in* $\mathrm{Supp}(\mu)$.

We note that since there are $d^{(n+1)j} - 1/d^j - 1 \approx d^{nj}$ points of period dividing j of a self-map $f : \mathbb{P}^n \to \mathbb{P}^n$ of degree d then as j goes to infinity the ratio

$$\frac{\sharp\{\text{points of period exactly } j\}}{\sharp\{\text{points of period dividing } j\}},$$

goes to one. From Briend and Duval [**BD99**] it also follows that as j goes to infinity then the ratio

$$\frac{\sharp\{\text{repelling periodic points of period dividing } j\}}{\sharp\{\text{points of period dividing } j\}},$$

also goes to one (since ν_j is normalized by dividing by d^{nj} which is asymptotically the number of points of period dividing j). Thus, in fact, for large j most periodic points of period dividing j are repelling of period exactly j and lie in $\mathrm{Supp}(\mu)$.

6. *Further stability conclusions on* $\mathbb{P}^n$

We can use Briend and Duval's work discussed in the previous section to extend our conclusions for a locally simply parameterized holomorphic family $f : \mathbb{P}^n \times M \to \mathbb{P}^n$ which is postcritically bounded by an open subset of $\mathbb{P}^n$.

For each $y \in M$ let R_y denote the set of all repelling periodic points of f_y in $\mathbb{P}^n$. For each $k \in \mathbb{N}$ let $R_{y,k}$ denote the set of all repelling periodic points of f_y whose period divides k. Also, let $S_y = R_y \cap \mathrm{Supp}(\mu_y)$ and let $S_{y,k} = R_{y,k} \cap \mathrm{Supp}(\mu_y)$ denote the set of repelling periodic points of f_y in $\mathrm{Supp}(\mu_y)$ and the set of repelling periodic points of f_y in $\mathrm{Supp}(\mu_y)$ with period dividing k respectively.

Given $y_1, y_2 \in M$ we have an obvious map $\tau_{y_1,y_2} : S_{y_1} \to R_{y_2}$ constructed as follows: given $p \in S_{y_1,k}$ let s_p be the unique Fatou section with $s(y_1) = p$. Then define $\tau_{y_1,y_2}(p) = s(y_2)$. That this map sends repelling periodic points of f_{y_1} in $\mathrm{Supp}(\mu_{y_1})$ to repelling periodic points of f_{y_2} is the content of Theorem 4.2. It also follows from that theorem that τ_{y_1,y_2} is injective and maps points of period k to points of period k. To show that the map τ_{y_1,y_2} is continuous assume that we are given a sequence of points $p_i \in S_{y_1}$ which converges to a point $p \in S_{y_1}$. Then for each i let s_{p_i} be the unique Fatou section with $s_{p_i}(y_1) = p_i$. Any subsequence of s_{p_i} has a convergent subsequence since the set of Fatou

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sections is compact, and the limit of that convergent subsequence must be the unique Fatou section s with $s(y_1) = p$. Thus by Lemma 4.3 the sequence s_{p_i} converges to s and so $\lim_{i\to\infty} \tau_{y_1,y_2}(p_i) = \lim_{i\to\infty} s_{p_i}(y_2) = s_p(y_2) = \tau_{y_1,y_2}(p)$. Hence τ_{y_1,y_2} is continuous.

THEOREM 6.1. *Given that the locally simply parameterized family f is postcritically bounded by an open subset of $\mathbb{P}^n$, if x is a repelling periodic point of f_y and x lies in* $\mathrm{Supp}(\mu_y)$ *then x does not leave* $\mathrm{Supp}(\mu_y)$ *as y is allowed to vary, or rather, if $\sigma(y) = (s(y), y)$ is the unique Fatou section through (x, y) then $s(y') \in \mathrm{Supp}(\mu_{y'})$ for all $y' \in M$.*

