Ergod. Th. & Dynam. Sys. (2010), **30**, 1239–1257
doi:10.1017/S014338570900056X
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Multiple equivalent matings with the aeroplane polynomial

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(*Received* 7 *December* 2008 *and accepted in revised form* 22 *April* 2009)

Dedicated to the memory of Adrien Douady

Abstract. We produce arbitrarily large equivalence classes of matings with the aeroplane polynomial. These are obtained via a slight generalization of the technique of proof of a similar result for Wittner captures.

1. *Introduction*

1.1. Mating of polynomials is a construction that was invented by Douady and Hubbard in the 1980s, soon after their seminal work on the structure of the Mandelbrot set for quadratic polynomials [**1, 2**]. The construction is of a rational map $f_1 \amalg f_2$ from any two given critically finite quadratic polynomials f_1 and f_2; there are some restrictions on (f_1, f_2), but these are (with hindsight) very natural and simple. The construction suggests that the intricacy of structure of the parameter space of quadratic polynomials is, at the very least, reproduced in the parameter space of rational maps. Given that the coordinates of a mating are polynomials, one might expect a product structure of some sort in the space of rational maps. However, it has been known for some time (from unpublished work of Adam Epstein) that this cannot be true in any strong topological sense. Nonetheless, the way in which the structure translates to the parameter space of rational maps is even more interesting than might at first appear. This has been one aspect of an extensive study; see, for example, [**4–8**]. In the current work, we avoid direct reference to such theory and, instead, present some examples. The construction of these examples is self-contained, although they were found during a larger exploration of the theory. We show that, even if we restrict to a single choice of f_1, namely the so-called aeroplane polynomial, the map

$$f_2 \mapsto f_1 \amalg f_2$$

is unboundedly many-to-one. Thus, a critically finite quadratic rational map can be realized as a mating of the aeroplane polynomial with another polynomial in arbitrarily many ways.

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We also obtain a similar result for *captures*. The capture construction probably first appeared in written form in the 1988 thesis of Wittner [**10**], but it had been studied earlier by the Hubbard group in the 1980s. (The term 'capture' is used here in a more restrictive sense than is sometimes the case; its precise definition will be given later.) The capture $\sigma_{\beta_x} \circ f$ is a critically finite quadratic rational map made from a critically finite quadratic polynomial f and a path β_x from the fixed critical point ∞ of f which crosses the Julia set of f just once—at a point x in the boundary of a preperiodic Fatou component—and ends at the precritical point in that Fatou component. As in the case of the mating construction, there are some extra, natural and easily stated restrictions. One of the constant refrains of [**4–8**] is that 'type III' maps, of which captures are examples, are technically easier to work with than 'type IV' maps, which include matings. It is therefore no surprise that the mating result is essentially obtained from the capture result. The way in which this is done is reminiscent of the argument in [**4**]; the proof is not difficult and may indicate that more general theoretical results can be obtained in a similar fashion.

1.2. *Lamination maps.* We use the notation and conventions of [**7**]. This means, in particular, that we work with the lamination map which is Thurston equivalent to the aeroplane polynomial, rather than with the aeroplane polynomial itself. We call this lamination map $s_{3/7}$. It preserves the unit disc, unit circle and exterior of the unit disc. On the exterior of the unit disc, $s_{3/7}(z) = z^2$; on the interior of the unit disc, $s_{3/7}$ preserves a lamination known as $L_{3/7}$, and also preserves the set of complementary components of $\bigcup L$ in the unit disc. This lamination is an example of a *quadratic invariant lamination*. It is a closed set of disjoint chords, or *leaves*, in the unit disc. *Invariance* of a lamination L in the unit disc means that:

- whenever there is a leaf of L joining z_1 and z_2, there is also a leaf of L joining z_1^2 and z_2^2;
- whenever there is a chord joining z_1 and z_2, there are points $\pm z_3$ and $\pm z_4$ with $z_3^2 = z_1$ and $z_4^2 = z_2$ such that there are leaves of L joining z_3 to z_4 and $-z_3$ to $-z_4$.

The label 3/7 on $L_{3/7}$ means that the (common) image of the two longest leaves of $L_{3/7}$ has one endpoint at $e^{2\pi i(3/7)}$. This image is known as the *minor leaf* of $L_{3/7}$. The other endpoint of the minor leaf is at $e^{2\pi i(4/7)}$. In what follows, we consider only invariant laminations in which every leaf has exactly two preimages. Such an invariant lamination is uniquely determined by its minor leaf. We therefore write $L_{3/7} = L_{4/7}$ and $s_{3/7} = s_{4/7}$. The minor leaf is periodic under $s_{3/7}$, of period 3. It is adjacent to a unique complementary component of $\bigcup L_{3/7}$ in the unit disc, also known as a *gap* of $\bigcup L_{3/7}$, which is also of period 3. The gap adjacent to the minor leaf has a connected preimage under $s_{3/7}$; this connected preimage contains 0 and is known as the *critical gap*, since it maps with degree two onto its image. It then maps homeomorphically on the rest of the cycle, and returns to itself under $s_{3/7}^3$ with degree two. This is consistent with $s_{3/7}$ having a unique critical point in the unit disc at 0, and with this critical point being of period 3. We choose $s_{3/7}$ like this.

More generally, L_p and s_p are uniquely determined in the same way, for any rational p in [0, 1] with odd denominator. So s_p is a critically periodic branched covering, with one fixed critical point (at ∞) and one periodic critical point of the same period as the period of p under $x \mapsto 2x \bmod 1$. There is a one-to-one correspondence between such maps s_p and

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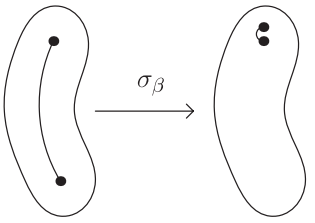


FIGURE 1. The homeomorphism σ_β.

critically periodic polynomials in the family $z^2 + c$ (for $c \in \mathbb{C}$), where the correspondence is defined by *Thurston equivalence*. In this paper, two critically finite branched coverings f_0 and f_1 with numbered postcritical sets are *(Thurston) equivalent* if there is a homotopy through critically finite branched coverings f_t such that the postcritical set $X(f_t)$ moves isotopically and the numbering of $X(f_0)$ and $X(f_1)$ is preserved by this isotopy. The stipulation that numbering be preserved is not universal, but it is convenient here, especially when we consider matings. For quadratic branched coverings with a fixed critical point, the requirement is superfluous because a fixed critical point moves isotopically through fixed critical points.

The *aeroplane polynomial* is the (unique) quadratic polynomial of the form $z^2 + c$ which is Thurston equivalent to $s_{3/7}$. The critical point 0 is of period 3. The parameter value c is real and in $(-2, -1)$, and it is the unique such value for which 0 is of period 3.

1.3. *Captures and matings.* For any path β, we define a homeomorphism σ_β, up to isotopy of a suitable type. The definition can be given for any path, but since we need it only for paths which are arcs, we shall state the definition only for such paths. This homeomorphism σ_β is the identity outside a suitably small disc neighbourhood of β and maps the startpoint of β to the endpoint, as shown in Figure 1.

A *capture path for* s_p is a path from ∞ to a non-periodic point x in the backward orbit of the (periodic) point 0 which crosses the unit circle just once, goes into a gap of L_p, and ends at x, with no intersections with L_p. A *capture by* s_p is then a branched covering of the form $\sigma_\beta \circ s_p$ for a capture path β. Such a branched covering is critically finite, with critical points at 0 and ∞ and critical values at $s_p(0)$ and x, the endpoint of β. Then 0 is periodic under $\sigma_\beta \circ s_p$, and the critical point x is eventually mapped by $\sigma_\beta \circ s_p$ to the periodic orbit of 0. We refer to such a degree-two branched covering, with one critical point periodic and the other non-periodic but in the backward orbit of the first, as being of *type III*.

