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On satellite knots

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1. *Introduction and results*

Throughout this paper we work in the smooth category and assume that all knots are oriented and consider two knots K_1 and K_2 to be equivalent if and only if there is an orientation preserving homeomorphism $h: S^3 \to S^3$ which carries K_1 onto K_2 so that their orientations match. We write $K_1 \cong K_2$ if K_1 and K_2 are equivalent and $-K$ denotes the knot obtained from K by inverting its orientation.

Let V be a standardly embedded solid torus in S^3 with the orientation induced from that of S^3 and K a knot in V, which cannot be contained in a 3-ball in V. For an orientation preserving embedding $f: V \hookrightarrow S^3$, we understand that $f(V)$ and $f(K)$ have orientations induced from those of V and K respectively via f. The pair (V, K) is called a pattern and the minimal geometric intersection number of K and a meridian disc of V is called the wrapping number of K in V and denoted by $\mathrm{wrap}_V(K)$.

The purpose of this paper is to give a complete answer to the following.

PROBLEM. *Let (V, K) be a pattern and $f: V \hookrightarrow S^3$ an orientation preserving embedding such that $f(V)$ is knotted. Determine up to isotopy the orientation preserving embeddings $g: V \hookrightarrow S^3$ such that $g(K) \cong f(K)$.*

Let C_V be an oriented core of a solid torus V in S^3. We choose a preferred meridian-longitude pair (m_V, l_V) of V so that $[l_V] = [C_V] \in H_1(V)$ and $lk(m_V, C_V) = 1$, where $lk(\alpha, \beta)$ denotes the linking number of α and β. Let $f: V \hookrightarrow S^3$ be an orientation preserving embedding, then we adopt $f(C_V)$ as an oriented core of $f(V)$. This determines a preferred meridian-longitude pair $(m_{f(V)}, l_{f(V)})$ of $f(V)$ so that $[l_{f(V)}] = [f(C_V)] \in H_1(f(V))$ and $lk(m_{f(V)}, f(C_V)) = 1$. Then we have an expression $[f(l_V)] = [l_{f(V)}] + n[m_{f(V)}] \in H_1(\partial f(V))$ for some integer n. We define the *twist number* of the embedding f to be this n and denote it by $\mathrm{twist}(f)$. Note that $[f(m_V)] = [m_{f(V)}]$. If $\mathrm{twist}(f) = 0$, the embedding f is said to be *faithful*.

When $\mathrm{wrap}_V(K) = 1$, from Schubert's unique factorization theorem [**9**] we see that $g(K) \cong f(K)$ if and only if $g(C_V) \cong f(C_V)$.

Here we shall prove

THEOREM 1·1. *Let (V, K) be a pattern with $\mathrm{wrap}_V(K) \geqslant 2$ and $f: V \hookrightarrow S^3$ an orientation preserving embedding such that $f(V)$ is knotted. Let $g: V \hookrightarrow S^3$ be an orientation preserving embedding which satisfies $g(K) \cong f(K)$. Then $g(C_V) \cong f(C_V)$, or $f(C_V) \cong K_0 \# K_1$ and $g(C_V) \cong (-K_0) \# K_1$, where K_0 and K_1 are the knots uniquely determined by f and the pattern (V, K). Furthermore, in any case* $\mathrm{twist}(f) = \mathrm{twist}(g)$.

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As a corollary we can estimate the number of isotopy classes of embeddings $V \hookrightarrow S^3$ which define equivalent satellite knots.

COROLLARY 1·2. *Let (V,K) and f be as in Theorem 1·1. Then up to isotopy there is at most one orientation preserving embedding $g: V \hookrightarrow S^3$ which is not isotopic to f and $g(K) \cong f(K)$.*

In Section 4 we shall give examples in which $f(K) \cong g(K)$ but f and g are not isotopic. In particular we can construct an example such that $f(K) \cong g(K)$ but $f(C_V)$ and $g(C_V)$ cannot be equivalent even in the weakest sense (see Example 4·2). Example 4·1 corresponds to the case in Theorem 1·1 where K_1 is trivial and Example 4·2 corresponds to the case where K_1 is non-trivial.

In the final section, we shall treat the special case where K is homologically essential in V. In this case we can prove the following refinement of Theorem 1·1.

THEOREM 1·3. *Let (V,K) be a pattern such that $\mathrm{wrap}_V(K) \geqslant 2$ and K is homologically essential in V, and let $f: V \hookrightarrow S^3$ be an orientation preserving embedding such that $f(V)$ is knotted. If an orientation preserving embedding $g: V \hookrightarrow S^3$ satisfies $g(K) \cong f(K)$, then g is isotopic to f.*

Theorem 1·1 follows from the theorem below, which we shall prove in Sections 2 and 3.

