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Thue equations that simultaneously fail the Hasse principle

BY PALOMA BENGOECHEA†

ETH Zürich, Department of Mathematics, Rämistrasse 101, 8092 *Zürich, Switzerland.*
e-mails: paloma.bengoechea@math.ethz.ch

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Abstract

We refine a previous construction by Akhtari and Bhargava so that, for every positive integer m, we obtain a positive proportion of Thue equations $F(x, y) = h$ that fail the integral Hasse principle simultaneously for every positive integer h less than m. The binary forms F have fixed degree ≥ 3 and are ordered by the absolute value of the maximum of the coefficients.

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1. *Introduction*

Let $F(x, y)$ be an irreducible binary form with integer coefficients of degree $n \geq 3$, and let m be a positive integer. Thue showed that the inequalities

$$|F(x, y)| \leq m, \tag{1}$$

known as *Thue inequalities*, have finitely many solutions in integers x and y. Mahler [6], Schmidt [8], Mueller–Schmidt [7], Thunder [9], recently the author [3], and others obtained upper bounds for the number of solutions.

In this article we will prove that, when the forms are ordered by height, a positive proportion have no integer solutions (x, y) to (1) but have p-adic solutions for all primes p to all the equations $F(x, y) = h$, where h is any positive integer less than m.

This paper is a refinement of [2], where a positive proportion of Thue equations $F(x, y) = m$ are shown to have solutions locally everywhere but not globally (i.e. they fail the integral Hasse principle). We refine the construction in [2] so that the Hasse principle fails simultaneously for all Thue equations $F(x, y) = h$, where $1 \leq h \leq m$.

THEOREM 1·1. *Let m be a positive integer. When integral binary forms of fixed degree* ≥ 3 *are ordered by height, a positive proportion of primitive maximal forms represent every positive integer less than m locally everywhere but not globally.*

A *maximal* form F is a form which is not a proper subform of any other form, i.e. F cannot be written as $G(ax + by, cx + dy)$ for any integer matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $|\det(A)| > 1$

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and any other binary form G. In Theorem 1·1 we exclude non-maximal forms because if F is a proper subform of G, then the number of integer solutions of F and G is the same.

The construction in [**2**] also works for cubic forms ordered by discriminant instead of the height, and for quartic forms ordered by the two generators of their rings of invariants, classically denoted by I, J (see [**1**]). The same can be done here; Theorem 1·1 holds for cubic forms ordered by discriminant and for quartic forms ordered by the invariants I, J.

The paper is organised as follows. In Section 2 we give some preliminaries, specially a result due to György (Theorem 2·1) that gives an upper bound for the number of solutions to Thue inequalities (1). In Section 3 we explain our strategy and how György's result will be relevant for the construction of forms satisfying Theorem 1·1. We also give an expicit lower bound for the proportion in Theorem 1·1 that depends on m and the fixed degree of the forms. In Section 4 we construct those forms. We prove that they represent every positive integer less than m locally everywhere, but not globally, in Sections 5 and 6 respectively.

2. *Preliminaries*

2·1. *Discriminant, height and equivalent forms*

For a binary form $F(x, y)$ that factors over $\mathbb{C}$ as

$$\prod_{i=1}^{n}(\alpha_i x - \beta_i y),$$

the discriminant $D(F)$ of F is given by

$$D(F) = \prod_{i<j} \left(\alpha_i\beta_j - \alpha_j\beta_i\right)^2.$$

Therefore, if we write

$$F(x, y) = a_n(x - \gamma_1 y)\cdots(x - \gamma_n y),$$

we have

$$D(F) = a_n^{2(n-1)} \prod_{i<j} \left(\gamma_i - \gamma_j\right)^2.$$

If we write $F(x, y) = a_n x^n + a_{n-1}x^{n-1}y + \cdots + a_1 x y^{n-1} + a_0 y^n$, the (naive) height of F, denoted by $H(F)$, is defined by

$$H(F) = \max\left(|a_n|, |a_{n-1}|, \ldots, |a_0|\right).$$

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and a binary form $F(x, y)$, we define the binary form F_A by

$$F_A(x, y) = F(ax + by, cx + dy).$$

Note that

$$D(F_A) = (\det(A))^{n(n-1)} D(F). \tag{2}$$

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If $A \in \mathrm{GL}(2, \mathbb{Z})$ we say that F_A is equivalent to F, and if $A \in \mathrm{GL}(2, \mathbb{Z}_p)$ we say that F_A is equivalent to F over $\mathbb{Z}_p$. Two forms that are equivalent (respectively equivalent over $\mathbb{Z}_p$) represent the same integers (respectively the same p-adic integers).

