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Monoidal categorification and quantum affine algebras

Masaki Kashiwara, Myungho Kim, Se-jin Oh and Euiyong Park

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Abstract

We introduce and investigate new invariants of pairs of modules M and N over quantum affine algebras $U_q'(\mathfrak{g})$ by analyzing their associated R-matrices. Using these new invariants, we provide a criterion for a monoidal category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules to become a monoidal categorification of a cluster algebra.

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1. Introduction

For an affine Kac–Moody algebra $\mathfrak{g}$, let $U_q'(\mathfrak{g})$ be the corresponding quantum affine algebra. Since the category $\mathscr{C}_{\mathfrak{g}}$ of finite-dimensional integrable representations over $U_q'(\mathfrak{g})$ has a rich structure including rigidity, it has been intensively studied in various fields of mathematics and physics (see for example [AK97, CP94, FR99, GV93, KS95, Nak01]). In particular, the representation theory for $\mathscr{C}_{\widehat{\mathfrak{sl}}_2}$ is well understood: every simple module in $\mathscr{C}_{\widehat{\mathfrak{sl}}_2}$ is isomorphic to a tensor product $S_1 \otimes S_2 \otimes \cdots \otimes S_r$ of simple modules, called *Kirillov–Reshetikhin* modules, satisfying that S_i and S_j *are in general position* [CP91]. A simple object S of a monoidal category is said to be *real* if $S \otimes S$ is simple, and to be *prime* if there exists no non-trivial factorization $S \simeq S_1 \otimes S_2$. Since Kirillov–Reshetikhin modules over $U_q'(\widehat{\mathfrak{sl}}_2)$ are prime and real, every simple module in $\mathscr{C}_{\widehat{\mathfrak{sl}}_2}$ is real and can be expressed as a tensor monomial of prime real simple modules. However, these phenomena cannot be expected for general $\mathfrak{g}$. In fact, there exist $\mathfrak{g}$ such that $\mathscr{C}_{\mathfrak{g}}$ contains non-real simple modules [Lec03].

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The cluster algebras were introduced by Fomin and Zelevinsky in [FZ02] for studying the upper global bases of quantum groups and total positivity [Kas90, Lus90] in the viewpoint of combinatorics. Since their introduction, numerous connections and applications have been discovered in diverse fields of mathematics including representation theory, tropical geometry, integrable system and Poisson geometry (see [BM06, FG06, FZ03, FR05, GSV03]).

The representation theory of quantum affine algebras and the cluster algebras are connected by the notion of *monoidal categorification*, introduced by Hernandez and Leclerc in [HL10]. A monoidal category $\mathcal{C}$ is called a monoidal categorification of a cluster algebra $\mathscr{A}$ if it satisfies:

(a) the Grothendieck ring $K(\mathcal{C})$ of $\mathcal{C}$ is isomorphic to $\mathscr{A}$; and

(b) each cluster monomial of $\mathscr{A}$ corresponds to a real simple object in $\mathcal{C}$, under the isomorphism.

(This definition is weaker than the original one.) Note that, by the *Laurent phenomenon* of $\mathscr{A}$ [FZ02], the *Laurent positivity* (proved by [LS15] in a general setting) follows immediately, if $\mathcal{C}$ is a monoidal categorification of $\mathscr{A}$.

The notion of a monoidal categorification is extended in [KKKO18] to quantum cluster algebras, a q-analogue of cluster algebras, which were introduced by Berenstein and Zelevinsky in [BZ05]. Unlike cluster algebras, cluster variables are not commutative but q-commutative, where the q-commutation relation is controlled by a skew-symmetric matrix L. In [GLS13a], Geiß, Leclerc and Schröer showed that the quantum unipotent coordinate algebra $A_q(\mathfrak{n}(w))$, associated with a symmetric quantum group $U_q(\mathfrak{g})$ and its Weyl group element w, has a skew-symmetric quantum cluster algebra structure (see [GY16] for the non-symmetric case). Using the quiver Hecke algebras R introduced by Khovanov and Lauda [KL09, KL11] and Rouquier [Rou08, Rou12] independently, the authors in [KKKO18] introduced certain monoidal subcategory $\mathcal{C}_w$ of the category R-gmod of finite-dimensional graded modules over R and proved that $\mathcal{C}_w$ gives a monoidal categorification for $A_q(\mathfrak{n}(w))$ in the following sense:

(i) $\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C}_w) \simeq A_{q^{1/2}}(\mathfrak{n}(w)) := \mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} A_q(\mathfrak{n}(w))$;

(ii) there exists a *quantum monoidal seed* $\mathscr{S} = (\{V_i\}_{i\in K}, L, \widetilde{B}, D)$ in $\mathcal{C}_w$, consisting of a strongly commuting family $\{V_i\}_{i\in K}$ of real simple modules in $\mathcal{C}_w$, the $K \times K$ $\mathbb{Z}$-valued matrix $L = (-\Lambda(V_i, V_j))_{i,j\in K}$, an exchange matrix $\widetilde{B}$ and a set D of *weights* of the modules V_i in the root lattice of $\mathfrak{g}$, such that $[\mathscr{S}] := (\{q^{m_i}[V_i]\}_{i\in K}, L, \widetilde{B})$ is a quantum seed of $A_{q^{1/2}}(\mathfrak{n}(w))$ for some $m_i \in \frac{1}{2}\mathbb{Z}$; (1.1)

(iii) $\mathscr{S}$ *admits successive mutations in all directions in* K^{ex}.

Here $\Lambda(V, W)$ denotes the degree of the R-matrix $\mathbf{r}_{V,W}$, constructed in [KKK18], which is a distinguished homomorphism from $V \circ W$ to $W \circ V$, where $V \circ W$ denotes the convolution product of V and W (see [KKKO18] for notations). Note that the condition (iii) in (1.1) is not easy to check since it is concerned with infinitely many mutations. In the first part of [KKKO18], it was proved that the conditions (ii) and (iii) in (1.1) are a consequence of the following condition:

(ii′) there exists an *admissible* monoidal seed $\mathscr{S} = (\{V_i\}_{i\in K}, \widetilde{B})$ in $\mathcal{C}_w$ such that $[\mathscr{S}] := (\{q^{m_i}[V_i]\}_{i\in K}, (-\Lambda(V_i, V_j))_{i,j\in K}, \widetilde{B})$ is a quantum seed of $A_{q^{1/2}}(\mathfrak{n}(w))$ for some $m_i \in \frac{1}{2}\mathbb{Z}$ (see Definition 6.3).

Here, the admissibility means that the monoidal seed admits the first step mutations. Thus, (ii′) implies that, to achieve a monoidal categorification, it suffices to check the existence of such M'_k only at the first mutation in each direction k.

On the whole flow of [KKKO18], the integer-valued invariants $\Lambda(V, W)$, $\widetilde{\Lambda}(V, W)$ and $\mathfrak{d}(V, W)$, arising from the $\mathbb{Z}$-grading structure of R and defined in [KKK18, KKKO18], provide important

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information in the representation theory of quiver Hecke algebra R. To name a few: (i) Λ provides information about whether the restriction of R-matrix $\mathbf{r}_{V,W}$ to $V' \circ W'$ vanishes or not for subquotients V' and W' of V and W respectively; (ii) Λ indicates the head and socle in the constituent of $V \circ W$; (iii) $\widetilde{\Lambda}(V,W) := \frac{1}{2}(\Lambda(V,W) + (\mathrm{wt}(V), \mathrm{wt}(W)))$ measures the degree shifts of $V \circ W$ from the *self-duality*; (iv) the non-negative integer $\mathfrak{d}(V,W) := \frac{1}{2}(\Lambda(V,W) + \Lambda(W,V))$ tells whether $V \circ W$ is simple or not, corresponding to $\mathfrak{d}(V,W) = 0$ or $\mathfrak{d}(V,W) > 0$, under suitable assumptions on V and W. However, in general, computing those values are quite difficult, and the second half of [KKKO18] is devoted to investigating several properties of those invariants.

After the success in the quiver Hecke algebras setting, it is natural to ask a criterion for monoidal categorification for subcategories of $\mathscr{C}_{\mathfrak{g}}$. There are monoidal subcategories $\mathscr{C}_N$ ($N \in \mathbb{Z}_{\geqslant 1}$), $\mathscr{C}_{\mathfrak{g}}^-$ and $\mathscr{C}_{\mathfrak{g}}^0$ of $\mathscr{C}_{\mathfrak{g}}$, introduced by Hernandez and Leclerc in [HL10, HL16] (see also [KKOP19b] for $\mathscr{C}_{\mathfrak{g}}^0$), whose Grothendieck rings $K(\mathcal{C})$ have cluster algebra structures, and which are conjectured to be monoidal categorifications of $K(\mathcal{C})$. The conjecture for $\mathscr{C}_{\mathfrak{g}}^0$ of affine types $A_n^{(t)}$ ($t = 1, 2$) and $B_n^{(1)}$ is proved in [KKOP19b] *indirectly* by using generalized quantum Schur–Weyl duality constructed in [KKK18, KKKO15b, KKO19], and for $\mathscr{C}_1$ and $\mathscr{C}_N$ ($N \in \mathbb{Z}_{\geqslant 1}$) of untwisted affine types ADE are proved in [HL10, HL13, Nak11] and [Qin17] respectively, by approaches different from [KKOP19b]. However, by the lack of $\mathbb{Z}$-grading structure on $U_q'(\mathfrak{g})$, one can *not* apply the framework in [KKKO18] to those categories for monoidal categorifications directly (see also [CW19, § 4]).

The aim of this paper is to give a criterion for a monoidal subcategory $\mathcal{C}$ of $\mathscr{C}_{\mathfrak{g}}$ to become a monoidal categorification of $K(\mathcal{C})$. We first introduce new invariants, denoted also by $\Lambda(M,N)$, $\widetilde{\Lambda}(M,N)$, $\Lambda^{\infty}(M,N)$ and $\mathfrak{d}(M,N)$, for a pair of modules M and N in $\mathscr{C}_{\mathfrak{g}}$, by analyzing R-matrices associated to $M \otimes N_z$.

We say that the universal R-matrix

$$R^{\mathrm{univ}}_{M,N_z} \colon \mathbf{k}((z)) \otimes_{\mathbf{k}[z^{\pm 1}]} (M \otimes N_z) \to \mathbf{k}((z)) \otimes_{\mathbf{k}[z^{\pm 1}]} (N_z \otimes M)$$

is *rationally renormalizable* if there exists $c_{M,N}(z) \in \mathbf{k}((z))^{\times}$ such that $R^{\mathrm{ren}}_{M,N_z} := c_{M,N}(z) R^{\mathrm{univ}}_{M,N_z}$ sends $M \otimes N_z$ to $N_z \otimes M$. In such a case, we can normalize $c_{M,N}(z) \in \mathbf{k}((z))^{\times}$ (up to multiplication by an element of $\mathbf{k}[z^{\pm 1}]^{\times}$) such that $R^{\mathrm{ren}}_{M,N_z}|_{z=x} \colon M \otimes N_x \to N_x \otimes M$ does not vanish at any $x \in \mathbf{k}^{\times}$. We call $c_{M,N}(z)$ the *renormalizing coefficient* of M and N. We define $\widetilde{\Lambda}(M,N)$ as the order of zero of $c_{M,N}(z)$ at $z = 1$. We then define $\Lambda(M,N)$, $\Lambda^{\infty}(M,N)$ and $\mathfrak{d}(M,N)$ similarly to $\widetilde{\Lambda}(M,N)$ (see Definition 3.6 for new invariants). Note that $\Lambda^{\infty}(M,N) = 2\widetilde{\Lambda}(M,N) - \Lambda(M,N)$ can be understood as a quantum affine analogue of $(\mathrm{wt}(V), \mathrm{wt}(W))$.

When M and N are simple modules in $\mathscr{C}_{\mathfrak{g}}$, $c_{M,N}(z)$ is the ratio $d_{M,N}(z)$ to $a_{M,N}(z)$, where $d_{M,N}(z)$ (respectively $a_{M,N}(z)$) denotes the *denominator* (respectively *universal coefficient*) of the *normalized R-matrix* $R^{\mathrm{norm}}_{M,N_z}(z)$ of M and N, computed in [AK97, DO94, KKK15, Oh15, OS19] for fundamental representations. Thus $\mathfrak{d}(M,N)$ can be interpreted as the degree of zero of $d_{M,N}(z) d_{N,M}(z^{-1})$ at $z = 1$ with the results in [AK97] (see § 2.2).

We next investigate several properties of the new invariants by using R-matrices and their coefficients, and prove that they play the same role in the representation theory for quantum affine algebras as the ones for quiver Hecke algebras do. Furthermore, new invariants provide more information arising from taking duals in $\mathscr{C}_{\mathfrak{g}}$, which *cannot* be obtained in the quiver Hecke algebra setting (see Remark 3.21). For instances, we have that:

- $\Lambda(M,N)$ and $\Lambda^{\infty}(M,N)$ can be expressed in terms of $\mathfrak{d}(M, \mathscr{D}^k N)$ for $k \in \mathbb{Z}$;
- $\Lambda^{\infty}(M,N) = \Lambda^{\infty}(N,M) = -\Lambda^{\infty}(M^*,N) = -\Lambda^{\infty}({}^*M,N)$;
- $\Lambda^{\infty}(M,N) = -\Lambda(M, \mathscr{D}^{2n}N) = \Lambda(M, \mathscr{D}^{-2n}N)$ for $n \gg 0$;

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- $\Lambda(M,N)=\Lambda(N^*,M)=\Lambda(N,{}^*M)$ and hence $\mathfrak{d}(M,N)=\frac{1}{2}(\Lambda(M,N)+\Lambda(N,M))=\frac{1}{2}(\Lambda(M,N)+\Lambda(M^*,N))$,

where N^* (respectively ${}^*N$ and $\mathscr{D}^k N$) denotes the left (respectively right and kth left) dual of N (see § 3).

With the new invariants at hand, we introduce the following notions: (a) a Λ-*seed* $\mathcal{S}_\Lambda$, a triple $\mathcal{S}_\Lambda=(\{X_i\}_{i\in K},L,\widetilde{B})$ consisting of a cluster $\{X_i\}_{i\in K}$, a skew-symmetric $K\times K$-matrix L and a $K\times K^{\mathrm{ex}}$-matrix $\widetilde{B}=(b_{jk})$ such that $(L\widetilde{B})_{ij}=2\delta_{ij}$; and (b) a cluster algebra *associated to* $\mathcal{S}_\Lambda$. Here the mutation rule for the pair $(L,\widetilde{B})$ associated to $\mathcal{S}_\Lambda$ is the same as the ones for quantum cluster algebras.

Finally, we introduce the notion of a Λ-*admissible monoidal seed* in a monoidal subcategory $\mathcal{C}$ of $\mathscr{C}_{\mathfrak{g}}$ by using the new invariants as follows. A monoidal seed $\mathscr{S}=(\{M_i\}_{i\in K},\widetilde{B})$ is said to be Λ-admissible if it satisfies:

(a) $(\Lambda^{\mathscr{S}}\widetilde{B})_{jk}=-2\delta_{jk}$ where $\Lambda^{\mathscr{S}}:=(\Lambda(M_i,M_j))_{i,j\in K}$;

(b) for each $k\in K^{\mathrm{ex}}$, there exists a real simple module M_k' in $\mathcal{C}$, corresponding to the mutated cluster variable X_k', satisfying $\mathfrak{d}(M_j,M_k')=\delta_{jk}$ and the short exact sequence

$$0\to\bigotimes_{b_{ik}>0}M_i^{\otimes b_{ik}}\to M_k\otimes M_k'\to\bigotimes_{b_{ik}<0}M_i^{\otimes(-b_{ik})}\to 0.$$

By employing the framework of [KKKO18, § 7] with new invariants and notions, we prove the main result of this paper.

Main Theorem (Theorem 6.10). *For a monoidal seed $\mathscr{S}=(\{M_i\}_{i\in K},\widetilde{B})$ in a monoidal subcategory $\mathcal{C}$ of $\mathscr{C}_{\mathfrak{g}}$, assume the following conditions.*

- *The Grothendieck ring $K(\mathcal{C})$ of $\mathcal{C}$ is isomorphic to the cluster algebra $\mathscr{A}$ associated to the initial seed $[\mathscr{S}]:=(\{[M_i]\}_{i\in K},\widetilde{B})$.*
- *The monoidal seed $\mathscr{S}$ is Λ-admissible.*

Then the category $\mathcal{C}$ is a monoidal categorification of the cluster algebra $\mathscr{A}$.

As consequences, we can obtain the following applications (Corollary 6.11).

(i) For $k\in K^{\mathrm{ex}}$ and the kth cluster variable module $\widetilde{M}_k$ of a monoidal seed $\widetilde{\mathscr{S}}$ obtained by successive mutations from the initial monoidal seed $\mathscr{S}$, we have $\mathfrak{d}(\widetilde{M}_k,\widetilde{M}_k')=1$.

(ii) Any monoidal cluster $\{\widetilde{M}_i\}_{i\in K}$ is a maximal real commuting family in $\mathcal{C}$ (see Definition 6.8).

In the forthcoming paper, we will apply the main theorem to certain monoidal subcategories $\mathcal{C}$ of $\mathscr{C}_{\mathfrak{g}}$ for providing monoidal categorifications.

This paper is organized as follows. We give the necessary background on quantum affine algebras, their representations, and R-matrices, their related coefficients in § 2. In §§ 3 and 4, we introduce new invariants for pairs of $U_q'(\mathfrak{g})$-modules by using R-matrices and investigate their properties. Especially, we will show the similarities of new invariants with the ones for quiver Hecke algebras in § 4. In § 5, we briefly recall the definition of cluster algebras with the consideration on Λ-seeds. In § 6, we prove our main result with newly introduced invariants and notions.

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2. Preliminaries

Convention 2.1.

(i) For a statement P, $\delta(P)$ is 1 or 0 according to whether P is true or not.

(ii) For a field $\mathbf{k}$, $a \in \mathbf{k}$ and $f(z) \in \mathbf{k}(z)$, we denote by $\mathrm{zero}_{z=a} f(z)$ the order of zero of $f(z)$ at $z = a$. In particular, $\mathrm{zero}_{z=a} f(z) = -k$ for $k \in \mathbb{Z}_{\geqslant 1}$ implies that $f(z)$ has a pole of order k at $z = a$.