THEOREM 2·1. *Let V be a knotted solid torus in S^3 and K a knot in V with $\mathrm{wrap}_V(K) \geqslant 2$. Let $h: V \hookrightarrow S^3$ be an orientation preserving embedding which satisfies $h(K) = K$. Then $h(C_V) \cong C_V$, or, in terms of a canonical decomposition (defined in Section 3), $C_V \cong K_0 \# K_1$ depending only on (V,K), $h(C_V) \cong (-K_0) \# K_1$. In any twist $(h) = 0$.*

The theory of companionships was introduced and studied by H. Schubert [**10**], and later developed to the theory of torus decompositions by Jaco–Shalen [**5**] and Johannson [**6**]. Our work here is motivated by Schubert's stimulating work, and Example 4·2 is the starting point for Theorem 1·1.

2. *Proof of Theorem 2·1: first step*

Before proving Theorem 2·1 we deduce Theorem 1·1 from it.

Suppose that $g(K) \cong f(K)$, so that we have an orientation preserving homeomorphism ϕ of S^3 which carries $g(K)$ to $f(K)$ such that their orientations match. Consider the orientation preserving embedding $h = \phi \circ g \circ f^{-1}: f(V) \hookrightarrow S^3$ sending $f(K)$ to $f(K)$. Regarding the pair $(f(V), h, f(K), f(C_V))$ as (V, h, K, C_V) in Theorem 2·1, we see that $h(f(C_V)) \cong f(C_V)$ and twist $(h) = 0$. These imply that $g(C_V) \cong f(C_V)$ and twist (f) = twist (g) by the definition of h.

In what follows we prove Theorem 2·1. To begin with we note

LEMMA 2·2 ([**11**]). *Under the assumption of Theorem 2·1, $h(V)$ is also knotted in S^3.*

Proof. If $h(V)$ is unknotted, then $h(K)$ is a preimage knot of K in the sense of [**11**]. By Theorem 1 in [**11**], we have $h(K) \ncong K$. This contradicts the assumption. ∎

Let W be a solid torus in V. We say that W has *property* (*) if the following conditions are satisfied.

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(1) W contains K in its interior,
(2) $\mathrm{wrap}_V(C_W) = 1$, and
(3) C_W is not a core of V.
In the rest of this section we prove

PROPOSITION 2·3. *Let V be a knotted solid torus in S^3 and K a knot in V with $\mathrm{wrap}_V(K) \geqslant 2$. Let $h: V \hookrightarrow S^3$ be an orientation preserving embedding which satisfies $h(K) = K$. If there is no solid torus $W(\subset V)$ satisfying the property (*), then $h(C_V) \cong \epsilon C_V$ ($\epsilon = \pm 1$) and $\mathrm{twist}(h) = 0$.*

Proof. We may assume that $h(N(K)) = N(K)$, and by Lemma 2·2 that $h(V)$ is knotted.

Let us consider the torus decomposition (Jaco–Shalen [**5**] and Johannson [**6**]) of $V - \mathrm{int}N(K)$ with decomposing tori $\{J_1, \ldots, J_n\}$, and denote the piece containing ∂V by P. Then P is either a cable space, a composing space (see [**5**]), or an anannular simple manifold. By Thurston's uniformization theorem [**8**], in the last case P admits a complete hyperbolic structure of finite volume in its interior.

SUBLEMMA 2·4. *P is not a composing space.*

Proof. If P is a composing space (i.e. (disc with holes) $\times S^1$), then the boundary component T' of P separating ∂V and $\partial N(K)$ bounds a solid torus with the property (*). This is a contradiction. ∎

Hence it turns out that $T = \partial V, J_1, \ldots, J_n$ can be extended to a family of tori $\mathfrak{T}$ defining the torus decomposition of $S^3 - \mathrm{int}N(K)$. In addition since $h(V - \mathrm{int}N(K))$ is homeomorphic to $V - \mathrm{int}N(K)$ and $h(V)$ is knotted, $h(T), h(J_1), \ldots, h(J_n)$ can also be extended to a family of tori which defines a torus decomposition of $S^3 - \mathrm{int}N(K)$. Applying the uniqueness of the torus decomposition, after an ambient isotopy of $S^3 - \mathrm{int}N(K)$, we may assume that $h(T), h(J_1), \ldots, h(J_n)$ also belong to the family of tori $\mathfrak{T}$.

In particular we have the following possibilities.

(1) $h(T) = T$.
(2) $h(T) \not\subset V - \mathrm{int}N(K)$.
(3) $h(T) \subset \mathrm{int}(V - \mathrm{int}N(K))$ (in which case $h(T) \in \{J_1, \ldots, J_n\}$).