In this paper we only consider captures by $s_{3/7}$, which, up to equivalence, are *captures by the aeroplane polynomial*, since they can equally well be described (up to equivalence) by capture paths in the dynamical plane of the aeroplane polynomial, that is, paths which cross the Julia set of the aeroplane polynomial just once. By the result of Tan Lei, or at least by the technique that led up to it ([**9**]; see also [**4**]), a capture $\sigma_\beta \circ s_{3/7}$ is Thurston equivalent to a quadratic rational map (which is unique up to Möbius conjugation) if and only if the endpoint x of the path is a point in the full orbit of 0 which is in the larger region

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of the unit disc bounded by the leaf of $L_{3/7}$ with endpoints at $e^{\pm 2\pi i(1/3)}$. This means that the crossing point by the capture path of the unit circle is at $e^{2\pi iy}$ for $y \in (-1/3, 1/3)$. In [**7**, §2.8], elementary equivalences between different captures by $s_{3/7}$ were discussed, for capture paths with the same endpoint. It was shown that any two captures $\sigma_{\beta_1} \circ s_{3/7}$ and $\sigma_{\beta_2} \circ s_{3/7}$ such that β_1 and β_2 have the same endpoint are Thurston equivalent, except when the common endpoint of the paths is in the region of the unit disc bounded between the leaf with endpoints $e^{\pm 2\pi i(1/3)}$ and the one with endpoints $e^{\pm 2\pi i(2/7)}$. In the exceptional case, $\sigma_{\beta_1} \circ s_{3/7}$ and $\sigma_{\beta_2} \circ s_{3/7}$ are still Thurston equivalent if β_1 and β_2 have the same endpoint and the S^1-crossing points $e^{2\pi iy_1}$ and $e^{2\pi iy_2}$ are such that y_1 and y_2 are both in $(2/7, 1/3)$ or both in $(2/3, 5/7)$. An example was given in [**7**, §9.2] of capture paths β_1 and β_2 with different endpoints, having crossing points $e^{2\pi iy_1}$ and $e^{2\pi iy_2}$ with $y_1 \in (2/7, 1/3)$ and $y_2 \in (2/3, 5/7)$, such that $\sigma_{\beta_1} \circ s_{3/7}$ and $\sigma_{\beta_2} \circ s_{3/7}$ are Thurston equivalent. It was also shown in the 'easy' main theorem of [**7**, §2.10 and §6.1] that captures with different endpoints are not Thurston equivalent except when the endpoints are in the region bounded by the leaves joining $e^{\pm 2\pi i(1/3)}$ and $e^{\pm 2\pi i(2/7)}$. (This is essentially a folklore result if the crossing point $e^{2\pi iy}$ satisfies $y \in (1/7, 2/7) \cup (5/7, 6/7)$, but not so if $y \in (-1/7, 1/7)$.) A detailed analysis of the *parameter capture map* for the aeroplane polynomial

$$[\beta] \mapsto [\sigma_\beta \circ s_{3/7}],$$

where $[\cdot]$ on the domain is suitable homotopy equivalence while $[\cdot]$ on the range is Thurston equivalence, will be given in [**8**]. The purpose of the present paper is simply to exhibit examples showing that the fibres of this capture map—that is, the sets of equivalent captures—can be arbitrarily large.

We also describe matings only up to Thurston equivalence. For (quadratic) lamination maps s_p and s_q, the *mating* $s_p \amalg s_q$ of s_p and s_q is the branched covering defined by

$$s_p \amalg s_q(z) = \begin{cases} s_p(z) & \text{if } |z| \le 1, \\ (s_q(z^{-1}))^{-1} & \text{if } |z| \ge 1. \end{cases}$$

Here we are considering only the case of $p = 3/7$. It follows from Tan Lei's theorem [**4, 9**] that $s_{3/7} \amalg s_q$ is Thurston equivalent to a quadratic rational map (unique up to Möbius conjugacy) if and only if $q \bmod 1 \in (-1/3, 1/3)$. As in the case of captures, it is essentially a folklore result that for $q_1 \in (1/7, 2/7) \cup (5/7, 6/7)$, branched coverings $s_{3/7} \amalg s_{q_1}$ and $s_{3/7} \amalg s_{q_2}$ are Thurston equivalent if and only if $s_{q_1} = s_{q_2}$, that is, $e^{2\pi iq_1}$ and $e^{2\pi iq_2}$ are endpoints of the same minor leaf (so that $L_{q_1} = L_{q_2}$). It is likely that the corresponding result for $q_1 \in (-1/7, 1/7)$ is true and can be proved from the analogous result for captures; it is hoped to include a proof of this in another paper in the near future. For $q_1 \in (2/7, 1/3) \cup (2/3, 5/7)$, the above statement is far from true, as this paper will show.

There is a way of describing lamination maps s_q and matings up to Thurston equivalence by means of homeomorphisms σ_β, which has been used in [**4, 5**] and will be useful here, too. First, we describe s_q up to Thurston equivalence, for any odd-denominator rational q in $(0, 1)$. Let s_0 denote the map $s_0(z) = z^2$. Suppose that $e^{2\pi iq}$ has period r under s_0. Let

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β be an arc from 0 to a point near $e^{2\pi iq}$ which does not intersect the unit circle. Let ζ be a similar path from 0 to $s_0^{r-1}(\mathrm{end}(\beta))$ which is in a small neighbourhood of $s_0^{-1}(\beta)$. Then the Thurston equivalence class of

$$\sigma_\zeta^{-1} \circ \sigma_\beta \circ s_0$$

is uniquely determined, assuming that sufficiently small neighbourhoods were chosen. We can choose β to intersect the minor leaf of L_q exactly once, and ζ so as not to intersect $L_q \cup S^1$ at all. It is then clear that $\sigma_\zeta^{-1} \circ \sigma_\beta \circ s_0$ preserves L_q, up to isotopy, and is Thurston equivalent to s_q.

For the similar description of matings, we can describe $s_p \amalg s_q$ as follows, again assuming that $e^{2\pi iq}$ is of period r under s_0. We take β to be a path from ∞ to a small neighbourhood of $e^{-2\pi iq}$ (not $e^{2\pi iq}$, because the identification with the unit disc is by $z \mapsto z^{-1}$) which does not cross the unit circle, and we let ζ be a path from ∞ to a small neighbourhood of $s_p^{r-1}(\mathrm{end}(\beta))$ which is in a small neighbourhood of $s_p^{-1}(\beta)$. Then

$$\sigma_\zeta^{-1} \circ \sigma_\beta \circ s_p$$

is Thurston equivalent to $s_p \amalg s_q$.

1.4. *The capture and mating theorems.* We shall prove the following.

THEOREM 1. (Capture theorem)

- *There is a pair of equivalent captures by the aeroplane polynomial, with the non-periodic critical point of preperiod* 13*, such that the capture paths have distinct endpoints but their S^1-crossing points are both in $\{e^{2\pi it} \mid t \in (2/7, 9/28)\}$.*
- *For each integer $n \geq 0$, there is a set of $n+2$ equivalent captures by the aeroplane polynomial, with the non-periodic critical point of preperiod $26 \cdot 2^n - 11$, such that all the capture paths have distinct endpoints but their S^1-crossing points are all in $\{e^{2\pi it} \mid t \in (2/7, 9/28)\}$.*

THEOREM 2. (Mating theorem) *For each integer $n \geq 0$, there is a set of $n+2$ equivalent matings $s_{3/7} \amalg s_q$ with the aeroplane polynomial such that the period is $30 \cdot 2^n - 12$ for the second critical point; moreover, $q \in (19/28, 5/7)$ for all such q.*

1.5. *Symbolic dynamics.* To identify points in S^1 and leaves and gaps of $L_{3/7}$, and also in order to do our calculations, we shall use the symbolic dynamics for $s_{3/7}$ used in [**7, 8**], with letters L_j and R_j, $1 \leq j \leq 3$, and C, BC and UC. The labelled regions are shown in Figure 2. The vertical lines separating regions join points $e^{\pm 2\pi i(r/14)}$ on the unit circle, for $1 \leq r \leq 6$. We use $D(L_j)$, $D(R_j)$ and $D(C)$ to denote the closures of the regions labelled by L_j, R_j and$C$, respectively. We define

$$D(UC) = D(C) \cap \{z \mid \mathrm{Im}(z) \geq 0\},$$
$$D(BC) = D(C) \cap \{z \mid \mathrm{Im}(z) \leq 0\}.$$

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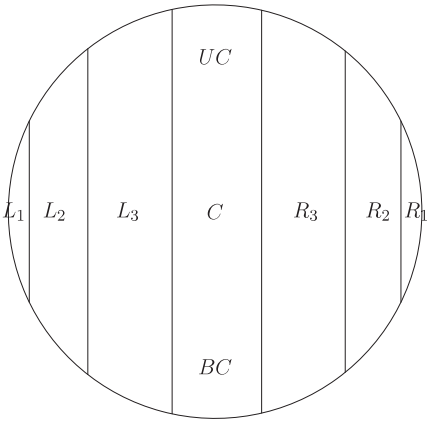


FIGURE 2. Regions in the unit disc.

