ON EQUIVALENCE RELATIONS INDUCED BY POLISH GROUPS

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Abstract. The motivation of this article is to introduce a kind of orbit equivalence relations which can well describe structures and properties of Polish groups from the perspective of Borel reducibility. Given a Polish group G, let $E(G)$ be the right coset equivalence relation $G^\omega/c(G)$, where $c(G)$ is the group of all convergent sequences in G. Let G be a Polish group. (1) G is a discrete countable group containing at least two elements iff $E(G) \sim_B E_0$; (2) if G is TSI uncountable non-Archimedean, then $E(G) \sim_B E_0^\omega$; (3) G is non-Archimedean iff $E(G) \leq_B =^+$; (4) if H is a CLI Polish group but G is not, then $E(G) \not\leq_B E(H)$; and (5) if H is a non-Archimedean Polish group but G is not, then $E(G) \not\leq_B E(H)$. The notion of α-l.m.-unbalanced Polish group for $\alpha < \omega_1$ is introduced. Let G and H be Polish groups, $0 < \alpha < \omega_1$. If G is α-l.m.-unbalanced but H is not, then $E(G) \not\leq_B E(H)$. For TSI Polish groups, the existence of Borel reduction is transformed into the existence of a well-behaved continuous mapping between topological groups. As its applications, for any Polish group G, let G_0 be the connected component of the identity element 1_G. Let G and H be two separable TSI Lie groups. If $E(G) \leq_B E(H)$, then there exists a continuous locally injective map $S : G_0 \to H_0$. Moreover, if G_0 and H_0 are abelian, S is a group homomorphism. In particular, for $c_0, e_0, c_1, e_1 \in \mathbb{N}$, $E(\mathbb{R}^{c_0} \times \mathbb{T}^{e_0}) \leq_B E(\mathbb{R}^{c_1} \times \mathbb{T}^{e_1})$ iff $e_0 \leq e_1$ and $c_0 + e_0 \leq c_1 + e_1$.

§1. Introduction. In recent years, a theory has been developed abundantly in descriptive set theory: using Borel reducibility to investigate the complexity of equivalence relations. Equivalence relations from different branches of mathematics have been systematically studied, and a large number of results on Borel reducibility or non-reducibility have been found. Polish groups are important tools in the study of equivalence relations. The orbit equivalence relations generated by the continuous (or Borel) group actions of Polish groups G on Polish spaces X (denoted by E_G^X) account for most of the various equivalence relations concerned by various branches of classical mathematics.

We let E be a Borel equivalence relation on a Polish space X. We say that E is *smooth* if E is Borel reducible to the equality relation of some Polish space; and say that E is *countable* if every E-class is countable. We call E *essentially countable* if there is a countable Borel equivalence relation F such that E is Borel reducible to F. Feldman–Moore Theorem [14, Theorem 7.1.4] asserts that, for every countable Borel equivalence relation E on a Polish space X, there is a countable group Γ and a Borel action of Γ on X such that $E = E_\Gamma^X$.

A very prevalent phenomenon is that the structural properties of a Polish group G can affect the Borel complexity of the orbit equivalence relations E_G^X. For instances,

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Solecki [28] proved that a Polish group G is compact if and only if E_G^X is smooth for any Polish G-space X; Kechris [22] proved that if G is a locally compact Polish group, then E_G^X is essentially countable for any Polish G-space X; Gao and Jackson [15] proved that if G is a countable discrete abelian group, then E_G^X is Borel reducible to E_0 for any Polish G-space X; Hjorth's turbulence theory [17] asserts that orbit equivalence relations induced by turbulent actions are not Borel reducible to $E_{S_\infty}^X$ for any Polish S_∞-space X; the orbit equivalence relation generated by a continuous action of a CLI Polish group on a Polish space X is pinned, which implies that any non-pinned Borel equivalence relation is not Borel reducible to E_G^X (see [21, Section 17]). Naturally, we ask the following problem.

PROBLEM 1. *Can we characterize structures and properties of Polish groups through the study of equivalence relations*?

Theoretically, this should be possible: for each kind of Polish groups, studying the properties of all/part of their orbit equivalence relations will inevitably reflect the properties of this kind of Polish groups. However, systematic research needs to find a specific orbit equivalence relation that can well reflect the properties of a given Polish group.

Given a countable discrete group Γ. There is a canonical continuous *shift action* of Γ on $\{0,1\}^\Gamma$ given by $(gp)(h) = p(g^{-1}h)$ for $p \in \{0,1\}^\Gamma$ and $g, h \in \Gamma$. The *free part* of the action is the set

$$(2)^\Gamma = \{p \in \{0,1\}^\Gamma : gp \neq p \text{ for all } 1_\Gamma \neq g \in \Gamma\}.$$

We let E_Γ be the corresponding orbit equivalence relation on $(2)^\Gamma$. Indeed, the Borel complexity of the relation E_Γ can reflect the properties of the group Γ. For instance, if E_Γ is Borel reducible to E_0, then Γ is amenable, i.e., there is a left-invariant finitely additive probability measure μ on Γ (cf. [14, Section 7.4]). For a special case of Γ, Thomas proved that E_Γ is Borel reducible to E_Λ if and only if Γ embeds into Λ (cf. [29, Corollary 3.8]). So in this special case, we can think that the equivalence relation E_Γ captures the algebraic structure of Γ very well.

The main purpose of this article is to attempt to provide a possible answer to Problem 1 for general Polish groups.

Given a Polish group G, we define an equivalence relation $E(G)$ on G^ω as: for $x, y \in G^\omega$,

$$xE(G)y \iff \lim_n x(n)y(n)^{-1} \text{ converges in } G.$$

We say that $E(G)$ is the *equivalence relation induced by* G. Indeed, $E(G)$ is the orbit equivalence relation induced by the action of left multiplication of $c(G)$ on G^ω, where $c(G)$ is the Polish group consisting of all convergent sequences in G.

For all Polish groups G, equivalence relations $E(G)$ are $\mathbf{\Pi}^0_3$.

The study of equivalence relations induced by Polish groups can distinguish Polish groups very well. For instance, we can compare them with some benchmark equivalence relations: E_0, E_0^ω and $=^+$ (the definitions of these benchmark equivalence relations are left for the next section).