2.1 Quantum affine algebras

Let $(\mathsf{A}, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$ be an *affine Cartan datum*. It consists of an affine Cartan matrix $\mathsf{A} = (a_{ij})_{i,j\in I}$ with a finite index set I, a free abelian group P of rank $|I| + 1$, called the *weight lattice*, a set $\Pi = \{\alpha_i \in \mathsf{P} \mid i \in I\}$ of linearly independent elements called *simple roots*, the group $\mathsf{P}^\vee := \mathrm{Hom}_{\mathbb{Z}}(\mathsf{P}, \mathbb{Z})$ called the *coweight lattice*, and a set $\Pi^\vee = \{h_i \mid i \in I\} \subset \mathsf{P}^\vee$ of *simple coroots*. Note that the pairing $\langle\ ,\ \rangle$ between $\mathsf{P}^\vee$ and P satisfies $\langle h_i, \alpha_j\rangle = a_{ij}$ for all $i, j \in I$, and for each $i \in I$, there exists $\Lambda_i \in \mathsf{P}$ such that $\langle h_j, \Lambda_i\rangle = \delta_{ij}$ for all $j \in I$. We choose such elements Λ_i and call them the *fundamental weights*. The free abelian group $\mathsf{Q} := \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subset \mathsf{P}$ is called the *root lattice*. Set $\mathsf{Q}_+ = \sum_{i\in I} \mathbb{Z}_{\geqslant 0}\alpha_i \subset \mathsf{Q}$. Similarly we set $\mathsf{Q}^\vee := \bigoplus_{i\in I} \mathbb{Z}h_i \subset \mathsf{P}^\vee$ and $\mathsf{Q}^\vee_+ := \sum_{i\in I} \mathbb{Z}_{\geqslant 0}h_i$.

We choose *the imaginary root* $\delta = \sum_{i\in I} \mathsf{a}_i\alpha_i \in \mathsf{Q}_+$ and *the center* $c = \sum_{i\in I} \mathsf{c}_i h_i \in \mathsf{Q}^\vee_+$ such that $\{\lambda \in \mathsf{P} \mid \langle h_i, \lambda\rangle = 0 \text{ for every } i \in I\} = \mathbb{Z}\delta$ and $\{h \in \mathsf{P}^\vee \mid \langle h, \alpha_i\rangle = 0 \text{ for every } i \in I\} = \mathbb{Z}c$ (see [Kac90, ch. 4]). We set $\mathsf{P}_{\mathrm{cl}} := \mathsf{P}/(\mathsf{P} \cap \mathbb{Q}\delta) \simeq \mathrm{Hom}(\mathsf{Q}^\vee, \mathbb{Z})$ and call it *the classical weight lattice*. We choose $\rho \in \mathsf{P}$ (respectively $\rho^\vee \in \mathsf{P}^\vee$) such that $\langle h_i, \rho\rangle = 1$ (respectively $\langle \rho^\vee, \alpha_i\rangle = 1$) for all $i \in I$.

Set $\mathfrak{h} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathsf{P}^\vee$. Then there exists a symmetric bilinear form $(\ ,\ )$ on $\mathfrak{h}^*$ satisfying

$$\langle h_i, \lambda\rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{for any } i \in I \text{ and } \lambda \in \mathfrak{h}^*.$$

We normalize the bilinear form $(\ ,\ )$ by

$$\langle c, \lambda\rangle = (\delta, \lambda) \quad \text{for any } \lambda \in \mathfrak{h}^*. \tag{2.1}$$

We denote by $\mathfrak{g}$ the *affine Kac–Moody algebra* associated with $(\mathsf{A}, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$ and by $\mathsf{W} := \langle r_i \mid i \in I\rangle \subset \mathrm{GL}(\mathfrak{h}^*)$ the *Weyl group* of $\mathfrak{g}$, where $r_i(\lambda) := \lambda - \langle h_i, \lambda\rangle\alpha_i$ for $\lambda \in \mathfrak{h}^*$. We will use the standard convention in [Kac90] to choose $0 \in I$ except $A^{(2)}_{2n}$-case, in which case we take the longest simple root as α_0. In particular, we have always $\mathsf{a}_0 = 1$, while $\mathsf{c}_0 = 2$ or 1 according to whether $\mathfrak{g} = A^{(2)}_{2n}$ or not.

We define $\mathfrak{g}_0$ to be the subalgebra of $\mathfrak{g}$ generated by the Chevalley generators e_i, f_i and h_i for $i \in I_0 := I \setminus \{0\}$ and W_0 to be the subgroup of W generated by r_i for $i \in I_0$. Note that $\mathfrak{g}_0$ is a finite-dimensional simple Lie algebra and W_0 contains the longest element w_0.

Let q be an indeterminate and $\mathbf{k}$ be the algebraic closure of the subfield $\mathbb{C}(q)$ in the algebraically closed field $\widehat{\mathbf{k}} := \bigcup_{m>0} \mathbb{C}((q^{1/m}))$. When we deal with quantum affine algebras, we regard $\mathbf{k}$ as the base field.

For $m, n \in \mathbb{Z}_{\geqslant 0}$ and $i \in I$, we define $q_i = q^{(\alpha_i,\alpha_i)/2}$ and

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[m-n]_i![n]_i!}.$$

Definition 2.2. The *quantum affine algebra* $U_q(\mathfrak{g})$ associated with an affine Cartan datum $(\mathsf{A}, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$ is the associative algebra over $\mathbf{k}$ with 1 generated by e_i, f_i $(i \in I)$ and q^h $(h \in \gamma\, \mathsf{P}^\vee)$ satisfying the following relations:

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(i) $q^0 = 1, q^h q^{h'} = q^{h+h'}$ for $h, h' \in \gamma\, \mathsf{P}^\vee$;
(ii) $q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i$, $q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i$ for $h \in \gamma^{-1}\mathsf{P}^\vee, i \in I$;
(iii) $e_i f_j - f_j e_i = \delta_{ij}((K_i - K_i^{-1})/(q_i - q_i^{-1}))$, where $K_i = q_i^{h_i}$;
(iv) $\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} = 0$ for $i \neq j$,

where $e_i^{(k)} = e_i^k/[k]_i!$ and $f_i^{(k)} = f_i^k/[k]_i!$.

Let us denote by $U_q^+(\mathfrak{g})$ (respectively $U_q^-(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by e_i (respectively f_i) for $i \in I$. We denote by $U_q'(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, K_i^{\pm 1}$ $(i \in I)$ and we call it *also* the quantum affine algebra. Throughout this paper, we mainly deal with $U_q'(\mathfrak{g})$.

We use the coproduct Δ of $U_q'(\mathfrak{g})$ given by

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i. \tag{2.2}$$

Let us denote by $^-$ the bar involution of $U_q'(\mathfrak{g})$ defined as follows:

$$q^{1/m} \to q^{-1/m}, \quad e_i \mapsto e_i, \quad f_i \mapsto f_i, \quad K_i \mapsto K_i^{-1}.$$

We denote by $\mathscr{C}_\mathfrak{g}$ the category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules; i.e., finite-dimensional modules M with a weight decomposition

$$M = \bigoplus_{\lambda \in \mathsf{P}_{\mathrm{cl}}} M_\lambda \quad \text{where } M_\lambda = \{u \in M \mid K_i u = q_i^{\langle h_i, \lambda \rangle} u\}.$$

Note that $\mathscr{C}_\mathfrak{g}$ is a monoidal category with the coproduct in (2.2). It is known that the Grothendieck ring $K(\mathscr{C}_\mathfrak{g})$ is a commutative ring. A simple module M in $\mathscr{C}_\mathfrak{g}$ contains a non-zero vector u of weight $\lambda \in \mathsf{P}_{\mathrm{cl}}$ such that: (i) $\langle h_i, \lambda \rangle \geqslant 0$ for all $i \in I_0$; (ii) all the weights of M are contained in $\lambda - \sum_{i \in I_0} \mathbb{Z}_{\geqslant 0} \mathrm{cl}(\alpha_i)$, where $\mathrm{cl}\colon \mathsf{P} \to \mathsf{P}_{\mathrm{cl}}$ denotes the canonical projection. Such a λ is unique and u is unique up to a constant multiple. We call λ the *dominant extremal weight* of M and u a *dominant extremal weight vector* of M.

For an integrable $U_q'(\mathfrak{g})$-module M, the *affinization* M_z of M is the $U_q(\mathfrak{g})$-module

$$M_z = \bigoplus_{\lambda \in \mathsf{P}} (M_z)_\lambda \quad \text{with } (M_z)_\lambda = M_{\mathrm{cl}(\lambda)}.$$

Here the actions e_i and f_i are defined in a way that they commute with the canonical projection $\mathrm{cl} : M_z \to M$.

We denote by $z_M\colon M_z \to M_z$ the $U_q'(\mathfrak{g})$-module automorphism of weight δ defined by $(M_z)_\lambda \xrightarrow{\sim} (M_z)_{\lambda+\delta}$. For $x \in \mathbf{k}^\times$, we define

$$M_x := M_z/(z_M - x)M_z.$$

We call x a *spectral parameter*. Note that, for a module M in $\mathscr{C}_\mathfrak{g}$ and $x \in \mathbf{k}^\times$, M_x is also contained in $\mathscr{C}_\mathfrak{g}$. The functor T_x defined by $T_x(M) = M_x$ is an endofunctor of $\mathscr{C}_\mathfrak{g}$ which commutes with tensor products.

Let us take a section $\iota\colon \mathsf{P}_{\mathrm{cl}} \rightarrowtail \mathsf{P}$ of $\mathrm{cl}\colon \mathsf{P} \to \mathsf{P}_{\mathrm{cl}}$ such that $\iota\mathrm{cl}(\alpha_i) = \alpha_i$ for all $i \in I_0$. For $u \in M_\lambda$ $(\lambda \in P_{\mathrm{cl}})$ and an indeterminate z, let us denote by $u_z \in (M_z)_{\iota(\lambda)}$ the element such that $\mathrm{cl}(u_z) = u$. With this notation, we have

$$e_i(u_z) = z^{\delta_{i,0}}(e_i u)_z, \quad f_i(u_z) = z^{-\delta_{i,0}}(f_i u)_z, \quad K_i(u_z) = (K_i u)_z.$$

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Then we have $M_z \simeq \mathbf{k}[z^{\pm 1}] \otimes M$, and the automorphism z_M on M_z corresponds to the multiplication of z on $\mathbf{k}[z^{\pm 1}] \otimes M$. Thus u_z is the element $1 \otimes u \in \mathbf{k}[z^{\pm 1}] \otimes M$ for $u \in M$. We also use M_{z_M} instead of M_z to emphasize z as the automorphism on M_z of weight δ.

For each $i \in I_0$, we set

$$\varpi_i := \gcd(\mathsf{c}_0, \mathsf{c}_i)^{-1} \mathrm{cl}(\mathsf{c}_0 \Lambda_i - \mathsf{c}_i \Lambda_0) \in \mathsf{P}_{\mathrm{cl}}.$$

Then $\mathsf{P}_{\mathrm{cl}}^0 := \{\lambda \in \mathsf{P}_{\mathrm{cl}} \mid \langle c, \lambda \rangle = 0\}$ is equal to $\bigoplus_{i \in I_0} \mathbb{Z}\varpi_i$. Moreover, for any $i \in I_0$, there exists a unique simple module $V(\varpi_i)$ in $\mathscr{C}_{\mathfrak{g}}$ satisfying certain conditions (see [Kas02, § 5.2]), which is called the *fundamental module of weight* ϖ_i. The dominant extremal weight of $V(\varpi_i)$ is ϖ_i.

For a $U_q'(\mathfrak{g})$-module M, we denote by $\overline{M} = \{\bar{u} \mid u \in M\}$ the $U_q'(\mathfrak{g})$-module defined by $x\bar{u} := \overline{\overline{x}u}$ for $x \in U_q'(\mathfrak{g})$. Then we have

$$\overline{M_a} \simeq (\overline{M})_{\bar{a}}, \quad \overline{M \otimes N} \simeq \overline{N} \otimes \overline{M}. \tag{2.3}$$

Note that $V(\varpi_i)$ is *bar-invariant*; i.e., $\overline{V(\varpi_i)} \simeq V(\varpi_i)$ (see [AK97, Appendix A]).

Remark 2.3 [AK97, § 1.3]. Let m_i be a positive integer such that

$$\mathsf{W}\pi_i \cap (\pi_i + \mathbb{Z}\delta) = \pi_i + \mathbb{Z}m_i\delta,$$

where π_i is an element of P such that $\mathrm{cl}(\pi_i) = \varpi_i$. We have $m_i = (\alpha_i, \alpha_i)/2$ in the case when $\mathfrak{g}$ is the dual of an untwisted affine algebra, and $m_i = 1$ otherwise. Then, for $x, y \in \mathbf{k}^\times$, we have

$$V(\varpi_i)_x \simeq V(\varpi_i)_y \quad \text{if and only if } x^{m_i} = y^{m_i}.$$

For a module M in $\mathscr{C}_{\mathfrak{g}}$, let us denote the right and the left dual of M by ${}^*M$ and M^*, respectively. That is, we have isomorphisms

$$\begin{aligned} &\mathrm{Hom}_{U_q'(\mathfrak{g})}(M \otimes X, Y) \simeq \mathrm{Hom}_{U_q'(\mathfrak{g})}(X, {}^*M \otimes Y), \quad \mathrm{Hom}_{U_q'(\mathfrak{g})}(X \otimes {}^*M, Y) \simeq \mathrm{Hom}_{U_q'(\mathfrak{g})}(X, Y \otimes M), \\ &\mathrm{Hom}_{U_q'(\mathfrak{g})}(M^* \otimes X, Y) \simeq \mathrm{Hom}_{U_q'(\mathfrak{g})}(X, M \otimes Y), \quad \mathrm{Hom}_{U_q'(\mathfrak{g})}(X \otimes M, Y) \simeq \mathrm{Hom}_{U_q'(\mathfrak{g})}(X, Y \otimes M^*), \end{aligned}$$

which are functorial in $U_q'(\mathfrak{g})$-modules X and Y.

Hence, we have the evaluation morphisms

$$M \otimes {}^*M \to \mathbf{1}, \quad M^* \otimes M \to \mathbf{1}$$

and the co-evaluation morphisms

$$\mathbf{1} \to {}^*M \otimes M, \quad \mathbf{1} \to M \otimes M^*.$$

Note the following (see [AK97, Appendix A]).

(i) For any module M in $\mathscr{C}_{\mathfrak{g}}$, we have

$$M^{**} \simeq M_{q^{-2(\delta,\rho)}} \quad \text{and} \quad {}^{**}M \simeq M_{q^{2(\delta,\rho)}}.$$

(ii) The duals of $V(\varpi_i)_x$ $(x \in \mathbf{k}^\times)$ satisfy

$$(V(\varpi_i)_x)^* \simeq V(\varpi_{i^*})_{(p^*)^{-1}x}, \quad {}^*(V(\varpi_i)_x) \simeq V(\varpi_{i^*})_{p^*x}, \tag{2.4}$$

where $p^* := (-1)^{\langle \rho^\vee, \delta \rangle} q^{\langle c, \rho \rangle}$ and $i^* \in I_0$ is defined by $\alpha_{i^*} = -w_0\, \alpha_i$.

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We say that a $U_q'(\mathfrak{g})$-module M is *good* if it has a *bar involution*, a crystal basis with *simple crystal graph*, and a *global basis* (see [Kas02] for the precise definition). It is known that the fundamental representations are good modules.

DEFINITION 2.4. We say that a $U_q'(\mathfrak{g})$ module M is *quasi-good* if

$$M \simeq V_c$$

for some good module V and $c \in \mathbf{k}^\times$.

Note that every quasi-good module is a simple $U_q'(\mathfrak{g})$-module. Moreover the tensor product $M^{\otimes k} := \underbrace{M \otimes \cdots \otimes M}_{k\text{-times}}$ for a quasi-good module M and $k \in \mathbb{Z}_{\geqslant 1}$ is again quasi-good.

For simple modules M and N in $\mathscr{C}_\mathfrak{g}$, we say that M and N *commute* or M commutes with N if $M \otimes N \simeq N \otimes M$. We say that M and N *strongly commute* or M *strongly commutes with* N if $M \otimes N$ is simple. When simple modules M and N strongly commute, they commute. Note that $M \otimes N$ is simple if and only if $N \otimes M$ is simple, since $K(\mathscr{C}_\mathfrak{g})$ is a commutative ring.

Also, when the simple modules M_t $(1 \leqslant t \leqslant m)$ strongly commute with each other, it is proved in [Her10] that

$$M_1 \otimes \cdots \otimes M_m \simeq M_{\sigma(1)} \otimes \cdots \otimes M_{\sigma(m)} \text{ is simple}$$

for every element σ in the symmetric group $\mathfrak{S}_m$ on m-letters. We say that a simple module L in $\mathscr{C}_\mathfrak{g}$ is *real* if L strongly commutes with itself, i.e., if $L \otimes L$ is simple. Note that quasi-good modules are real.

2.2 *R*-matrices, universal and renormalizing coefficients

In this subsection, we review the notion of R-matrices on $U_q'(\mathfrak{g})$-modules and their coefficients by following mainly [Kas02, § 8] and [AK97, Appendices A and B]. Let us choose the *universal R-matrix* in the following way. Take a basis $\{P_\nu\}_\nu$ of $U_q^+(\mathfrak{g})$ and a basis $\{Q_\nu\}_\nu$ of $U_q^-(\mathfrak{g})$ dual to each other with respect to a suitable coupling between $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$. Then for $U_q'(\mathfrak{g})$-modules M and N define

$$R_{M,N}^{\mathrm{univ}}(u \otimes v) = q^{(\mathrm{wt}(u),\mathrm{wt}(v))} \sum_\nu P_\nu v \otimes Q_\nu u, \tag{2.5}$$

so that $R_{M,N}^{\mathrm{univ}}$ gives a $U_q'(\mathfrak{g})$-linear homomorphism from $M \otimes N$ to $N \otimes M$ provided that the infinite sum has a meaning.

For modules M and N in $\mathscr{C}_\mathfrak{g}$, it is known that $R_{M,N_z}^{\mathrm{univ}}$ converges in the z-adic topology. Hence, it induces a morphism of $\mathbf{k}((z)) \otimes U_q'(\mathfrak{g})$-modules

$$R_{M,N_z}^{\mathrm{univ}} \colon \mathbf{k}((z)) \otimes_{\mathbf{k}[z^{\pm 1}]} (M \otimes N_z) \longrightarrow \mathbf{k}((z)) \otimes_{\mathbf{k}[z^{\pm 1}]} (N_z \otimes M). \tag{2.6}$$

Moreover, $R_{M,N_z}^{\mathrm{univ}}$ is an isomorphism.

It is known that R^{univ} satisfies the following properties: the following diagram commutes

$$\begin{array}{ccc} \mathbf{k}((z)) \otimes_{\mathbf{k}[z^{\pm 1}]} (M \otimes N \otimes L_z) & \xrightarrow{R_{M\otimes N,L_z}^{\mathrm{univ}}} & \mathbf{k}((z)) \otimes_{\mathbf{k}[z^{\pm 1}]} (L_z \otimes M \otimes N) \\ & {\scriptstyle M\otimes R_{N,L_z}^{\mathrm{univ}}} \searrow \quad \nearrow {\scriptstyle R_{M,L_z}^{\mathrm{univ}} \otimes N} & \\ & \mathbf{k}((z)) \otimes_{\mathbf{k}[z^{\pm 1}]} (M \otimes L_z \otimes N) & \end{array} \tag{2.7}$$

for L, M, N in $\mathscr{C}_\mathfrak{g}$.