2·2. *Number of solutions to Thue inequalities*

The upper bounds for the number of solutions to Thue inequalities (1) obviously depend on m. However, if m is bounded above in terms of the absolute value of the discriminant of F, then one can find bounds that are independent of m. The following is [**5**, Corollary 3] with $\vartheta = 1/3$.[1]

THEOREM 2·1 (Györy). *Let $F(x, y) \in \mathbb{Z}[x, y]$ be a binary form of degree $n \geq 3$ and m be an integer satisfying*

$$|D(F)| \geq (3^n m)^{6(n-1)}. \tag{3}$$

The number of solutions to $|F(x, y)| \leq m$ in co-prime integers x, y is at most $32n + 11$.

A solution (x, y) to (1) is called *primitive* if x and y are co-prime integers.

2·3. *Finding solutions of Thue equations over $\mathbb{Z}_p$*

In order to find solutions of Thue equations or inequalities over $\mathbb{Z}_p$, most of the time we will find a solution over $\mathbb{F}_p$ and lift it to $\mathbb{Z}_p$ using Hensel's Lemma: if $(\alpha, \beta) \in \mathbb{Z}_p^2$ satisfies $F(\alpha, \beta) \equiv 0 \pmod{p}$ and either $F'_x(\alpha, \beta) \not\equiv 0$ or $F'_y(\alpha, \beta) \not\equiv 0 \pmod{p}$, then there exists $(\alpha_0, \beta_0) \in \mathbb{Z}_p^2$ such that $(\alpha_0, \beta_0) \equiv (\alpha, \beta) \pmod{p}$ and $F(\alpha_0, \beta_0) = 0$ over $\mathbb{Z}_p$.

3. *Strategy*

Let $m \geq 1$, $n \geq 3$ and set $g = (n-1)(n-2)/2$. Let

$$M = \max(m, (2g+1)^2, 17)$$

and denote by $p_1, \ldots, p_k$ all the primes less than M. We have that

$$2^k - 32n - 11 \geq 1. \tag{4}$$

Indeed, for $n \geq 4$,

$$k \geq \frac{M}{\log M} \geq \frac{(2g+1)^2}{2\log(2g+1)} \geq 32n + 11,$$

and for $n = 3$,

$$2^k \geq 2^7 \geq 32 \cdot 3 + 12.$$

Let X be a positive integer such that

$$X > \frac{2^{(n+1)k} \prod_{i=1}^{k} p_i^{n-1}}{(n+1)^{k/2}},$$

[1]Note that similar results to Theorem 2·1 were given previously in [**4**] for the number of all solutions, not necessarily primitive.

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and define

$$\tilde{X} := X \frac{(n+1)^{k/2}}{2^{(n+1)k} \prod_{i=1}^{k} p_i^{n-1}} \geq 1.$$

Let S be the set of maximal primitive integral binary forms of degree n that satisfy the conditions:

(I) they represent every positive integer $\leq m$ locally everywhere;
(II) they do not represent any positive integer $\leq m$ globally.

Let $\tilde{S}$ be the set of primitive integral binary forms F of degree n that satisfy (3) and the following three conditions:

(i) F has Galois group S_n (in particular F is irreducible and maximal);
(ii) F has two simple roots in $\mathbb{F}_{p_i}$ for all $i = 1, \ldots, k$;
(iii) for each prime $p > M$, F does not factor as $cM(x, y)^r$ modulo p for any $r > 1$ and any binary form $M(x, y)$ and constant $c > 0$.