Under the map $s_{3/7}$, the regions are mapped as follows:

$$D(R_1),\ D(L_1) \to D(R_1) \cup D(R_2),$$
$$D(R_2),\ D(L_2) \to D(R_3) \cup D(C),$$
$$D(R_3),\ D(L_3) \to D(L_3) \cup D(L_2),$$
$$D(C) \to D(L_1).$$

Each $D(R_j)$ or $D(L_j)$ is mapped homeomorphically by $s_{3/7}$ onto its image, while $D(C)$ is mapped with degree two onto $D(L_1)$ by $s_{3/7}$. Each of $D(BC)$ and $D(UC)$ is also mapped onto $D(L_1)$.

If $w = w_0 \cdots w_n$ is a word in the letters L_j, R_j, BC, UC and C, then we define

$$D(w) = \bigcap_{k=0}^{n} s_{3/7}^{-k}(D(w_k)).$$

We can make a similar definition for infinite words, in which case $D(w)$ is a point on the unit circle S^1, a leaf of $L_{3/7}$, or the closure of a gap of $L_{3/7}$. We define

$$Z_m = Z_m(s_{3/7}) = s_{3/7}^{-m}(\{0, s_{3/7}(0), s_{3/7}^2(0)\})$$

and

$$Z_\infty = \bigcup_{m \geq 0} Z_m.$$

If z is the point of Z_∞ of lowest preperiod in $D(w)$, then we write $z = p(w)$ and, sometimes, $D(z)$ for $D(w)$.

We shall often consider words which contain only the letters L_3, L_2 and R_3, possibly with C at the end. For any such word w, $D(w)$ is bounded in the unit disc by vertical chords. For these words, we shall use the notation $v < w$ if $D(v)$ is to the right of $D(w)$ in the unit disc. We shall avoid this notation in the case where v is a prefix of w or vice versa. We shall also sometimes write $v \leq w$ to mean that $v < w$ or $v = w$. For such a word w,

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we shall use $\beta(w)$ to stand for a capture path ending at $p(w)$ and crossing into $D(w)$ on the upper unit circle. The exact crossing point does not matter, because all such paths are homotopic via homotopies that fix the forward orbit of $p(w)$.

2. *Proofs*

2.1. *The basic idea.*

The following lemma provides the basic idea. The rest of the proof of the capture and mating theorems is obtained from variations on this idea.

LEMMA. *Let*

$$u_0 = L_3L_2R_3L_3L_2C,$$
$$v_0 = L_3L_2R_3L_3(L_2R_3)^2,$$
$$w_0 = L_3L_2R_3L_3^5.$$

Then the captures $\sigma_{\beta(w_0u_0)} \circ s_{3/7}$ and $\sigma_{\beta(v_0u_0)} \circ s_{3/7}$ are Thurston equivalent.

Proof. Let us write $s_{3/7}$ simply as s from now on. We also omit the index 0 and simply write v, w and u for v_0, w_0 and u_0. So take $x = p(wu)$ and $y = p(vu)$. Then $D(y)$ is to the right of $D(x)$, although both are in $D(L_3L_2R_3L_3)$. Note, also, that $p(u)$ is in the forward orbit under s of x and y, and that there is no other point in the forward orbit of y which is between $D(y)$ and $D(x)$. So now we take the subdisc D' of the unit disc which is bounded by $D(x)$ and $D(y)$. It does not matter whether we choose D' to contain $D(x)$ (or $D(y)$) or not, because we are only concerned with the isotopy class with respect to the strict forward orbit of y, but let us choose it to be strictly between $D(x)$ and $D(y)$. Choose a point $z \in Z_\infty$ whose word starts with L_3 and contains only the letters L_3, L_2 and R_3 such that z is not in $\bigcup_{n=0}^{\infty} s^{-n}(D' \cup D(y) \cup D(x))$ and $D(z)$ is to the right of $D(y)$. We are now going to use $\sigma_{\beta(z)} \circ s$ as a base for the construction. The exact position of z does not matter.

Let α be the simple loop based at z which crosses out of the gap containing z on the upper unit circle, crosses back to trace round D' in an anticlockwise direction, crosses out again on the upper boundary of the unit disc, then goes back across the upper unit circle into the gap containing z and, finally, returns to z. See Figure 3.

Let

$$O(z) = \{s^m(z) \mid m \geq 0\},$$

and similarly for any other point instead of z. Note that $p(u) = s^8(y) = s^8(x)$ is of preperiod 5 under s, and so $O(p(u)) \subset (\sigma_{\beta(z)} \circ s)^{-5}(O(z))$ and, for any $m \geq 5$, we have $s^{5-m}(O(p(u)) \subset (\sigma_{\beta(z)} \circ s)^{-m}(O(z))$. Now, for any $m \geq 5$, we can find a homeomorphism

$$\psi_m : (\overline{\mathbb{C}}, (\sigma_{\beta(z)} \circ s)^{-m}(O(z))) \to (\overline{\mathbb{C}}, (\sigma_{\beta(z)} \circ s)^{-m}(O(z)))$$

such that

$$(\sigma_{\beta(z)} \circ s, (\sigma_{\beta(z)} \circ s)^{-m}(O(z))) \simeq_{\psi_m} (\sigma_\alpha \circ \sigma_{\beta(z)} \circ s, (\sigma_{\beta(z)} \circ s)^{-m}(O(z))), \qquad (2.1.1)$$

where the notation $\simeq_{\psi_m}$ (which is also used in [**4–8**]) means that $\psi_m \circ \sigma_{\beta(z)} \circ s \circ \psi_m^{-1}$ and $\sigma_\alpha \circ \sigma_{\beta(z)} \circ s$ are homotopic via a homotopy through critically finite branched coverings

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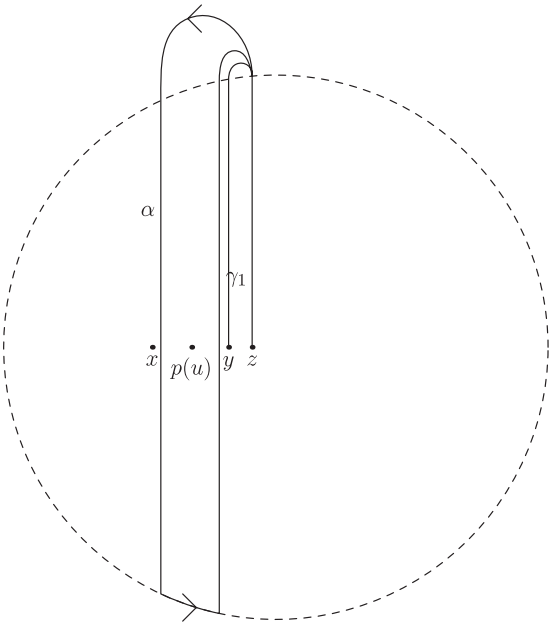


FIGURE 3. The loop α based at z.

