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On the Structure of Dense Triangle-Free Graphs

STEPHAN BRANDT

FB Mathematik & Informatik, WE 2, Freie Universität Berlin,
Arnimallee 3, 14195 Berlin, Germany
(e-mail: brandt@math.fu-berlin.de)

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As a consequence of an early result of Pach we show that every maximal triangle-free graph is either homomorphic with a member of a specific infinite sequence of graphs or contains the Petersen graph minus one vertex as a subgraph. From this result and further structural observations we derive that, if a (not necessarily maximal) triangle-free graph of order n has minimum degree $\delta \geqslant n/3$, then the graph is either homomorphic with a member of the indicated family or contains the Petersen graph with one edge contracted. As a corollary we get a recent result due to Chen, Jin and Koh. Finally, we show that every triangle-free graph with $\delta > n/3$ is either homomorphic with C_5 or contains the Möbius ladder. A major tool is the observation that every triangle-free graph with $\delta \geqslant n/3$ has a unique maximal triangle-free supergraph.

1. Introduction and main results

We investigate properties of simple, finite and undirected graphs. For elementary concepts and notation the reader is referred to introductory literature, for instance [17]. A graph is *maximal* triangle-free if it contains no triangle but joining any two non-adjacent vertices by an edge creates a triangle. Note that a triangle-free graph of order $n \geqslant 3$ is maximal if and only if it has diameter two. We investigate the structure of maximal triangle-free graphs and of triangle-free graphs with large minimum degree. It will turn out that the latter graphs are not too different from maximal triangle-free graphs since every such graph has a unique maximal triangle-free supergraph with which it shares many common properties. To state our results we need to introduce the notion of graph homomorphisms, as well as a sequence of specific triangle-free graphs which plays an important role in the investigation of dense triangle-free graphs and which has been rediscovered several times over the years. As far as the author knows the sequence goes back to Andrásfai in 1962 [1].

Let $\Gamma_1 = K_2$ and $\Gamma_i = \overline{C_{3i-1}^{i-1}}$ for $i \geqslant 2$, that is, Γ_i is the complement of the $(i-1)$st power of a cycle of length $3i-1$. As usual, the ith power of a graph G is the graph

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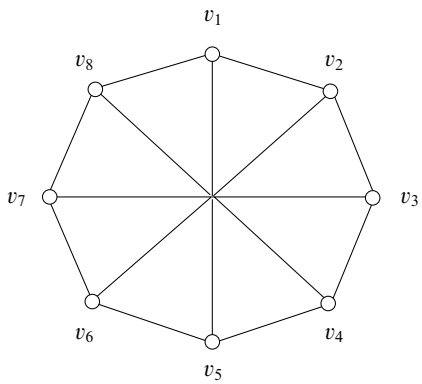


Figure 1 The Möbius ladder Γ_3

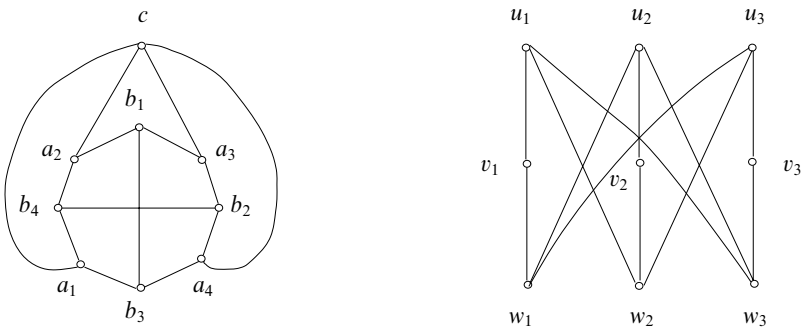


Figure 2 The edge-contracted Petersen graph H_{10}/e (left) and the vertex deleted Petersen graph $H_{10} - v$ (right) with certain labellings

obtained by joining any pair of vertices at distance at most i in G. So $\Gamma_2 = C_5$ and Γ_3 is the Möbius ladder (with four spokes – see Figure 1). A homomorphism from a graph G to H is a mapping $\sigma : V(G) \to V(H)$ such that, for every edge vw of G, the edge $\sigma(v)\sigma(w)$ is present in H. Graph homomorphisms are a refinement of graph colouring, since a graph is k-colourable if and only if it is homomorphic with K_k. Note that if H is triangle-free then G must be triangle-free as well, and if G is maximal triangle-free and H is triangle-free then $\sigma(u) = \sigma(v)$ implies $N_G(u) = N_G(v)$.