For each $F \in \tilde{S}$, we will construct $2^k - 32n - 11$ forms $G_j \in S$ such that, if $H(F) < \tilde{X}$, then $H(G_j) < X$. Thus, if we denote by $h(S, H)$ the number of forms in S with height H, we will have

$$\sum_{0<H<X} h(S, H) \geq (2^k - 32n - 11) \sum_{0<H<\tilde{X}} h(\tilde{S}, H). \tag{5}$$

The right-hand side of (5) can be estimated as in [2, theorem 5·3]. [2] We have:

$$\sum_{0<H<\tilde{X}} h(\tilde{S}, H) \sim 2^n \tilde{X}^{n+1} \prod_{p} \mu_p(\tilde{S}), \tag{6}$$

where $\mu_p(\tilde{S})$ denotes the p-adic density of $\tilde{S}$ in the space of integral binary forms of degree n. By (5) and (6),

$$\sum_{0<H<X} h(S, H) \gg (2^k - 32n - 11) 2^n \left(\frac{(n+1)^{k/2}}{2^{(n+1)k} \prod_{i=1}^{k} p_i^{n-1}} \right)^{n+1} X^{n+1}, \tag{7}$$

where the implicit constant depends only on X.

The estimation (7) already gives a positive proportion. Using that $k \leq M/\log M$ and $\prod_{i=1}^{k} p_i \leq 4^M$, we have

$$\frac{(n+1)^{k/2}}{2^{(n+1)k} \prod_{i=1}^{k} p_i^{n-1}} \gg \frac{1}{8^{Mn}}.$$

[2] The estimation is done using Ekedahl's sieve (we refer to the references within [2] for an exposition) together with the facts that 100% of forms satisfy (i) and that conditions (ii) and (iii) are described by a suitable set of congruences.

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For $p \in \{p_1, \ldots, p_k\}$, the density of forms satisfying (ii) is greater than

$$\frac{\frac{p(p-1)}{2}(p^{n-1} - p^{n-2})}{p^{n+1}}.$$

If we set $p = p_i$, the first term in the numerator counts the possibilities for two simple roots in $\mathbb{F}_{p_i}$ satisfying condition (ii), and the second term bounds the possibilities for a binary form of degree $n-2$. The denominator counts the total number of binary forms modulo p by considering the p possible values for each of the $n+1$ coefficients. Hence the density of forms satisfying (ii) is greater than

$$\frac{1}{2}\Big(\frac{p-1}{p}\Big)^2.$$

For $p > M$, the density of forms satisfying (iii), i.e. that do not factor modulo p as $cM(x, y)^r$ for any constant c, any binary form M and any $r > 1$, is

$$\begin{aligned} 1 - \frac{p-1}{p^{n+1}} \sum_{\substack{r \mid n \\ r < n}} p^{r+1} &\geq 1 - \frac{p-1}{p^n} \sum_{r=0}^{[\frac{n}{2}]} p^r \\ &= 1 - \frac{p^{[\frac{n}{2}]+1} - 1}{p^n} \\ &\geq 1 - \frac{1}{p^{[\frac{n}{2}]+1}}. \end{aligned}$$

The factor $p-1$ counts the possible values for c and $[\cdot]$ denotes the integer part. Therefore,

$$\begin{aligned} \prod_p \mu_p(\tilde{S}) &\geq \prod_{i=1}^{k} \frac{1}{2}\Big(\frac{p_i - 1}{p_i}\Big)^2 \prod_{p>M} 1 - \frac{p^{[\frac{n}{2}]+1} - 1}{p^n} \\ &\geq \frac{1}{2^k} \prod_{i=2}^{k+1} \Big(\frac{i-1}{i}\Big)^2 \prod_{p>M} 1 - \frac{1}{p^{[\frac{n}{2}]+1}} \\ &\geq \frac{1}{2^M (M+1)^2} \prod_{p>M} 1 - \frac{1}{p^{[\frac{n}{2}]+1}}. \end{aligned}$$

We conclude that

$$\sum_{0<H<X} h(S, H) \gg \frac{2^n}{8^{Mn(n+1)}(M+1)^2} \prod_{p>M} 1 - \frac{1}{p^{[\frac{n}{2}]+1}}.$$