which all take the same values on $(\sigma_{\beta(z)} \circ s)^{-m}(O(z))$. In particular, $\sigma_{\beta(z)} \circ s$ and $\sigma_\alpha \circ \sigma_{\beta(z)} \circ s$ are Thurston equivalent. Of course, Thurston equivalence is an equivalence relation, while $\simeq_{\psi_m}$ is not. Nevertheless, this notation gives useful information. Similarly to [**7**, §§3.3, 6.3 or 8.1], we can define ψ_m by induction on m, setting ψ_5 to be the identity and, for $m \geq 5$, defining ψ_{m+1} in terms of ψ_m by

$$\psi_m \circ \sigma_{\beta(z)} \circ s = \sigma_\alpha \circ \sigma_{\beta(z)} \circ s \circ \psi_{m+1}$$

together with the property that ψ_{m+1} is isotopic to ψ_m via an isotopy that is the identity on $(\sigma_{\beta(z)} \circ s)^{-5}(O(z))$, which contains $O(p(u))$. It is then also true that ψ_m and ψ_{m+1} are isotopic via an isotopy which is the identity on $(\sigma_{\beta(z)} \circ s)^{-m}(O(z))$. Defining ξ_m by

$$\psi_{m+1} = \xi_m \circ \psi_m,$$

we have

$$\sigma_{\beta(z)} \circ s = \sigma_\alpha \circ \sigma_{\beta(z)} \circ s \circ \xi_5$$

and, for $m \geq 5$,

$$\sigma_\alpha \circ \sigma_{\beta(z)} \circ s \circ \xi_{m+1} = \xi_m \circ \sigma_\alpha \circ \sigma_{\beta(z)} \circ s.$$

We then have

$$\psi_m = \xi_{m-1} \circ \cdots \circ \xi_5.$$

We need ψ_m for m equal to the preperiod of y (i.e. $m = 13$ in this case, which is also the preperiod of x), up to isotopy preserving $O(y) \subset s^{5-m}(O(p(u))) \subset (\sigma_{\beta(z)} \circ s)^{-m}(O(z))$. For such an m, we write $\psi_m = \psi$. Now, each ξ_k is a composition of disc-exchange. We see

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this first for $\xi_5 = \psi_6$, which is a single disc exchange whose support is an arbitrarily small neighbourhood of

$$s^{-1}(\beta(z) \cup \alpha \cup D'),$$

that is, approximately the union of $s^{-1}(D')$ and the diagonal in the exterior of the unit disc connecting these. The disc exhanges interchanges the components of $s^{-1}(D')$, tracing a path halfway round the boundary of the supporting disc in a *clockwise* direction. This ensures that $\xi_5^{-1} = \psi_6^{-1}$ exchanges discs, tracing a path halfway round the boundary of the supporting disc in an *anticlockwise* direction. Similarly, we have

$$\operatorname{supp}(\xi_{k+1}) = (\sigma_\alpha \circ \sigma_{\beta(z)} \circ s)^{-1}(\operatorname{supp}(\xi_k)).$$

We only need to consider the components of the support of ξ_k which intersect $O(y)$. Note that the support of ξ_5 does not intersect α, although it does intersect $\beta(z)$ near ∞. We shall see inductively that for all $6 \le k \le 12$, the components of ξ_k which intersect $O(y)$ do not intersect $\alpha \cup \beta(z)$ but do intersect $O(x)$. So we have

$$\operatorname{supp}(\xi_6) = (\sigma_{\beta(z)} \circ s)^{-1}(\operatorname{supp}(\xi_5))$$

and, for $6 \le k \le 12$,

$$\operatorname{supp}(\xi_{k+1}) = s^{-1}(\operatorname{supp}(\xi_k)).$$

We can describe the disc exchanges using the symbolic dynamics. The disc exchange of ξ_5 is described by

$$L_3 D'(\text{bot}) \leftrightarrow (\text{top}) R_3 D'.$$

The support of ξ_5 thus contains just the points $p(L_3 u)$ and $p(R_3 u)$ in $O(x)$ and $O(y)$, respectively. See Figure 4.

We have the following components in the support of ξ_k for $5 \le k \le 12$. In each case, two discs $u_1 D'$ and $u_2 D'$ are connected by an arc in the exterior of the unit disc. The labels 'top' and 'bot' indicate whether the arc joins onto the disc $u_j D'$ on the upper or lower half of the unit circle. We then need only consider the components of the support of ξ_k which intersect $O(y)$. Now, $p(u)$ is the only point in D' in the forward orbit of y (and of x), and it turns out that a single component of the support of ξ_k contains the points in the backward orbit of both x and y which map forward under s^{k+1} to u. For $6 \le k \le 11$, the action of ξ_k is represented by

$$\begin{gathered}
L_3^2 D'(\text{top}) \leftrightarrow (\text{bot}) L_2 R_3 D', \\
L_3^3 D'(\text{bot}) \leftrightarrow (\text{bot}) R_3 L_2 R_3 D', \\
L_3^4 D'(\text{top}) \leftrightarrow (\text{top}) L_2 R_3 L_2 R_3 D', \\
L_3^5 D'(\text{bot}) \leftrightarrow (\text{bot}) L_3 L_2 R_3 L_2 R_3 D', \\
R_3 L_3^5 D'(\text{bot}) \leftrightarrow (\text{bot}) R_3 L_3 L_2 R_3 L_2 R_3 D', \\
L_2 R_3 L_3^5 D'(\text{top}) \leftrightarrow (\text{top}) L_2 R_3 L_3 L_2 R_3 L_2 R_3 D'
\end{gathered}$$

and, finally, for $k = 12$,

$$v D'(\text{bot}) \leftrightarrow (\text{bot}) w D'.$$

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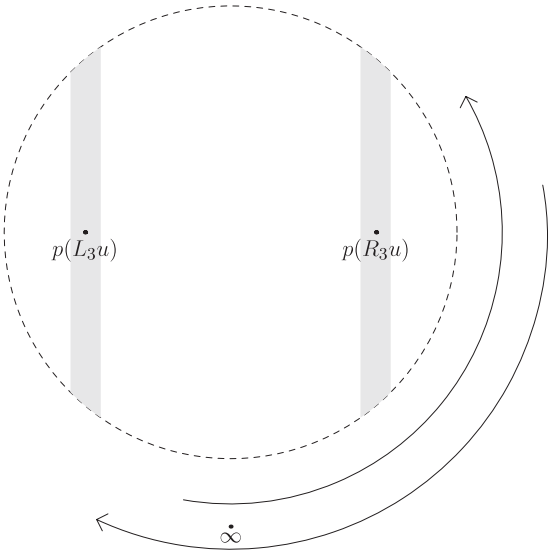


FIGURE 4. The disc exchange of ξ_5.

The support of this last component contains both x and y, and interchanges them. Define the path γ_1 by

$$\beta(y) = \beta(z) * \gamma_1,$$

where $*$ denotes the usual composition of paths. So the endpoints of γ_1 are z and y. We may take this as a pointwise definition of γ_1, but we are only interested in γ_1 up to homotopy preserving $O(y)$ and $O(z)$. Up to such a homotopy, the path γ_1 is an arc which crosses out of the gap containing z across the upper unit circle, goes into the gap containing y, then crosses the upper unit circle again, and ends at y. We write

$$\gamma_2 = \alpha * \psi(\gamma_1).$$

Then

$$\sigma_{\beta(y)} \circ s \simeq_{\text{identity}} \sigma_{\gamma_1} \circ \sigma_{\beta(z)} \circ s \simeq_{\psi} \sigma_{\gamma_2} \circ \sigma_{\beta(z)} \circ s.$$

None of the disc exchanges in the support of ψ intersects γ_1, except for the one in the support of ξ_{12} which interchanges the discs $D(x)$ and $D(y)$, with connecting arc along the lower unit circle. So the endpoints of γ_2 are z and x. Because this disc exchange (like all of them) is in a clockwise direction, $\psi(\gamma_1)$ passes to the left of $D(y)$ while α passes to the right of vD_1', where $D_1' \subset D'$ is defined by $vD_1' = vD' \setminus D'$. However, $vD_1' \cap O(x) = \emptyset$. Note that when discs are exchanged, upper boundaries go to upper boundaries and lower boundaries to lower. So $\psi(\gamma_1)$ hooks round wD_1', but $wD_1' \cap O(x) = \emptyset$ because $vD_1' \cap O(y) = \emptyset$. Therefore

$$\beta(z) * \gamma_2 = \beta(z) * \alpha * \psi(\gamma_1) = \beta(x) \text{ rel } O(x)$$

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and

$$\sigma_{\gamma_2} \circ \sigma_{\beta(z)} \circ s \simeq_{\psi} \sigma_{\beta(x)} . \circ s. \tag{2.1.2}$$

□

2.2. *Extensions of the basic idea.* Write

$$a = L_3(L_2R_3)^2 \quad \text{and} \quad b = L_3^5.$$

It is clear that the pair (a, b) plays a key role in the basic lemma of §2.1. Note that

$$v_0 = L_3L_2R_3a \quad \text{and} \quad w_0 = L_3L_2R_3b.$$

The following properties hold.