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THEOREM 1.1. *Let G be a Polish group, then we have*:

(1) *G is a discrete countable group containing at least two elements iff $E(G) \sim_B E_0$*;
(2) *if G is TSI uncountable non-Archimedean, then $E(G) \sim_B E_0^\omega$*;
(3) *G is non-Archimedean iff $E(G) \leq_B =^+$*.

Recently, the authors [10] use the Borel complexity of $E(G)$ to characterize TSI non-Archimedean Polish groups.

THEOREM [10, Theorem 1.3]. *Let G be a Polish group. Then the following are equivalent*:

(1) *G is TSI non-Archimedean*;
(2) *$E(G) \leq_B E_0^\omega$*;
(3) *$E(G) \leq_B \mathbb{R}^\omega/c_0$*.

In particular, $E(G) \sim_B E_0^\omega$ iff G is TSI uncountable non-Archimedean.

Many notions about Borel reductions can be applied in the research on equivalence relations induced by Polish groups. For instance, using the notion of right embedability introduced by Lupini and Panagiotopoulos [25], we get the following.

THEOREM 1.2. *Let G be a non-CLI Polish group and H a CLI Polish group. Then $E(G) \not\leq_B E_H^Y$ for any Polish H-space Y.*
In particular, since $c(H)$ is also CLI, we have $E(G) \not\leq_B E(H)$.

Similarly, using the notion of turbulence introduced by Hjorth [17] and c_0-equality introduced by Farah [13], we get the following.

THEOREM 1.3. *Let G and H be two Polish groups. If H is non-Archimedean but G is not, then $E(G) \not\leq_B E_H^Y$ for any Polish H-space Y.*
In particular, since $c(H)$ is also non-Archimedean, we have $E(G) \not\leq_B E(H)$.

Recently, similar result concerning TSI Polish groups is obtained by the authors [10].

THEOREM [10, Theorem 1.2]. *Let G, H be two Polish groups. If H is TSI but G is not, then $E(G) \not\leq_B E(H)$.*

More attractive results appear in the study of CLI Polish groups. Using the notion of α-unbalanced relations introduced by Allison and Panagiotopoulos [1], we define the notion of α-l.m.-unbalanced Polish groups for $\alpha < \omega_1$ in Section 5.1 ("l.m." stands for left multiplication.) and obtain the following.

THEOREM 1.4. *Let G, H be Polish groups, $0 < \alpha < \omega_1$. If G is α-l.m.-unbalanced but H is not, then $E(G) \not\leq_B E(H)$.*

We will also present examples of α-l.m.-unbalanced groups. Recall that a topological group G is said to be distal if $1_G \notin \overline{\{ghg^{-1} : g \in G\}}$ for all $h \neq 1_G \in G$. It is known that all TSI Polish groups and all nilpotent Polish groups are distal (c.f. [27]).

THEOREM 1.5. *Let G be a Polish group.*

(1) *If G is not distal, then G is 1-l.m.-unbalanced. In particular, if G is locally compact, then G is not distal iff it is 1-l.m.-unbalanced.*

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(2) *If G is locally compact, then G is not 2-l.m.-unbalanced.*
(3) *Let Λ be an infinite countable discrete group, and $\alpha < \omega_1$. Then G is α-l.m.-unbalanced iff the wreath product $\Lambda \wr G$ is $(\alpha+1)$-l.m.-unbalanced.*

Therefore, we can find a sequence (G_n) of CLI Polish groups such that

$$E(G_0) <_B \cdots <_B E(G_n) <_B E(G_{n+1}) <_B \ldots.$$

Unfortunately, we have not found any ω-l.m.-unbalanced CLI Polish group so far, though we know that S_∞ is α-l.m.-unbalanced for any ordinal α.

The most refined result appears in the study of TSI Polish groups. Applying the tool of additive reduction and ultrafilter limit, the existence of Borel reduction is transformed into the existence of a well-behaved continuous mapping between Polish groups.

Let G be a Polish group. We define equivalence relation $E_*(G)$ on G^ω as:

$$xE_*(G)y \iff \lim_n x(0)x(1)\ldots x(n)y(n)^{-1}\ldots y(1)^{-1}y(0)^{-1} \text{ converges}$$

for $x, y \in G^\omega$. It is clear that $E(G) \sim_B E_*(G)$ (cf. [9, Proposition 2.2]). It turns out that, for TSI Polish groups, $E_*(G)$ is a more convenient research object than $E(G)$.

THEOREM 1.6 (Pre-rigid Theorem). *Let G, H be two TSI Polish groups. Suppose G is connected, H is locally compact and the interval $[0,1]$ embeds in H. Then $E(G) \leq_B E(H)$ iff there exists a continuous map $S : G \to H$ with $S(1_G) = 1_H$ such that, for $x, y \in G^\omega$, if $\lim_n d_G(x(n), y(n)) = 0$, then*

$$xE_*(G)y \iff S^\omega(x)E_*(H)S^\omega(y),$$

where the map $S^\omega : X^\omega \to Y^\omega$ is defined as: for $x \in X^\omega$ and $n \in \omega$,

$$S^\omega(x)(n) = S(x(n)).$$

To illustrate the strength of this theorem, we recall some notions. For any Polish group G, let G_0 be the connected component of the identity element 1_G, which is a closed normal subgroup of G. It is known that, if G is locally compact TSI, then G_0 is the product of a compact connected Polish group and a group $\mathbb{R}^n$. Now we apply Pre-rigid Theorem on Lie groups as follows.

THEOREM 1.7. *Let G and H be two separable TSI Lie groups. If $E(G) \leq_B E(H)$, then there exists a continuous locally injective map $S : G_0 \to H_0$.*

In particular, if G_0, H_0 are abelian, then S is a group homomorphism.

As a corollary, we have $\dim(G) \leq \dim(H)$. Furthermore, if $\dim(G) = \dim(H)$ and G_0 is compact, then H_0 is also compact and G_0 is a covering space of H_0, and hence the corresponding Lie algebras of G_0 and H_0 are isomorphic. In particular, we obtain a complete answer for abelian connected Lie groups. Recall that $\mathbb{T}$ is the multiplicative group of all complex numbers with modulus 1.