Close relatives of the Petersen graph play a central role in this paper. Let H_{10} denote the Petersen graph, H_{10}/e the Petersen graph with one edge contracted, $H_{10} - v$ the Petersen graph with one vertex deleted (see Figure 2) and H_{11} the Grötzsch graph (see Figure 3). Now we can state our main results.

Theorem 1.1. *Let G be a maximal triangle-free graph. Then G is not homomorphic with any graph Γ_i if and only if G contains $H_{10} - v$ as a subgraph.*

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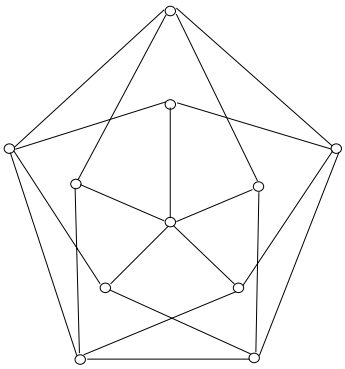


Figure 3 The Grötzsch graph H_{11}

Theorem 1.2. *Let G be a triangle-free graph of order n with minimum degree $\delta \geqslant n/3$. Then G is not homomorphic with any graph Γ_i if and only if G contains H_{10}/e as a subgraph.*

As a consequence, we can easily derive the main result of Chen, Jin and Koh [9] from Theorem 1.2, which only covers triangle-free graphs satisfying the slightly stronger requirement $\delta > n/3$. Note that the structural freedom seems to increase considerably when lowering the degree requirement from $\delta > n/3$ to $\delta \geqslant n/3$.

Corollary 1.3 (Chen, Jin and Koh [9]). *Let G be a triangle-free graph of order n with minimum degree $\delta > n/3$. Then G is not homomorphic with a graph in Γ_i if and only if G contains the Grötzsch graph H_{11} as a subgraph.*

This result distinguishes the 4-chromatic and the 3-colourable triangle-free graphs with $\delta > n/3$, since the Grötzsch graph is 4-chromatic and the graphs in Γ_i are 3-colourable. Note that our approach is much simpler than the original proof by Chen, Jin and Koh, which involves many lengthy calculations.

It can be seen by an easy counting argument that every triangle-free graph with $\delta > 2n/7$ is either bipartite (*i.e.*, homomorphic with K_2) or contains C_5 as a subgraph. If $\delta(G) > n/3$ we can prove the next step.

Theorem 1.4. *Let G be a triangle-free graph with $\delta(G) > n/3$. Then G is not homomorphic with C_5 if and only if G contains Γ_3 as a subgraph.*

The result generalizes a result of Häggkvist [12, Theorem 3], saying that every triangle-free graph with $\delta > 3n/8$ is homomorphic with $\Gamma_2 = C_5$, since by an easy counting argument (see [12, Lemma 5]) every triangle-free graph which contains Γ_3 has minimum degree $\delta \leqslant 3n/8$. Moreover, it generalizes another result of Chen, Jin and Koh [9, Theorem 6], who essentially proved this statement for maximal triangle-free graphs. The degree bound is best possible, since the lexicographic product $(H_{10}/e)[\overline{K}_r]$ (see, *e.g.*, [17, p. 370]

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for the definition of the lexicographic product) has minimum degree $n/3$, but is neither homomorphic with C_5 nor does it contain Γ_3.

It would be interesting to know which bound on the minimum degree implies that a triangle-free graph is homomorphic with Γ_3 or contains Γ_4 as a subgraph. The bound we can hope for is $\delta > 10n/29$, in view of the Grötzsch graph H_{11}.

2. Background and history

There are many results showing that, in triangle-free graphs with $\delta > n/3$, several graph parameters are strongly interrelated (see [4, Theorems 2.1, 2.2, 3.3 and Lemma 3.1] and [5]). Moreover, Goddard and Kleitman [11, Theorem 1] observed an interesting structural property of maximal triangle-free graphs satisfying the degree bound.