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Let M and N be non-zero modules in $\mathscr{C}_{\mathfrak{g}}$. If there exists $a(z) \in \mathbf{k}((z))$ such that

$$a(z)R^{\mathrm{univ}}_{M,N_z}(M \otimes N_z) \subset N_z \otimes M,$$

then we say that $R^{\mathrm{univ}}_{M,N_z}$ is *rationally renormalizable*. In this case, we can choose $c_{M,N}(z) \in \mathbf{k}((z))^\times$ as $a(z)$ such that, for any $x \in \mathbf{k}^\times$, the specialization of $R^{\mathrm{ren}}_{M,N_z} := c_{M,N}(z)R^{\mathrm{univ}}_{M,N_z} \colon M \otimes N_z \to N_z \otimes M$ at $z = x$

$$R^{\mathrm{ren}}_{M,N_z}|_{z=x} \colon M \otimes N_x \to N_x \otimes M$$

does not vanish. Such R^{ren}_{M,N_z} and $c_{M,N}(z)$ are unique up to multiplication by an element of $\mathbf{k}[z^{\pm 1}]^\times = \bigsqcup_{n\in\mathbb{Z}} \mathbf{k}^\times z^n$. We call $c_{M,N}(z)$ the *renormalizing coefficient*.

We write

$$\mathbf{r}_{M,N} := R^{\mathrm{ren}}_{M,N_z}|_{z=1} \colon M \otimes N \to N \otimes M,$$

and call it *R-matrix*. The R-matrix $\mathbf{r}_{M,N}$ is well defined up to a constant multiple when $R^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable. By the definition, $\mathbf{r}_{M,N}$ never vanishes.

Now assume that M and N are simple $U'_q(\mathfrak{g})$-modules in $\mathscr{C}_{\mathfrak{g}}$. Then $\mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (M \otimes N_z)$ is a simple $\mathbf{k}(z) \otimes U'_q(\mathfrak{g})$-module [Kas02, Proposition 9.5].

Furthermore, we have the following. Let u and v be dominant extremal weight vectors of M and N, respectively. Then there exists $a_{M,N}(z) \in \mathbf{k}[[z]]^\times$ such that

$$R^{\mathrm{univ}}_{M,N_z}(u \otimes v_z) = a_{M,N}(z)(v_z \otimes u).$$

Then $R^{\mathrm{norm}}_{M,N_z} := a_{M,N}(z)^{-1}R^{\mathrm{univ}}_{M,N_z}|_{\mathbf{k}(z)\otimes_{\mathbf{k}[z^{\pm 1}]}(M\otimes N_z)}$ induces a unique $\mathbf{k}(z) \otimes U'_q(\mathfrak{g})$-module isomorphism

$$R^{\mathrm{norm}}_{M,N_z} \colon \mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (M \otimes N_z) \xrightarrow{\sim} \mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (N_z \otimes M)$$

satisfying

$$R^{\mathrm{norm}}_{M,N_z}(u \otimes v_z) = v_z \otimes u.$$

We call $a_{M,N}(z)$ the *universal coefficient* of M and N, and $R^{\mathrm{norm}}_{M,N_z}$ the *normalized R-matrix*.

Similarly there exists a unique $\mathbf{k}(z) \otimes U'_q(\mathfrak{g})$-module isomorphism

$$R^{\mathrm{norm}}_{M_z,N} \colon \mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (M_z \otimes N) \xrightarrow{\sim} \mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (N \otimes M_z)$$

satisfying

$$R^{\mathrm{norm}}_{M,N}(u_z \otimes v) = v \otimes u_z.$$

Note that $R^{\mathrm{norm}}_{M_z,N} = T_z \circ R^{\mathrm{norm}}_{M,N_w}$ with $w = 1/z$. Here, for $x \in \mathbf{k}(z)$, the functor T_x is the endofunctor of the category of $\mathbf{k}(z) \otimes U'_q(\mathfrak{g})$-modules L given by $T_x(L) = L_x$.

Let $d_{M,N}(z) \in \mathbf{k}[z]$ be a monic polynomial of the smallest degree such that the image of $d_{M,N}(z)R^{\mathrm{norm}}_{M,N_z}(M \otimes N_z)$ is contained in $N_z \otimes M$. We call $d_{M,N}(z)$ the *denominator of* $R^{\mathrm{norm}}_{M,N_z}$. Then we have

$$R^{\mathrm{ren}}_{M,N_z} = d_{M,N}(z)R^{\mathrm{norm}}_{M,N_z} \colon M \otimes N_z \longrightarrow N_z \otimes M \tag{2.8}$$

up to multiplication by an element of $\mathbf{k}[z^{\pm 1}]^\times$.

Hence, the universal R-matrix $R^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable and we have

$$R^{\mathrm{ren}}_{M,N_z} = a_{M,N}(z)^{-1}d_{M,N}(z)R^{\mathrm{univ}}_{M,N_z} \quad \text{and} \quad c_{M,N}(z) = \frac{d_{M,N}(z)}{a_{M,N}(z)} \tag{2.9}$$

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up to multiplication by an element of $\mathbf{k}[z^{\pm 1}]^\times$ (see [KKK18, Example 2.3] and [DO94] for instances of R-matrices).

Since $\mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (M \otimes N_z)$ is a simple $\mathbf{k}(z) \otimes U_q'(\mathfrak{g})$-module [Kas02, Proposition 9.5], we have

$$\mathrm{Hom}_{\mathbf{k}[z^{\pm 1}] \otimes U_q'(\mathfrak{g})}(M \otimes N_z, N_z \otimes M) = \mathbf{k}[z^{\pm 1}] R^{\mathrm{ren}}_{M,N_z}. \tag{2.10}$$

Similarly there exists a $\mathbf{k}[z^{\pm 1}] \otimes U_q'(\mathfrak{g})$-linear homomorphism $R^{\mathrm{ren}}_{M_z,N} \colon M_z \otimes N \to N \otimes M_z$ such that

$$\mathrm{Hom}_{\mathbf{k}[z^{\pm 1}] \otimes U_q'(\mathfrak{g})}(M_z \otimes N, N \otimes M_z) = \mathbf{k}[z^{\pm 1}] R^{\mathrm{ren}}_{M_z,N}. \tag{2.11}$$

The homomorphism $R^{\mathrm{ren}}_{M_z,N}$ is unique up to multiplication by an element of $\mathbf{k}[z^{\pm 1}]^\times$. We have

$$R^{\mathrm{ren}}_{M_z,N} = d_{M,N}(z^{-1})\, R^{\mathrm{norm}}_{M_z,N} \mod \mathbf{k}[z^{\pm 1}]^\times. \tag{2.12}$$

In particular, we have

$$R^{\mathrm{ren}}_{N_z,M} \circ R^{\mathrm{ren}}_{M,N_z} = d_{M,N}(z) d_{N,M}(z^{-1})\, \mathrm{id}_{M \otimes N_z} \mod \mathbf{k}[z^{\pm 1}]^\times. \tag{2.13}$$

THEOREM 2.5 ([AK97, Cha10, Kas02, KKKO15a]; see also [KKK18, Theorem 2.2]).

(i) *For good modules M and N, the zeroes of $d_{M,N}(z)$ belong to $\mathbb{C}[[q^{1/m}]]q^{1/m}$ for some $m \in \mathbb{Z}_{>0}$.*

(ii) *For simple modules M and N such that one of them is real, M_x and N_y strongly commute to each other if and only if $d_{M,N}(z)d_{N,M}(1/z)$ does not vanish at $z = y/x$.*

(iii) *Let M_k be a good module with a dominant extremal vector u_k of weight λ_k, and $a_k \in \mathbf{k}^\times$ for $k = 1, \dots, t$. Assume that a_j/a_i is not a zero of $d_{M_i,M_j}(z)$ for any $1 \leqslant i < j \leqslant t$. Then the following statements hold.*

  (a) *$(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ is generated by $u_1 \otimes \cdots \otimes u_t$.*

  (b) *The head of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ is simple.*

  (c) *Any non-zero submodule of $(M_t)_{a_t} \otimes \cdots \otimes (M_1)_{a_1}$ contains the vector $u_t \otimes \cdots \otimes u_1$.*

  (d) *The socle of $(M_t)_{a_t} \otimes \cdots \otimes (M_1)_{a_1}$ is simple.*

  (e) *Let $\mathbf{r} : (M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t} \to (M_t)_{a_t} \otimes \cdots \otimes (M_1)_{a_1}$ be the specialization of $R^{\mathrm{norm}}_{M_1,\dots,M_t} := \prod_{1 \leqslant j < k \leqslant t} R^{\mathrm{norm}}_{M_j,M_k}$ at $z_k = a_k$. Then the image of $\mathbf{r}$ is simple and it coincides with the head of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ and also with the socle of $(M_t)_{a_t} \otimes \cdots \otimes (M_1)_{a_1}$.*

(iv) *For a simple integrable $U_q'(\mathfrak{g})$-module M, there exists a finite sequence*

$$((i_1, a_1), \dots, (i_t, a_t)) \in I_0 \times \mathbf{k}^\times \tag{2.14}$$

which satisfies the following condition: for any $\sigma \in \mathfrak{S}_t$ such that

$$d_{V(\varpi_{i_{\sigma(k)}}),V(\varpi_{i_{\sigma(k')}})}(a_{\sigma(k')}/a_{\sigma(k)}) \neq 0 \quad \text{for } 1 \leqslant k < k' \leqslant t,$$

M is isomorphic to the head of $V(\varpi_{i_{\sigma(1)}})_{a_{\sigma(1)}} \otimes \cdots \otimes V(\varpi_{i_{\sigma(t)}})_{a_{\sigma(t)}}$.

Moreover, such a sequence $((i_1, a_1), \dots, (i_t, a_t))$ is unique up to permutation. In particular, M has the dominant extremal weight $\sum_{k=1}^t \varpi_{i_k}$.

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From the above theorem, for each simple module M in $\mathscr{C}_{\mathfrak{g}}$, we can associate a multiset of pairs $\{(i_k, a_k) \in I_0 \times \mathbf{k}^\times\}_{1\leqslant k\leqslant t}$ satisfying the conditions in Theorem 2.5(iv). We call $\{(i_k, a_k) \in I_0 \times \mathbf{k}^\times\}_{1\leqslant k\leqslant t}$ of M *the multipair associated to* M, and write

$$M = S((i_1, a_1), \dots, (i_t, a_t)).$$

PROPOSITION 2.6. *Let M and N be non-zero modules in $\mathscr{C}_{\mathfrak{g}}$, and $a \in \mathbf{k}^\times$ such that $R^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable. Then we have*

$$\begin{aligned} c_{M,N}(z) &= c_{M^*,N^*}(z) = c_{{}^*M,{}^*N}(z), \\ c_{M_a,N}(z) &= c_{M,N}(a^{-1}z), \quad c_{M,N_a}(z) = c_{M,N}(az). \end{aligned}$$

Proof. The first assertion follows from $(R^{\mathrm{univ}}_{M,N_z})^* = R^{\mathrm{univ}}_{M^*,N_z^*}$: that is,

$$\begin{array}{ccc} (N_z \otimes M)^* & \xrightarrow[(R^{\mathrm{univ}}_{M,N_z})^*]{} & (M \otimes N_z)^* \\ \big| \wr & & \big| \wr \\ M^* \otimes N_z^* & \xrightarrow[R^{\mathrm{univ}}_{M^*,N_z^*}]{} & (N_z)^* \otimes M^* \end{array}$$

commutes [FR92]. The second follows from the first and the others are trivial. □

PROPOSITION 2.7 [AK97, (A14), (A15), Proposition A.1, Lemma C.15]. *Let M and N be simple modules in $\mathscr{C}_{\mathfrak{g}}$.*

(i) *We have*

$$\begin{aligned} a_{M,N}(z) &= a_{M^*,N^*}(z) = a_{{}^*M,{}^*N}(z), \\ d_{M,N}(z) &= d_{M^*,N^*}(z) = d_{{}^*M,{}^*N}(z), \\ a_{M,N}(z) &= a_{M_x,N_x}(z), \quad d_{M,N}(z) = d_{M_x,N_x}(z) \quad \text{for any } x \in \mathbf{k}^\times. \end{aligned} \tag{2.15}$$

(ii) *We have* $a_{M,N}(z)a_{{}^*M,N}(z) \equiv d_{M,N}(z)/d_{N,{}^*M}(z^{-1}) \bmod \mathbf{k}[z^{\pm 1}]^\times$.

We set

$$\varphi(z) := \prod_{s=0}^{\infty}(1 - \widetilde{p}^{\,s}z) \in \mathbf{k}[[z]] \subset \widehat{\mathbf{k}}[[z]]. \tag{2.16}$$

Here $\widetilde{p} := p^{*\,2} = q^{2\langle c,\rho\rangle} = q^{2\sum_{i\in I}\mathsf{c}_i}$. We have

$$\varphi(z) = \sum_{m=0}^{\infty}(-1)^m \frac{\widetilde{p}^{\,m(m-1)/2}}{\prod_{k=1}^{m}(1-\widetilde{p}^{\,k})} z^m.$$

For $i, j \in I_0$, set

$$\begin{aligned} a_{i,j}(z) &:= a_{V(\varpi_i),V(\varpi_j)}(z), \\ d_{i,j}(z) &:= d_{V(\varpi_i),V(\varpi_j)}(z). \end{aligned} \tag{2.17}$$

Then the universal coefficient $a_{i,j}(z)$ is obtained as follows (see [AK97, Appendix A]):

$$a_{i,j}(z) \equiv \frac{\prod_\mu \varphi(p^* y_\mu z)\varphi(p^* \overline{y_\mu} z)}{\prod_\nu \varphi(x_\nu z)\varphi(p^{*2}\overline{x_\nu} z)} \quad \bmod \mathbf{k}[z^{\pm 1}]^\times, \tag{2.18}$$

where

$$d_{i,j}(z) = \prod_\nu (z - x_\nu) \quad \text{and} \quad d_{i^*,j}(z) = \prod_\mu (z - y_\mu).$$

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Example 2.8. For the fundamental representations $V(\varpi_i)$ over

$$U'_q(A^{(1)}_{n-1}) \quad (i \in I_0 = \{1, \dots, n-1\}),$$

the denominators $d_{i,j}(z) := d_{V(\varpi_i),V(\varpi_j)}(z)$ and the universal coefficients of $a_{i,j}(z) := a_{V(\varpi_i),V(\varpi_j)}(z)$ are given as follows:

$$d_{i,j}(z) = \prod_{s=1}^{\min(i,j,n-i,n-j)} (z - (-q)^{2s+|i-j|}) \quad \text{and} \quad a_{i,j}(z) \equiv \frac{[\,|i-j|\,]\,[2n-|i-j|\,]}{[i+j]\,[2n-i-j]} \mod \mathbf{k}[z^{\pm 1}]^{\times},$$

where $[a] := \varphi((-q)^a z)$. Note that $p^* = (-q)^n$ in this case.

Remark 2.9. The denominators of the normalized R-matrices $d_{i,j}(z)$ and hence the universal coefficients $a_{i,j}(z)$ were calculated in [AK97, DO94, KKK15, Oh15] for the classical affine types and in [OS19] for the exceptional affine types (see also [FH15, Fuj18, KKMM$^+$92, KMOY07, Yam98]).

LEMMA 2.10 [KKKO15a, Lemma 3.10]. *Let M_k be a module in $\mathscr{C}_{\mathfrak{g}}$ $(k = 1, 2, 3)$. Let X be a $U'_q(\mathfrak{g})$-submodule of $M_1 \otimes M_2$ and Y a $U'_q(\mathfrak{g})$-submodule of $M_2 \otimes M_3$ such that $X \otimes M_3 \subset M_1 \otimes Y$ as submodules of $M_1 \otimes M_2 \otimes M_3$. Then there exists a $U'_q(\mathfrak{g})$-submodule N of M_2 such that $X \subset M_1 \otimes N$ and $N \otimes M_3 \subset Y$.*

PROPOSITION 2.11 [KKKO15a, Corollary 3.11].

(i) *Let M_k be a module in $\mathscr{C}_{\mathfrak{g}}$ $(k = 1, 2, 3)$, and let $\varphi_1 \colon L \to M_2 \otimes M_3$ and $\varphi_2 \colon M_1 \otimes M_2 \to L'$ be non-zero morphisms. Assume further that M_2 is a simple module. Then the composition*

$$M_1 \otimes L \xrightarrow{M_1 \otimes \varphi_1} M_1 \otimes M_2 \otimes M_3 \xrightarrow{\varphi_2 \otimes M_3} L' \otimes M_3$$

does not vanish.

(ii) *Let M, N_1 and N_2 be non-zero modules in $\mathscr{C}_{\mathfrak{g}}$, and assume that $R^{\mathrm{univ}}_{N_k,M_z}$ is rationally renormalizable for $k = 1, 2$. Then $R^{\mathrm{univ}}_{N_1 \otimes N_2,M_z}$ is rationally renormalizable, and we have*

$$\frac{c_{N_1,M}(z) c_{N_2,M}(z)}{c_{N_1 \otimes N_2,M}(z)} \in \mathbf{k}[z^{\pm 1}].$$

If we assume further that M is simple, then we have

$$c_{N_1 \otimes N_2,M}(z) \equiv c_{N_2,M}(z) c_{N_1,M}(z) \mod \mathbf{k}[z^{\pm 1}]^{\times}$$

and the following diagram commutes up to a constant multiple.

$$N_1 \otimes N_2 \otimes M \xrightarrow{N_1 \otimes \mathbf{r}_{N_2,M}} N_1 \otimes M \otimes N_2 \xrightarrow{\mathbf{r}_{N_1,M} \otimes N_2} M \otimes N_1 \otimes N_2 \qquad (2.19)$$

(iii) *Let M, N_1 and N_2 be non-zero modules in $\mathscr{C}_{\mathfrak{g}}$, and assume that $R^{\mathrm{univ}}_{M,(N_k)_z}$ is rationally renormalizable for $k = 1, 2$. Then $R^{\mathrm{univ}}_{M,(N_1 \otimes N_2)_z}$ is rationally renormalizable, and we have*

$$\frac{c_{M,N_1}(z) c_{M,N_2}(z)}{c_{M,N_1 \otimes N_2}(z)} \in \mathbf{k}[z^{\pm 1}].$$

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If we assume further that M is simple, then we have

$$c_{N_1\otimes N_2,M}(z) \equiv c_{N_2,M}(z)c_{N_1,M}(z) \mod \mathbf{k}[z^{\pm 1}]^\times$$

and the following diagram commutes up to a constant multiple.

$$\begin{array}{ccccc} M\otimes N_1\otimes N_2 & \xrightarrow[\mathbf{r}_{M,N_1}\otimes N_2]{} & N_1\otimes M\otimes N_2 & \xrightarrow[N_1\otimes \mathbf{r}_{M,N_2}\otimes N_2]{} & N_1\otimes N_2\otimes M \end{array} \qquad \overset{\mathbf{r}_{M,\ N_1\otimes N_2}}{\frown} \tag{2.20}$$

Proof. Part (i) and the commutativity of (2.19) are nothing but [KKKO15a, Corollary 3.11]. Since

$$\begin{array}{ccc} \mathbf{k}((z))\otimes_{\mathbf{k}[z^{\pm 1}]}(N_1\otimes N_2\otimes M_z) & \xrightarrow{R^{\mathrm{univ}}_{N_1\otimes N_2,\ M}} & \mathbf{k}((z))\otimes_{\mathbf{k}[z^{\pm 1}]}(M_z\otimes N_1\otimes N_2) \\ & {\scriptstyle N_1\otimes R^{\mathrm{univ}}_{N_2,M_z}}\searrow \quad \nearrow{\scriptstyle R^{\mathrm{univ}}_{N_1,M_z}\otimes N_2} & \\ & \mathbf{k}((z))\otimes_{\mathbf{k}[z^{\pm 1}]}(N_1\otimes M_z\otimes N_2) & \end{array}$$

commutes, the diagram

$$\begin{array}{ccccc} N_1\otimes N_2\otimes M_z & \xrightarrow[N_1\otimes c_{N_2,M}(z)R^{\mathrm{univ}}_{N_2,M_z}]{} & N_1\otimes M_z\otimes N_2 & \xrightarrow[c_{N_1,M}(z)R^{\mathrm{univ}}_{N_1,M_z}\otimes N_2]{} & M_z\otimes N_1\otimes N_2 \end{array} \qquad \overset{c_{N_2,M}(z)c_{N_1,M}(z)R^{\mathrm{univ}}_{N_1\otimes N_2,\ M_z}}{\frown}$$

commutes. Hence $R^{\mathrm{univ}}_{N_1\otimes N_2,M_z}$ is rationally renormalizable, and we have $c_{N_1,M}(z)c_{N_2,M}(z) \in \mathbf{k}[z^{\pm 1}]c_{N_1\otimes N_2,M}(z)$.