THEOREM 1.8. *Let $c_0, e_0, c_1, e_1 \in \mathbb{N}$, then $E(\mathbb{R}^{c_0} \times \mathbb{T}^{e_0}) \leq_B E(\mathbb{R}^{c_1} \times \mathbb{T}^{e_1})$ iff $e_0 \leq e_1$ and $c_0 + e_0 \leq c_1 + e_1$.*

Recently, the aforementioned results have been vigorously generalized by the authors [10]. A completely metrizable topological group G is called a *pro-Lie group*

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if every open neighborhood of 1_G contains a normal subgroup N such that G/N is a Lie group (cf. [18, Definition 1]). By [20, Theorem 3.6], every locally compact TSI Polish group is a pro-Lie group. The following is a Rigid Theorem concerning locally compact connected TSI Polish groups and TSI pro-Lie groups.

THEOREM [10, Theorem 6.9]. *Let G be a locally compact connected TSI Polish group, H a TSI pro-Lie Polish group. Then $E(G) \leq_B E(H)$ iff there exists a continuous homomorphism $S : G \to H$ such that* $\ker(S)$ *is non-Archimedean.*

We also get a few results for disconnected Lie groups. Let $\mathrm{Inn}_G(G_0)$ be the group of all $\iota_u : G_0 \to G_0$ with $\iota_u(g) = ugu^{-1}$ for $u \in G$.

THEOREM 1.9. *Let G be a separable TSI Lie group with $G_0 = \mathbb{R}$. Define $\iota_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ as $\iota_{\mathbb{R}}(t) = -t$. Then we have*:

(1) $E(G) \sim_B E(\mathbb{R}) \iff \mathrm{Inn}_G(G_0) = \{\mathrm{id}_{\mathbb{R}}\}$;
(2) $E(G) \sim_B E(\mathbb{Z}_2 \ltimes \mathbb{R}) \iff \mathrm{Inn}_G(G_0) = \{\mathrm{id}_{\mathbb{R}}, \iota_{\mathbb{R}}\}$.

Similar results of $G_0 = \mathbb{T}$ in place of $\mathbb{R}$ are also valid.

In the end, the groups $\mathbb{T}_p$ of p-adic solenoids for all $p \geq 2$ are considered. Let $N(p)$ be the set of all prime divisors of p.

THEOREM 1.10. *Let $p, q \geq 2$ be natural numbers, then we have*:

(1) $E(\mathbb{T}_p) \leq_B E(\mathbb{T}_q) \iff N(p) \supseteq N(q)$;
(2) $E(\mathbb{R}) <_B E(\mathbb{T}_p) <_B E(\mathbb{T})$.

The authors also generalized the above theorem to P-adic solenoids, where P is a sequence of natural numbers ≥ 2 (see [9, Theorem 3.2]). Moreover, the Borel reducibility among $E(G)$'s between $E(\mathbb{R})$ and $E(\mathbb{T})$ are extremely complicated that the partial ordered set $P(\omega)/\mathrm{Fin}$ embeds into them (see [9, Theorem 3.6]).

This article is organized as follows. In Section 2, we recall some notions in descriptive set theory and Polish groups. In Section 3, we define the equivalence relations induced by Polish groups, and prove Theorem 1.1 and some basic facts. In Section 4, we prove Theorems 1.2 and 1.3. In Section 5, we define the notion of α-l.m.-unbalanced groups and prove Theorems 1.4 and 1.5. In Section 6, we present many results on TSI Polish groups and prove Theorems 1.6–1.10. Finally, in Section 7, we give some further open questions and additional remarks.

§2. Preliminaries. We say that a topological space is *Polish* if it is separable and admits a compatible complete metric. Let X, Y be two Polish spaces, E and F equivalence relations on X and Y, respectively. We say that a map $\theta : X \to Y$ is a *Borel reduction* of E to F if θ is Borel and for $x, y \in X$,

$$xEy \iff \theta(x)F\theta(y).$$

If such a Borel reduction exists, we say that E is *Borel reducible to* F, denoted by $E \leq_B F$. We also denote by $E \sim_B F$ for $E \leq_B F$ and $F \leq_B E$; and denote by $E <_B F$ for $E \leq_B F$ but $F \not\leq_B E$. For more details on Borel reducibility and descriptive set theory, we refer to [14, 21, 23].

The identity element of a group G is denoted by 1_G. We say that a topological group is a *Polish group* if its underlying topology is Polish. Given a Polish group G and a

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Polish space X, an action of G on X, denoted by $G \curvearrowright X$, is a map $a : G \times X \to X$ satisfying that $a(1_G, x) = x$ and $a(gh, x) = a(g, a(h, x))$ for $g, h \in G$ and $x \in X$. The pair (X, a) is called a *Polish (Borel) G-space* if a is a continuous (Borel) map. Throughout this article, we write gx in place of $a(g, x)$ for the sake of brevity. For each $x \in X$, the *G-orbit* of x, denoted by $[x]$ or Gx, is the set $\{gx : g \in G\}$. A subset A of X is *G-invariant* if $Gx \subseteq A$ for each $x \in A$. The *orbit equivalence relation*, denoted by E_G^X, is defined as

$$xE_G^X y \iff [x] = [y] \iff \exists g \in G\, (gx = y).$$

Becker and Kechris proved that, for any Borel G-space (X, a), there is a Polish topology τ on X such that τ generates the Borel structure of X and (X, a) is a Polish G-space (cf. [14, Theorem 4.4.6]). This implies that $E_G^X = E_G^{(X,\tau)}$. Therefore, the concepts of Borel G-space and Polish G-space are equivalent under the perspective of Borel reducibility.

Note that any discrete countable group is Polish. The *infinite permutation group* S_∞ is the group of all permutations of ω equipped with the pointwise convergence topology. A Polish group G is *non-Archimedean* if it has a neighborhood base of 1_G consisting of open subgroups. Becker and Kechris proved that a Polish group is non-Archimedean iff it is topologically isomorphic to a closed subgroup of S_∞ (cf. [4, Theorem 1.5.1] or [14, Theorem 2.4.1]). A metric d on a group G is *left-invariant* if $d(gh, gk) = d(h, k)$ for all $g, h, k \in G$. Similarly, we can define *right-invariant* metric. We say that d is *two-sided invariant* if it is both left and right-invariant. A compatible two-sided invariant metric on a Polish group is necessarily complete (cf. [4, Corollary 1.2.2]). We say that a Polish group G is CLI if it admits a left-invariant compatible complete metric, and that G is TSI if it admits a two-sided invariant compatible metric.