(1) The words v_0 and w_0 contain only the letters L_3, L_2 and R_3, and u_0 contains only the letters L_3, L_2 and R_3 apart from the last letter C.

(2) We have

$$v_0 < u_0 < w_0,$$

where u_0 is the only suffix u of v_0u_0 which satisfies

$$v_0u_0 < u < w_0u_0,$$

and w_0u_0 is obtained from v_0u_0 by changing the word a preceding u_0 in v_0u_0 to b.

These are essentially the only properties of u_0, v_0 and w_0 that were used in the proof in §2.1. We want words v_0 and w_0 such that $D(v_0)$ and $D(w_0)$ are in the subset of the unit disc bounded by the leaves of $L_{3/7}$ with endpoints at $e^{\pm 2\pi i(2/7)}$ and $e^{\pm 2\pi i(1/3)}$. This will be true provided that the common prefix starts with $L_3^{2i+1}L_2$ for some $i \geq 0$.

The above observation motivates the following. We choose the words c and d to minimize the preperiods and periods in the capture and mating theorems for all $n \geq 1$. Of course, other choices are possible. Let a and b be as above and let v_0, w_0 and u_0 be as in §2.1. In addition, define

$$\begin{aligned} c &= L_3^3L_2C, \\ d &= L_3^2a, \\ u_0' &= L_3L_2R_3c, \\ t_0 &= L_3L_2R_3d \end{aligned}$$

and, for $k \geq 1$, define v_k, w_k, t_k and u_k inductively by

$$\begin{aligned} v_k &= v_{k-1}t_{k-1}a, \\ w_k &= v_{k-1}t_{k-1}b, \\ u_k &= v_{k-1}t_{k-1}c, \\ t_k &= v_{k-1}t_{k-1}d. \end{aligned}$$

We write $|v|$ for the number of letters in a word v. Note that, for $k \geq 1$,

$$\begin{aligned} |v_k| + |t_k| &= 2(|v_{k-1}| + |t_{k-1}|) + 12, \\ |v_k| - |t_k| &= 2(|v_{k-1}| - |t_{k-1}|) - 2; \end{aligned}$$

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so

$$|v_n| = 2|v_{n-1}| + 5 = 13 \cdot 2^n - 5,$$
$$|t_n| = 2|t_{n-1}| + 7 = 17 \cdot 2^n - 7$$

and, for $n \geq 1$ (i.e. not including $n = 0$),

$$|u_n| = 13 \cdot 2^n - 6.$$

Thus we have

$$|v_n| + |u_n| = 26 \cdot 2^n - 11$$

for all $n \geq 1$, and

$$|v_n| + |t_n| = 30 \cdot 2^n - 12$$

for all $n \geq 0$. This will give the numbers in the capture and mating theorems.

LEMMA 2.3. *v_k, u_k and w_k satisfy the conditions* (1) *and* (2) *stated above.*

Proof. Each of a, b, c, d and u_0 has an odd number of letters L_3 and L_2. Hence, the same is true for v_0, w_0 and t_0, which are obtained by prefixing a, b and d by $L_3L_2R_3$. Therefore, by induction, the same will be true of v_k, w_k, u_k and t_k for all $k \geq 0$. Since the words v_k, w_k, u_k and t_k for $k > 0$ are obtained from a, b, c and d by adjoining the common prefix $v_{k-1}t_{k-1}$, which has an even number of letters L_3 and L_2, and since

$$a < d < c < b,$$

we have

$$v_0 < t_0 < u_0' < w_0$$

and, for all $k > 0$,

$$v_k < t_k < u_k < w_k. \tag{2.3.1}$$

Then, for $k > 1$, since $t_{k-1} < u_{k-1}$, we have $v_{k-1}u_{k-1} < v_{k-1}t_{k-1}$ and

$$v_{k-1}u_{k-1} < v_ku_k.$$

Now, the only condition that remains to be checked is that u_k is the only suffix u of v_ku_k satisfying $v_ku_k < u < w_ku_k$. Since $v_{k-1}t_{k-1}$ is a common prefix of v_k and w_k, we only need to consider proper suffixes of v_ku_k which start with $v_{k-1}t_{k-1}$. To do this, we prove a more general result about occurrences of $v_{k-1}t_{k-1}$.

LEMMA 2.4. *The only occurrences of $v_{k-1}t_{k-1}$ in*

$$v_kt_k, \quad t_kav_k \quad \text{or} \quad t_kau_k \tag{2.4.1}$$

are those visible in the definitions of v_k, t_k and u_k above, that is, at the start of these words.

Proof. We prove this by induction on k. We can easily check, by inspection, that the statement is true for $k = 1$. Suppose that it holds for $k - 1$. Now, $v_{k-1}t_{k-1}$ starts with v_{k-1}, which starts with $v_{k-2}t_{k-2}$. Note that

$$v_kt_k = v_{k-1}t_{k-1}av_{k-1}t_{k-1}d,$$
$$t_kav_k = v_{k-1}t_{k-1}dat_{k-1}v_{k-1}a,$$
$$t_kau_k = v_{k-1}t_{k-1}dav_{k-1}t_{k-1}c.$$

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So any occurrence of $v_{k-1}t_{k-1}$ in any of the words of (2.4.1) contains a $v_{k-2}t_{k-2}$, which must be in one of the corresponding words for $k-2$ and must coincide with the start of one of the words v_{k-1}, t_{k-1} or u_{k-1}. Then, by inspection, we see that the only possibility for $v_{k-1}t_{k-1}$ in any of the words of (2.4.1) is at the start of one of the subwords v_k, t_k or u_k. □

We then have the following corollary.

COROLLARY 2.5. *For v_k, w_k and u_k defined as above, the captures $\sigma_{\beta(w_ku_k)} \circ s_{3/7}$ and $\sigma_{\beta(v_ku_k)} \circ s_{3/7}$ are Thurston equivalent.*

Proof. Define $y = p(v_ku_k)$ and $x = p(w_ku_k)$ in a similar way as in §2.1, and define α, D', γ_1 and γ_2 exactly as in §2.1. As before, choose z to the right of $D(y)$, to the left of $D(C)$ and outside $\bigcup_{n\geq 0} s^{-n}(D' \cup D(x) \cup D(y))$. Let $m = |v_k| = 13 \cdot 2^k - 5$, and let S be the local inverse of s^m defined by v_k, so that $S(p(u_k)) = x$. Then $S(D' \cup D(x) \cup D(y))$ intersects $D' \cup D(x) \cup D(y)$, but, by Lemma 2.3, it does not contain any points of $O(x)$ apart from x itself. Thus we can choose z arbitrarily close to $\bigcup_{n\geq 0} S^n(D' \cup D(x) \cup D(y))$ and on the right of this set. We then define the sequences of homeomorphisms ψ_m and ξ_ℓ as in §2.1. We need to consider the effect of disc exchanges in the support of ξ_ℓ on γ_1, as in §2.1. We need only consider those which pull back to ones that intersect γ_1, and which also intersect $O(y)$. The first few disc exchanges are as in §2.1. The rest are all of the form $u'aD' \leftrightarrow u'bD'$ where u' is a prefix of u_k and, also, $u'au_k$ is a suffix of v_ku_k, i.e. $u'a$ is a suffix of v_k. By the conditions imposed on z, it is never in either of the sets $u'aD'$ or $u'bD'$, and is only strictly between them if D' is as well. Thus, γ_1 does not intersect the disc exchange given by $u'aD' \leftrightarrow u'bD'$ if y is not in $u'aD' \cup u'bD'$ and y is not between $u'aD'$ and $u'bD'$. So there is no intersection, except possibly when $u'a$ is a suffix of $v_k = v_{k-1}t_{k-1}a$ and u' is a prefix of v_k; next, we consider when this can happen. Take the largest i such that v_it_i is a (not necessarily proper) prefix of u' and, hence, also a prefix of v_k. Then, by Lemma 2.4, the only occurrences of v_it_i in $v_{k-1}t_{k-1}$ are those in the subwords v_{i+1} and t_{i+1} which occur in the construction of v_{k-1} and t_{k-1}. So the only possibility is that $u' = v_{k-1}t_{k-1}$ and $u'a = v_k$. This gives the disc exchange $v_kD' \leftrightarrow w_kD'$, which has exactly the right effect on γ_1, as in §2.1. □