We further divide (3) into two cases.

(3)(i) $h(T)$ separates T and $\partial N(K)$.
(3)(ii) $h(T)$ does not separate T and $\partial N(K)$.

Claim. Cases (2) and (3)(i) cannot occur.

Verification of Claim. If (2) occurs, then $h(V - \mathrm{int}N(K)) = h(V) - \mathrm{int}N(K)$ contains $V - \mathrm{int}N(K)$ in its interior. We remark that the torus decomposition of $V - \mathrm{int}N(K)$ is given by $\{J_1, \ldots, J_n\}$; meanwhile the family of tori defining the torus decomposition of $h(V - \mathrm{int}N(K)) = h(V) - \mathrm{int}N(K)$ contains at least $\{T, J_1, \ldots, J_n\}$. Considering the cardinal number of them, we get a contradiction by the uniqueness of torus decomposition.

Similarly if (3)(i) occurs, $V - \mathrm{int}N(K)$ contains $h(V - \mathrm{int}N(K)) = h(V) - \mathrm{int}N(K)$ in its interior and $h(T)$ coincides with one of $J_1, \ldots, J_n$, say J_i. Thus the torus

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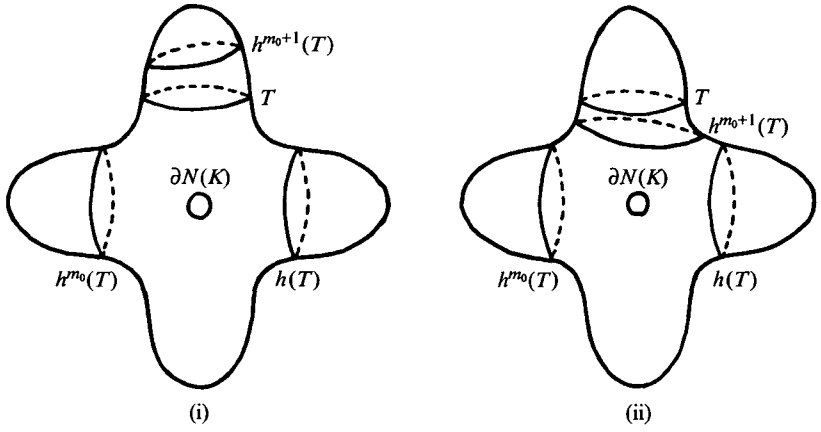


Fig. 2·1.

decomposition of $h(V-\operatorname{int}N(K)) = h(V)-\operatorname{int}N(K)$ is defined by a subfamily of $\{J_1, \ldots, J_{i-1}, J_{i+1}, \ldots, J_n\}$. On the other hand $V-\operatorname{int}N(K)$ is decomposed by $\{J_1, \ldots, J_n\}$. Considering again the cardinal number of these families, we have a contradiction by the uniqueness of the torus decomposition. ∎

We now consider case (1), where $h(V) = V$; this implies that $h(C_V) \cong \pm C_V$.

LEMMA 2·5. $[h(l_V)] = \epsilon[l_V] = [l_{h(V)}]$ $(\epsilon = \pm 1)$.

Proof. We note that $h(P) = P$ and P is not a composing space. Hence P is a cable space or a hyperbolic manifold. First we suppose that P is a cable space. On the boundary component $\partial V(\subset P)$, we consider two simple loops m_V and t, where t denotes an oriented regular fibre of P. It is easy to check that $[t] \neq \pm[m_V]$. Since a Seifert fibration of P is unique up to isotopy, we have $[h(t)] = \epsilon[t]$. Moreover because $h: V \to h(V) = V$ is an orientation preserving homeomorphism, $[h(m_V)] = \epsilon[m_V]$ $(= [m_{h(V)}]$ in our convention). It follows that $[h(l_V)]$ is $\epsilon[l_V]$ $(= [l_{h(V)}]$ in our convention).

Now we suppose that P is hyperbolic. Then the mapping class group $\pi_0(\mathrm{Diff}(P))$ of P is finite ([**6**] 27·1), so $(h|_P)^N$ is isotopic to the identity for some integer $N > 0$. Hence $[h(l_V)]$ is $\epsilon[l_V]$ $(= [l_{h(V)}]$ in our convention). ∎

For the remainder of this section we treat case (3)(ii). We remark that $h: V \to h(V)$ is an orientation preserving homeomorphism and that $h(T)$ is contained in $\operatorname{int}V$. It follows that $h(T) \in \{J_1, \ldots, J_n\}$ and if $h(J_i)$ is contained in V again, then we have $h(J_i) \in \{T, J_1, \ldots, J_n\}$. In this way we can consider $h^k(T)$ for some k.