Now we recall some benchmark equivalence relations in the research of Borel reducibility. The equivalence relation E_0 on 2^ω is defined as

$$xE_0 y \iff \exists m\, \forall n > m\, (x(n) = y(n)).$$

If E is an equivalence relation on a Polish space X, then we define equivalence relations E^ω and E^+ on X^ω as

$$xE^\omega y \iff \forall n\, (x(n) E y(n)),$$

$$xE^+ y \iff \forall n\, \exists m\, (x(n) E y(m)) \wedge \forall m\, \exists n\, (x(n) E y(m)).$$

We pay special attention to E_0^ω and $\mathrm{id}(\omega^\omega)^+$ (denoted by $=^+$ for brevity).

Let G and Λ be two groups, and ϕ a homomorphism from Λ to $\mathrm{Aut}(G)$, the group of automorphisms on G. Recall that the *semi product* $\Lambda \ltimes_\phi G$ is the set $\Lambda \times G$ equipped with group operation as: for $(\lambda_1, g_1), (\lambda_2, g_2) \in \Lambda \times G$,

$$(\lambda_1, g_1)(\lambda_2, g_2) = (\lambda_1 \lambda_2, g_1 \phi(\lambda_1)(g_2)).$$

Note that, if G is a Polish group, Λ is a countable discrete group, and $\phi(\lambda)$ is a continuous automorphism on G for each λ, then $\Lambda \ltimes_\phi G$ equipped with the product topology on $\Lambda \times G$ is also a Polish group and G is topologically isomorphic to the open normal subgroup $\{1_\Lambda\} \times G$. We also recall that the *wreath product* $\Lambda \wr G$ is the

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set $\Lambda \times G^\Lambda$ equipped with group operation as: for $(\lambda_1, \chi_1), (\lambda_2, \chi_2) \in \Lambda \times G^\Lambda$,

$$(\lambda_1, \chi_1)(\lambda_2, \chi_2) = (\lambda_1\lambda_2, \chi)$$

with $\chi(\lambda) = \chi_1(\lambda)\chi_2(\lambda_1^{-1}\lambda)$ for $\lambda \in \Lambda$. If Λ is countable and G is Polish, $\Lambda \wr G$ equipped the product topology of $\Lambda \times G^\Lambda$ is also a Polish group.

Given two sets X, Y and a map $f : X \to Y$, we define a map $f^\omega : X^\omega \to Y^\omega$ as: for $x \in X^\omega$ and $n \in \omega$,

$$f^\omega(x)(n) = f(x(n)).$$

§3. Equivalence relations induced by Polish groups.

DEFINITION 3.1. Let G be a Polish group. We define an equivalence relation $E(G)$ on G^ω as

$$xE(G)y \iff \lim_n x(n)y(n)^{-1} \text{ converges in } G$$

for $x, y \in G^\omega$. We say that $E(G)$ is the *equivalence relation induced by* G. Moreover, we define a subgroup of G^ω as

$$c(G) = \{x \in G^\omega : \lim_n x(n) \text{ converges in } G\}.$$

Then we have

$$xE(G)y \iff xy^{-1} \in c(G) \iff c(G)x = c(G)y.$$

It is easy to see that $E(G)$ is a $\mathbf{\Pi}^0_3$ equivalence relation on G^ω.

Let d be a compatible metric on G. We define the supremum metric as $d_u(x, y) = \sup\{d(x(n), y(n)) : n \in \omega\}$ for $x, y \in G^\omega$. For any sequence (x_k) in G^ω and any $x \in G^\omega$, we use $x_k \rightrightarrows x$ to denote $d_u(x_k, x) \to 0$, i.e., (x_k) converges to x uniformly (respect to d_u).

THEOREM 3.2. *Let G be a Polish group, d a complete compatible metric on G, and let d_u be the supremum metric on $c(G)$. Then $(c(G), d_u)$ is also a Polish group.*

Furthermore, let d' be any compatible metric on G and d'_u the supremum metric of d', then $(c(G), d_u)$ and $(c(G), d'_u)$ have the same topology.

PROOF. (1) First, we prove that d_u is a Polish metric on $c(G)$. Let (γ_m) be a d_u-Cauchy sequence in $c(G)$, and let $g_m = \lim_n \gamma_m(n)$. Then for each $n \in \omega$, $(\gamma_m(n))$ is a d-Cauchy sequence. So there exists an $x \in G^\omega$ such that $\lim_m \gamma_m(n) = x(n)$ for each $n \in \omega$. It is clear that (g_m) is also a d-Cauchy sequence, so $g_m \to g$ for some $g \in G$. From the property of uniform convergence, we have

$$\lim_n x(n) = \lim_n \lim_m \gamma_m(n) = \lim_m \lim_n \gamma_m(n) = \lim_m g_m = g.$$

So $x \in c(G)$, and hence d_u is complete.

(2) Secondly, we prove that $(c(G), d_u)$ is a topological group. Let $\gamma, \eta \in c(G)$ and let $(\gamma_m), (\eta_m)$ be two sequences in $c(G)$. Put $g = \lim_n \gamma(n)$, $h = \lim_n \eta(n)$, $g_m = \lim_n \gamma_m(n)$, and $h_m = \lim_n \eta_m(n)$ for each $m \in \omega$. Now suppose $d_u(\gamma_m, \gamma) \to 0$ and $d_u(\eta_m, \eta) \to 0$. We only need to show that $d_u(\gamma_m\eta_m^{-1}, \gamma\eta^{-1}) \to 0$.

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For any $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that, for any $g', h' \in G$, if $d(g', g) < \delta_1$ and $d(h', h) < \delta_1$, then we have $d(g'(h')^{-1}, gh^{-1}) < \varepsilon/2$. For this $\delta_1 > 0$, there exists an $N_1 \in \omega$ such that, for $n > N_1$, we have $d(\gamma(n), g) < \delta_1/2$ and $d(\eta(n), h) < \delta_1/2$. It follows that $d(\gamma(n)\eta(n)^{-1}, gh^{-1}) < \varepsilon/2$. There also exists an $M_1 \in \omega$ such that, for $m > M_1$, we have $d_u(\gamma_m, \gamma) < \delta_1/2$ and $d_u(\eta_m, \eta) < \delta_1/2$. So

$$d(\gamma_m(n), g) \leq d(\gamma_m(n), \gamma(n)) + d(\gamma(n), g) \leq d_u(\gamma_m, \gamma) + d(\gamma(n), g) < \delta_1,$$

and also $d(\eta_m(n), h) < \delta_1$. It follows that $d(\gamma_m(n)\eta_m(n)^{-1}, gh^{-1}) < \varepsilon/2$. Then we have

$$d(\gamma_m(n)\eta_m(n)^{-1}, \gamma(n)\eta(n)^{-1}) < \varepsilon \quad (\forall m > M_1, \forall n > N_1).$$