Proof. Assume that there exists $y' \in M$ with $x' = s(y') \notin \mathrm{Supp}(\mu_{y'})$. Then consider the map $\tau_{y,y'} : S_y \to R_{y'}$. This map is continuous and $\tau_{y,y'}(x) = x'$ with $x' \notin \mathrm{Supp}(\mu_{y'})$. Thus choose an open neighborhood $U \Subset \mathbb{P}^n \setminus \mathrm{Supp}(\mu_{y'})$ of x'. Since U is open there exists V an open neighborhood of x in $\mathbb{P}^n$ with $\tau(V \cap S_y) \subset U$. We know that $\mu_y(V) > 0$ since V is open and contains $x \in \mathrm{Supp}(\mu_y)$.

Now $d^{nk} \approx d^{(n+1)k} - 1/(d^k - 1)$ = number of periodic points of period dividing $k \geq |R_{y',k}|$. But $|S_{y',k}|/d^{nk}$ converges to the probability measure $\mu_{y'}$ from the version of Briend and Duval's result given as Corollary 5.1 above. Therefore $\lim_{k\to\infty} |R_{y',k} \setminus S_{y',k}|/d^{nk}$ converges to zero.

Putting this all together gives the following string of inequalities:

$$0 < \lim_{k\to\infty} \frac{1}{d^{nk}} |V \cap S_{y,k}| \leq \lim_{k\to\infty} \frac{1}{d^{nk}} |U \cap R_{y',k}| \leq \mu_{y'}(\mathbb{P}^n \setminus \mathrm{Supp}(\mu_{y'})).$$

This is a contradiction because $\mu_{y'}(\mathbb{P}^n \setminus \mathrm{Supp}(\mu_{y'})) = 0$. □

We see next that in such a family, any point in the Julia set for one member behaves dynamically comparably to some point in the Julia set of each other member of the family.

PROPOSITION 6.1. *Given that the locally simply parameterized family f is postcritically bounded by an open subset of $\mathbb{P}^n$ then given any point x in the Julia set of some f_y there exists a Fatou section σ through (x, y). Moreover this section is a limit of repelling periodic sections which all remain in the Julia set.*

Proof. Assume that $x_0 \in \mathbb{P}^n$ is a point which is in $\mathrm{Supp}(\mu_{y_0})$ for some $y_0 \in M$. By Briend and Duval's result (Corollary 5.1 in this paper) we can find some sequence p_i of repelling periodic points in $\mathrm{Supp}(\mu_y)$ which converge to x. Let $\sigma_i(y) = (s_i(y), y)$ be the unique Fatou section through the point (y_0, p_i) (i.e. $s_i(y_0) = p_i$). By replacing the sequence p_i (and hence the sequences σ_i and s_i) with a subsequence if necessary we can assume that σ_i converges to a Fatou section σ. Writing $\sigma(y) = (s(y), y)$, then $s(y_0) = x_0$. Since $s_i(y) \in \mathrm{Supp}(\mu_y)$ for each $y \in M$ and since $\mathrm{Supp}(\mu_y)$ is closed for each $y \in M$ we conclude that $s(y) \in \mathrm{Supp}(\mu_y)$ for each $y \in M$. □

In a locally simply parameterized family postcritically bounded by an open set we can conclude that the support of μ is disjoint from the critical set for all members of the family as long as it is true for one member of the family.

THEOREM 6.2. *Given that the locally simply parameterized family $f : \mathbb{P}^n \times M \to \mathbb{P}^n$ is postcritically bounded by an open subset of $\mathbb{P}^n$ and if the Julia set* $\mathrm{Supp}(\mu_{y_0})$ *intersects the*

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critical set $\mathrm{Crit}(f_{y_0})$ *for some member of the family then this intersection is persistent throughout the family in the sense that if* $x_0 \in \mathrm{Supp}(\mu_{y_0}) \cap \mathrm{Crit}(f_{y_0})$ *then there is a Fatou section* $\sigma(y) = (s(y), y)$ *through* (x_0, y_0) *such that* $s(y) \in \mathrm{Supp}(\mu_y) \cap \mathrm{Crit}(f_y)$ *for all* $y \in M$.