LEMMA 2·6. $h^m(T) = T$, *for some integer* $m > 0$.

Proof. Since the family $\{J_1, \ldots, J_n\}$ consists of only finitely many tori, we can easily find a positive integer m_0 such that $h^i(T)$ $(1 \leqslant i \leqslant m_0)$ is contained in $\operatorname{int}V$ and does not separate T and $\partial N(K)$, and $h^{m_0+1}(T)$ does not have this property. Now for a contradiction we assume that $h^{m_0+1}(T) \neq T$. Then $h^{m_0+1}(T)$ is contained in $S^3 - V$ (see Figure 2·1 (i)), or contained in $V-\operatorname{int}N(K)$ and separates T $(= \partial V)$ and $\partial N(K)$ (see Figure 2·1 (ii)). In both cases, since $h^{m_0+1}(T)$ and T belong to $\mathfrak{T}$, they are not parallel.

If the former case occurs, then using the orientation preserving homeomorphism $h^{-1}: h(V) \to V$, we can obtain a family of mutually non-parallel incompressible tori $\{h^{-i}(T); 1 \leqslant i \leqslant s\}$. We note that we can take any integer $s > 0$. This contradicts Haken's finiteness theorem ([**3**], [**4**]).

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If we have the latter case, then using the orientation preserving homeomorphism $h: V \to h(V)$, we can obtain a family of mutually non-parallel incompressible tori $\{h^{m_0+i}(T); 1 \leqslant i \leqslant s\}$. Here we can take any integer $s > 0$. This is a contradiction again.

It follows that $h^{m_0+1}(T) = T$. ▌

Let V_i be the solid torus bounded by the torus $T_i = h^i(T)$, and $E_i = S^3 - \operatorname{int} V_i$. Let (m_i, l_i) be a preferred meridian-longitude pair of V_i $(i = 1, \ldots, m-1)$. We assume here $[m_1] = [m_{h(V)}] \in H_1(\partial h(V))$.

LEMMA 2·7. *$h|_{T_i}: T_i \to T_{i+1}$ $(i = 1, \ldots, m-1)$ satisfies $[h|_{T_i}(m_i)] = \epsilon_i[m_{i+1}]$ $(\epsilon_i = \pm 1)$, where we put $T_m = T$.*

Proof. Since the non-trivial knot exterior $E_i = S^3 - \operatorname{int} V_i$ is contained in $\operatorname{int} V$ and T_i $(= h^i(T) = \partial V_i)$ is compressible in V, it follows that T_i is compressible in $V - \operatorname{int} E_i$. Hence T_i has a compressing disc $D_i \subset V - \operatorname{int} E_i$. Thus we can take a meridian disc D_i of V_i which is contained in V. Therefore $h|_{T_i}(m_i)$ is also a meridian of V_{i+1}, that is, $[h|_{T_i}(m_i)] = \epsilon_i[m_{i+1}]$. ▌

Since $h(E_i) = E_{i+1}$, we can get an orientation preserving homeomorphism $h|_{E_{m-1}} \circ \ldots \circ h|_{E_1}$ from $E_1 = S^3 - \operatorname{int} h(V)$ to $E_m = S^3 - \operatorname{int} V$. By Lemma 2·7, this homeomorphism sends $[m_1] = [m_{h(V)}]$ to $\epsilon_1 \ldots \epsilon_{m-1}[m_V]$. Extending this homeomorphism, we get an orientation preserving homeomorphism of S^3 which sends $h(C_V)$ to $\pm C_V$ as claimed in Proposition 2·3.

We now prove that $\operatorname{twist}(h) = 0$, that is, $[h(l_V)] = [l_{h(V)}]$.

We consider an orientation preserving homeomorphism $h|_{V - \operatorname{int}(E_1 \cup \ldots \cup E_{m-1})}$: $V - \operatorname{int}(E_1 \cup \ldots \cup E_{m-1}) \to V - \operatorname{int}(E_1 \cup \ldots \cup E_{m-1})$. We use the same symbol h to denote its restriction.

Now let us assume $[h(l_V)] = [l_{h(V)}] + \alpha[m_{h(V)}] = [l_1] + \alpha[m_1]$.