If M is simple, then (i) implies that $c_{N_2,M}(z)c_{N_1,M}(z)R^{\mathrm{univ}}_{N_1\otimes N_2,\ M_z}$ never vanishes at any $z = a \in \mathbf{k}^\times$. Hence $c_{N_2,M}(z)c_{N_1,M}(z)R^{\mathrm{univ}}_{N_1\otimes N_2,\ M_z} \equiv R^{\mathrm{ren}}_{N_1\otimes N_2,\ M_z} \mod \mathbf{k}[z^{\pm 1}]^\times$, which implies $c_{N_1\otimes N_2,M}(z) \equiv c_{N_2,M}(z)c_{N_1,M}(z) \mod \mathbf{k}[z^{\pm 1}]^\times$.

The proof of (iii) is similar. □

PROPOSITION 2.12. *Let M and N be modules in $\mathscr{C}_\mathfrak{g}$, and let M' and N' be a non-zero subquotient of M and N, respectively. Assume that $R^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable. Then $R^{\mathrm{univ}}_{M',N'_z}$ is rationally renormalizable, and $c_{M,N}(z)/c_{M'.N'}(z) \in \mathbf{k}[z^{\pm 1}]$.*

Proof. We shall show that $R^{\mathrm{univ}}_{M',N_z}$ is rationally renormalizable and $c_{M,N}(z)/c_{M'.N}(z) \in \mathbf{k}[z^{\pm 1}]$ for a non-zero quotient M' of M. We have a commutative diagram

$$\begin{array}{ccc} \mathbf{k}((z))\otimes_{\mathbf{k}[z^{\pm 1}]}(M\otimes N_z) & \xrightarrow{c_{M,N}(z)R^{\mathrm{univ}}_{M,N_z}} & \mathbf{k}((z))\otimes_{\mathbf{k}[z^{\pm 1}]}(N_z\otimes M) \\ \downarrow & & \downarrow \\ \mathbf{k}((z))\otimes_{\mathbf{k}[z^{\pm 1}]}(M'\otimes N_z) & \xrightarrow{c_{M,N}(z)R^{\mathrm{univ}}_{M',N_z}} & \mathbf{k}((z))\otimes_{\mathbf{k}[z^{\pm 1}]}(N_z\otimes M') \end{array}$$

which induces the following.

$$\begin{array}{ccc} M\otimes N_z & \xrightarrow{R^{\mathrm{ren}}_{M,N_z}} & N_z\otimes M \\ \downarrow & & \downarrow \\ M'\otimes N_z & \xrightarrow{c_{M,N}(z)R^{\mathrm{univ}}_{M',N_z}} & N_z\otimes M' \end{array}$$

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Hence $R^{\mathrm{univ}}_{M',N_z}$ is rationally renormalizable and $c_{M,N}(z) \in c_{M'.N}(z)\mathbf{k}[z^{\pm 1}]$.

Similarly $R^{\mathrm{univ}}_{M',N_z}$ is rationally renormalizable and $c_{M,N}(z)/c_{M'.N}(z) \in \mathbf{k}[z^{\pm 1}]$ for any non-zero submodule of M' of M, and hence for any non-zero subquotient of M' of M.

We can argue similarly for non-zero subquotients N' of N. □

THEOREM 2.13 [KKKO15a]. *Let M and N be simple modules in $\mathscr{C}_{\mathfrak{g}}$ and assume that one of them is real. Then we have that:*

(i) $\mathrm{Hom}(M \otimes N, N \otimes M) = \mathbf{k}\,\mathbf{r}_{M,N}$;
(ii) *$M \otimes N$ and $N \otimes M$ have simple socles and simple heads;*
(iii) *moreover, $\mathrm{Im}(\mathbf{r}_{M,N})$ is isomorphic to the head of $M \otimes N$ and the socle of $N \otimes M$;*
(iv) *$M \otimes N$ is simple whenever its head and its socle are isomorphic to each other.*

Note that (i) is not proved in [KKKO15a] but it can be proved similarly to the quiver Hecke algebra case given in [KKKO18, Theorem 2.11].

For modules M and N in $\mathscr{C}_{\mathfrak{g}}$, we denote by $M \nabla N$ and $M \Delta N$ the head and the socle of $M \otimes N$, respectively.

3. New invariants for pairs of modules

In this section, we introduce new invariants for pairs of $U'_q(\mathfrak{g})$-modules by using R-matrices and investigate their properties. These invariants have similar properties to those in the quiver Hecke algebra case.

Recall that

$$\widetilde{p} := p^{*2} = q^{2(c,\rho)} \quad \text{and} \quad \varphi(z) = \prod_{s \in \mathbb{Z}_{\geqslant 0}} (1 - \widetilde{p}^{\,s} z) \in \mathbf{k}[[z]].$$

We set

$$\widetilde{p}^{\,S} := \{\widetilde{p}^{\,k} \mid k \in S\} \quad \text{for a subset } S \text{ of } \mathbb{Z}.$$

DEFINITION 3.1. We define the subset $\mathcal{G}$ of $\mathbf{k}((z))^{\times}$ as follows:

$$\mathcal{G} := \left\{ cz^m \prod_{a \in \mathbf{k}^{\times}} \varphi(az)^{\eta_a} \,\middle|\, \begin{array}{c} c \in \mathbf{k}^{\times}, m \in \mathbb{Z}, \\ \eta_a \in \mathbb{Z} \text{ vanishes except finitely many } a \end{array} \right\}. \tag{3.1}$$

Note that $\mathcal{G}$ forms a group with respect to the multiplication. We have $\mathbf{k}(z)^{\times} \subset \mathcal{G}$. Note also that for $f(z) = cz^m \prod_{a \in \mathbf{k}^{\times}} \varphi(az)^{\eta_a}$, $\{\eta_a\}_{a \in \mathbf{k}^{\times}}$ is determined by $f(z)$ since

$$\frac{f(z)}{f(\widetilde{p}z)} = (\widetilde{p})^{-m} \prod_{a \in \mathbf{k}^{\times}} (1 - az)^{\eta_a}.$$

PROPOSITION 3.2. *Let M and N be modules in $\mathscr{C}_{\mathfrak{g}}$. If $R^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable, then the renormalizing coefficient $c_{M,N}(z)$ belongs to $\mathcal{G}$.*

Proof. Let us take a simple submodule M' of M and a simple submodule N' of N. Then, Proposition 2.12 implies that $c_{M,N}(z)/c_{M'N'}(z) \in \mathbf{k}(z)^{\times} \subset \mathcal{G}$. Hence the assertion follows from the following lemma. □

LEMMA 3.3. *For simple modules M and N in $\mathscr{C}_{\mathfrak{g}}$, the universal coefficient $a_{M,N}(z)$ as well as the renormalizing coefficient $c_{M,N}(z)$ is contained in $\mathcal{G}$.*

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Proof. Let us write $M = S((i_1, a_1), \dots, (i_t, a_t))$ and $N = S((j_1, b_1), \dots, (j_{t'}, b_{t'}))$. When $t+t' = 2$, $a_{M,N}(z)$ is nothing but $a_{i,j}(b_1/a_1 z)$ in (2.18), and our assertion holds. Then the induction on $t + t'$ proceeds by Propositions 2.11 and 2.12. □

For each subset S of $\mathbb{Z}$, we can construct a group homomorphism from $\mathcal{G}$ to the additive group $\mathbb{Z}$ by associating the sum of exponents η_a such that $a \in \widetilde{p}^S$. For instance, by taking S as $\mathbb{Z}$ or $\mathbb{Z}_{\leqslant 0}$, we define the group homomorphisms

$$\widetilde{\mathrm{Deg}} \colon \mathcal{G} \to \mathbb{Z} \quad \text{and} \quad \mathrm{Deg}^\infty \colon \mathcal{G} \to \mathbb{Z},$$

by

$$\widetilde{\mathrm{Deg}}(f(z)) = \sum_{a \in \widetilde{p}^{\mathbb{Z}_{\leqslant 0}}} \eta_a \quad \text{and} \quad \mathrm{Deg}^\infty(f(z)) = \sum_{a \in \widetilde{p}^{\mathbb{Z}}} \eta_a,$$

for $f(z) = cz^m \prod \varphi(az)^{\eta_a} \in \mathcal{G}$. As their linear combination, we introduce the group homomorphism

$$\mathrm{Deg} \colon \mathcal{G} \to \mathbb{Z} \quad \text{by } \mathrm{Deg} = 2\widetilde{\mathrm{Deg}} - \mathrm{Deg}^\infty,$$

namely,

$$\mathrm{Deg}(f(z)) = \sum_{a \in \widetilde{p}^{\mathbb{Z}_{\leqslant 0}}} \eta_a - \sum_{a \in \widetilde{p}^{\mathbb{Z}_{>0}}} \eta_a. \tag{3.2}$$

Recall Convention 2.1(ii).

LEMMA 3.4. *Let $f(z) \in \mathcal{G}$.*

(i) *If $f(z) \in \mathbf{k}(z)$, then we have*

$$\widetilde{\mathrm{Deg}}(f(z)) = \mathrm{zero}_{z=1} f(z), \quad \mathrm{Deg}^\infty(f(z)) = 0, \quad \textit{and} \quad \mathrm{Deg}(f(z)) = 2\mathrm{zero}_{z=1} f(z).$$

(ii) *If $g(z),\ h(z) \in \mathcal{G}$ satisfy $g(z)/h(z) \in \mathbf{k}[z^{\pm 1}]$, then $\mathrm{Deg}(h(z)) \leqslant \mathrm{Deg}(g(z))$.*

(iii) *We have $\mathrm{Deg}^\infty f(z) = -\mathrm{Deg}(f(\widetilde{p}^n z)) = \mathrm{Deg}(f(\widetilde{p}^{-n} z))$ for $n \gg 0$.*

(iv) *If $\mathrm{Deg}^\infty(f(cz)) = 0$ for any $c \in \mathbf{k}^\times$, then $f(z) \in \mathbf{k}(z)$.*

Proof. We may assume $f(z) = \prod_{a \in \mathbf{k}^\times} \varphi(az)^{\eta_a}$.

(i) For $a \notin \widetilde{p}^{\mathbb{Z}}$, it is obvious. For $a \in \widetilde{p}^{\mathbb{Z}}$, we have

$$\begin{aligned} \widetilde{\mathrm{Deg}}(1 - az) &= \widetilde{\mathrm{Deg}}(\varphi(az)/\varphi(\widetilde{p}az)) \\ &= \delta(a \in \widetilde{p}^{\mathbb{Z}_{\leqslant 0}}) - \delta(\widetilde{p}a \in \widetilde{p}^{\mathbb{Z}_{\leqslant 0}}) = \delta(a = 1) = \mathrm{zero}_{z=1}(1 - az) \end{aligned}$$

and

$$\mathrm{Deg}^\infty(1 - az) = \mathrm{Deg}^\infty(\varphi(az)/\varphi(\widetilde{p}az)) = 1 - 1 = 0.$$

(ii) This follows from (i).

(iii) We have

$$\mathrm{Deg}(f(\widetilde{p}^n z)) = \sum_{a\widetilde{p}^n \in \widetilde{p}^{\mathbb{Z}_{\leqslant 0}}} \eta_a - \sum_{a\widetilde{p}^n \in \widetilde{p}^{\mathbb{Z}_{>0}}} \eta_a.$$

Hence we have $\mathrm{Deg}(f(\widetilde{p}^n z)) = -\sum_{a \in \widetilde{p}^{\mathbb{Z}}} \eta_a$ if $n \gg 0$ and $\mathrm{Deg}(f(\widetilde{p}^n z)) = \sum_{a \in \widetilde{p}^{\mathbb{Z}}} \eta_a$ if $n \ll 0$.

(iv) By the assumption, we can easily see that $f(z)$ is a product of functions of the form $\varphi(az)/\varphi(\widetilde{p}^m az)$ $(a \in \mathbf{k}^\times,\ m \in \mathbb{Z})$. Then the result follows from $\varphi(az)/\varphi(\widetilde{p}^m az) \in \mathbf{k}(z)$. □

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Remark 3.5. Any $f(z) \in \mathcal{G}$ extends to a meromorphic function on

$$\{(z, q^{1/\ell}) \in \mathbb{C} \times \mathbb{C}; |q^{1/\ell}| < \varepsilon\}$$

for some $\ell \in \mathbb{Z}_{>0}$ and $\varepsilon > 0$. Hence $\mathrm{zero}_{z=\widetilde{p}^k} f(z)$, the order of zero of $f(z)$ at $z = \widetilde{p}^k$, makes sense for any $k \in \mathbb{Z}$. Then one has $\widetilde{\mathrm{Deg}}(f(z)) = \mathrm{zero}_{z=1} f(z)$.

Using the homomorphisms Deg, $\widetilde{\mathrm{Deg}}$ and Deg^∞, we define the new invariants for a pair of modules M, N in $\mathscr{C}_\mathfrak{g}$ such that $R^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable.

DEFINITION 3.6. For non-zero modules M and N in $\mathscr{C}_\mathfrak{g}$ such that $R^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable, we define the integers $\Lambda(M,N)$, $\widetilde{\Lambda}(M,N)$ and $\Lambda^\infty(M,N)$ as follows:

$$\Lambda(M,N) = \mathrm{Deg}(c_{M,N}(z)), \quad \widetilde{\Lambda}(M,N) = \widetilde{\mathrm{Deg}}(c_{M,N}(z)), \quad \Lambda^\infty(M,N) = \mathrm{Deg}^\infty(c_{M,N}(z)).$$

Hence, we have

$$\widetilde{\Lambda}(M,N) = \tfrac{1}{2}(\Lambda(M,N) + \Lambda^\infty(M,N)). \tag{3.3}$$

LEMMA 3.7. *For any simple modules M, N in $\mathscr{C}_\mathfrak{g}$ and $x \in \mathbf{k}^\times$, we have*

$$\begin{aligned}
\Lambda(M,N) &= \Lambda(M^*,N^*) = \Lambda({}^*M,{}^*N) = \Lambda(M_x,N_x),\\
\widetilde{\Lambda}(M,N) &= \widetilde{\Lambda}(M^*,N^*) = \widetilde{\Lambda}({}^*M,{}^*N) = \widetilde{\Lambda}(M_x,N_x),\\
\Lambda^\infty(M,N) &= \Lambda^\infty(M^*,N^*) = \Lambda^\infty({}^*M,{}^*N) = \Lambda^\infty(M_x,N_x).
\end{aligned}$$

Proof. These assertions follow from Proposition 2.6. □

LEMMA 3.8. *Let M and N be non-zero modules in $\mathscr{C}_\mathfrak{g}$.*

(i) *If M and N are simple, then we have $\Lambda^\infty(M,N) = \mathrm{Deg}^\infty(c_{M,N}(z)) = -\mathrm{Deg}^\infty(a_{M,N}(z))$.*
(ii) *If $R^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable, then*

$$\Lambda^\infty(M,N) = -\Lambda(M,N_{\widetilde{p}^n}) = \Lambda(M,N_{\widetilde{p}^{-n}}) \quad \text{for } n \gg 0.$$

Proof. (i) This follows from $a_{M,N}(z)c_{M,N}(z) \in \mathbf{k}(z)$ and Lemma 3.4(ii).
(ii) This follows from $c_{M,N_{\widetilde{p}^n}}(z) = c_{M,N}(\widetilde{p}^n z)$ and Lemma 3.4(iii). □

PROPOSITION 3.9. *Let M and N be modules in $\mathscr{C}_\mathfrak{g}$, and let M' and N' be a non-zero subquotient of M and N, respectively. Assume that $R^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable. Then $R^{\mathrm{univ}}_{M',N'_z}$ is rationally renormalizable, and*

$$\Lambda(M',N') \leqslant \Lambda(M,N) \quad \text{and} \quad \Lambda^\infty(M',N') = \Lambda^\infty(M,N).$$

Proof. These assertions follow from Proposition 2.12 and Lemma 3.4. □

LEMMA 3.10. *Let M, N and L be non-zero modules in $\mathscr{C}_\mathfrak{g}$.*

(i) *If $R^{\mathrm{univ}}_{M,L_z}$ and $R^{\mathrm{univ}}_{N,L_z}$ are rationally renormalizable, then $R^{\mathrm{univ}}_{M\otimes N,L_z}$ is rationally renormalizable and*

$$\Lambda(M\otimes N,L) \leqslant \Lambda(M,L) + \Lambda(N,L) \quad \text{and} \quad \Lambda^\infty(M\otimes N,L) = \Lambda^\infty(M,L) + \Lambda^\infty(N,L).$$

If we assume further that L is simple, then the equality holds instead of the inequality.

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(ii) *If* $R^{\mathrm{univ}}_{L,M_z}$ *and* $R^{\mathrm{univ}}_{L,N_z}$ *are rationally renormalizable, then* $R^{\mathrm{univ}}_{L,\,(M\otimes N)_z}$ *is rationally renormalizable and*

$$\Lambda(L, M\otimes N) \leqslant \Lambda(L,M)+\Lambda(L,N) \quad \text{and} \quad \Lambda^{\infty}(L, M\otimes N) = \Lambda^{\infty}(L,M)+\Lambda^{\infty}(L,N).$$

If we assume further that L *is simple, then the equality holds instead of the inequality.*

Proof. These assertions follow from Proposition 2.11. □

PROPOSITION 3.11. *Let* M, N *and* L *be non-zero modules in* $\mathscr{C}_{\mathfrak{g}}$, *and let* S *be a non-zero subquotient of* $M\otimes N$.