For this $\varepsilon > 0$, there exists a $\delta_2 > 0$ such that, for any $g', h' \in G$ and any $n \leq N_1$, if $d(g', \gamma(n)) < \delta_2$ and $d(h', \eta(n)) < \delta_2$, then $d(g'(h')^{-1}, \gamma(n)\eta(n)^{-1}) < \varepsilon$. Then there exists an $M_2 \in \omega$ such that $d_u(\gamma_m, \gamma) < \delta_2$ and $d_u(\eta_m, \eta) < \delta_2$ for $m > M_2$. Thus we have

$$d(\gamma_m(n)\eta_m(n)^{-1}, \gamma(n)\eta(n)^{-1}) < \varepsilon \quad (\forall m > M_2, \forall n \leq N_1).$$

Therefore, $d_u(\gamma_m\eta_m^{-1}, \gamma\eta^{-1}) \leq \varepsilon$ for all $m > \max\{M_1, M_2\}$. It follows that $d_u(\gamma_m\eta_m^{-1}, \gamma\eta^{-1}) \to 0$.

It is trivial that $(c(G), d_u)$ is separable, so it is a Polish group.

(3) Let d' be a compatible metric on G and d'_u the supremum metric (respect to d') on $c(G)$. Let $\gamma \in c(G)$ and let (γ_m) be a sequence in $c(G)$, we only need to show that: $d_u(\gamma_m, \gamma) \to 0$ iff $d'_u(\gamma_m, \gamma) \to 0$. Let $\lim_n \gamma(n) = g$ and $\lim_n \gamma_m(n) = g_m$ for each $m \in \omega$. Suppose $d_u(\gamma_m, \gamma) \to 0$. Following similar arguments in the proof of $d_u(\gamma_m\eta_m^{-1}, \gamma\eta^{-1}) \to 0$ in (2), we can prove that $d'_u(\gamma_m, \gamma) \to 0$. And vice versa. ⊣

Therefore, $E(G)$ is the orbit equivalence relation induced by $c(G) \curvearrowright G^\omega$, the action of left multiplication of $c(G)$ on G^ω.

PROPOSITION 3.3. *Let G be a Polish group. If G is CLI (TSI, or non-Archimedean), so is $c(G)$.*

PROOF. If G is CLI, let d be a left-invariant compatible complete metric on G. Then it is trivial to check that the supremum metric d_u is also a left-invariant compatible complete metric on $c(G)$, so $c(G)$ is CLI too.

If G is TSI, the same arguments show that $c(G)$ is TSI.

If G is non-Archimedean, let (G_n) be a sequence of open subgroups of G which forms a neighborhood base of 1_G, and let d be a compatible metric on G and d_u the supremum metric on $c(G)$. Find an $\varepsilon_n > 0$ such that

$$\{g \in G : d(g, 1_G) < \varepsilon_n\} \subseteq G_n.$$

Let $V_n = \{\gamma \in c(G) : d_u(\gamma, 1_{c(G)}) < \varepsilon_n\}$. Then for any $\eta \in G_n^\omega \cap c(G)$, we have

$$\eta \in \eta V_n \subseteq G_n^\omega \cap c(G).$$

So $G_n^\omega \cap c(G)$ is an open subgroup of $c(G)$. It is clear that $(G_n^\omega \cap c(G))$ is a neighborhood base of $1_{c(G)}$, so $c(G)$ is non-Archimedean. ⊣

PROPOSITION 3.4. *Let G, H be two Polish groups. If G is topologically isomorphic to a closed subgroup of H, then $E(G) \leq_B E(H)$.*

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PROOF. It follows from the definitions of $E(G)$ and $E(H)$. ⊣

THEOREM 3.5.

(1) *Let G be a discrete countable group containing at least two elements, then $E(G) \sim_B E_0$.*
(2) *Let G be an uncountable Polish group, then $E_0^\omega \leq_B E(G)$.*
(3) *Let G be a TSI non-Archimedean uncountable Polish group, then $E(G) \sim_B E_0^\omega$.*
(4) *Let G be a non-Archimedean Polish group, then $E(G) \leq_B =^+$.*
(5) *$E(S_\infty) \sim_B =^+$.*

PROOF. (1) Let G be a discrete countable group containing at least two elements. For $x, y \in G^\omega$, we have

$$xE(G)y \iff xy^{-1} \in c(G) \iff \exists m\, \exists g \in G\, \forall n > m\, (x(n)y(n)^{-1} = g).$$

For all $m \in \omega$, we define

$$\begin{aligned} xF_m y &\iff \exists g \in G\, \forall n > m\, (x(n)y(n)^{-1} = g) \\ &\iff \forall n > m\, (x(n)y(n)^{-1} = x(n+1)y(n+1)^{-1}). \end{aligned}$$

Thus each F_m is a closed equivalence relation on G^ω, and hence is smooth (cf. [14, Proposition 5.4.7]). Note that $E(G) = \bigcup_m F_m$ and $F_m \subseteq F_{m+1}$ for each $m \in \omega$. So $E(G)$ is hypersmooth (cf. [14, Definition 8.1.1]). It is clear that each orbit of $E(G)$ is countable. By Dougherty–Jackson–Kechris' theorem (cf. [14, Theorem 8.1.5]), we have $E(G) \leq_B E_0$.

Fix a $g_0 \in G$ with $g_0 \neq 1_G$. For $a \in 2^\omega$, we define $\theta(a) \in G^\omega$ as

$$\theta(a)(n) = \begin{cases} g_0, & n = 2k, a(k) = 1, \\ 1_G, & \text{otherwise.} \end{cases}$$

Then θ witnesses that $E_0 \leq_B E(G)$. Therefore, we have $E(G) \sim_B E_0$.

(2) Let G be uncountable, then we can find a sequence (g_i) in G such that $\lim_i g_i = 1_G$ with all $g_i \neq 1_G$ for $i \in \omega$. Fix a bijection $\langle \cdot, \cdot \rangle$ from $\omega \times \omega$ to ω. For $\alpha \in (2^\omega)^\omega$, we define $\vartheta(\alpha) \in G^\omega$ as

$$\vartheta(\alpha)(\langle i, j \rangle) = \begin{cases} g_i, & j = 2k, \alpha(i)(k) = 1, \\ 1_G, & \text{otherwise.} \end{cases}$$

Then ϑ witnesses that $E_0^\omega \leq_B E(G)$.