To obtain equivalences between more than two captures, we need a slight generalization. For $n > k$, define $w_{k,n}$, $u_{k,n}$ and $t_{k,n}$ by replacing every occurrence of a which precedes a word v_k in v_n (or u_n) by b. Note that by Lemma 2.4 applied to $v_{k-1}t_{k-1}$, every occurrence of v_k in these longer words is indeed preceded by a, since $d = L_3^2a$.

LEMMA 2.6. *The suffixes u of v_nu_n which satisfy $v_nu_n < u < w_{k,n}u_{k,n}$ are precisely those that start with v_k or t_k.*

Proof. We consider only the $k = 0$ case, since the proofs for the other cases are exactly analogous. It is clear that we only need to consider prefixes which start with v_0 or t_0. Every suffix u which starts with t_0 certainly satisfies $v_n < u < w_{0,n}$, because $w_{0,n}$ starts with w_0 and v_n starts with v_0. So there is only a question about suffixes which start with v_0, and all of these start with v_0t_0 and hence with either $v_1 = v_0t_0a$ or $t_1 = v_0t_0d$, or $u_1 = v_0t_1c$ if $n = 1$. Therefore, we are reduced to considering suffixes which start with v_1. Continuing

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in the same way, we see that all suffixes which start with v_r for any $1 \le r < n$ or with u_n have the required property. □

The following result completes the proof of the theorem for captures.

LEMMA 2.7. *$\sigma_{\beta(w_{k,n}u_{k,n})} \circ s_{3/7}$ and $\sigma_{\beta(v_nu_n)} \circ s_{3/7}$ are Thurston equivalent.*

Proof. We consider $n > k \ge 0$, since the lemma in §2.1 covers the case $k = n = 0$ while Corollary 2.5 covers $k = n$ in general. We focus on the case of $k = 0$, since the other cases are treated exactly analogously for any $k < n$. By Lemma 2.6, the points in the forward orbit of $p(v_nu_n)$ that are between $p(v_nu_n)$ and $p(w_{0,n}u_n)$ are precisely the points $p(u)$ for all suffixes u of v_nu_n which start with v_0 or t_0.

Once again, we proceed much as in 2.1 and the proof of Corollary 2.5. We similarly define

$$x = p(w_{0,n}u_{0,n}), \quad y = p(v_nu_n),$$

and take D' to be the region strictly between $D(w_{0,n})$ and $D(v_nu_n)$. Choose z as in Corollary 2.5, but with S being the local inverse defined by v_k. This is still possible, by using Lemma 2.6 rather than Lemma 2.3. Let α be the anticlockwise loop round D' based at z, define γ_1 by $\beta(z) * \gamma_1 = \beta(y)$, and then define ξ_k and ψ_m as in §2.1. Just like in §2.1, we want to prove that for $\psi = \psi_m$ and $m = |v_n| + |u_n| - 1 = 26 \cdot u^n - 11$,

$$\beta(x) = \beta(z) * \alpha * \psi(\gamma_1) \text{ rel } O(x). \tag{2.7.1}$$

By the same argument as in Corollary 2.5, the only disc exchanges $u'aD' \leftrightarrow u'bD'$ which have to be considered are those for which $O(y) \cap u'aD' \neq \emptyset$ and $y \in D(u')$. By Lemma 2.6, these are disc exchanges for which $u'aw$ is a suffix of v_nu_n, where w is a suffix of v_nu_n that starts with v_0 or t_0 and u' is also a prefix of v_nu_n. Using Lemma 2.4, we see that the possibilities are that $u'aw$ must be a suffix of v_nu_n which starts with v_i or t_i for some $i > 0$ or (if $n = 1$) with u_1. All such disc exchanges are properly contained in D' and do not intersect $\alpha \cup \beta(z)$.

This time, in contrast to §2.1 and Corollary 2.5, there are many disc exchanges intersecting D' and γ_1 along the lower unit circle and the upper unit circle. The occurrences of v_0t_0 in v_nu_n are separated by strings of the form d^ra for a variable integer $r \ge 0$. The number of letters L_3 and L_2 in d^ra is odd or even, depending on whether r is even or odd; but each of v_0 and t_0 has an odd number of letters L_3 and L_2, and, apart from the very first occurrence of v_0, each occurrence of v_0 or t_0 in v_nu_n is always preceded by a. Now, every exchange ξ_j along the lower unit circle introduces a double hook in $\psi_{j+1}(\gamma_1)$, and every exchange ξ_j along the upper unit circle is such that $\psi_{j+1}(\gamma_1)$ and $\psi_j(\gamma_1)$ have the same number of hooks and the last hook on $\psi_{j+1}(\gamma_1)$ is a reduction of the last hook on $\psi_j(\gamma_1)$. Figure 5 shows $\psi_{\ell_1}(\gamma_1)$ where ℓ_1 is the first ℓ for which $\psi_\ell(\beta(y)) \neq \beta(y)$, so that $\ell_1 = |v_k| = 13 \cdot 2^k - 5$ (see §2.2). Note that $\psi_{\ell_1}(y)$ is to the right of x. If ℓ_2 is the first $\ell > \ell_1$ for which $\psi_\ell(\gamma_1) \neq \psi_{\ell_1}(\gamma_1)$, so that $\ell_2 = 13 \cdot 2^{k+1} - 5$, then $\psi_{\ell_2}(y)$ is to the right of x and the last hook is round x.

In order to obtain (2.7.1), we need the upward-pointing hooks on $\psi(\gamma_1)$, i.e. those which have the hook bridging the upper unit circle, to be homotopically trivial relative to $O(x)$. This is true because all the upward-pointing hooks are round subsets of $\psi(D_1')$,

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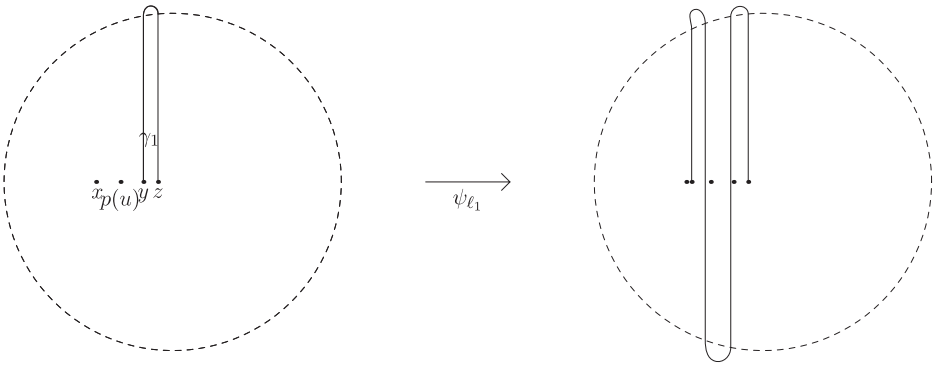


FIGURE 5. The homeomorphism ψ_{ℓ_1} applied to γ_1.

where (as in §2.1) $D_1' = v_0 D' \setminus D'$ is to the right of D' and $D_1' \cap O(y) = \emptyset$ by Lemma 2.6. Therefore, $\psi(D_1') \cap O(x) = \emptyset$. So we do have (2.7.1) and, as in §2.1, this gives the desired equivalence. □

2.8. *Proof of the mating theorem.* Following on from the capture theorem, the proof of the mating theorem is completed by the next lemma.