Since E_i $(i = 1, \ldots, m-1)$ contains an orientable surface F_i with $\partial F_i = l_i$, we have $[h(l_1)] = \epsilon_1[l_2], \ldots, [h(l_{m-1})] = \epsilon_{m-1}[l_V]$. In addition we have

$$[h(m_V)] = [m_{h(V)}] = [m_1]$$

and by Lemma 2·7 $[h(m_1)] = \epsilon_1[m_2], \ldots, [h(m_{m-1})] = \epsilon_{m-1}[m_V]$. Hence

$$[h^m(m_V)] = \epsilon_1 \ldots \epsilon_{m-1}[m_V]$$

and $[h^m(l_V)] = \epsilon_1 \ldots \epsilon_{m-1}([l_V] + \alpha[m_V])$. We use the torus decomposition of

$$V - \operatorname{int}(E_1 \cup \ldots \cup E_{m-1}) - \operatorname{int} N(K),$$

and let P be the piece which contains ∂V.

SUBLEMMA 2·8. *P is not a composing space.*

Proof. This is essentially the same as that of Sublemma 2·4.

If P is a composing space, then a boundary component T' of P separating ∂V and $\partial N(K)$ bounds a solid torus W. It is easy to see that W satisfies property $(*)$, a contradiction. ▌

Thus P is a cable space or a hyperbolic manifold. We remark that $h^m(P) = P$. Applying the argument in the proof of Lemma 2·5 to the homeomorphism $h^m: P \to P$, we see that $[h^m(l_V)] = \epsilon[l_V]$ and hence $[h(l_V)] = [l_{h(V)}]$. This completes the proof of Proposition 2·3. ▌

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3. *Proof of Theorem* 2·1: *second step* (*Canonical decompositions of companion knots*)

In this section we treat the case where a solid torus W ($\subset \operatorname{int} V$) satisfying property (*) exists.

We say that a solid torus W ($\subset \operatorname{int} V$) satisfying property (*) is (*)-*minimal* if there is no further solid torus W' ($\subset \operatorname{int} W$) satisfying property (*) for W.

LEMMA 3·1. *If there is a solid torus satisfying property* (*) *in* V, *then there exists a* (*)-*minimal solid torus* W *in* V, *unique up to isotopy.*

Proof. The existence of such a solid torus is ensured by Haken's finiteness theorem.

Let W_1 and W_2 be two (*)-minimal solid tori in V. Then the component of the torus decomposition of $V - \operatorname{int} W_i$ ($i = 1, 2$) containing ∂V is a composing space, which also contains ∂W_i. The (*)-minimality of W_i implies that ∂W_i belongs to a family of tori which defines a torus decomposition of $V - \operatorname{int} N(K)$. It follows from the uniqueness of the torus decomposition that we can isotope ∂W_2 so that $\partial W_2 = \partial W_1$. This implies also that by an isotopy W_2 can be deformed into W_1, and completes the proof. ∎

Let W be a (*)-minimal solid torus in V. Let C_W be an oriented core of W satisfying $[C_W] = [C_V] \in H_1(V)$. Then we have a description $C_W \cong C_V \# k$ for some non-trivial knot k (see Figure 3·1).

Remark 3·2. By Lemma 3·1, (V, K) uniquely determines knots C_V, C_W and k, which are related by $C_W \cong C_V \# k$.

Now we define a canonical decomposition of C_V which depends only on the pair (V, K).

If there is no solid torus W in V satisfying property (*), then we put $K_0 = C_V$ and a canonical decomposition of C_V is defined to be $C_V \cong K_0$.

If there is a solid torus W in V satisfying property (*), then we take a (*)-minimal solid torus W and choose an orientation of C_W so that $[C_W] = [C_V] \in H_1(V)$; then $C_W \cong C_V \# k$ for some knot k (see Figure 3.1). Let $k \cong k_1 \# \ldots \# k_n$ be a prime decomposition of k. We delete all the invertible factors and all the pairs (k_i, k_j) with $k_i \cong -k_j$. Then we get $k_1 \# \ldots \# k_m$ (re-indexing if necessary). We put $K_1 \cong -(k_1 \# \ldots \# k_m)$. (Possibly K_1 is a trivial knot.) If C_V has an expression $C_V \cong K_0 \# K_1$ for some K_0, then we define a canonical decomposition of C_V to be $C_V \cong K_0 \# K_1$. If C_V admits no such expression we put $K_0 = C_V$ and define the canonical decomposition of C_V to be $C_V \cong K_0$.

Remark 3·3. By Remark 3·2 a canonical decomposition of C_V depends only on (V, K).

Since W is a (*)-minimal solid torus, we can apply Proposition 2·3 to W and see that $h(C_W) \cong \epsilon C_W$ ($\epsilon = \pm 1$) and $h|_W : W \to S^3$ is faithful.

LEMMA 3·4. (i) *If* $h(C_V) \ncong C_V$, *then in terms of the canonical decomposition* $C_V \cong K_0 \# K_1$, *we have* $h(C_V) \cong (-K_0) \# K_1$.