The investigation of structural aspects of dense triangle-free graphs has an interesting history of research progress obtained while dealing with false conjectures. The history started with a result of Andrásfai, Erdős and Sós [2], who proved – as a special case of a result for graphs of given clique number (see also [6] for a simple proof of the general result) – that every triangle-free graph with $\delta > 2n/5$ is bipartite, that is, homomorphic with $\Gamma_1 = K_2$. Motivated by a construction of triangle-free graphs with $\delta > (1 - o(1))n/3$ with arbitrarily large chromatic number, due to Hajnal (see [10]), and unaware of a 4-chromatic triangle-free graph with $\delta > n/3$, Erdős and Simonovits [10] conjectured that every triangle-free graph with $\delta > (1 + o(1))n/3$ is 3-colourable. This conjecture was refuted by Häggkvist [12], who found a sequence of 4-chromatic graphs with $\delta = 10n/29$ homomorphic with the Grötzsch graph. Recently, with the help of the computer program Vega, Brandt and Pisanski [7] found an infinite sequence of homomorphically minimal subgraphs of triangle-free 4-chromatic graphs with $\delta > n/3$.

On the positive side, Häggkvist [12] proved, as mentioned above, that every triangle-free graph with $\delta > 3n/8$ is homomorphic with $\Gamma_2 = C_5$ and conjectured that every triangle-free graph with $\delta > 10n/29$ is homomorphic with Γ_3. This, in turn, was refuted by Jin [13], who observed that the graphs Γ_i for $4 \leqslant i \leqslant 8$ are counterexamples. He proved that every graph with $\delta > 10n/29$ is homomorphic with Γ_9 and therefore 3-colourable. On the other hand, Jin conjectured that there are triangle-free graphs with minimum degree exceeding $n/3$ and arbitrarily large chromatic number [14].

3. Methods and tools

There are two central features which make our proofs much simpler than those of Chen, Jin and Koh: the existence of a unique maximal triangle-free supergraph for any triangle-free graph with $\delta \geqslant n/3$, which allows us to focus our attention on maximal triangle-free graphs, and the forbidden subgraph characterization in Theorem 1.1 of the triangle-free graphs for which every independent vertex set is contained in the neighbourhood of a vertex.

Let us start with the first property, the existence of a unique maximal triangle-free supergraph. Closure concepts in graph theory proved to be useful tools mainly in connection with cycle lengths in graphs (see [3] and [16] for two particularly powerful closure

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concepts). Here we introduce a closure concept for triangle-free graphs with a slightly different flavour: we construct a supergraph $\mathrm{cl}_\Delta(G)$ by consecutively adding edges joining vertices of distance $\geqslant 3$ until the resulting graph has diameter 2. In general, the result is far from being unique. But there is an easy characterization of those triangle-free graphs G for which $\mathrm{cl}_\Delta(G)$ is unique and it is not difficult to show that this property is satisfied if $\delta(G) \geqslant n/3$.

Closely related to this closure is the *dist-2-graph*. For a given graph $G = (V, E)$, the edge set E_2 of the dist-2-graph $G_2 = (V, E_2)$ consists of the pairs of vertices that are at distance 2 in G. Note that a maximal triangle-free graph is the complement of its dist-2-graph. We will now characterize the triangle-free graphs that have a unique closure (in the labelled sense) via their dist-2-graphs.

Theorem 3.1. *Let G be a triangle-free graph. Then $\mathrm{cl}_\Delta(G)$ is uniquely determined if and only if $\overline{G_2}$ is triangle-free. In this case, $\mathrm{cl}_\Delta(G) = \overline{G_2}$.*

Proof. The edges appearing in some triangle-free supergraph of G are those joining vertices at distance 1 or at distance $\geqslant 3$ in G, that is, exactly those of $\overline{G_2}$. So, if $\overline{G_2}$ is triangle-free we obtain that $\mathrm{cl}_\Delta(G) = \overline{G_2}$ is unique. Conversely, if $\overline{G_2}$ contains a triangle x, y, z, then at least two of its edges, say xy, xz, are not in G. So G has a maximal triangle-free supergraph G' containing xy and yz and another one, $G'' \neq G'$, containing xz. □

The graph $\overline{G_2}$ is easily derived from the square of the adjacency matrix of G, so it is algorithmically easy to decide whether $\mathrm{cl}_\Delta(G)$ is unique (see [5]). For $\delta(G) \geqslant n/3$ this is always the case.