(i) *Assume that* $R^{\mathrm{univ}}_{M,L_z}$ *and* $R^{\mathrm{univ}}_{N,L_z}$ *are rationally renormalizable. Then* $R^{\mathrm{univ}}_{S,L_z}$ *is rationally renormalizable and*

$$\Lambda(S, L) \leqslant \Lambda(M,L)+\Lambda(N,L) \quad \text{and} \quad \Lambda^{\infty}(S, L) = \Lambda^{\infty}(M,L)+\Lambda^{\infty}(N,L).$$

(ii) *Assume that* $R^{\mathrm{univ}}_{L,M_z}$ *and* $R^{\mathrm{univ}}_{L,N_z}$ *are rationally renormalizable. Then* $R^{\mathrm{univ}}_{L,S_z}$ *is rationally renormalizable and*

$$\Lambda(L, S) \leqslant \Lambda(L,M)+\Lambda(L,N) \quad \text{and} \quad \Lambda^{\infty}(L, S) = \Lambda^{\infty}(L,M)+\Lambda^{\infty}(L,N).$$

Proof. These assertions follow from Propositions 3.9 and 3.8. □

COROLLARY 3.12. *For simple modules*

$$M = S((i_1,a_1),\dots,(i_\ell,a_\ell)) \quad \text{and} \quad N = S((j_1,b_1),\dots,(j_{\ell'},b_{\ell'})) \quad \text{in } \mathscr{C}_{\mathfrak{g}},$$

we have

$$\Lambda^{\infty}(M,N) = \sum_{1\leqslant\nu\leqslant\ell,\,1\leqslant\mu\leqslant\ell'} \Lambda^{\infty}(V(\varpi_{i_\nu})_{a_\nu}, V(\varpi_{j_\mu})_{b_\mu}).$$

Example 3.13. Take $L = M = V(\varpi_1)_{(-q)^{-2}}$ and $N = V(\varpi_1)$ over $U'_q(A^{(1)}_2)$ where $p^* = (-q)^3$ and $\widetilde{p} = q^6$. Then we have

$$c_{M,L}(z) = \frac{[2][-2]}{[0][6]}, \quad c_{N,L}(z) = \frac{[0][-4]}{[-2][4]} \quad \text{and hence } c_{M,L}(z)c_{N,L}(z) = \frac{[2][-4]}{[6][4]}.$$

On the other hand, we have $M\nabla N = V(\varpi_2)_{(-q)^{-1}}$ and

$$c_{M\nabla N,L}(z) = \frac{[2][-4]}{[0][4]}.$$

Thus we have

$$\widetilde{\Lambda}(M,L)+\widetilde{\Lambda}(N,L) = (-1)+1 = 0, \quad \widetilde{\Lambda}(M\nabla N, L) = -1$$

and hence

$$\Lambda(M,L)+\Lambda(N,L)-\Lambda(M\nabla N,L) = 2 \quad \text{and} \quad c_{M\nabla N,L}(z)\times(1-z) = c_{M,L}(z)c_{N,L}(z).$$

DEFINITION 3.14 (see Corollary 3.19). For simple modules M and N in $\mathscr{C}_{\mathfrak{g}}$, we define $\mathfrak{d}(M,N)$ by

$$\mathfrak{d}(M,N) = \tfrac{1}{2}(\Lambda(M,N)+\Lambda(M^*,N)).$$

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Now we will prove that $\mathfrak{d}(M,N)$ is non-negative integer. In order to do that, we need some preparation.

LEMMA 3.15. *For simple modules M and N in $\mathscr{C}_{\mathfrak{g}}$, we have*

$$c_{M,N}(z)c_{M^*,N}(z) \equiv d_{M,N}(z)d_{N,M}(z^{-1})$$

and

$$\frac{c_{M,N}(z)}{c_{M,N}(\widetilde{p}z)} \equiv \frac{d_{M,N}(z)d_{N,M}(z^{-1})}{d_{M^*,N}(z)d_{N,M^*}(z^{-1})}$$

up to multiplication by an element of $\mathbf{k}[z^{\pm 1}]^{\times}$.

Proof. By Proposition 2.7(ii), we have

$$a_{M^*,N}(z)a_{M,N}(z) \equiv \frac{d_{M^*,N}(z)}{d_{N,M}(z^{-1})} \mod \mathbf{k}[z^{\pm 1}]^{\times}.$$

Recall that $c_{M,N}(z) = d_{M,N}(z)/a_{M,N}(z)$. Then we have

$$\begin{aligned} c_{M,N}(z)c_{M^*,N}(z) &= \frac{d_{M,N}(z)}{a_{M,N}(z)} \times \frac{d_{M^*,N}(z)}{a_{M^*,N}(z)} \\ &\equiv d_{M,N}(z) \times d_{M^*,N}(z) \times \frac{d_{N,M}(z^{-1})}{d_{M^*,N}(z)} \\ &\equiv d_{M,N}(z) \times d_{N,M}(z^{-1}) \mod \mathbf{k}[z^{\pm 1}]^{\times}. \end{aligned}$$

Thus we have

$$\frac{c_{M,N}(z)}{c_{M,N}(\widetilde{p}z)} = \frac{c_{M,N}(z)c_{M^*,N}(z)}{c_{M^*,N}(z)c_{M^{**},N}(z)} \equiv \frac{d_{M,N}(z)d_{N,M}(z^{-1})}{d_{M^*,N}(z)d_{N,M^*}(z^{-1})} \mod \mathbf{k}[z^{\pm 1}]^{\times}. \qquad \square$$

PROPOSITION 3.16. *For simple modules M and N in $\mathscr{C}_{\mathfrak{g}}$, we have*

$$\mathfrak{d}(M,N) = \mathrm{zero}_{z=1}(d_{M,N}(z)d_{N,M}(z^{-1})). \tag{3.4}$$

In particular,

$$\mathfrak{d}(M,N) \in \mathbb{Z}_{\geqslant 0},$$

and

$$\mathfrak{d}(M,N) = \mathfrak{d}(N,M). \tag{3.5}$$

Proof. By the preceding lemma,

$$\begin{aligned} 2\mathfrak{d}(M,N) &= \mathrm{Deg}(c_{M,N}(z)c_{M^*,N}(z)) = \mathrm{Deg}(d_{M,N}(z)d_{N,M}(z^{-1})) \\ &= 2\mathrm{zero}_{z=1}(d_{M,N}(z)d_{N,M}(z^{-1})). \end{aligned}$$

Here the last equality follows from Lemma 3.4(ii). The other assertions follow from (3.4). $\square$

COROLLARY 3.17. *Let M and N be simple modules in $\mathscr{C}_{\mathfrak{g}}$. Assume that one of them is real. Then M and N strongly commute if and only if $\mathfrak{d}(M,N) = 0$.*

Proof. The corollary follows from Proposition 3.16 and Theorem 2.5(ii). $\square$

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For $k \in \mathbb{Z}$ and a module M in $\mathscr{C}_{\mathfrak{g}}$, we define

$$\mathscr{D}^k(M) := \begin{cases} (\cdots(M^* \underbrace{)^*\cdots)^*}_{(-k)\text{-times}} & \text{if } k < 0, \\ \underbrace{{}^*(\cdots(}_{k\text{-times}} {}^*M)\cdots) & \text{if } k \geqslant 0. \end{cases}$$

Proposition 3.18. *For simples M and N in $\mathscr{C}_{\mathfrak{g}}$, we have*

$$\Lambda(M,N) = \Lambda(N^*,M) = \Lambda(N,{}^*M).$$

Proof. We shall prove $\Lambda(M,N) = \Lambda(N^*,M)$. The other equality follows from Lemma 3.7.

By (3.5), we have

$$\begin{aligned} &\Lambda(M,N) + \Lambda(M^*,N) = \Lambda(N,M) + \Lambda(N^*,M) \\ &\quad\Longleftrightarrow\ \Lambda(M,N) - \Lambda(N^*,M) = \Lambda(N,M) - \Lambda(M^*,N). \end{aligned}$$

Set

$$K(M,N) := \Lambda(M,N) - \Lambda(N^*,M).$$

Then we have $K(M,N) = K(N,M)$ and

$$K(M^*,N) = \Lambda(M^*,N) - \Lambda(N^*,M^*) \underset{(\star)}{=} \Lambda(M^*,N) - \Lambda(N,M) = -K(N,M) = -K(M,N),$$

where $(\star)$ follows from Lemma 3.7. Hence we have

$$K(M,N) = K(\mathscr{D}^{2n}(M),N) \quad \text{for any } n \in \mathbb{Z}.$$

Note that, for $n \gg 0$, we have

$$\Lambda(\mathscr{D}^{2n}(M),N) = \Lambda(M_{\widetilde{p}^n},N) = \Lambda(M,N_{\widetilde{p}^{-n}}) = \begin{cases} \Lambda^\infty(M,N) & \text{if } n \gg 0, \\ -\Lambda^\infty(M,N) & \text{if } n \ll 0, \end{cases} \tag{3.6a}$$

$$\Lambda(N^*,\mathscr{D}^{2n}(M)) = \Lambda(N^*,M_{\widetilde{p}^n}) = \begin{cases} -\Lambda^\infty(N^*,M) & \text{if } n \gg 0, \\ \Lambda^\infty(N^*,M) & \text{if } n \ll 0. \end{cases} \tag{3.6b}$$

Thus, for $n \gg 0$, we have $K(\mathscr{D}^{2n}(M),N) = -K(\mathscr{D}^{-2n}(M),N)$, which implies $K(M,N) = -K(M,N)$. Finally, we conclude that

$$K(M,N) = 0. \qquad \square$$

Corollary 3.19. *For any simple modules M and N in $\mathscr{C}_{\mathfrak{g}}$, we have*

$$\mathfrak{d}(M,N) = \tfrac{1}{2}(\Lambda(M,N) + \Lambda(N,M)).$$

Corollary 3.20. *For any real simple M in $\mathscr{C}_{\mathfrak{g}}$, we have*

$$\Lambda(M,M) = 0.$$

Proof. By Corollaries 3.17, 3.19 and the assumption that M is real simple, we have

$$0 = 2\mathfrak{d}(M,M) = \Lambda(M,M) + \Lambda(M,M),$$

which implies our assertion. $\square$

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Remark 3.21. The formula in Proposition 3.18 holds also for objects in the rigid monoidal category $\widetilde{\mathcal{C}}_w$ (see [KKOP19a]). Indeed, we have

$$\mathrm{Hom}(N^* \otimes M_z, M_z \otimes N^*) \simeq \mathrm{Hom}(M_z \otimes N, N \otimes M_z)$$

and hence their generators $R^{\mathrm{norm}}_{N^*,M_z}$ and $R^{\mathrm{norm}}_{M_z,N}$ have the same homogeneous degree.

PROPOSITION 3.22. *For simple modules M and N in $\mathscr{C}_{\mathfrak{g}}$, we have:*

(i) $\Lambda(M,N) = \sum_{k\in\mathbb{Z}} (-1)^{k+\delta(k<0)} \mathfrak{d}(M, \mathscr{D}^k N)$;
(ii) $\Lambda^\infty(M,N) = \sum_{k\in\mathbb{Z}} (-1)^k \mathfrak{d}(M, \mathscr{D}^k N)$.

Proof. Write $c_{M,N}(z) \equiv \prod \varphi(az)^{\eta_a} \bmod \mathbf{k}[z^{\pm 1}]^\times$. Then we have

$$\frac{c_{M,N}(z)}{c_{M,N}(\widetilde{p}z)} \equiv \prod (1-az)^{\eta_a}.$$

and hence

$$\begin{aligned}
\eta_{\widetilde{p}^k} &= \mathrm{zero}_{z=\widetilde{p}^{-k}}\left(\frac{c_{M,N}(z)}{c_{M,N}(\widetilde{p}z)}\right) = \mathrm{zero}_{z=1}\left(\frac{c_{M,N}(\widetilde{p}^{-k}z)}{c_{M,N}(\widetilde{p}^{-k+1}z)}\right) \\
&\underset{(\star)}{=} \mathfrak{d}(M, N_{\widetilde{p}^{-k}}) - \mathfrak{d}(M^*, N_{\widetilde{p}^{-k}}) \\
&= \mathfrak{d}(M, \mathscr{D}^{-2k}N) - \mathfrak{d}(M, \mathscr{D}^{-2k+1}N).
\end{aligned}$$

Here $(\star)$ follows from Lemma 3.15 and Proposition 3.16.

Thus we have

$$\begin{aligned}
\Lambda(M,N) &= \sum_{k\in\mathbb{Z}} (-1)^{\delta(k<0)} \eta_{\widetilde{p}^k} \\
&= \sum_{k\in\mathbb{Z}} (-1)^{\delta(k<0)} (\mathfrak{d}(M, \mathscr{D}^{2k}N) - \mathfrak{d}(M^*, \mathscr{D}^{2k}N)) \\
&= \sum_{k\in\mathbb{Z}} (-1)^{\delta(k<0)} (\mathfrak{d}(M, \mathscr{D}^{2k}N) - \mathfrak{d}(M, \mathscr{D}^{2k+1}N)) \\
&= \sum_{k\in\mathbb{Z}} (-1)^{k+\delta(k<0)} \mathfrak{d}(M, \mathscr{D}^k N),
\end{aligned}$$

which imply the first assertion. Similarly, we have

$$\Lambda^\infty(M,N) = \sum_{k\in\mathbb{Z}} (\mathfrak{d}(M, \mathscr{D}^{-2k}N) - \mathfrak{d}(M, \mathscr{D}^{-2k+1}N)) = \sum_{k\in\mathbb{Z}} (-1)^k \mathfrak{d}(M, \mathscr{D}^k N). \qquad \square$$

The following corollary is a direct consequence of Proposition 3.22 and (3.5).

COROLLARY 3.23. *For simple modules M and N in $\mathscr{C}_{\mathfrak{g}}$, we have:*

(1) $\Lambda^\infty(M,N) = \Lambda^\infty(N,M)$;
(2) $\Lambda^\infty(M,N) = -\Lambda^\infty(M^*,N) = -\Lambda^\infty({}^*M,N)$.

Proof. Sine $\mathfrak{d}(M,N) = \mathfrak{d}(\mathscr{D}^k M, \mathscr{D}^k N)$, we have

$$\begin{aligned}
\Lambda^\infty(M,N) &= \sum_{k\in\mathbb{Z}} (-1)^k \mathfrak{d}(M, \mathscr{D}^k N) = \sum_{k\in\mathbb{Z}} (-1)^k \mathfrak{d}(\mathscr{D}^k M, N) \\
&= \sum_{k\in\mathbb{Z}} (-1)^k \mathfrak{d}(N, \mathscr{D}^k M) = \Lambda^\infty(N,M).
\end{aligned}$$

Hence the first assertion follows. The second assertion follows similarly. $\square$

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4. Further properties of the invariants

We start this section with the following proposition, which can be understood as a quantum affine analogue of [KKKO18, Proposition 3.2.8].

PROPOSITION 4.1. *Let N_1, N_2 and M be non-zero modules in $\mathscr{C}_\mathfrak{g}$ and let $f\colon N_1 \to N_2$ be a morphism. We assume that $R^{\mathrm{univ}}_{N_k,M_z}$ is rationally renormalizable for $k = 1, 2$.*

(i) *If f does not vanish, then $c_{N_1,M}(z)/c_{N_2,M}(z) \in \mathbf{k}(z)$ and*

$$\Lambda(N_1, M) - \Lambda(N_2, M) = 2\mathrm{zero}_{z=1}\left(\frac{c_{N_1,M}(z)}{c_{N_2,M}(z)}\right).$$

(ii) *If $\Lambda(M, N_1) = \Lambda(M, N_2)$, then the following diagram is commutative.*

$$\begin{array}{ccc} N_1 \otimes M & \xrightarrow{\mathbf{r}_{N_1,M}} & M \otimes N_1 \\ {\scriptstyle f\otimes M}\downarrow & & \downarrow{\scriptstyle M\otimes f} \\ N_2 \otimes M & \xrightarrow{\mathbf{r}_{N_2,M}} & M \otimes N_2 \end{array}$$

(iii) *If $\Lambda(N_1, M) > \Lambda(N_2, M)$, then the composition*

$$N_1 \otimes M \xrightarrow{\mathbf{r}_{N_1,M}} M \otimes N_1 \xrightarrow{M\circ f} M \otimes N_2$$

vanishes.

(iv) *If $\Lambda(N_1, M) < \Lambda(N_2, M)$, then the composition*

$$N_1 \otimes M \xrightarrow{f\circ M} N_2 \otimes M \xrightarrow{\mathbf{r}_{N_2,M}} M \otimes N_2$$

vanishes.

Although we do not write them, similar statements hold for $c_{M,N_k}(z)$ and $M \otimes N_k$.

Proof. Without loss of generality, we may assume that f is non-zero.

(i) Proposition 2.12 implies that $c_{N_1,M}(z)/c_{N_2,M}(z) \in \mathbf{k}(z)$. Hence we have by Lemma 3.4 that

$$2\mathrm{zero}_{z=1}\left(\frac{c_{N_1,M}(z)}{c_{N_2,M}(z)}\right) = \mathrm{Deg}\left(\frac{c_{N_1,M}(z)}{c_{N_2,M}(z)}\right) = \Lambda(N_1, M) - \Lambda(N_2, M).$$

Set $t = \mathrm{zero}_{z=1}(c_{N_1,M}(z)/c_{N_2,M}(z))$. Then we can write $g(z)c_{N_1,M}(z) = h(z)(z-1)^t c_{N_2,M}(z)$ for some $t \in \mathbb{Z}$ and $g(z), h(z) \in \mathbf{k}[z]$ which do not vanish at $z = 1$.

If $t \geqslant 0$, then we have the following commutative diagram.

$$\begin{array}{ccc} N_1 \otimes M_z & \xrightarrow{g(z)R^{\mathrm{ren}}_{N_1,M_z}} & M_z \otimes N_1 \\ {\scriptstyle f\otimes M_z}\downarrow & & \downarrow{\scriptstyle M_z\otimes f} \\ N_2 \otimes M_z & \xrightarrow{h(z)(z-1)^t R^{\mathrm{ren}}_{N_2,M_z}} & M_z \otimes N_2 \end{array} \tag{4.1}$$

(ii) Since $t = 0$, by specializing $z = 1$ in the above diagram, we obtain the commutativity of (4.1).

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(iii) Since $t > 0$, the homomorphism $h(z)(z-1)^t R^{\mathrm{ren}}_{N_2,M_z}$ vanishes at $z = 1$. Hence we have

$$(M \otimes f) \circ \mathbf{r}_{N_1,M} = (z-1)^t R^{\mathrm{ren}}_{N_2,M_z}|_{z=1} \circ (f \otimes M) = 0,$$

as desired.