4. *Construction of suitable forms*

4·1. *Construction of 2^k primitive forms*

Let $F(x, y) = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n \in \mathbb{Z}[x, y]$ be a primitive binary form of degree $n \geq 3$. Suppose that α is a simple integer root of $F(x, 1)$ modulo a prime p;

$$F(\alpha, 1) \equiv 0 \pmod{p} \quad \text{with } \alpha \in \mathbb{Z}. \tag{8}$$

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Consider the form

$$F\begin{pmatrix} p & \alpha \\ 0 & 1 \end{pmatrix}(x, y) = F(px + \alpha y, y),$$

and write $F(px + \alpha y, y) = e_0 x^n + e_1 x^{n-1} y + \cdots + e_n y^n$. The coefficients e_i are given by

$$e_{n-j} = p^j \sum_{i=0}^{n-j} a_i \alpha^{n-i-j} \binom{n-i}{j}. \tag{9}$$

Claim 4·1. The content of $F(px + \alpha y, y)$ is p.

Proof. Any divisor of the content should divide $e_0 = p^n a_0$, and thus it should divide either p or a_0. By induction on the formula (9), any divisor of the content should then divide p or all coefficients $a_0, \ldots, a_n$. Since F is primitive, any divisor of the content should divide p. If $j \geq 1$, clearly e_{n-j} is divisible by p. By (8) we also have

$$e_n = F(\alpha, 1) \equiv 0 \pmod{p}.$$

Since $e_{n-1} = pF'_x(\alpha, 1)$ and α is a simple root modulo p, we have that $e_{n-1} \not\equiv 0 \pmod{p^2}$. So the content is exactly p.

Hence the form

$$F_\alpha(x, y) := \frac{1}{p} F(px + \alpha y, y) \tag{10}$$

has integral coefficients and content 1. For later, note that it is congruent to

$$y^{n-1} L(x, y) \pmod{p} \tag{11}$$

where $L(x, y) = F'_x(\alpha, 1)x + (yF(\alpha, 1))/p$ satisfies $F'_x(\alpha, 1) \not\equiv 0 \pmod{p}$.

Note also that we can choose the integer α so that

$$-\frac{p-1}{2} \leq \alpha \leq \frac{p-1}{2}.$$

Then we have, by (9),

$$\begin{aligned} |e_{n-j}| &\leq p^j H(F) \left(\frac{p-1}{2}\right)^{n-j} \binom{n+1}{j+1} \\ &< p^n H(F) \frac{2^{n+1}}{\sqrt{n+1}}, \end{aligned}$$

where in the second inequality we used Stirling's approximation. Therefore,

$$H(F_\alpha) < p^{n-1} \frac{2^{n+1}}{\sqrt{n+1}} H(F).$$

Now let $F \in S$. In particular, F has two (distinct) simple roots modulo p_i for all $i = 1, \ldots, k$. Applying the procedure described above with $p = p_1$, we obtain F_α and F_β, where α, β are simple roots modulo p_1. The forms F_α and F_β have again two distinct simple roots modulo p_i for $i = 2, \ldots, k$, since the transformation matrices from F to F_α and F_β

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have determinant p_1, which is a unit in $\mathbb{F}_{p_i}$. Hence we can apply the procedure described above to F_α and F_β with $p = p_2$ and obtain four new forms with integral coefficients and content 1. Applying the whole procedure with each prime p_i, we obtain 2^k primitive integral binary forms of degree n, that we denote by G_j ($j = 1, \dots, 2^k$). Their discriminant and height satisfy respectively

$$D(G_j) = \left(\prod_{i=1}^{k} p_i\right)^{n-1} D(F), \tag{12}$$

$$H(G_j) \le \left(\prod_{i=1}^{k} p_i\right)^{n-1} \frac{2^{(n+1)k}}{(n+1)^{k/2}} H(F).$$

Since F is irreducible over $\mathbb{Q}$ and all of the matrix actions involved are rational, the forms G_j are also irreducible over $\mathbb{Q}$.