(3) Let G be TSI non-Archimedean uncountable. By (2), we only need to show $E(G) \leq_B E_0^\omega$. From [16, Theorem 1.1], G is isomorphic to a closed subgroup of a product $\prod_k \Gamma_k$, where each Γ_k is a discrete countable group. So, by Proposition 3.4, it suffices to prove $E(\prod_k \Gamma_k) \leq_B E_0^\omega$. For $x, y \in (\prod_k \Gamma_k)^\omega$, we have

$$\begin{aligned} xE(\textstyle\prod_k \Gamma_k)y &\iff xy^{-1} \in c(\textstyle\prod_k \Gamma_k) \\ &\iff \forall k\, (x(\cdot)(k)y(\cdot)(k)^{-1} \in c(\Gamma_k)) \\ &\iff \forall k\, (x(\cdot)(k)E(\Gamma_k)y(\cdot)(k)). \end{aligned}$$

Then $E(\prod_k \Gamma_k) \leq_B E_0^\omega$ follows from (1).

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(4) From [14, Theorem 2.4.1] and Proposition 3.4, it suffices to prove $E(S_\infty) \leq_B =^+$. For $x, y \in (S_\infty)^\omega$, we have

$$\begin{aligned} xE(S_\infty)y &\iff xy^{-1} \in c(S_\infty) \\ &\iff \exists g \in S_\infty \, \forall k \, \exists m \, \forall n > m \, ((x(n)y(n)^{-1})(k) = g(k)) \\ &\iff \exists g \in S_\infty \, \forall k \, \exists m \, \forall n > m \, (x(n)^{-1}(g(k)) = y(n)^{-1}(k)). \end{aligned}$$

We define $\theta'(x) \in (\omega^\omega)^\omega$ as $\theta'(x)(k)(n) = x(n)^{-1}(k)$ for $n, k \in \omega$. Then

$$xE(S_\infty)y \iff \exists g \in S_\infty \, \forall k \, (\theta'(x)(g(k))E_0\theta'(y)(k)).$$

By [14, Exercise 8.3.4], this implies that $E(S_\infty) \leq_B E_0^+$. It is well known that $E_0^+ \leq_B =^+$ (cf. [14, Theorem 7.4.10 and Exercise 8.3.3]), so we have $E(S_\infty) \leq_B =^+$.

(5) By [14, Lemma 10.3.4] and (4), we only need to show that $=^+ \restriction Y \leq_B E(S_\infty)$, where $Y = \{\alpha \in (2^\omega)^\omega : \forall n, m \, (n \neq m \Rightarrow \alpha(n) \neq \alpha(m))\}$. Let (p_i) be the strictly increasing enumeration of all prime numbers. Define a map $\varphi : Y \to (\omega^\omega)^\omega$ with

$$\varphi(\alpha)(n)(k) = \begin{cases} p_0^{\alpha(n)(0)} \cdots p_k^{\alpha(n)(k)}, & k \geq n, \\ 0, & \text{otherwise}, \end{cases}$$

for $\alpha \in Y$ and $n, k \in \omega$. It is clear that

$$\varphi(\alpha)(n)E_0\varphi(\beta)(m) \iff \alpha(n) = \beta(m).$$

Define a map $\rho : (\omega^\omega)^\omega \to (\omega^\omega)^\omega$ by induction on n as

$$\rho(x)(k)(n) = \begin{cases} x(n)(k), & x(n)(k) \notin \{\rho(x)(k)(i) : i < n\}, \\ \min \omega \setminus \{\rho(x)(k)(i) : i < n\}, & \text{otherwise}, \end{cases}$$

for $x \in (\omega^\omega)^\omega$ and $n, k \in \omega$. Note that $\varphi(\alpha)(n)(k) = 0$ for all $n > k$, so $\rho(\varphi(\alpha))(k) \in S_\infty$ for all $\alpha \in Y$ and $k \in \omega$. Furthermore, for any $n \in \omega$, $\alpha(0), \ldots, \alpha(n)$ are pairwise distinct elements in 2^ω, so for large enough k, we have that $\varphi(\alpha)(0)(k), \ldots, \varphi(\alpha)(n)(k)$ are pairwise distinct natural numbers. It follows that

$$\forall n \, \forall^\infty k \, (\rho(\varphi(\alpha))(k)(n) = \varphi(\alpha)(n)(k)).$$

Now we define a map $\vartheta' : Y \to (S_\infty)^\omega$ as

$$\vartheta'(\alpha)(k) = \rho(\varphi(\alpha))(k)^{-1} \in S_\infty$$

for $x \in Y$ and $k \in \omega$. Then for any $\alpha, \beta \in Y$, we have

$$\begin{aligned} &\vartheta'(\alpha)E(S_\infty)\vartheta'(\beta) \\ \iff &\exists g \in S_\infty \, \forall n \, \forall^\infty k \, ((\rho(\varphi(\alpha))(k)^{-1}\rho(\varphi(\beta))(k))(n) = g(n)) \\ \iff &\exists g \in S_\infty \, \forall n \, \forall^\infty k \, (\rho(\varphi(\alpha))(k)(g(n)) = \rho(\varphi(\beta))(k)(n)) \\ \iff &\exists g \in S_\infty \, \forall n \, \forall^\infty k \, (\varphi(\alpha)(g(n))(k) = \varphi(\beta)(n)(k)) \\ \iff &\exists g \in S_\infty \, \forall n \, (\varphi(\alpha)(g(n))E_0\varphi(\beta)(n)) \\ \iff &\exists g \in S_\infty \, \forall n \, (\alpha(g(n)) = \beta(n)) \\ \iff &\alpha =^+ \beta. \end{aligned}$$

Thus $=^+ \restriction Y \leq_B E(S_\infty)$. So $E(S_\infty) \sim_B =^+$. ⊣

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Furthermore, the authors [10] get the following stronger result.

THEOREM 3.6 [10, Theorem 1.3]. *Let G be a Polish group. Then the following are equivalent*:

(1) *G is TSI non-Archimedean*;
(2) $E(G) \leq_B E_0^\omega$;
(3) $E(G) \leq_B \mathbb{R}^\omega/c_0$.