Corollary 3.2. *Let G be a triangle-free graph of order n. If $\delta(G) \geqslant n/3$ then $\mathrm{cl}_\Delta(G)$ is uniquely determined.*

Proof. Suppose that $\overline{G_2}$ has a triangle x, y, z with $xy, xz \notin E(G)$. Since no pair of these vertices has distance 2 in G, the neighbourhoods $N_G(x)$, $N_G(y)$, $N_G(z)$ are disjoint. Moreover, x is contained in none of the sets, since x has distance $\geqslant 3$ to y and z, so $d_G(x) + d_G(y) + d_G(z) \leqslant n - 1$. □

Note that $\mathrm{cl}_\Delta(G)$ shares many structural properties with G. It was shown in [5] that every independent set U with $|U| > n/3$ of G is an independent set of $\mathrm{cl}_\Delta(G)$ if $\delta(G) \geqslant n/3$, so the independence number $\alpha(G) = \alpha(\mathrm{cl}_\Delta(G))$. Combined with the main result of [4], saying that the length of a longest cycle of a triangle-free graph with $\delta > n/3$ equals $\min\{n, 2(n-\alpha)\}$, this implies that if $\delta > n/3$ the length of a longest cycle remains unchanged in the closure as well.

Pach [15] investigated triangle-free graphs for which every independent set is contained in the neighbourhood of a vertex, and proved the following result, which is formulated here with regard to our needs (see also the simpler proof of this result due to Brouwer [8]).

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Theorem 3.3 (Pach [15]). *Every independent set of a triangle-free graph G is contained in the neighbourhood of a vertex if and only if G is maximal triangle-free and homomorphic with a graph Γ_i.*

We will first give the simple proof of Theorem 1.1 based on Pach's result.

Proof of Theorem 1.1. First suppose that G contains $H_{10}-v$ as a subgraph. Note that in $H_{10}-v$ no two vertices can be identified by a homomorphism without creating a triangle. So, if G is homomorphic with a graph Γ_i then $H_{10}-v$ is a subgraph of Γ_i as well. Now consider the subgraph induced by the degree 3 vertices of $H_{10}-v$. This subgraph is an induced 6-cycle in any triangle-free supergraph of $H_{10}-v$, so G is not homomorphic with Γ_i, since it is easy to see that Γ_i does not contain an induced 6-cycle.

Now suppose that G is not homomorphic with any graph Γ_i. By Theorem 3.3, G contains an independent set U which is not contained in the neighbourhood of a vertex. Take a minimal subset $U'=\{u_1,\dots,u_p\}$ of U which is not contained in the neighbourhood of a vertex. Since G has diameter 2 we have $p\geqslant 3$. By the minimality of U', for each $u_i\in U'$ there is a vertex w_i, which is adjacent to each vertex of U' except u_i. So the set $S=\{u_1,u_2,u_3,w_1,w_2,w_3\}$ induces a 6-cycle in G. Again by the diameter condition there must be distinct vertices v_1,v_2,v_3 such that v_i is adjacent to u_i and w_i. So the resulting subgraph is indeed $H_{10}-v$. □

We cannot hope for a larger substructure since $H_{10}-v$ is the largest common subgraph of the Petersen graph H_{10} and H_{10}/e, which are both maximal triangle-free. On the other hand the property of containing $H_{10}-v$ in Theorem 1.1 can be replaced by the property of containing an induced C_6. Note that in view of Theorems 3.3 and 1.1 the following four statements are equivalent for a triangle-free graph G:

(1) every independent set of G is contained in the neighbourhood of a vertex,
(2) G is maximal triangle-free and homomorphic with a graph Γ_i,
(3) G is maximal triangle-free and does not contain $H_{10}-v$ as a subgraph,
(4) G is maximal triangle-free and does not contain an induced 6-cycle.

4. Proofs

Now let us turn to the proof of Theorem 1.2. This is based on Corollary 3.2 and on a technical lemma which will be applied repeatedly. For a subgraph isomorphic to H_{10}/e we assume the labelling of Figure 2 to be given and we calculate the indices modulo 4.