By construction, each solution to an equation

$$G_j(x, y) = h \tag{13}$$

gives a solution to the equation

$$F(x, y) = h \prod_{i=1}^{k} p_i. \tag{14}$$

Note that, with the notation (10), if (x_0, y_0) is a solution to $F_\alpha(x, y) = h$ and $F_\beta(x, y) = h$ simultaneously, since $\alpha \neq \beta$, then $(px_0 + \alpha y_0, y_0)$ and $(px_0 + \beta y_0, y_0)$ are two different solutions to $F(x, y) = ph$. Also, if $F_\alpha(x, y) = h$ has two different solutions (x_0, y_0) and (x_0', y_0'), then clearly $(px_0 + \alpha y_0, y_0)$ and $(px_0' + \alpha y_0, y_0)$ are two different solutions to $F(x, y) = ph$. It follows that a solution to two equations

$$G_j(x, y) = h, \qquad G_{j'}(x, y) = h \qquad (j \neq j')$$

gives at least two different solutions to (14).

4·2. *The forms are not equivalent*

Any two G_j, $G_{j'}$ as above are of the form F_γ and $F_{\gamma'}$ with

$$\gamma = \begin{pmatrix} \prod_{i=1}^{k} p_i & \sum_{i=1}^{k} u_i \prod_{\ell=1}^{i-1} p_\ell \\ 0 & 1 \end{pmatrix}, \qquad \gamma' = \begin{pmatrix} \prod_{i=1}^{k} p_i & \sum_{i=1}^{k} u_i' \prod_{\ell=1}^{i-1} p_\ell \\ 0 & 1 \end{pmatrix},$$

where u_i, u_i' are simple roots modulo p_i. Any transformation B such that $F_{\gamma B} = F_{\gamma'}$ satisfies $B = \gamma^{-1} A \gamma'$, where A stabilises F. Since F has Galois group S_n, its stabilizer is trivial and so

$$B = \gamma^{-1}\gamma' = \begin{pmatrix} 1 & \frac{\sum_{i=1}^{k} (-u_i + u_i') \prod_{\ell=1}^{i-1} p_\ell}{\prod_{i=1}^{k} p_i} \\ 0 & 1 \end{pmatrix}.$$

Using inductively that $p_i \mid (u_i - u_i')$ only if $u_i = u_i'$ for $i = 1, \dots, k$, we conclude that the term in the top right of the matrix is an integer only if $\gamma = \gamma'$. Otherwise, $B \notin \mathrm{GL}(2, \mathbb{Z})$.

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4·3. *The forms are maximal*

Suppose that a form G_j is not maximal, so there is an integer matrix A with $\det(A) > 1$ and $\det(A) \mid \operatorname{disc}(G_j)$. In particular, there is a prime p such that $p \mid \det(A)$ and $p \mid \operatorname{disc}(G_j)$. For $p > p_k$, up to a unit constant every form G_j is equivalent to F over $\mathbb{Z}_p$, and since F is maximal, G_j is also maximal over $\mathbb{Z}_p$, so $p \nmid \det(A)$. Now let $p \in \{p_1, \ldots, p_k\}$. In the construction process of G_j, there is a form $K(x, y)$ that by (11) satisfies

$$K(x, y) \equiv y^{n-1}(\ell_1 x + \ell_2 y) \pmod{p} \tag{15}$$

with $p \nmid \ell_1$. By construction, G_j is equivalent to K over $\mathbb{Z}_p$ up to a unit constant. Since $\operatorname{disc}(K) \equiv \ell_1^{2(n-1)} \not\equiv 0 \pmod{p}$, $p \nmid \operatorname{disc}(G_j)$.

5. *All forms represent locally every integer $\leq m$*

Let h be a positive integer, $h \leq m$. In this section we will show that every form G_j represents h over $\mathbb{Z}_p$ for all prime p.

5·1. *At primes $\leq M$*

Let p be a prime in $\{p_1, \ldots, p_k\}$. As in Section 4·3, we consider the form $K(x, y)$ in the construction of G_j that by (11) satisfies

$$K(x, y) \equiv y^{n-1}(\ell_1 x + \ell_2 y) \pmod{p} \tag{16}$$

with $p \nmid \ell_1$.