In particular, $E(G) \sim_B E_0^\omega$ iff G is TSI uncountable non-Archimedean.

COROLLARY 3.7. *Let Λ be a discrete countable group, G a Polish group containing an element h_0 with $1_G \notin \overline{\{gh_0g^{-1} : g \in G\}}$. Then $E(G \times \Lambda) \sim_B E(G)$.*

PROOF. Proposition 3.4 gives $E(G) \leq_B E(G \times \Lambda)$. It is trivial that $E(G \times \Lambda) \sim_B E(G) \times E(\Lambda)$. By Theorem 3.5(1), $E(\Lambda) \leq_B E_0$, so we only need to show that $E(G) \times E_0 \leq_B E(G)$. For $x \in G^\omega$ and $a \in 2^\omega$, define $\theta(x, a) \in G^\omega$ as

$$\theta(x,a)(n) = \begin{cases} x(k)h_0, & n = 2k, a(k) = 1, \\ x(k), & n = 2k, a(k) = 0 \text{ or } n = 2k+1. \end{cases}$$

It is trivial to check that θ is a Borel reduction of $E(G) \times E_0$ to $E(G)$. ⊣

§4. Non-CLI Polish groups and non non-Archimedean Polish groups.

4.1. Non-CLI Polish groups. Lupini and Panagiotopoulos introduced a notion of right embedability for proving that an orbit equivalence relation is not Borel reducible to any E_H^Y, where H is a CLI Polish group.

DEFINITION 4.1 (Becker [3], Lupini–Panagiotopoulos [25]). Let G be a Polish group, (g_n) a sequence in G. We say that (g_n) is *left Cauchy* if $\lim_{m,n} g_m^{-1} g_n = 1_G$, and say that (g_n) is *right Cauchy* if $\lim_{m,n} g_m g_n^{-1} = 1_G$.

Let X be a Polish G-space and $x, y \in X$. We say that x is *left embeddable* into y, denoted by $x \overset{l}{\to} y$, if there exists a left Cauchy sequence (g_n) in G such that $g_n x \to y$, and say that x is *right embeddable* into y, denoted by $x \overset{r}{\to} y$, if there exists a right Cauchy sequence (h_n) in G such that $h_n y \to x$.

Clearly, (g_n) is right Cauchy iff (g_n^{-1}) is left Cauchy. Let d be a left-invariant compatible metric on G, (g_n) a sequence in G. It is worth noting that (g_n) is left Cauchy iff it is d-Cauchy. It was proved by Becker that the relation of left embeddability is a preorder, and is invariant (cf. [3]). Similar arguments show that the relation of right embeddability is a preorder, and is invariant too. So we can write $[x] \overset{l}{\to} [y]$ and $[x] \overset{r}{\to} [y]$ in place of $x \overset{l}{\to} y$ and $x \overset{r}{\to} y$, respectively.

THEOREM 4.2. *Let X be a Polish G-space and $x, y \in X$. Then $x \overset{l}{\to} y$ implies $x \overset{r}{\to} y$.*

PROOF. Suppose $x \overset{l}{\to} y$. Then there exists a left Cauchy sequence (g_n) in G such that $g_n x \to y$. Let d be a compatible metric on G and ρ a compatible metric on X.

By the continuity of the action of G on X, for any $k \in \omega$, there exists a $\delta > 0$ such that, for any $g \in G$, if $d(g, 1_G) < \delta$, then $\rho(gx, x) < 2^{-(k+1)}$. Since (g_n) is left Cauchy, for this $\delta > 0$, there exists an $m_k \in \omega$ such that, for any $n > m_k$, we have $d(g_{m_k}^{-1} g_n, 1_G) < \delta$, and hence $\rho(g_{m_k}^{-1} g_n x, x) < 2^{-(k+1)}$. By $g_n x \to y$, we have

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$g_{m_k}^{-1} g_n x \to g_{m_k}^{-1} y$. There exists an $n > m_k$ such that $\rho(g_{m_k}^{-1} g_n x, g_{m_k}^{-1} y) < 2^{-(k+1)}$. So we have $\rho(g_{m_k}^{-1} y, x) < 2^{-k}$. Without loss of generality, we can assume that (m_k) is strictly increasing. Let $h_k = g_{m_k}^{-1}$ for each k. Then (h_k) witnesses that $x \xrightarrow{r} y$. ⊣

Suppose that X is a Polish G-space, and $x, y \in X$. Lupini and Panagiotopoulos defined a Becker-embedding game $Emb(x, y)$ (see [25, Definition 2.2]). We say that x is Becker embeddable into y and write $x \preceq_B y$ if Player II has a winning strategy for the game Emb(x, y). They showed the notion of Becker-embeddability is actually equivalent to the notion of right embedability, i.e., $x \preceq_B y$ iff $x \xrightarrow{r} y$ [25, Lemma 2.4].

THEOREM 4.3. *Let G be a non-CLI Polish group and H a CLI Polish group. Then for any Polish H-space Y, we have $E(G) \not\leq_B E_H^Y$.*

PROOF. Let d be a left-invariant compatible metric on G, and let d_u be the supremum metric on $c(G)$. Since G is not CLI, d is not complete. So there exists a d-Cauchy sequence (g_n) in G such that (g_n) diverges. For $m \in \omega$, we define $\gamma_m \in c(G)$ as

$$\gamma_m(n) = \begin{cases} g_n, & n < m, \\ g_m, & n \geq m. \end{cases}$$

For $m < k$, we have $d_u(\gamma_m^{-1}\gamma_k, 1_{c(G)}) = \max\{d(g_m, g_n) : m \leq n \leq k\}$. So (γ_m) is left Cauchy, and hence (γ_m^{-1}) is right Cauchy.

We define $z \in G^\omega$ as $z(n) = g_n$ for each $n \in \omega$. Then $z \notin c(G)$. We claim that for any $c(G)$-invariant dense G_δ subset C of G^ω there exist $x, y \in C$ with different orbits such that $x \preceq_B y$. Then by [25, Theorem 2.9] we have $E(G) \not\leq_B E_H^Y$.