(iv) Since $t < 0$, we have the following commutative diagram.

$$\begin{array}{ccc} N_1 \otimes M_z & \xrightarrow{g(z)(z-1)^{-t} R^{\mathrm{ren}}_{N_1,M_z}} & M_z \otimes N_1 \\ {\scriptstyle f\otimes M_z}\downarrow & & \downarrow{\scriptstyle M_z \otimes f} \\ N_2 \otimes M_z & \xrightarrow{h(z) R^{\mathrm{ren}}_{N_2,M_z}} & M_z \otimes N_2 \end{array} \tag{4.2}$$

Since $g(z)(z-1)^{-t} R^{\mathrm{ren}}_{N_1,M_z}$ vanishes at $z = 1$, we obtain the desired result. □

From the above proposition, we can show that the new invariants share similar properties with the one for quiver Hecke algebras studied in [KKKO18, § 3.2]. We will collect such properties. Since the proofs are similar, we sometimes omit the proofs.

PROPOSITION 4.2. *Let L, M and N be simple modules. Then we have*

$$\mathfrak{d}(S, L) \leqslant \mathfrak{d}(M, L) + \mathfrak{d}(N, L) \tag{4.3}$$

for any simple subquotient S of $M \otimes N$. Moreover, when L is real, the following conditions are equivalent.

(a) *L strongly commutes with M and N.*

(b) *Any simple subquotient S of $M \otimes N$ commutes with L and satisfies*

$$\Lambda(S, L) = \Lambda(M, L) + \Lambda(N, L).$$

(c) *Any simple subquotient S of $M \otimes N$ commutes with L and satisfies*

$$\Lambda(L, S) = \Lambda(L, M) + \Lambda(L, N).$$

LEMMA 4.3. *Let L, M and N be simple modules in $\mathscr{C}_{\mathfrak{g}}$, and assume that L is real.*

(i) *If L strongly commutes with N, then the diagram*

$$\begin{array}{ccc} (M \otimes N) \otimes L & \xrightarrow{\mathbf{r}_{M\otimes N,L}} & L \otimes (M \otimes N) \\ \downarrow & & \downarrow \\ (M \nabla N) \otimes L & \xrightarrow{\mathbf{r}_{M \nabla N,L}} & L \otimes (M \nabla N) \end{array}$$

commutes and

$$\Lambda(M \nabla N, L) = \Lambda(M, L) + \Lambda(N, L).$$

(ii) *If L strongly commutes with M, then the diagram*

$$\begin{array}{ccc} L \otimes (M \otimes N) & \xrightarrow{\mathbf{r}_{L,M\circ N}} & (M \otimes N) \otimes L \\ \downarrow & & \downarrow \\ L \otimes (M \nabla N) & \xrightarrow{\mathbf{r}_{L,M \nabla N}} & (M \nabla N) \otimes L \end{array}$$

commutes and

$$\Lambda(L, M \nabla N) = \Lambda(L, M) + \Lambda(L, N).$$

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Corollary 4.4. *Let L, M N be non-zero modules in $\mathscr{C}_{\mathfrak{g}}$. Assume that L is real. Then we have the following.*

(i) *If L^* and M strongly commute, then*

$$\Lambda(M \nabla N, L) = \Lambda(M, L) + \Lambda(N, L).$$

(ii) *If L and N^* strongly commute, then*

$$\Lambda(L, M \nabla N) = \Lambda(L, M) + \Lambda(L, N).$$

Proof. (i) We have

$$\Lambda(M \nabla N, L) \underset{(*)}{=} \Lambda(L^*, M \nabla N) \underset{(**)}{=} \Lambda(L^*, M) + \Lambda(L^*, N) = \Lambda(M, L) + \Lambda(N, L),$$

where $(*)$ follows from Proposition 3.18 and $(**)$ from Lemma 4.3. The proof of (ii) is similar.

$$\Lambda(L, M \nabla N) = \Lambda(M \nabla N, {}^*L) = \Lambda(M, {}^*L) + \Lambda(N, {}^*L) = \Lambda(L, M) + \Lambda(L, N). \qquad \square$$

Proposition 4.5. *Let M and N be non-zero modules in $\mathscr{C}_{\mathfrak{g}}$ and assume that M is real.*

(i) *Assume that N has a simple socle, $R^{\mathrm{univ}}_{N,M}$ is rationally renormalizable and the diagram*

$$\begin{array}{ccc} \mathrm{soc}(N) \otimes M & \xrightarrow{\mathbf{r}_{\mathrm{soc}(N),M}} & M \otimes \mathrm{soc}(N) \\ \downarrow & & \downarrow \\ N \otimes M & \xrightarrow{\mathbf{r}_{N,M}} & M \otimes N \end{array}$$

commutes up to a non-zero constant multiple. Then $M \Delta \mathrm{soc}(N)$ is isomorphic to the socle of $M \otimes N$. In particular, $M \otimes N$ has a simple socle.

(ii) *Assume that N has a simple head, $R^{\mathrm{univ}}_{M,N}$ is rationally renormalizable and the diagram*

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\mathbf{r}_{M,N}} & N \otimes M \\ \downarrow & & \downarrow \\ M \otimes \mathrm{hd}(N) & \xrightarrow{\mathbf{r}_{M,\mathrm{hd}(N)}} & \mathrm{hd}(N) \otimes M \end{array}$$

commutes up to a non-zero constant multiple. Then $M \nabla \mathrm{hd}(N)$ is equal to the simple head of $M \otimes N$.

Proof. Let S be an arbitrary simple submodule of $M \otimes N$. Then we have the following commutative diagram

$$\begin{array}{ccc} S \otimes M_z & \xrightarrow{f(z)(z-1)^m R^{\mathrm{ren}}_{S \otimes M_z}} & M_z \otimes S \\ \downarrow & & \downarrow \\ M \otimes N \otimes M_z & \xrightarrow{R^{\mathrm{ren}}_{M \otimes N, M_z}} & M_z \otimes M \otimes N \end{array}$$

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for some $f(z) \in \mathbf{k}(z)$ which is regular and does not vanish at $z = 1$ and $m \in \mathbb{Z}_{\geqslant 0}$. By specializing at $z = 1$, we have a commutative diagram (up to a constant multiple).

$$\begin{array}{ccc} S \otimes M & \longrightarrow & M \otimes S \\ \downarrow & & \downarrow \\ M \otimes N \otimes M & \xrightarrow{M \otimes \mathbf{r}_{N,M}} & M \otimes M \otimes N \end{array}$$

Here, we use the fact that $\mathbf{r}_{M\otimes N,M} = (\mathbf{r}_{M,M} \otimes N) \circ (M \otimes \mathbf{r}_{N,M})$ and $\mathbf{r}_{M,M}$ is equal to $\mathrm{id}_{M\otimes M}$ up to a non-zero constant multiple, because M is real.

It follows that $S \otimes M \subset M \otimes (\mathbf{r}_{N,M})^{-1}(S)$. Hence there exists a submodule K of N such that $S \subset M \otimes K$ and $K \otimes M \subset (\mathbf{r}_{N,M})^{-1}(S)$ by Lemma 2.10. Hence $K \neq 0$ and $\mathrm{soc}(N) \subset K$ by the assumption. Hence $\mathbf{r}_{N,M}(\mathrm{soc}(N) \otimes M) \subset \mathbf{r}_{N,M}(K \otimes M) \subset S$. Since $\mathbf{r}_{N,M}(\mathrm{soc}(N) \otimes M)$ is non-zero by the assumption, we have $\mathbf{r}_{N,M}(\mathrm{soc}(N) \otimes M) = S$. Thus we obtain the desired result for the first assertion.

The second assertion can be proved similarly. □

PROPOSITION 4.6. *Let L, M and N be simple modules. We assume that L is real and one of M and N is real.*

(i) *If $\Lambda(L, M \nabla N) = \Lambda(L, M) + \Lambda(L, N)$, then $L \otimes M \otimes N$ has a simple head and $N \otimes M \otimes L$ has a simple socle.*

(ii) *If $\Lambda(M \nabla N, L) = \Lambda(M, L) + \Lambda(N, L)$, then $M \otimes N \otimes L$ has a simple head and $L \otimes N \otimes M$ has a simple socle.*

(iii) *If $\mathfrak{d}(L, M \nabla N) = \mathfrak{d}(L, M) + \mathfrak{d}(L, N)$, then $L \otimes M \otimes N$ and $M \otimes N \otimes L$ have simple heads, and $N \otimes M \otimes L$ and $L \otimes N \otimes M$ have simple socles.*

PROPOSITION 4.7. *Let M and N be simple modules. Assume that one of them is real and $\mathfrak{d}(M, N) = 1$. Then we have an exact sequence*

$$0 \to M \Delta N \to M \otimes N \to M \nabla N \to 0.$$

In particular, $M \otimes N$ has length 2.

Proof. By Theorem 2.13, Proposition 3.16 and (2.13), we can apply the same argument in the proof of [KO19, Lemma 7.3]. □

DEFINITION 4.8. For simple modules M and M' in $\mathscr{C}_{\mathfrak{g}}$, we say that they are *simply linked* if $\mathfrak{d}(M, M') = 1$.

PROPOSITION 4.9. *Let X, Y, M and N be simple modules in $\mathscr{C}_{\mathfrak{g}}$. Assume that there is an exact sequence*

$$0 \to X \to M \otimes N \to Y \to 0,$$

and $X \otimes N$ and $Y \otimes N$ are simple.

(i) *If $X \otimes N \not\simeq Y \otimes N$, then N is a real simple module.*

(ii) *If M is real, then N is a real simple module.*

LEMMA 4.10. *Let $\{M_i\}_{1\leqslant i\leqslant n}$ and $\{N_i\}_{1\leqslant i\leqslant n}$ be a pair of commuting families of real simple modules in $\mathscr{C}_{\mathfrak{g}}$. We assume that:*

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(a) $\{M_i \nabla N_i\}_{1\leqslant i\leqslant n}$ *is a commuting family of real simple modules;*
(b) $M_i \nabla N_i$ *commutes with* N_j *for any* $1 \leqslant i, j \leqslant n$.

Then we have

$$\Big(\bigotimes_{1\leqslant i\leqslant n} M_i\Big) \nabla \Big(\bigotimes_{1\leqslant j\leqslant n} N_j\Big) \simeq \bigotimes_{1\leqslant i\leqslant n} (M_i \nabla N_i).$$

Theorem 4.11. *Let M and N be simple modules. We assume that M is real. Then we have the equalities in the Grothendieck group $K(\mathscr{C}_{\mathfrak{g}})$:*

(i) $[M\otimes N] = [M \nabla N] + \sum_k [S_k]$ *with simple modules S_k such that* $\Lambda(M, S_k) < \Lambda(M, M \nabla N) = \Lambda(M, N)$;
(ii) $[M \otimes N] = [M \Delta N] + \sum_k [S_k]$ *with simple modules S_k such that* $\Lambda(S_k, M) < \Lambda(M \Delta N, M) = \Lambda(N, M)$;
(iii) $[N \otimes M] = [N \nabla , M] + \sum_k [S_k]$ *with simple modules S_k such that* $\Lambda(S_k, M) < \Lambda(N \nabla M, M) = \Lambda(N, M)$;
(iv) $[N\otimes M] = [N \Delta M] + \sum_k [S_k]$ *with simple modules S_k such that* $\Lambda(M, S_k) < \Lambda(M, N \Delta M) = \Lambda(M, N)$.

In particular, $M \nabla N$ as well as $M \Delta N$ appears only once in the Jordan–Hölder series of $M \otimes N$ in $\mathscr{C}_{\mathfrak{g}}$.

Proof. We shall prove only statement (iii). The other statements are proved similarly. First remark that $\Lambda(N \nabla M, M) = \Lambda(N, M) + \Lambda(M, M) = \Lambda(N, M)$ by Lemma 4.3 and Corollary 3.20.

Let

$$N \otimes M = K_0 \supset K_1 \supset \cdots \supset K_\ell \supset K_{\ell+1} = 0$$

be a Jordan–Hölder series of $N \otimes M$. Then we have $K_0/K_1 \simeq N \nabla M$. Let us consider the renormalized R-matrix $R^{\mathrm{ren}}_{N\otimes M,M_z} = (R^{\mathrm{ren}}_{N,M_z} \otimes M) \circ (N \otimes R^{\mathrm{ren}}_{M,M_z})$

$$N \otimes M \otimes M_z \xrightarrow{N\otimes R^{\mathrm{ren}}_{M,M_z}} N \otimes M_z \otimes M \xrightarrow{R^{\mathrm{ren}}_{N,M_z}\otimes M} M_z \otimes N \otimes M.$$

Then $R^{\mathrm{ren}}_{N\otimes M,M_z}$ sends $K_k \otimes M_z$ to $M_z \otimes K_k$ for any k. By evaluating the above diagram at $z = 1$, we obtain the following.

$$\begin{array}{ccc} N \otimes M \otimes M & \xrightarrow{\mathbf{r}_{N,M}\otimes M} & M \otimes N \otimes M \\ \uparrow & & \uparrow \\ K_1 \otimes M & \longrightarrow & M \otimes K_1 \end{array}$$

Since $\mathrm{Im}(\mathbf{r}_{N,M}\colon N \otimes M \to M \otimes N) \simeq (N \otimes M)/K_1$, we have $\mathbf{r}_{M,N}(K_1) = 0$. Hence, $R^{\mathrm{ren}}_{N\otimes M,M_z}$ sends $K_1 \otimes M_z$ to $(M_z \otimes K_1) \cap (z-1)(M_z \otimes (N \otimes M)) = (z-1)(M_z \otimes K_1)$. Thus $(z-1)^{-1} R^{\mathrm{ren}}_{N\otimes M,M_z}|_{K_1\otimes M_z}$ is well defined. Hence, we have

$$\Lambda(K_1, M) \leqslant \Lambda(N \otimes M, M) - 1 = \Lambda(N, M) - 1.$$

Hence we have $\Lambda(K_k/K_{k+1}, M) \leqslant \Lambda(K_1, M) < \Lambda(N, M)$ for $k \geqslant 1$ by Proposition 3.9. □

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Corollary 4.12. *Let M and N be simple modules in $\mathscr{C}_{\mathfrak{g}}$. We assume that one of them is real and $M \otimes N$ is not simple. We write*

$$[M \otimes N] = [M \nabla N] + [M \Delta N] + \sum_k [S_k]$$

with simple modules S_k in the Grothendieck ring $K(\mathscr{C}_{\mathfrak{g}})$. Then we have the following.

(i) *If M is real, then we have $\Lambda(M, M \Delta N) < \Lambda(M, N)$, $\Lambda(M \nabla N, M) < \Lambda(N, M)$ and $\Lambda(M, S_k) < \Lambda(M, N)$, $\Lambda(S_k, M) < \Lambda(N, M)$.*

(ii) *If N is real, then we have $\Lambda(N, M \nabla N) < \Lambda(N, M)$, $\Lambda(M \Delta N, N) < \Lambda(M, N)$ and $\Lambda(N, S_k) < \Lambda(N, M)$, $\Lambda(S_k, N) < \Lambda(M, N)$.*

The following theorem is a $U'_q(\mathfrak{g})$-analogue of [KK19, Theorem 4.1].

Theorem 4.13. *Let X be a simple module and M a real simple module in $\mathscr{C}_{\mathfrak{g}}$. If $[X] = [M]\phi$ for some ϕ in $K(\mathscr{C}_{\mathfrak{g}})$, then $X \simeq M \otimes Y$ for some simple module Y in $\mathscr{C}_{\mathfrak{g}}$ which strongly commutes with M.*

Proof. We may assume that

$$\phi = \sum_{i \in K} [Y_i] - \sum_{j \in K'} [Z_j],$$

where Y_i and Z_j are simple modules in $\mathscr{C}_{\mathfrak{g}}$ and there is no pair $(i, j) \in K \times K'$ such that $Y_i \simeq Z_j$. It follows that

$$[X] + \sum_{j \in K'} [M \otimes Z_j] = \sum_{i \in K} [M \otimes Y_i] \quad \text{and} \quad [X] + \sum_{j \in K'} [Z_j \otimes M] = \sum_{i \in K} [Y_i \otimes M]$$

in $K(\mathscr{C}_{\mathfrak{g}})$. Take i_0 such that $\Lambda(M, Y_{i_0}) = \max\{\Lambda(M, Y_i) \mid i \in K\}$. For any $j \in K'$, the head $M \nabla Z_j$ appears as a subquotient of some $M \otimes Y_i$. Since $M \nabla Z_j \not\simeq M \nabla Y_i$, we have

$$\Lambda(M, Z_j) = \Lambda(M, M \nabla Z_j) < \Lambda(M, M \nabla Y_i) = \Lambda(M, Y_i) \leqslant \Lambda(M, Y_{i_0}).$$

Since any simple subquotient S of $M \otimes Z_j$ satisfies

$$\Lambda(M, S) \leqslant \Lambda(M, Z_j) < \Lambda(M, Y_{i_0}) = \Lambda(M, M \nabla Y_{i_0}),$$

we conclude that $M \nabla Y_{i_0}$ does not appear in $M \otimes Z_j$ for any $j \in K'$. Hence

$$X \simeq M \nabla Y_{i_0}.$$

In particular, we have

$$\Lambda(M, Y_i) \leqslant \Lambda(M, Y_{i_0}) = \Lambda(M, M \nabla Y_{i_0}) = \Lambda(M, X) \quad \text{for any } i \in K.$$

Take i_1 such that $\Lambda(Y_{i_1}, M) = \max\{\Lambda(Y_i, M) \mid i \in K\}$. For any $j \in K'$, the head $Z_j \nabla M$ appears as a subquotient of some $Y_i \otimes M$. Since $Z_j \nabla M \not\simeq Y_i \nabla M$, we have

$$\Lambda(Z_j, M) = \Lambda(Z_j \nabla M, M) < \Lambda(Y_i \nabla M, M) = \Lambda(M, Y_i) \leqslant \Lambda(Y_{i_1}, M).$$

Thus, by the same reasoning as above, we have

$$X \simeq Y_{i_1} \nabla M \simeq M \Delta Y_{i_1}.$$

In particular, we have

$$\Lambda(Y_{i_0}, M) \leqslant \Lambda(Y_{i_1}, M) = \Lambda(Y_{i_1} \nabla M, M) = \Lambda(X, M) = \Lambda(M \nabla Y_{i_0}, M).$$

Hence, if M and Y_{i_0} do not strongly commute, the inequality $\Lambda(Y_{i_0}, M) \leqslant \Lambda(M \nabla Y_{i_0}, M)$ contradicts Corollary 4.12(i). Thus M and Y_{i_0} strongly commute and hence $X \simeq \mathrm{M} \otimes Y_{i_0}$. □

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DEFINITION 4.14 (Cf. [KK19, Definition 2.5]). A sequence $(L_1, \dots, L_r)$ of real simple modules in $\mathscr{C}_{\mathfrak{g}}$ is called a *normal sequence* if the composition of the R-matrices

$$\begin{aligned}\mathbf{r}_{L_1,\dots,L_r} &:= \prod_{1 \leqslant i < k \leqslant r} \mathbf{r}_{L_i,L_k} \\ &= (\mathbf{r}_{L_{r-1},L_r}) \circ \cdots \circ (\mathbf{r}_{L_2,L_r} \circ \cdots \circ \mathbf{r}_{L_2,L_3}) \circ (\mathbf{r}_{L_1,L_r} \circ \cdots \circ \mathbf{r}_{L_1,L_2}) \\ &\quad : L_1 \otimes \cdots \otimes L_r \longrightarrow L_r \otimes \cdots \otimes L_1\end{aligned}$$

does not vanish.