Set $p = p_i$, so that $K(x, y)$ is obtained in the ith step of the construction. Using (16) it is clear that, for each $y_0 = 1, \ldots, p-1$, the equation

$$K(x, y) = h \prod_{\ell=i+1}^{k} p_\ell$$

has a solution modulo p which is not a root of the derivative $K'_x(x, y) \equiv y^{n-1}\ell_1 \pmod{p}$. By Hensel's Lemma, we can lift the solution over $\mathbb{Z}_p$. Now by construction, $K(x, y)$ and $G_j(x, y) \prod_{\ell=i+1}^{k} p_\ell$ are equivalent over $\mathbb{Z}_p$, so the equation $G_j(x, y) = h$ has also a solution over $\mathbb{Z}_p$.

5·2. *At primes $> M$*

Let p be a prime greater than M. In particular, $p > h$ and $p > n$, so $p \nmid hn$. Consider the plane curve $hz^n = G_j(x, y)$ defined over $\mathbb{F}_p$. It follows from (iii) and the irreducibility of G_j, that G_j is irreducible also over $\mathbb{F}_p$. This ensures that $G_j(x, y) = 0$ does not have any solution over $\mathbb{F}_p$ and hence the curve is smooth (we use here that $p \nmid hn$). Let N be the number of points over $\mathbb{F}_p$ on the curve. Since $p > (2g+1)^2$ (where $g = (n-1)(n-2)/2$ was introduced earlier), the Hasse-Weil bound

$$|N - (p+1)| \leq 2g\sqrt{p}$$

gives a smooth point (x_0, y_0, z_0) with $z_0 \neq 0$ over $\mathbb{F}_p$. This point gives a solution $(x_0 z_0^{-1}, y_0 z_0^{-1})$ to $G_j(x, y) \equiv h \pmod{p}$ that can be lifted over $\mathbb{Z}_p$ by Hensel's Lemma.

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6. *Many forms do not represent globally any integer* $\leq m$

Let h be a positive integer, $h \leq m$. Suppose that $G_j(x, y) = h$ has a solution $(x_0, y_0) \in \mathbb{Z}^2$. By construction,

$$G_j = \left(\prod_{i=1}^{k} p_i^{-1}\right) F \begin{pmatrix} p_1 & u_1 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} p_k & u_k \\ 0 & 1 \end{pmatrix}, \tag{17}$$

where u_i are simple roots modulo p_i. It follows that

$$(X_0, y_0) := \begin{pmatrix} p_1 & u_1 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} p_k & u_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

is a solution to

$$F(x, y) = h \prod_{i=1}^{k} p_i. \tag{18}$$

Hence a solution to the inequality

$$G_j(x, y) \leq m \tag{19}$$

gives a solution to

$$F(x, y) \leq m \prod_{i=1}^{k} p_i \tag{20}$$

and actually a primitive solution (a non-primitive solution $(a\tilde{X}_0, a\tilde{y}_0)$ to (20) with $(\tilde{X}_0, \tilde{y}_0) = 1$ gives a primitive solution to $F(x, y) \leq \frac{m}{a^n} \prod_{i=1}^{k} p_i$). Suppose that (x_0, y_0) is also a solution to $G_{j'}(x, y) = h'$, with $j \neq j'$, $0 < h' \leq m$. Then $G_{j'}$ is of the form (17) with $u_1', \ldots, u_k'$ instead of $u_1, \ldots, u_k$ and

$$(X_0', y_0) := \begin{pmatrix} p_1 & u_1' \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} p_k & u_k' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

is a solution to

$$F(x, y) = h' \prod_{i=1}^{k} p_i.$$

If $h = h'$, then we already know (see Section 4.1) that (x_0, y_0) gives at least two solutions to (18) and hence to (20). If $h \neq h'$, then $F(X_0', y_0) \neq F(X_0, y_0)$ so $X_0 \neq X_0'$, and hence also in this case (x_0, y_0) gives two solutions to (20). Therefore, a solution to ℓ inequalities (19) (with $j \in \{1, \ldots, 2^k\}$) gives at least ℓ primitive solutions to (20). By Theorem 2·1, the inequality (20) has at most $32n + 11$ primitive solutions. So there are at least $2^k - 32n - 11$ forms G_j with no solution to (19).

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