Proof. Assume that $x_0 \in \mathbb{P}^n$ is a point which is both in the critical set of f_{y_0} and in $\mathrm{Supp}(\mu_{y_0})$ for some $y_0 \in M$. By the previous lemma there is a Fatou section $\sigma(y) = (s(y), y)$ through (x_0, y_0) which remains in the Julia set and is the limit of a sequence $\sigma_i(y) = (s_i(y), y)$ of repelling periodic sections in the Julia sets of the family. We will show that the set $I = \{y \in M \mid s(y) \text{ is in the critical set of } f_y\}$ is both open and closed in M.

If we write the self-map $\dot{f} = (f, \mathrm{id}_M)$ about the point (x_0, y_0) using local coordinates on $\mathbb{P}^n$ we find that

$$\det d\dot{f} = \det \begin{bmatrix} D_x f & D_y f \\ 0 & I_m \end{bmatrix}.$$

so that the critical set of $\dot{f}$ is exactly the union of the critical sets of f_y in $\mathbb{P}^n \times \{y\}$ as y varies over all of M. Also, the critical set of $\dot{f}$ is locally given by the vanishing of a single holomorphic function.

Now the critical set C of $\dot{f}$ is a closed analytic variety of $\mathbb{P}^n \times M$ so it intersects $\sigma(M)$ in a closed subset of $\mathbb{P}^n \times M$. The projection from $\sigma(M)$ to M is a biholomorphism, hence I is closed.

Now none of the repelling periodic sections σ_i constructed above can intersect the critical set C of $\dot{f}$ (i.e. you cannot have a repelling periodic point with a zero eigenvalue). Thus if y_1 is a point of I then let $x_1 = s(y_1)$ and choose an open neighborhood $U \times V \Subset \mathbb{P}^n \times M$ of (x_1, y_1) for which the critical set C has a holomorphic defining function $h\colon U \times V \to \mathbb{C}$, and where V is biholomorphic to a polydisk. Shrink V if necessary so that $s(V) \Subset U$. For i sufficiently large we must have $s_i(V) \subset U$ and thus $\sigma_i(V) \subset U \times V$. Then for sufficiently large i the sequence $h \circ \sigma_i : V \to \mathbb{C}$ is never zero (since C is disjoint from the image of each σ_i), however $h \circ \sigma(y_1) = 0$. By the Hurwitz theorem we conclude that $h \circ \sigma$ is identically zero, thus $s(V) \subset C$ and so $V \subset I$. Hence if y_1 is an arbitrary point of I then there is an open polydisk V which contains y_1 and lies in I. Thus I is both open and closed, hence all of M. □

7. *Final remarks*

To keep a point in the Julia set from bifurcating all that is needed is for a Fatou section through that point to comprise entirely repelling points. We have shown that there are Fatou sections through every point of $\mathrm{Supp}(\mu_y)$ and it is already known [**BD99**] that the characteristic exponents of any μ_y are positive. It seems that in a family which is postcritically bounded by an open set one should be able to prove a non-bifurcation result for 'most' points in the Julia set, instead of just the repelling periodic points.

One also sees that for postcritically bounded families, because the measure μ_y is a sort of limit of the repelling periodic points and because we have shown that the repelling periodic points in the support of one μ_y never bifurcate nor leave $\mathrm{Supp}(\mu_y)$ as y is allowed to vary then one can 'follow' μ_y by following the repelling periodic points.

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Lastly we note that independently the author (Robertson 2002, unpublished manuscript) and Maegawa [**Mae05**] have given conditions under which one can guarantee that a holomorphic self-map of $\mathbb{P}^n$ is uniformly repelling on a neighborhood of the Julia set. This provides examples of postcritically bounded families and thereby shows that they exist.

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