The following two lemmas can be proved by the same arguments in [KK19, § 2.3] with Λ.

LEMMA 4.15. *If $(L_1, \dots, L_r)$ is a normal sequence of real simple modules in $\mathscr{C}_{\mathfrak{g}}$, then the image of $\mathbf{r}_{L_1,\dots,L_r}$ is simple and coincides with the head of $L_1 \circ \cdots \circ L_r$ and also with the socle of $L_r \circ \cdots \circ L_1$,*

LEMMA 4.16. *Let $(L_1, \dots, L_r)$ be a sequence of real simple modules in $\mathscr{C}_{\mathfrak{g}}$. Then the following three conditions are equivalent:*

(a) *$(L_1, \dots, L_r)$ is a normal sequence;*
(b) *$(L_2, \dots, L_r)$ is a normal sequence and*

$$\Lambda(L_1, \mathrm{hd}(L_2 \otimes \cdots \otimes L_r)) = \sum_{2 \leqslant j \leqslant r} \Lambda(L_1, L_j);$$

(c) *$(L_1, \dots, L_{r-1})$ is a normal sequence and*

$$\Lambda(\mathrm{hd}(L_1 \otimes \cdots \otimes L_{r-1}), L_r) = \sum_{1 \leqslant j \leqslant r-1} \Lambda(L_j, L_r).$$

LEMMA 4.17. *For real simple modules L, M and N in $\mathscr{C}_{\mathfrak{g}}$, (L, M, N) is a normal sequence if either L and M strongly commute or L and N^* strongly commute.*

Proof. The first case follows from Lemma 4.3, and the second case from Corollary 4.4. □

COROLLARY 4.18. *For real simple modules L, M and N in $\mathscr{C}_{\mathfrak{g}}$, (L^*, M, N) is a normal sequence if and only if (M, N, L) is a normal sequence.*

Proof. Proposition 3.18 implies that

$$\Lambda(L^*, M) + \Lambda(L^*, N) - \Lambda(L^*, M \nabla N) = \Lambda(M, L) + \Lambda(N, L) - \Lambda(M \nabla N.L).$$

Then our assertion follows from Lemma 4.16 since (M, N) is a normal sequence. □

5. Cluster algebras

In this section, we briefly recall the definition of cluster algebra with little modifications. For more detail, we refer the reader to [BZ05, FZ02]. Fix a countable index set $K = K^{\mathrm{ex}} \sqcup K^{\mathrm{fr}}$ which decomposes into subset K^{ex} of exchangeable indices and a subset K^{fr} of frozen indices.

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Let $\widetilde{B} = (b_{ij})_{(i,j)\in K\times K^{\mathrm{ex}}}$ be an integer-valued matrix such that the following hold.

$$\begin{aligned}&\text{(a) For each } j \in K^{\mathrm{ex}}, \text{ there exist finitely many } i \in K \text{ such that } b_{ij} \neq 0.\\ &\text{(b) The } \textit{principal part } B := (b_{ij})_{i,j\in K^{\mathrm{ex}}} \text{ is skew-symmetric.}\end{aligned} \tag{5.1}$$

We extend the definition of b_{ij} for $(i,j) \in K \times K$ by

$$b_{ij} = -b_{ji} \quad \text{if } i \in K^{\mathrm{ex}} \text{ and } j \in K \quad \text{and} \quad b_{ij} = 0 \text{ for } i,j \in K^{\mathrm{fr}},$$

so that $(b_{ij})_{i,j\in K}$ is skew-symmetric.

To the matrix $\widetilde{B}$, we associate the quiver $\mathfrak{Q}_{\widetilde{B}}$ such that the set of vertices is K and the number of arrows from $i \in K$ to $j \in K$ is $\max(0, b_{ij})$. Then, $\mathfrak{Q}_{\widetilde{B}}$ satisfies the following.

$$\begin{aligned}&\text{(a) The set of vertices of } \mathfrak{Q}_{\widetilde{B}} \text{ are labeled by } K.\\ &\text{(b) The quiver } \mathfrak{Q}_{\widetilde{B}} \text{ does not have any loop, any 2-cycle nor arrow between frozen}\\ &\quad\text{vertices.}\\ &\text{(c) Each exchangeable vertex } v \text{ of } \mathfrak{Q}_{\widetilde{B}} \text{ has } \textit{finite degree}\text{; that is, the number of arrows}\\ &\quad\text{incident with } v \text{ is finite.}\end{aligned} \tag{5.2}$$

Conversely, for a given quiver satisfying (5.2), we can associate a matrix $\widetilde{B}$ by

$$b_{ij} := (\text{the number of arrows from } i \text{ to } j) - (\text{the number of arrows from } j \text{ to } i). \tag{5.3}$$

Then $\widetilde{B}$ satisfies (5.1).

Let $L = (\lambda_{ij})_{i,j\in K}$ be a skew-symmetric integer-valued $K \times K$-matrix. We say that L is *compatible* with $\widetilde{B}$ with a positive integer $d \in \mathbb{Z}_{\geqslant 1}$, if

$$\sum_{k\in K} \lambda_{ik} b_{kj} = \delta_{i,j} d \quad \text{for each } i \in K \text{ and } j \in K^{\mathrm{ex}}.$$

Let $\{X_i\}$ be the set of mutually commuting indeterminates.

DEFINITION 5.1. For a commutative ring A, we say that a triple $\mathcal{S}_\Lambda = (\{x_i\}_{i\in K}, L, \widetilde{B})$ is a Λ-*seed* in A if we have the following.

(a) There exists an injective algebra homomorphism $\mathbb{Z}[X_i]_{i\in K}$ into A such that $X_i \mapsto x_i$.
(b) The pair $(L, \widetilde{B})$ is a compatible pair with respect to $d \in \mathbb{Z}_{\geqslant 1}$. *In this paper, we always assume that* $d = 2$.

For a Λ-seed $\mathcal{S}_\Lambda = (\{x_i\}_{i\in K}, L, \widetilde{B})$, we call the set $\{x_i\}_{i\in K}$ the *cluster* of $\mathcal{S}_\Lambda$ and its elements the *cluster variables*. An element of the form $x^{\mathbf{a}} (\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^{\oplus K})$ is called a *cluster monomial*, where

$$x^{\mathbf{c}} := \prod_{k\in K} x_{i_k}^{c_{i_k}} \quad \text{for } \mathbf{c} = (c_i)_{i\in K} \in \mathbb{Z}^{\oplus K}.$$

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Let $\mathcal{S}_\Lambda = (\{x_i\}_{i\in K}, L, \widetilde{B})$ be a Λ-seed in a field $\mathfrak{K}$ of characteristic 0. For each $k \in K^{\mathrm{ex}}$, we define:

$$\begin{aligned}
&\text{(a)}\quad \mu_k(L)_{ij} = \begin{cases} -\lambda_{kj} + \sum_{t\in K} \max(0, -b_{tk})\lambda_{tj} & \text{if } i = k, j \neq k, \\ -\lambda_{ik} + \sum_{t\in K} \max(0, -b_{tk})\lambda_{it} & \text{if } i \neq k, j = k, \\ \lambda_{ij} & \text{otherwise;} \end{cases} \\
&\text{(b)}\quad \mu_k(\widetilde{B})_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + (-1)^{\delta(b_{ik}<0)} \max(b_{ik}b_{kj}, 0) & \text{otherwise;} \end{cases} \\
&\text{(c)}\quad \mu_k(x)_i = \begin{cases} x^{\mathbf{a}'} + x^{\mathbf{a}''} & \text{if } i = k, \\ x_i & \text{if } i \neq k, \end{cases}
\end{aligned} \tag{5.4}$$

where $\mathbf{a}' := (a'_i)_{i\in K}$ and $\mathbf{a}'' := (a''_i)_{i\in K} \in \mathbb{Z}^{\oplus K}$ are defined as

$$a'_i = \begin{cases} -1 & \text{if } i = k, \\ \max(0, b_{ik}) & \text{if } i \neq k, \end{cases} \qquad a''_i = \begin{cases} -1 & \text{if } i = k, \\ \max(0, -b_{ik}) & \text{if } i \neq k. \end{cases}$$

Then the triple

$$\mu_k(\mathcal{S}_\Lambda) := (\{\mu_k(x)_i\}_{k\in K}, \mu_k(L), \mu_k(\widetilde{B}))$$

becomes a new Λ-seed in $\mathfrak{K}$ and we call it the *mutation* of $\mathcal{S}_\Lambda$ at k.

The *cluster algebra* $\mathscr{A}(\mathcal{S}_\Lambda)$ *associated to the* Λ*-seed* $\mathcal{S}_\Lambda$ is the $\mathbb{Z}$ -subalgebra of the field $\mathfrak{K}$ generated by all the cluster variables in the Λ-seeds obtained from $\mathcal{S}_\Lambda$ by all possible successive mutations.

A *cluster algebra structure associated to a* Λ*-seed* $\mathcal{S}_\Lambda$ on a $\mathbb{Z}$-algebra A is a family $\mathscr{F}$ of Λ-seeds in A such that we have the following.

(a) For any Λ-seed $\mathcal{S}_\Lambda$ in $\mathscr{F}$, the cluster algebra $\mathscr{A}(\mathcal{S}_\Lambda)$ is isomorphic to A.
(b) Any mutation of a Λ-seed in $\mathscr{F}$ is in $\mathscr{F}$.
(c) For any pair $\mathcal{S}_\Lambda$, $\mathcal{S}'_\Lambda$ of Λ-seeds in $\mathscr{F}$, $\mathcal{S}'_\Lambda$ can be obtained from $\mathcal{S}_\Lambda$ by a finite sequence of mutations.

Note that the definition of cluster algebra associated to a Λ-seed is designed for the Grothendieck ring $K(\mathscr{C}_\mathfrak{g})$ of $\mathscr{C}_\mathfrak{g}$ and can be understood as an intermediate one between a cluster algebra and a quantum cluster algebra. When we ignore L in each Λ-seed $\mathcal{S}_\Lambda$, we recover the definition of cluster algebra.

6. Monoidal categorification

In this section, we construct a $U'_q(\mathfrak{g})$-analogue of [KKKO18, § 7]. *From now on,* $\mathcal{C}$ *is a full subcategory of* $\mathscr{C}_\mathfrak{g}$ *stable under taking tensor products, subquotients and extensions.* Note that $K(\mathcal{C})$ has a $\mathbb{Z}$-basis consisting of the isomorphism classes of simple modules.

DEFINITION 6.1. A *monoidal seed in* $\mathcal{C}$ is a pair $\mathscr{S} = (\{M_i\}_{i\in K}, \widetilde{B})$ consisting of a strongly commuting family $\{M_i\}_{i\in K}$ of real simple modules in $\mathcal{C}$ and an integer-valued $K \times K^{\mathrm{ex}}$-matrix $\widetilde{B} = (b_{ij})_{(i,j)\in K\times K^{\mathrm{ex}}}$ satisfying the conditions in (5.1).

For $i \in K$, we call M_i the ith *cluster variable module* of $\mathscr{S}$.

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For a monoidal seed $\mathscr{S} = (\{M_i\}_{i\in K}, \widetilde{B})$, let $\Lambda^{\mathscr{S}} = (\Lambda_{ij}^{\mathscr{S}})_{i,j\in K}$ be the skew-symmetric matrix given by $\Lambda_{ij}^{\mathscr{S}} = \Lambda(M_i, M_j)$.

DEFINITION 6.2. For $k \in K^{\mathrm{ex}}$, we say that a monoidal seed $\mathscr{S} = (\{M_i\}_{i\in K}, \widetilde{B})$ *admits a mutation in direction* k if there exists a simple object $M_k' \in \mathcal{C}$ such that we have the following.

(a) There exist exact sequences in $\mathcal{C}$

$$0 \to \bigotimes_{b_{ik}>0} M_i^{\otimes b_{ik}} \to M_k \otimes M_k' \to \bigotimes_{b_{ik}<0} M_i^{\otimes(-b_{ik})} \to 0,$$
$$0 \to \bigotimes_{b_{ik}<0} M_i^{\otimes(-b_{ik})} \to M_k' \otimes M_k \to \bigotimes_{b_{ik}>0} M_i^{\otimes b_{ik}} \to 0.$$

(b) The pair $\mu_k(\mathscr{S}) := (\{M_i\}_{i\neq k} \cup \{M_k'\}, \mu_k(\widetilde{B}))$ is a monoidal seed in $\mathcal{C}$.

Condition (b) is equivalent to saying that M_k' is real and strongly commuting with M_i for any $i \in K \setminus \{k\}$.

DEFINITION 6.3. A monoidal seed $\mathscr{S} = (\{M_i\}_{i\in K}, \widetilde{B})$ is called *admissible* if, for each $k \in K^{\mathrm{ex}}$, there exists a simple object M_k' of $\mathcal{C}$ such that there is an exact sequence in $\mathcal{C}$

$$0 \to \bigotimes_{b_{ik}>0} M_i^{\otimes b_{ik}} \to M_k \otimes M_k' \to \bigotimes_{b_{ik}<0} M_i^{\otimes(-b_{ik})} \to 0, \tag{6.1}$$

and M_k' commutes with M_i for any $i \neq k$.

Note that M_k' is uniquely determined by k and $\mathscr{S}$. Indeed, it follows from $M_k \nabla M_k' \simeq \bigotimes_{b_{ik}<0} M_i^{\otimes(-b_{ik})}$ and [KKKO15a, Corollary 3.7].

It is evident that a monoidal seed which admits a mutation at all $k \in K^{\mathrm{ex}}$ is admissible. Indeed, the converse is true.

PROPOSITION 6.4. *Let $\mathscr{S} = (\{M_i\}_{i\in K}, \widetilde{B})$ be an admissible monoidal seed in $\mathcal{C}$ and $k \in K^{\mathrm{ex}}$, Let M_k' be as in Definition 6.3. Then we have the following properties.*

(i) *The monoidal seed $\mathscr{S}$ admits a mutation in direction k. In particular, M_k' is a real simple object.*

(ii) *For any $j \in K$, we have $(\Lambda^{\mathscr{S}}\ \widetilde{B})_{jk} = -2\delta_{jk}\mathfrak{d}(M_k, M_k')$.*

(iii) *For any $j \in K$, we have*

$$\Lambda(M_j, M_k') = -\Lambda(M_j, M_k) - \sum_{b_{ik}<0} \Lambda(M_j, M_i) b_{ik}, \tag{6.2}$$
$$\Lambda(M_k', M_j) = -\Lambda(M_k, M_j) + \sum_{b_{ik}>0} \Lambda(M_i, M_j) b_{ik}. \tag{6.3}$$

Proof. (i) The reality of M_k' follows from the exact sequence (6.1) by applying Proposition 4.9(ii) to the case

$$M = M_k, \quad N = M_k', \quad X = \bigotimes_{b_{ik}>0} M_i^{\otimes b_{ik}} \quad \text{and} \quad Y = \bigotimes_{b_{ik}<0} M_i^{\otimes(-b_{ik})}.$$

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Note that $N \otimes M$ has the same length as the one of $M \otimes N$, that is 2. Since $N \Delta M \simeq M \nabla N \simeq Y$ and $N \nabla M \simeq M \Delta N \simeq X$, we have an exact sequence $0 \to Y \to N \otimes M \to X \to 0$.

Property (iii) follows from

$$\Lambda(M_j, M_k) + \Lambda(M_j, M'_k) = \Lambda(M_j, M_k \nabla M'_k) = \Lambda\Bigg(M_j, \bigotimes_{b_{ik}<0} M_i^{\otimes(-b_{ik})}\Bigg)$$
$$= \sum_{b_{ik}<0} \Lambda(M_j, M_i)(-b_{ik})$$

and

$$\Lambda(M_k, M_j) + \Lambda(M'_k, M_j) = \Lambda(M'_k \nabla M_k, M_j) = \Lambda\Bigg(\bigotimes_{b_{ik}>0} M_i^{\otimes b_{ik}}, M_j\Bigg)$$
$$= \sum_{b_{ik}>0} \Lambda(M_i, M_j) b_{ik}.$$

Property (ii) follows from property (iii) as follows:

$$\begin{aligned} 2\delta_{jk}\mathfrak{d}(M_k, M'_k) &= 2\mathfrak{d}(M_j, M'_k) = \Lambda(M_j, M'_k) + \Lambda(M'_k, M_j) \\ &= -\Lambda(M_j, M_k) - \Lambda(M_k, M_j) - \sum_{b_{ik}<0} \Lambda(M_j, M_i) b_{ik} + \sum_{b_{ik}>0} \Lambda(M_i, M_j) b_{ik} \\ &= -2\mathfrak{d}(M_j, M_k) - \sum_{b_{ik}<0} \Lambda(M_j, M_i) b_{ik} - \sum_{b_{ik}>0} \Lambda(M_j, M_i) b_{ik} \\ &= -\sum_{i \in K} \Lambda(M_j, M_i) b_{ik}. \end{aligned}$$

□

DEFINITION 6.5. We say that a monoidal seed $\mathscr{S} = (\{M_i\}_{i \in K}, \widetilde{B})$ is Λ-admissible if $\mathscr{S}$ is an admissible monoidal seed and $(-\Lambda^{\mathscr{S}}, \widetilde{B})$ is compatible with 2; i.e.,

$$(\Lambda^{\mathscr{S}} \widetilde{B})_{jk} = -2\delta_{jk} \quad \text{for } (j,k) \in K \times K^{\mathrm{ex}}.$$

Note that the compatibility condition is equivalent to saying that $\mathfrak{d}(M_k, M'_k) = 1$ for any $k \in K^{\mathrm{ex}}$.

For a monoidal seed $\mathscr{S} = (\{M_i\}_{i \in K}, \widetilde{B})$ in $\mathcal{C}$, we define the triple $[\mathscr{S}]_\Lambda$ in $K(\mathcal{C})$ by

$$[\mathscr{S}]_\Lambda := (\{[M_i]\}_{i \in K}, -\Lambda^{\mathscr{S}}, \widetilde{B}).$$

If $\mathscr{S}$ is a Λ-admissible monoidal seed, then $[\mathscr{S}]_\Lambda$ is a Λ-seed.

The following lemma immediately follows from Proposition 6.4.

LEMMA 6.6. *Let $\mathscr{S} = (\{M_i\}_{i \in K}, \widetilde{B})$ be a Λ-admissible monoidal seed, and $k \in K^{\mathrm{ex}}$. Then we have*

$$\mu_k([\mathscr{S}]_\Lambda) = [\,\mu_k(\mathscr{S})\,]_\Lambda.$$

In particular, $(-\Lambda^{\mu_k(\mathscr{S})}, \mu_k(\widetilde{B}))$ is compatible with 2.