(ii) $\operatorname{twist}(h) = 0$.

Proof. We recall that $C_W \cong C_V \# k$ and $h(C_W) \cong \epsilon C_W$ ($\epsilon = \pm 1$). Hence

$$h(C_V \# k) \cong \epsilon(C_V \# k).$$

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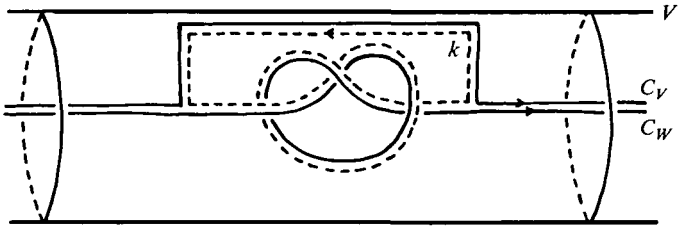


Fig. 3·1.

Moreover since the knot k is contained in a 3-ball in V, $h(C_V \# k) \cong h(C_V) \# k$. Thus we have $h(C_V) \# k \cong \epsilon(C_V \# k)$.

If $\epsilon = 1$, then $h(C_V) \# k \cong C_V \# k$ and so $h(C_V) \cong C_V$. Therefore $\epsilon = -1$ and $h(C_V) \# k \cong (-C_V) \# (-k)$. Let $k \cong k_1 \# \ldots \# k_n$ be a prime decomposition of k. Then $h(C_V) \# k_1 \# \ldots \# k_n \cong (-C_V) \# (-k_1) \# \ldots \# (-k_n)$. If k_l is invertible, that is, $k_l \cong -k_l$, then we can delete k_l from both sides, and do so for as many l as possible. In addition if a pair (k_i, k_j), $i \neq j$, satisfies $k_i \cong -k_j$, then we have simultaneously $k_j \cong -k_i$. Then we can delete $k_i \cong -k_j$, and $k_j \cong -k_i$ from both sides and do so as often as possible. As a result we have $h(C_V) \# k_1 \# \ldots \# k_m \cong (-C_V) \# (-k_1) \# \ldots \# (-k_m)$, where we change indices if necessary. Note that K_1 in the canonical decomposition of C_V is this $(-k_1) \# \ldots \# (-k_m)$, and we have $h(C_V) \# (-K_1) \cong (-C_V) \# K_1$. Since the prime factorization of $h(C_V) \# (-K_1)$ is unique, it follows that $-C_V \cong (-K_1) \# K'$ and $h(C_V) \cong K_1 \# K''$ for some K' and K'', and furthermore $K' \cong K''$. Putting here $K_0 = -K' \cong -K''$ and $C_V \cong (-K') \# K_1 = K_0 \# K_1$, which is just a canonical decomposition, we have $h(C_V) \cong (-K_0) \# K_1$. This establishes (i).

For (ii), we first note that

$$[l_V] = [l_W] \in H_1(V - \operatorname{int} W) \quad \text{and} \quad [l_{h(V)}] = [l_{h(W)}] \in H_1(h(V) - \operatorname{int} h(W)).$$

We note also that $[h(l_W)] = [l_{h(W)}]$. It follows that in

$$H_1(h(V) - \operatorname{int} h(W)) = H_1(h(V - \operatorname{int} W))$$

we have $[h(l_V)] = [h(l_W)] = [l_{h(W)}] = [l_{h(V)}]$ and hence

$$[h(l_V)] = [l_{h(V)}] \in H_1(\partial h(V)),$$

that is, $\operatorname{twist}(h) = 0$. This completes the proof. ∎

Combining Proposition 2·3 and Lemma 3·4, we get Theorem 2·1, and hence Theorem 1·1. Corollary 1·2 follows immediately from Theorem 1·1 together with

PROPOSITION 3·5. *Let $f, g: V \hookrightarrow S^3$ be orientation preserving embeddings. Then f and g are isotopic if and only if $f(C_V) \cong g(C_V)$ and* $\operatorname{twist}(f) = \operatorname{twist}(g)$.

Proof. The 'only if' part is obvious. The 'if' part follows from the fact that $\operatorname{Diff}(S^1 \times D^2 \operatorname{rel} \partial(S^1 \times D^2))$ is contractible [**2**]. (Here we use only the connectivity of the above space, originally proved by Cerf [**1**].) ∎

4. *Examples*

A pattern (V, K) is said to be *symmetric* if (V, K) admits an orientation preserving homeomorphism $\psi: V \to V$ which satisfies $[\psi(C_V)] = -[C_V] \in H_1(V)$ and $\psi(K) = K$. Let s be the π-rotation along the axis L as shown in Figure 4.1.