Lemma 4.1. *Let G be a triangle-free graph of order n with $\delta(G)\geqslant n/3$. If the subgraph of $\mathrm{cl}_\Delta(G)$ induced by a subset $S\subseteq V(G)$ is H_{10}/e then G contains vertices y,z and indices i,j of different parity with $ya_i, yb_i, yb_{i+1}, za_j, zb_j, zb_{j+1}\in E(G)$, unless $\delta(G)=n/3$ and G contains all edges joining vertices of $S\setminus\{c\}$ in $\mathrm{cl}_\Delta(G)$.*

Proof. Let S be a subset with the indicated properties. If $\delta(G)=n/3$ and all edges of $\mathrm{cl}_\Delta(G)$ joining vertices of $S\setminus\{c\}$ are present in G, then there is nothing to prove. So assume

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that $\delta(G) > n/3$ or one of the edges, say e', is missing in G. First note that the subgraphs of $\mathrm{cl}_\Delta(G)$ induced by $S_1 = \{a_1, a_3, b_1, b_2, b_3, b_4\}$ and by $S_2 = \{a_2, a_4, b_1, b_2, b_3, b_4\}$ are both isomorphic with C_6 (see Figure 2). Since the vertices a_i form an independent set we may assume that some vertex b_j is an end-vertex of e'. The other end-vertex of e' has in G at most one neighbour in S_1 and in S_2. Since $\delta \geqslant n/3$ and not all vertices have exactly two neighbours in S_1 nor in S_2, there must be a vertex y having at least (and therefore by Corollary 3.2 exactly) three neighbours in S_1 and a vertex z having three neighbours in S_2, which have the required properties. □

Now we are ready to prove Theorem 1.2 and Corollary 1.3. We will frequently switch between the graph G itself and the unique maximal triangle-free supergraph $\mathrm{cl}_\Delta(G)$.

Proof of Theorem 1.2. If G contains H_{10}/e, and therefore also $H_{10} - v$, then G is not homomorphic with any graph Γ_i by Theorem 1.1. So suppose that G is not homomorphic with any graph Γ_i. Then, by the transitivity of the homomorphism relation, $\mathrm{cl}_\Delta(G)$ is also not homomorphic with a graph Γ_i. By Theorem 1.1, $\mathrm{cl}_\Delta(G)$ contains $H_{10} - v$ as a subgraph.

Consider the 6-cycle C of $\mathrm{cl}_\Delta(G)$ induced by the degree 3 vertices u_1, u_2, u_3, w_1, w_2, w_3 of $H_{10} - v$ (see Figure 2). By Corollary 3.2, the vertices u_i and w_i must have a common neighbour v_i in G, otherwise the edge $u_i w_i$ would be present in some closure. Since $\delta(G) \geqslant n/3$ and not all vertices v_i can have exactly 3 neighbours in S, there must be a vertex v having in G at least 4 neighbours in S. Since every independent set of 4 vertices of $H_{10} - v$ has the form $\{u_i, w_i, v_j, v_k\}$ with $\{i, j, k\} = \{1, 2, 3\}$, the set $S' = S \setminus \{v_i\} \cup \{v\}$ induces H_{10}/e in $\mathrm{cl}_\Delta(G)$ and induces in G all edges of H_{10}/e except possibly some edges of C.

For convenience, relabel the vertices of S' according to the labelling of H_{10}/e in Figure 2. We claim that G contains an induced 6-cycle which is an induced 6-cycle in $\mathrm{cl}_\Delta(G)$ as well. If all the edges of H_{10}/e are present in G, this is certainly the case. Otherwise, by Lemma 4.1, there are vertices y, z adjacent to a_i, b_i, b_{i+1} and a_j, b_j, b_{j+1} in G, where i and j are of different parity. So the subgraph of G induced by $S' \cup \{y, z\}$ is H_{11} with some edges of the induced 6-cycle C of $\mathrm{cl}_\Delta(G)$ possibly missing. Since for every induced 6-cycle C there is another edge disjoint induced 6-cycle C' in H_{11}, we obtain that G and $\mathrm{cl}_\Delta(G)$ contain a common induced 6-cycle C'. By repeating the argument of the second paragraph of this proof with C' in place of C, we obtain that G contains H_{10}/e. □