LEMMA. *Continuing with the same notation, in addition let $r_{k,n}$ be obtained from t_n by replacing every occurrence of a in t_n before an occurrence of v_k or t_k by b. Let $e^{2\pi i q_{k,n}}$ be the lower end of the leaf of $L_{3/7}$ coded by*

$$(v_n t_n)^\infty$$

if $k = n+1$, and by

$$(w_{k,n} r_{k,n})^\infty$$

if $0 \le k \le n$. Then all the matings $s_{3/7} \sqcup s_{q_{k,n}}$ are Thurston equivalent.

Proof. We start by considering the case where $k = 0 = n$. Write $q_{0,0}$ as q_0 and $q_{1,0}$ as q_1. Note that $r_0 = t_0$. As in §2.1, we drop the index 0 and write

$$\begin{aligned} v = v_0 &= L_3 L_2 R_3 L_3 (L_2 R_3)^2, \\ u = u_0' &= L_3 L_2 R_3 L_3^3 L_2 C, \\ w = w_0 &= L_3 L_2 R_3 L_3^5, \\ t = t_0 &= L_3 L_2 R_3 L_3^3 (L_2 R_3)^2. \end{aligned}$$

Note that the common length of vt and wt is 18; so in this particular case we are establishing an equivalence between matings with the aeroplane for which the second critical point has period 18.

We define

$$x = p(wtwu)$$

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and

$$y = p(vtvu).$$

Then x and y have preperiod 33. As before, we write $s_{3/7} = s$.

Choose a path $\zeta(y)$ from ∞ to $p(R_3vu)$ which crosses the unit circle just once at the upper boundary of the gap of $L_{3/7}$ containing $p(R_3vu)$, then stays in this gap, and ends at $p(R_3vu)$. Then, using the description of $s \amalg s_{q_1}$ up to Thurston equivalence given at the end of §1.3,

$$s \amalg s_{q_1} \simeq \sigma_{\zeta(y)}^{-1} \circ \sigma_{\beta(y)} \circ s.$$

This is because the gap of $L_{3/7}$ containing the point $s^j(y)$, for $0 \le j \le 17$, is closer to $\exp(2\pi i 2^j q_1)$ than any other point $s^i(y)$ for $0 \le i \le 17$. Defining γ_1 by $\beta(z) * \gamma_1 = \beta(y)$ as before, we also have

$$\sigma_{\zeta(y)}^{-1} \circ \sigma_{\beta(y)} \circ s \simeq_{\text{identity}} \sigma_{\zeta(y)}^{-1} \circ \sigma_{\gamma_1} \circ \sigma_{\beta(z)} \circ s.$$

Similarly, define $\zeta(x)$ ending at $p(R_3wu)$ so that

$$s \amalg s_{q_0} \simeq \sigma_{\zeta(x)}^{-1} \circ \sigma_{\beta(x)} \circ s.$$

We are going to proceed much as in the previous proofs—in particular, as in §2.1. So let D' be the subset of the unit disc between $D(x)$ and $D(y)$, and let $z \in Z_\infty$ be such that $D(z)$ is to the right of $D(x)$ and

$$O(z) \cap (D' \cup D(x) \cup D(y)) = \emptyset.$$

Let α be the anticlockwise loop round D' based at z. As in §2.1, we have a homeomorphism $\psi = \psi_{33}$ such that

$$(\sigma_{\beta(z)} \circ s, (\sigma_{\beta(z)} \circ s)^{-33}(O(z))) \simeq_\psi (\sigma_\alpha \circ \sigma_{\beta(z)} \circ s, (\sigma_{\beta(z)} \circ s)^{-33}(O(z))).$$

We have

$$\sigma_{\zeta(y)}^{-1} \circ \sigma_{\gamma_1} \circ \sigma_{\beta(z)} \circ s \simeq_\psi \sigma_{\psi(\zeta(y))}^{-1} \circ \sigma_{\psi(\gamma_1)} \circ \sigma_\alpha \circ \sigma_{\beta(z)} \circ s.$$

Now the postcritical set $P(y)$ of $\sigma_{\zeta(y)}^{-1} \circ \sigma_{\beta(y)} \circ s$ is

$$\{0, s(0), s^2(0)\} \cup \{s^i(y) \mid 0 \le y < 18\}.$$

Note that it does not include $p(vu)$, and, similarly, the postcritical set $P(x)$ of $\sigma_{\zeta(x)}^{-1} \circ \sigma_{\beta(x)} \circ s$ does not include $p(wu)$. So, to prove the lemma, we only need to prove that

$$\beta(z) * \alpha * \psi(\gamma_1) = \beta(x) \text{ rel } P(x), \tag{2.8.1}$$

$$\psi(\zeta(y)) = \zeta(x) \text{ rel } P(x). \tag{2.8.2}$$

The proof of (2.8.1) is much the same as the proofs in §2.1 and of Lemma 2.7. The support of ξ_ℓ intersects $\psi_\ell(\beta(y))$ only for $\ell = 14$ and $\ell = 32$. In both cases, just one component of the support of ξ_ℓ intersects $\psi_\ell(\beta(y))$ and, restricted to this component, ξ_ℓ exchanges two discs. The homeomorphism ξ_{14} interchanges a disc containing $p(vu)$ and $p(vtvu)$ with a disc containing $p(wu)$ and $p(wtvu)$, with $p(vu)$ and $p(vtvu)$ mapped to $p(wu)$ and $p(wtvu)$, respectively, and ξ_{32} interchanges a disc containing $p(wtvu)$ with a disc

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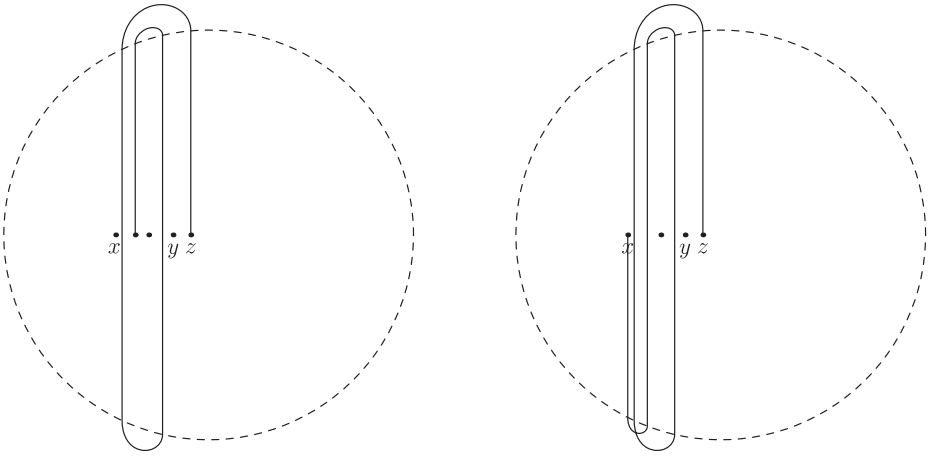


FIGURE 6. $\alpha * \psi_{15}(\beta(y))$ and $\alpha * \psi_{33}(\beta(y))$.