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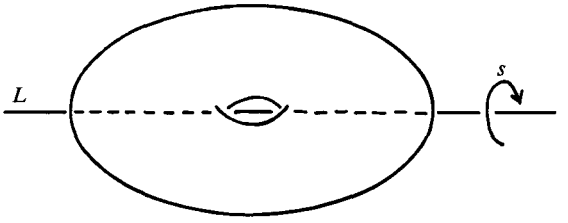


Fig. 4·1.

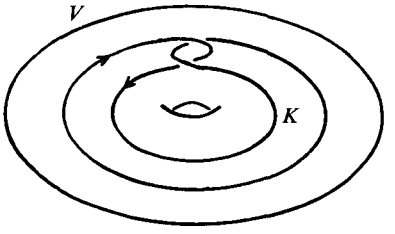


Fig. 4·2.

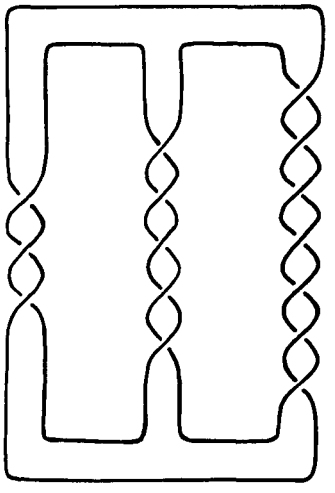


Fig. 4·3.

The pattern (V,K) given by Figure 4·2 is symmetric. In fact, for some homeomorphism ϕ isotopic to the identity $\phi \circ s$ gives the symmetry of (V,K).

For any symmetric pattern we can observe:

EXAMPLE 4·1. *Let (V,K) be a symmetric pattern and $f: V \hookrightarrow S^3$ an orientation preserving embedding such that $f(C_V)$ is a non-invertible knot (i.e. $f(C_V) \ncong -f(C_V)$).*

Then for two embeddings f and $g = f \circ s$, we have $f(K) \cong g(K)$ and $f(C_V) \ncong g(C_V)$. In particular f and g are not isotopic.

Proof. By the definition of g we have $g(C_V) \cong -f(C_V)$. Hence $f(C_V) \ncong g(C_V)$. Using the symmetry of (V,K) we can verify that $f(K) \cong g(K)$. ∎

To get a concrete example we may take the pattern given by Figure 4·2 and the embedding $f: V \hookrightarrow S^3$ so that $f(C_V)$ is the pretzel knot $K(3,5,7)$ (see Figure 4·3), which is known to be non-invertible [**12**].

Example 4·1 corresponds to the case in Theorem 1·1 where K_1 is trivial.

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EXAMPLE 4·2. *Let (V, k) be a pattern such that* $\mathrm{wrap}_V(k) = 1$ *and* $k \cong K(3,5,7)$ *in* S^3. *(For simplicity we assume that* $[k] = [C_V] \in H_1(V)$.*) Let (V, K) be a pattern such that K is the untwisted double of k. Let f (respectively g) be an orientation preserving embedding from V into S^3 so that*

$$f(C_V) \cong (-k) \# (-k) \quad (\textit{respectively } g(C_V) \cong k \# (-k)).$$

Then for two embeddings f and g, we have $f(K) \cong g(K)$. But $f(C_V)$ and $g(C_V)$ cannot be equivalent even in the weakest sense.

Proof. We note that since $k \cong K(3,5,7)$ is non-invertible and non-amphicheiral, no two of k, $-k$, k^*, $-k^*$ are equivalent, where k^* denotes the mirror image of k (see [**12**]). In addition k is of genus one and so it is prime. Suppose that there is a homeomorphism of S^3 carrying $f(C_V)$ onto $\epsilon g(C_V)$ $(\epsilon = \pm 1)$. Then we have $(-k) \# (-k) \cong (-\epsilon k) \# (\epsilon k)$ or $(-k) \# (-k) \cong (-\epsilon k^*) \# (\epsilon k^*)$. In any case $-k \cong k$ or $k \cong k^*$ must hold by Schubert's unique factorization theorem. This is a contradiction. Therefore $f(C_V)$ and $g(C_V)$ are not equivalent in the weakest sense.

To see that $f(K) \cong g(K)$, we note that $f(K)$ is the untwisted double of $(-k) \# (-k) \# k$ and $g(K)$ is the untwisted double of $k \# (-k) \# k \cong -((-k) \# (-k) \# k)$. The equation follows from using the symmetry of the pattern given by Figure 4·2. ∎

In this example, the canonical decomposition of $f(C_V)$ is given by

$$f(C_V) \cong (-k) \# (-k)$$

so that K_1 in Theorem 1·1 can be non-trivial.