Proof of Corollary 1.3. If G is not homomorphic with a graph Γ_i then by Theorem 1.2 it must contain H_{10}/e as a subgraph. By Lemma 4.1 applied to the subgraph H_{10}/e, the graph G contains H_{11}, as claimed. □

Note that in Corollary 1.3 we have proven the main result of Chen, Jin and Koh [9] as stated in their abstract. In the paper they prove the slightly stronger but somewhat more technical result [9, Theorem 3], saying that any graph with $\delta > n/3$ which is triangle-free and does not contain H_{11}, and which is edge-maximal subject to these properties, is either homomorphic with Γ_{i-1} or contains Γ_i as a subgraph for every $i \geqslant 2$. This can be easily derived from our results. By Corollary 1.3 such a graph is maximal triangle-free

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and homomorphic with some graph Γ_i. Choose i to be the smallest integer such that G is homomorphic with Γ_i. Since Γ_i is a subgraph of Γ_j whenever $j \geqslant i$, we get by the transitivity of homomorphism that G is also homomorphic with Γ_j. If the homomorphism $\sigma : V(G) \to V(\Gamma_i)$ is surjective then G contains Γ_i since both graphs are maximal triangle-free. Otherwise, G is homomorphic with $\Gamma_i - v$, and, since $\Gamma_i - v$ is homomorphic with Γ_{i-1}, we get that G is homomorphic with Γ_{i-1}, a contradiction to the choice of i.

Finally, we will give the simple proof of Theorem 1.4.

Proof of Theorem 1.4. If G contains H_{11} as a subgraph then G contains Γ_3, since H_{11} contains Γ_3. Otherwise, by Corollary 1.3 the graph G is homomorphic with Γ_i for some i. Let j be the smallest integer, such that G is homomorphic with Γ_j. If $j \leqslant 2$ then we are done, so suppose $j \geqslant 3$. As above we may assume that the homomorphism $\sigma : V(G) \to V(\Gamma_j)$ is surjective, otherwise G is homomorphic with Γ_{j-1}. By adding all edges to G, joining vertices v, w with $\sigma(v)\sigma(w) \in E(\Gamma_j)$ we get a maximal triangle-free supergraph $\mathrm{cl}_\Delta(G)$ which is unique by Corollary 3.2. The graph $\mathrm{cl}_\Delta(G)$ contains Γ_j as a subgraph and therefore Γ_3 as well.

Let G' be a subgraph of G which induces Γ_3 in $\mathrm{cl}_\Delta(G)$ and has, subject to this property, as many edges as possible. We will show that G' is isomorphic with Γ_3. Note that there are two edge orbits in Γ_3, the outer cycle and the diagonals in Figure 1. Suppose that an edge of the outer cycle, v_1v_2 say, is missing in G'. By the uniqueness of $\mathrm{cl}_\Delta(G)$ there must be a common neighbour x of v_1 and v_3 in G, otherwise we could add the edge v_1v_3. It is easy to check that $x \notin V(G')$. So the subgraph of $\mathrm{cl}_\Delta(G)$ induced by $(V(G') \setminus \{v_2\}) \cup \{x\}$ contains Γ_3 with the diagonal xv_6 deleted as a subgraph. Now consider the 6 vertices $S = \{v_1, x, v_3, v_7, v_6, v_5\}$ forming a 6-cycle in $\mathrm{cl}_\Delta(G)$. Since $\delta(G) > n/3$ there must be a vertex $y \in V(G)$ (possibly x or v_6) having 3 neighbours in S. But then, by replacing x or v_6 by y, we get a subgraph G'' which induces Γ_3 in $\mathrm{cl}_\Delta(G)$ and has more edges than G', a contradiction to the choice of G'. Finally, if a diagonal, say v_2v_6, is missing, then some vertex y has 3 neighbours in $S' = \{v_1, v_2, v_3, v_5, v_6, v_7\}$ and, again, replacing v_2 or v_6 by y we get a subgraph with more edges than G' inducing Γ_3 in $\mathrm{cl}_\Delta(G)$. So, indeed, G contains Γ_3 as a subgraph. □

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