DEFINITION 6.7. A category $\mathcal{C}$ is called a Λ-*monoidal categorification* of a cluster algebra A if:

(1) the Grothendieck ring $K(\mathcal{C})$ is isomorphic to A;

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(2) there exists a monoidal seed $\mathscr{S} = (\{M_i\}_{i\in K}, \widetilde{B})$ in $\mathcal{C}$ such that

$$[\mathscr{S}]_\Lambda := (\{[M_i]\}_{i\in K}, -\Lambda^{\mathscr{S}}, \widetilde{B})$$

is the initial Λ-seed of A and $\mathscr{S}$ admits successive mutations in all possible directions.

DEFINITION 6.8. A family of real simple modules $\{M_i\}_{i\in K}$ in $\mathcal{C}$ is called a *maximal real commuting family in* $\mathcal{C}$ if it satisfies both the following.

(a) The modules $\{M_i\}_{i\in K}$ are mutually strongly commuting.
(b) If a simple module X strongly commutes with all the M_i, then X is isomorphic to $\bigotimes_{i\in K} M^{\otimes a_i}$ for some $\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^{\oplus K}$.

The following theorem can be proved similarly to its quiver Hecke algebra version [KK19, Theorem 2.4].

THEOREM 6.9. *If $\mathscr{S} = (\{M_i\}_{i\in K}, \widetilde{B})$ is a monoidal seed of $\mathcal{C}$ and induces a Λ-seed of A, then $\{M_i\}_{i\in K}$ is a maximal real commuting family in $\mathcal{C}$.*

After the introduction of new invariants for pairs of modules in $\mathscr{C}_{\mathfrak{g}}$, the following theorem can be proved similarly to the one in [KKKO18, Theorem 7.1.3] with a small modification. Since it is one of the principal results of this paper, we repeat its proof.

THEOREM 6.10. *Let $\mathscr{S} = (\{M_i\}_{i\in K}, \widetilde{B})$ be a Λ-admissible monoidal seed in $\mathcal{C}$, and set*

$$[\mathscr{S}]_\Lambda := (\{[M_i]\}_{i\in K}, -\Lambda^{\mathscr{S}}, \widetilde{B}).$$

We assume that

$$\text{the algebra } K(\mathcal{C}) \text{ is isomorphic to } \mathscr{A}([\mathscr{S}]_\Lambda). \tag{6.4}$$

Then, for each $x \in K^{\mathrm{ex}}$, the monoidal seed $\mu_x(\mathscr{S})$ is Λ-admissible in $\mathcal{C}$.

Proof. Set $N_i := \mu_x(M)_i$ and $b'_{ij} := \mu_x(\widetilde{B})_{ij}$ for $i \in K$ and $j \in K^{\mathrm{ex}}$, i.e.,

$$\mu_x(\mathscr{S}) = (\{N_i\}_{i\in K}, (b'_{i,j})_{(i,j)\in K\times K^{\mathrm{ex}}}).$$

By Definition 6.3, it is enough to show that, for any $y \in K^{\mathrm{ex}}$, there exists a simple module $M''_y \in \mathcal{C}$ such that there is a short exact sequence

$$0 \longrightarrow \bigotimes_{b'_{iy}>0} N_i^{\otimes b'_{iy}} \longrightarrow N_y \otimes M''_y \longrightarrow \bigotimes_{b'_{iy}<0} N_i^{\otimes(-b'_{iy})} \longrightarrow 0 \tag{6.5}$$

and

$$\mathfrak{d}(N_i, M''_y) = 0 \quad \text{for } i \neq y.$$

If $x = y$, then $b'_{iy} = -b_{ix}$ and hence $M''_y = M_x$ satisfies the desired condition.

Assume that $x \neq y$ and $b_{xy} = 0$. Then $b'_{iy} = b_{iy}$ for any i and $N_i = M_i$ for any $i \neq x$. Hence $M''_y = \mu_y(M)_y$ satisfies the desired condition.

We will show the assertion in the case $b_{xy} > 0$. We omit the proof of the case $b_{xy} < 0$ because it can be shown in a similar way.

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Recall that we have

$$b'_{iy} = \begin{cases} b_{iy} + b_{ix}b_{xy} & \text{if } b_{ix} > 0, \\ b_{iy} & \text{if } b_{ix} \leqslant 0 \end{cases} \tag{6.6}$$

for $i \in K$ different from x and y.

Set

$$\begin{aligned} &M'_x := \mu_x(M)_x, \quad M'_y := \mu_y(M)_y, \\ &C := \bigotimes_{b_{ix}>0} M_i^{\otimes b_{ix}}, \quad S := \bigotimes_{b_{ix}<0,\ i\neq y} M_i^{\otimes -b_{ix}}, \\ &P := \bigotimes_{b_{iy}>0, i\neq x} M_i^{\otimes b_{iy}}, \quad Q := \bigotimes_{b'_{iy}<0,\ i\neq x} M_i^{\otimes -b'_{iy}}, \\ &A := \bigotimes_{b'_{iy}\leqslant 0,\ b_{ix}>0} M_i^{\otimes b_{ix}b_{xy}} \otimes \bigotimes_{b_{iy}<0,\ b'_{iy}>0,\ b_{ix}>0} M_i^{\otimes -b_{iy}} \\ &\quad \simeq \bigotimes_{b_{iy}<0,\ b_{ix}>0} M_i^{\otimes \min(b_{ix}b_{xy}, -b_{iy})}, \\ &B := \bigotimes_{b_{iy}\geqslant 0,\ b_{ix}>0} M_i^{\otimes b_{ix}b_{xy}} \otimes \bigotimes_{b'_{iy}>0,\ b_{iy}<0,\ b_{ix}>0} M_i^{\otimes b'_{iy}}. \end{aligned}$$

Set

$$L := (M'_x)^{\otimes b_{xy}}, \quad V := M_x^{\otimes b_{xy}}$$

and

$$X := \bigotimes_{b_{iy}>0} M_i^{\otimes b_{iy}} \simeq M_x^{\otimes b_{xy}} \otimes P = V \otimes P, \quad Y := \bigotimes_{b_{iy}<0} M_i^{\otimes -b_{iy}} \simeq Q \otimes A.$$

Then (6.5) reads as

$$0 \longrightarrow B \otimes P \longrightarrow M_y \otimes M''_y \longrightarrow L \otimes Q \longrightarrow 0. \tag{6.7}$$

Note that we have

$$0 \to C \to M_x \otimes M'_x \to M_y^{\otimes b_{xy}} \otimes S \to 0, \tag{6.8}$$

$$0 \to X \to M_y \otimes M'_y \to Y \to 0. \tag{6.9}$$

Taking the tensor products of $L = (M'_x)^{\otimes b_{xy}}$ and (6.9), we obtain

$$\begin{aligned} &0 \longrightarrow L \otimes X \longrightarrow L \otimes (M_y \otimes M'_y) \longrightarrow L \otimes Y \longrightarrow 0, \\ &0 \longrightarrow X \otimes L \longrightarrow (M_y \otimes M'_y) \otimes L \longrightarrow Y \otimes L \longrightarrow 0. \end{aligned}$$

Since L commutes with M_y, we have

$$\begin{aligned} \Lambda(L, Y) &= \Lambda(L, M_y \nabla M'_y) \\ &= \Lambda(L, M_y) + \Lambda(L, M'_y) = \Lambda(L, M_y \otimes M'_y). \end{aligned}$$

On the other hand, we have

$$\Lambda(M'_x, X) - \Lambda(M'_x, Y) = \Lambda\Big(M'_x, \bigotimes_{b_{iy}>0} M_i^{\otimes b_{iy}}\Big) - \Lambda\Big(M'_x, \bigotimes_{b_{iy}<0} M_i^{\otimes -b_{iy}}\Big)$$

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$$
\begin{aligned}
&= \sum_{b_{iy}>0} \Lambda(M'_x, M_i) b_{iy} - \sum_{b_{iy}<0} \Lambda(M'_x, M_i)(-b_{iy}) \\
&= \sum_{i\in K} \Lambda(M'_x, M_i) b_{iy} = \sum_{i\neq x} \Lambda(M'_x, M_i) b_{iy} + \Lambda(M'_x, M_x) b_{xy} \\
&= \sum_{i\neq x} \Lambda(M'_x, M_i)(b'_{iy} - \delta(b_{ix} > 0) b_{ix} b_{xy}) + \Lambda(M'_x, M_x) b_{xy} \\
&= \sum_{i\neq x} \Lambda(M'_x, M_i) b'_{iy} - \sum_{b_{ix}>0} \Lambda(M'_x, M_i) b_{ix} b_{xy} + \Lambda(M'_x, M_x) b_{xy} \\
&\underset{(\star)}{=} 0 - \Lambda\Big(M'_x, \bigotimes_{b_{ix}>0} M_i^{\otimes b_{ix}}\Big) b_{xy} + \Lambda(M'_x, M_x) b_{xy} \\
&= \Big(-\Lambda(M'_x, \bigotimes_{b_{ix}>0} M_i^{\otimes b_{ix}}) + \Lambda(M'_x, M_x)\Big) b_{xy} \\
&= (-\Lambda(M'_x, M'_x \nabla M_x) + \Lambda(M'_x, M_x)) b_{xy} \\
&= (-\Lambda(M'_x, M'_x) - \Lambda(M'_x, M_x) + \Lambda(M'_x, M_x)) b_{xy} = 0.
\end{aligned}
$$

Note that we have used the compatibility of $(\Lambda(\mu_x(M)_i, \mu(M)_j))_{i,j}$ and $\mu_x(B)$ when we derive the equality $(\star)$.

Since $L = (M'_x)^{\otimes b_{xy}}$, the equality $\Lambda(M'_x, X) = \Lambda(M'_x, Y)$ implies

$$\Lambda(L, X) = \Lambda(L, Y) = \Lambda(L, M_y \otimes M'_y).$$

Hence the following diagram is commutative by Proposition 4.1(ii).

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L \otimes X & \longrightarrow & L \otimes (M_y \otimes M'_y) & \longrightarrow & L \otimes Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathbf{r}_{L,X}} & & \downarrow{\scriptstyle \mathbf{r}_{L,M_y\otimes M'_y}} & & \downarrow{\scriptstyle \mathbf{r}_{L,Y}} & & \\
0 & \longrightarrow & X \otimes L & \longrightarrow & (M_y \otimes M'_y) \otimes L & \longrightarrow & Y \otimes L & \longrightarrow & 0
\end{array}
$$

Note that since $L = (M'_x)^{\otimes b_{xy}}$ commutes with Q and A, $\mathbf{r}_{L,Y}$ is an isomorphism. Hence we have

$$\mathrm{Im}(\mathbf{r}_{L,Y}) \simeq L \otimes Y.$$

Therefore we obtain an exact sequence

$$0 \longrightarrow \mathrm{Im}(\mathbf{r}_{L,X}) \longrightarrow \mathrm{Im}(\mathbf{r}_{L,M_y\otimes M'_y}) \longrightarrow L \circ Y \longrightarrow 0. \tag{6.10}$$

On the other hand, $\mathbf{r}_{L,M_y\otimes M'_y}$ decomposes by Proposition 2.11 as follows.

$$
\begin{array}{ccccc}
 & & \overset{\mathbf{r}_{L,M_y\otimes M'_y}}{\frown} & & \\
L \otimes M_y \otimes M'_y & \xrightarrow[\mathbf{r}_{L,M_y}\otimes M'_y]{\sim} & M_y \otimes L \otimes M'_y & \xrightarrow[M_y\otimes \mathbf{r}_{L,M'_y}]{} & M_y \otimes M'_y \otimes L
\end{array}
$$

Since $L = (M'_x)^{\otimes b_{xy}}$ commutes with M_y, the homomorphisms $\mathbf{r}_{L,M_y} \otimes M'_y$ is an isomorphism and hence we have

$$\mathrm{Im}(\mathbf{r}_{L,M_y\otimes M'_y}) \simeq M_y \otimes (L \nabla M'_y).$$

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Similarly, $\mathbf{r}_{L,X}$ decomposes as follows.

$$L \otimes V \otimes P \xrightarrow[\mathbf{r}_{L,V}\otimes P]{} V \otimes L \otimes P \xrightarrow[V\otimes \mathbf{r}_{L,P}]{\sim} V \otimes P \otimes L, \qquad L \otimes V \otimes P \xrightarrow{\mathbf{r}_{L,X}} V \otimes P \otimes L$$

Since L commutes with P, the homomorphism $V \otimes \mathbf{r}_{L,P}$ is an isomorphism and hence we have

$$\mathrm{Im}(\mathbf{r}_{L,X}) \simeq (L \nabla V) \otimes P \simeq ((M'_x)^{\otimes b_{xy}} \nabla M_x^{\otimes b_{xy}}) \otimes P.$$

On the other hand, Lemma 4.10 implies that

$$(M'_x)^{\otimes b_{xy}} \nabla M_x^{\otimes b_{xy}} \simeq (M'_x \nabla M_x)^{\otimes b_{xy}} \simeq C^{\otimes b_{xy}} \simeq B \otimes A,$$

and hence we obtain

$$\mathrm{Im}(\mathbf{r}_{L,X}) \simeq (B \otimes P) \otimes A.$$

Thus the exact sequence (6.10) becomes the exact sequence in $\mathcal{C}$:

$$0 \longrightarrow (B \otimes P) \otimes A \longrightarrow M_y \otimes (L \nabla M'_y) \longrightarrow (L \otimes Q) \otimes A \longrightarrow 0. \tag{6.11}$$

Thus we obtain the identity in $K(\mathscr{C}_{\mathfrak{g}})$:

$$[M_y]\,[L \nabla M'_y] = ([B \otimes P] + [L \otimes Q])[A].$$

On the other hand, the hypothesis (6.4) implies that there exists $\phi \in K(\mathcal{C})$ corresponding to $\mu_y\mu_x([M])$ so that it satisfies

$$[M_y]\phi = [B \otimes P] + [L \otimes Q]. \tag{6.12}$$

Hence, in $K(\mathcal{C})$, we have

$$[M_y]\phi[A] = ([B \otimes P] + [L \otimes Q])[A] = [M_y][L \otimes M'_y].$$

Since $K(\mathcal{C})$ is an integral domain, we conclude that

$$\phi[A] = [L \nabla M'_y].$$

By Theorem 4.13, there exists a simple module M''_y such that $\phi = [M''_y]$, since A is real simple.

Now (6.12) implies

$$[M_y \otimes M''_y] = [B \otimes P] + [L \otimes Q].$$

Hence there exists an exact sequence

$$0 \longrightarrow W \longrightarrow M_y \otimes M''_y \longrightarrow Z \longrightarrow 0,$$

where $W = B \otimes P$ and $Z = L \otimes Q$ or $W = L \otimes Q$ and $Z = B \otimes P$.

Since $\Lambda(M_y, L \otimes Q) - \Lambda(M_y, B \otimes P) = -\sum_{b'_{iy}} \Lambda(M_y, M_i) b'_{iy} = 2\mathfrak{d}(M_y, M''_y) > 0$, we have

$$0 \longrightarrow B \otimes P \longrightarrow M_y \otimes M''_y \longrightarrow L \otimes Q \longrightarrow 0,$$

by Corollary 4.12.

Now it remains to prove that:

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(a) M''_y strongly commutes with M_i $(i \neq x, y)$ and M'_x;
(b) M''_y is real simple.

Take M as one of M_i $(i \neq x, y)$ and M'_x. Then $M''_y \otimes M$ is of length less than or equal to 2, since $M_y \otimes M''_y \otimes M$ is of length 2:

$$0 \longrightarrow H \otimes M \longrightarrow M_y \otimes M''_y \otimes M \longrightarrow G \otimes M \longrightarrow 0,$$

where $H := B \otimes Q \simeq \bigotimes_{b'_{iy}>0} M_i^{\otimes b'_{iy}}$ and $G := L \otimes Q$.

Assume that $M''_y \otimes M$ is of length 2:

$$0 \longrightarrow U \longrightarrow M''_y \otimes M \longrightarrow V \longrightarrow 0. \tag{6.13}$$

By taking tensor $M_y\otimes$ to (6.13), we have

$$[M_y][U] = [H][M].$$

Note that $[H \otimes M] = [\bigotimes_{b'_{iy}>0} M_i^{\otimes b'_{iy}}][M]$. Since $K(\mathcal{C})$ has a cluster algebra structure, the cluster variables $[M_y]$ and $[M]$ are prime [GLS13b]. However, $[M_y]$ does not divide either $[H]$ nor $[M]$ which contradicts $[M_y][U] = [H][M]$. Thus we obtain (a).

By (a), M''_y strongly commute with $L \otimes Q$ and $B \otimes P$, and hence M''_y is real simple by Proposition 4.9. It completes the proof of Theorem 6.9. □

Corollary 6.11. *Let $\mathscr{S} = (\{M_i\}_{i\in K}, \widetilde{B})$ be a Λ-admissible monoidal seed in $\mathcal{C}$. Under the assumption (6.4), $\mathcal{C}$ is a Λ-monoidal categorification of the cluster algebra $\mathscr{A}([\mathscr{S}]_\Lambda)$. Furthermore, the following statements hold.*

(i) *The monoidal seed $\mathscr{S} = (\{M_i\}_{i\in K}, \widetilde{B})$ admits successive mutations in all directions.*
(ii) *Any cluster monomial in $K(\mathcal{C})$ is the isomorphism class of a real simple object in $\mathcal{C}$.*
(iii) *Any cluster monomial in $K(\mathcal{C})$ is a Laurent polynomial of the initial cluster variables with coefficient in $\mathbb{Z}_{\geqslant 0}$.*
(iv) *For $k \in K^{\mathrm{ex}}$ and the kth cluster variable module $\widetilde{M}_k$ of a monoidal seed $\widetilde{\mathscr{S}}$ obtained by successive mutations from the initial monoidal seed $\mathscr{S}$, we have*

$$\mathfrak{d}(\widetilde{M}_k, \widetilde{M}'_k) = 1.$$

Here $\widetilde{M}'_k$ is the kth cluster variable module of $\mu_k(\widetilde{\mathscr{S}})$.
(v) *Any monoidal cluster $\{\widetilde{M}_i\}_{i\in K}$ is a maximal real commuting family.*

Remark 6.12. In [KKO19, KKOP19b], we constructed several examples of monoidal subcategories $\mathcal{C}$ of $\mathscr{C}_{\mathfrak{g}}$ such that $\mathcal{C}$ is a Λ-monoidal categorification of $K(\mathcal{C})$. In those papers, we employed the method of generalized Schur–Weyl duality functors. Namely, we constructed a monoidal functor $\mathcal{F}\colon \mathcal{C}_{\mathrm{QHA}} \to \mathcal{C}$ such that:

(a) $\mathcal{C}_{\mathrm{QHA}}$ is a certain monoidal category related with a quiver Hecke algebra;
(b) $\mathcal{C}_{\mathrm{QHA}}$ is a monoidal categorification of the cluster algebra $K(\mathcal{C}_{\mathrm{QHA}})$;
(c) $\mathcal{F}$ induces an isomorphism $K(\mathcal{C}_{\mathrm{QHA}}) \xrightarrow{\sim} K(\mathcal{C})$.

Further examples of monoidal categorifications obtained from Theorem 6.10 will be given in a forthcoming paper.

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