Example 4·2 shows that the theorem in [**7**] and hence also the corollary in [**11**] are not true.

5. *Satellite knots with homologically essential patterns*

In Section 4 we gave examples in which $f(K) \cong g(K)$ but f and g are not isotopic. In both examples, K is homologically trivial in V, that is $[K] = 0 \in H_1(V)$. The purpose of this final section is to prove that in all other cases f and g are isotopic.

THEOREM 1·3. *Let (V, K) be a pattern such that* $\mathrm{wrap}_V(K) \geqslant 2$ *and K is homologically essential in V. Let $f: V \hookrightarrow S^3$ be an orientation preserving embedding such that $f(V)$ is knotted. Suppose that $g: V \hookrightarrow S^3$ is an orientation preserving embedding which satisfies $g(K) \cong f(K)$. Then g is isotopic to f.*

As in the proof of Theorem 1·1 it suffices to prove the following proposition corresponding to Theorem 2·1. Under the assumption of Theorem 1·3, Theorem 2·1 can be improved as follows.

PROPOSITION 5·1. *Let V be a knotted solid torus in S^3 and K a knot in V such that* $\mathrm{wrap}_V(K) \geqslant 2$ *and* $[K] \neq 0 \in H_1(V)$. *Let $h: V \hookrightarrow S^3$ be an orientation preserving embedding which satisfies $h(K) = K$. Then $h(C_V) \cong C_V$ and* $\mathrm{twist}(h) = 0$.

Proof. We explain only the main difference from the proof of Theorem 2·1.

In the proof of Proposition 2·3, if case (1) occurs, then we have $h(V) = V$. Since $h(K) = K$, the homological essentiality of K in V implies $h(C_V) \cong C_V$.

Also in case (3)(ii), the homological essentiality of K in V implies that the

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homeomorphism $h|_{E_{m-1}} \circ \ldots \circ h|_{E_1} : E_1 \to E_m$ sends $[m_1] = [m_{h(V)}]$ to $[m_V]$. Hence this homeomorphism can be extended to an orientation preserving homeomorphism of S^3 which carries $h(C_V)$ to C_V. It follows that we can conclude $h(C_V) \cong C_V$ in Proposition 2·3.

In the second step, instead of Lemma 3·4 we can prove $h(C_V) \cong C_V$ as follows.

Let W be a $(*)$-minimal solid torus in V. As in the proof of Lemma 3·3, we have $C_W \cong C_V \# k$ for some knot k. Applying the result in the first step for W, we get $h(C_V \# k) \cong C_V \# k$. This implies that $h(C_V) \# k \cong C_V \# k$ and hence $h(C_V) \cong C_V$.

As for the twist number of h, we already know twist $(h) = 0$ by Theorem 2·1. This completes the proof. ∎

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REFERENCES

[1] J. CERF. *Sur les difféomorphismes de la sphère de dimension trois* ($\Gamma_4 = 0$). Lectures Notes in Math. **53** (Springer-Verlag, 1968).
[2] A. HATCHER. A proof of the Smale Conjecture. Diff$(S^3) \simeq O(4)$. *Ann. of Math.* **117** (1983), 553–607.
[3] J. HEMPEL. 3-*manifolds*. Ann. of Math. Studies 86 (Princeton Univ. Press, 1976).
[4] W. JACO. *Lectures on three manifold topology*. Conference board of Math. Science, Regional Conference Series in Math. **43** (Amer. Math. Soc., 1980).
[5] W. JACO and P. SHALEN. Seifert fibered spaces in 3-manifolds. *Mem. Amer. Math. Soc.* **220** (1979).
[6] K. JOHANNSON. *Homotopy equivalences of 3-manifolds with boundaries*. Lectures Notes in Math. **761** (Springer-Verlag, 1979).
[7] M. KOUNO. On knots with companions. *Kobe J. Math.* **2** (1985), 143–148.
[8] J. MORGAN and H. BASS. *The Smith conjecture* (Academic Press, 1984).
[9] H. SCHUBERT. Die eindeutige Zerlegbarkeit eines Knoten in Primknoten. *Sitzungsber. Akad. Wiss. Heidelberg*, math. nat. KI. 1949, 3. Abh., 57–104.
[10] H. SCHUBERT. Knoten und Vollringe. *Acta Math.* **90** (1953), 131–286.
[11] T. SOMA. On preimage knots in S^3. *Proc. Amer. Math. Soc.* **100** (1987), 589–592.
[12] H. F. TROTTER. Non-invertible knots exist. *Topology* **2** (1964), 275–280.

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