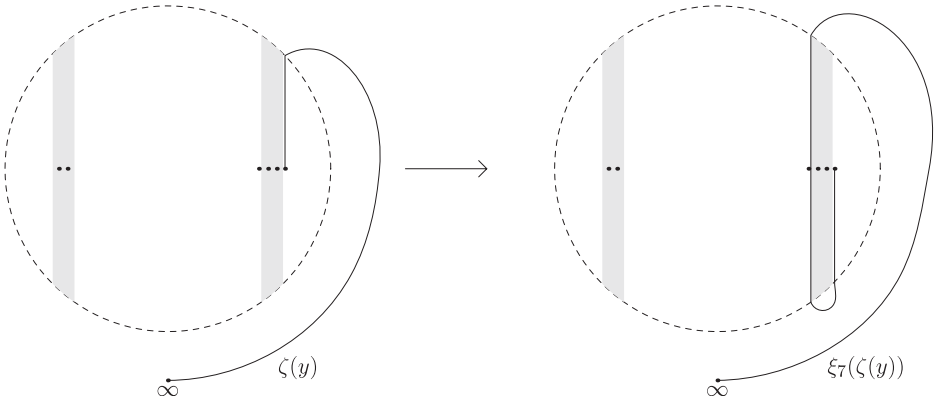


FIGURE 7. The action of $\xi_7 = \psi_8$ on $\zeta(y)$.

containing $p(wtwu)$. Figure 6 shows $\alpha * \psi_{15}(\beta(y))$ and $\alpha * \psi_{33}(\beta(y))$. The endpoint of $\psi_{15}(\beta(y))$ is $p(wtvu)$ and the endpoint of $\psi_{33}(\beta(y))$ is $p(wtwu) = x$. Both of these paths loop round $p(wu)$ as shown (but because of space constraints, this point is not labelled). Since $p(wu) = s^{18}(p(x))$ is not in $P(x)$, this does not matter, and we have (2.8.1) in the $n = 0$ case.

Now we consider (2.8.2). The action of $\xi_7 = \psi_8$ on $\zeta(y)$ is as shown in Figure 7. Note that the support of ξ_7 intersects $\zeta(y)$ but does not contain $p(R_3vu)$ since $p(vu) \notin D'$; this is because we chose u with $t < u$, and therefore $vu < vt$.

The two discs in the support of ξ_7 interchange in a clockwise direction round ∞, and the action on $\zeta(y)$ is such that $\xi_7(\zeta(y))$ hooks round the nearby component of $s^{-1}(D')$. The points $p(L_3u)$ and $p(R_3u)$ in $s^{-1}(D')$ are interchanged by ξ_7. Then there is a composition of disc exchanges similar to those in §2.1. The support of ξ_ℓ does not intersect $\psi_8(\zeta(y))$ for $7 < \ell < 14$ but does intersect it for $\ell = 15$. The endpoint of $\psi_{16}(\zeta(y))$ is then at $p(R_3wu)$.

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For $\ell \geq 16$, the discs in the support of ξ_ℓ contain only points of preperiod higher than 16, and 16 is the preperiod of $p(R_3wu)$. The discs are small enough that any such disc which intersects $\psi_{16}(\zeta(y))$ does not intersect $P(x)$. In particular, $\psi_{33}(\zeta(y))$ is homotopic to $\psi_{16}(y)$ relative to $P(x)$. This gives (2.8.2) in the case where $k = 0$. The $k = n$ case, for any $n \geq 0$, of (2.8.1) and (2.8.2) is treated in exactly the same way.

So now we consider the case where $n > k \geq 0$. Write

$$y = p(v_n t_n v_n u_n), \quad x = p(w_{k,n} r_{k,n} w_{k,n} u_{k,n}).$$

There is a composition of disc exchanges contributing to both $\psi(\gamma_1)$ and $\psi(\zeta(y))$. We consider the disc exchanges affecting γ_1, which are similar to those in the proof of Lemma 2.7. However, there is no exchange for the word a preceding $v_n u_n$ in $v_n t_n v_n u_n$. As in the $k = n = 0$ case, there is a difference from the proof of Lemma 2.7, because this time the disc $v_k D_1'$ contains the point $p(v_n u_n)$. This, however, is the only point in $O(y)$ that $v_k D_1'$ contains, and it is not in $P(y)$. So we have a picture similar to Figure 6, and (2.8.1) holds.

For $\zeta(y)$, as before, the first disc exchange which intersects $\zeta(y)$ does not move the endpoint but does create a hook round $R_3 D'$ exactly as in the picture for the $k = n = 0$ case. The next exchange—the inverse image of the first one affecting γ_1—is along the lower unit circle and reduces the size of this hook, replacing it by one of the same shape. As in Lemma 2.7, and as for $\beta(y)$, the disc exchanges which affect $\psi_\ell(\zeta(y))$ then alternate between a group of exchanges on the upper and on the lower unit circle. Finally, for $m = 26 \cdot 2^n - 11$, we have $\psi_m(\zeta(y)) = \zeta(x)$ up to homotopy preserving $P(x)$, and we have $\psi_m(\zeta(y)) = \psi(\zeta(y))$ up to homotopy preserving $P(x)$. Thus (2.8.2) holds as before. □

3. *Remarks*

The pair $(a, b) = (L_3(L_2R_3)^2, L_3^5)$ can be varied to

$$(L_3(L_2R_3)^{2j}, L_3(L_2R_3)^{2j-2}L_3^4)$$

for any $j \geq 1$, or to

$$(L_3(L_2R_3)^{2j-1}L_3^2L_2R_3, L_3(L_2R_3)^{2j}L_3^2)$$

for any $j \geq 1$.

Let us call such an ordered pair (a, b) an *exchangeable pair*. Note that each a and each b has an odd number of letters L_3 and L_2. Then, if we examine the proof of 2.7, which works more generally than §2.1, we see that if $y = p(eau)$, then $\sigma_{\beta(y)} \circ s$ and $\sigma_{\beta(x)} \circ s$ are Thurston equivalent for some $x \in D(eb)$ whenever the following conditions hold.

(1) $D(e) \subset D(L_3^{2\ell+1}L_2)$ for some $\ell \geq 0$.
(2) e consists only of the letters L_3, L_2 and R_3, with an even number of letters L_3 and L_2.
(3) u consists only of the letters L_3, L_2 and R_3, apart from the last letter C.
(4) No suffix u' of eau satisfies $D(u') \subset D(eb)$.
(5) $eau < u < eb$, but no suffix u' of eau with $|u'| > |u|$ satisfies $eau < u' < eb$.

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(6) Any suffix u' of u with $|u'| < |u|$ which satisfies $eaebu < u' < ebu$ must also satisfy the stronger condition $eau < u' < ebu$.

(7) For any suffix u' of u which satisfies $eau < u' < eb$, the prefix of u' in u is a_1, for some exchangeable pair (a_1, b_1).

(8) If $e'a'$ is a proper suffix of e for some exchangeable pair (a', b') such that $D(eau)$ is between $D(e'a'eau)$ and $D(e'b'eau)$, then e' has an even number of letters L_3 and L_2. Note that this then implies that $e'a'eau < eau < e'b'eau$.

Once these conditions are satisfied, we can define u_1 to be obtained from u upon replacing a_1 by the second coordinate b_1 of the exchangeable pair, for every suffix u' of u with $eau < u < eb$, where a_1 is the first coordinate of the exchangeable pair which precedes u'. Then $\sigma_{\beta(y)} \circ s$ and $\sigma_{\beta(x)} \circ s$ are Thurston equivalent, where $x = p(ebu_1)$. The idea in Lemma 2.7 is to have a word v which can be written in the form eau in many different ways.

This clearly gives a lot of choice, but it is unsatisfactory in that the conditions imposed on u are only satisfied by a zero-density set of words as the length tends to infinity. One would like to be able to say that for a fixed choice of e, and for u in a set of postive density, there exists $x \in D(eb)$ such that $\sigma_{\beta(y)} \circ s$ and $\sigma_{\beta(x)} \circ s$ are Thurston equivalent. Such questions will be addressed in [**8**].

I am not sure whether the growth rate of n equivalent matings or captures with critical point of period or preperiod of $O(2^n)$ is the best possible. A related question concerns the growth of the length of words v which can be written as eau for n different choices of (e, a, u) that satisfy the conditions listed above. This growth seems to be of $O(2^n)$. Once again, it is hoped that this issue can be properly addressed in [**8**].

Acknowledgement. I should like to thank the referee for full and prompt engagement with the paper, and for some helpful comments.

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