Fix a $c(G)$-invariant dense G_δ subset C of G^ω. Since $z^{-1}C$ is also dense G_δ, $C \cap z^{-1}C \neq \emptyset$. Let $x \in C \cap z^{-1}C$ and $y = zx$, then $x, y \in C$ and $[x] \neq [y]$. Note that $\gamma_m^{-1} y \to z^{-1} y = x$, so $x \xrightarrow{r} y$. Thus $x \preceq_B y$. ⊣

COROLLARY 4.4. *Let H be a CLI Polish group, then $E(S_\infty) \not\leq_B E(H)$.*

PROOF. It well known that S_∞ is not CLI (cf. [14, Example 2.2.7]). And by Proposition 3.3, $c(H)$ is CLI. ⊣

4.2. Non non-Archimedean Polish groups. Hjorth founded the theory of turbulence for proving that an orbit equivalence relation is not Borel reducible to $E_{S_\infty}^Y$ for any Polish S_∞-space Y. We omit the definition of turbulent actions since it is very complicated and is not used in this article, one can find it in [14, Section 10].

DEFINITION 4.5 (Farah [13, Definition 3.1]). If (X_n, d_n), $n \in \omega$ is a sequence of finite metric spaces, let $D = D(\langle X_n, d_n \rangle)$ be the equivalence relation on $\prod_n X_n$ defined as

$$xDy \iff \lim_n d_n(x(n), y(n)) = 0.$$

The equivalence relations of this form are called *c_0-equalities*.

By [13, Lemma 3.4], every c_0-equality is an orbit equivalence relation induced by a continuous Polish group action on $\prod_n X_n$.

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For a finite metric space (X, d) and $\varepsilon \in \mathbb{R}_+$, we define

$$\delta(\varepsilon, X) = \min\{\delta : \exists x_0, x_1, \dots, x_n \, (d(x_0, x_n) \geq \varepsilon \wedge \forall i < n \, (d(x_i, x_{i+1}) < \delta))\}.$$

LEMMA 4.6. *Let G be a Polish group. If G is not non-Archimedean, then there exists a turbulent c_0-equality D such that $D \leq_B E(G)$.*

PROOF. Let d be a left-invariant compatible metric on G. Since G is not non-Archimedean, there exists an $\varepsilon_0 > 0$ such that no open subgroup of G is contained in $V = \{g \in G : d(1_G, g) < \varepsilon_0\}$. For each $n \in \omega$, we define $V_n = \{g \in G : d(1_G, g) < 2^{-n}\}$. Then $V_n^{-1} = V_n$ and $G_n = \bigcup_m V_n^m$ is an open subgroup of G. Thus $G_n \not\subseteq V$ for each $n \in \omega$, and hence there is an integer $m_n > 0$ such that $V_n^{m_n} \not\subseteq V$. So we can find $1_G = g_{n,0}, g_{n,1}, \dots, g_{n,m_n} \in G$ such that $g_{n,i}^{-1} g_{n,i+1} \in V_n$ for each $i < m_n$ and $g_{n,m_n} \notin V$.

Let $X_n = \{g_{n,i} : 0 \leq i \leq m_n\}$ for each $n \in \omega$. Then (X_n, d) is a finite metric space. We define $\theta : \prod_n X_n \to G^\omega$ as, for $x \in \prod_n X_n$ and each $k \in \omega$,

$$\theta(x)(k) = \begin{cases} x(n)^{-1}, & k = 2n, \\ 1_G, & k = 2n+1. \end{cases}$$

It is trivial to check that θ is a Borel reduction of $D(X_n, d)$ to $E(G)$.

Note that $\delta(\varepsilon_0, X_n) \leq 2^{-n} \to 0$ as $n \to \infty$. By [13, Theorem 3.7(a)], there exists a turbulent c_0-equality D such that $D \leq_B D(X_n, d) \leq_B E(G)$. ⊣

Hjorth proved that orbit equivalence relations from turbulent actions are not Borel reducible to any S_∞ orbit equivalence relation (see [14, Corollary 10.4.3]). Using this result, we can prove the following theorem.

THEOREM 4.7. *Let G and H be two Polish groups. If H is non-Archimedean but G is not, then $E(G) \not\leq_B E_H^Y$ for any Polish H-space Y.*

In particular, we have $E(G) \not\leq_B E(S_\infty)$.

PROOF. Since H is topologically isomorphic to a closed subgroup of S_∞, by [14, Theorem 3.5.2], we only need to consider the case of $H = S_\infty$.

Then the result follows from the preceding lemma.

By Proposition 3.3, $c(S_\infty)$ is non-Archimedean, so $E(G) \not\leq_B E(S_\infty)$. ⊣

COROLLARY 4.8. *Let G be a Polish group, then $E(G) \leq_B =^+$ iff G is non-Archimedean.*

PROOF. It follows from Theorem 3.5(4) and the preceding theorem. ⊣

§5. Non-TSI Polish groups and α-unbalanced relations.

5.1. Unbalanced relations.

DEFINITION 5.1 (Allison–Panagiotopoulos [1, Definition 1.1]). Let X be a Polish G-space and let $x, y \in X$. We write $x \leftrightsquigarrow y$ provided that, for any open $V \ni 1_G$ and open set U of X with $U \cap ([x] \cup [y]) \neq \emptyset$, there exist $g^x, g^y \in G$ with $g^x x, g^y y \in U$, so that

$$g^y y \in \overline{V(g^x x)} \text{ and } g^x x \in \overline{V(g^y y)}.$$

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It is clear that $\leftrightsquigarrow$ is symmetric, and for any $g, h \in G$, we have that $x \leftrightsquigarrow y \iff gx \leftrightsquigarrow hy$. So we can write $[x] \leftrightsquigarrow [y]$ whenever $x \leftrightsquigarrow y$.

LEMMA 5.2. *Let G be a Polish group and $c(G) \curvearrowright G^\omega$ the action of left multiplication. For $x, y \in G^\omega$ we have*

$$x \leftrightsquigarrow y \iff \exists (\gamma_m), (\eta_m) \in c(G)^\omega \, (\gamma_m x y^{-1} \eta_m \rightrightarrows 1_{G^\omega}).$$

PROOF. Let d be a compatible metric on G and d_u the supremum metric on G^ω. Note that $1_{c(G)}(n) = 1_{G^\omega}(n) = 1_G$ for each $n \in \omega$. Let $V_m = \{\gamma \in c(G) : d_u(\gamma, 1_{c(G)}) < 2^{-m}\}$ for $m \in \omega$. Then (V_m) is a neighborhood basis of $1_{c(G)}$. Note that V_m is dense in $\{x \in G^\omega : d_u(x, 1_{G^\omega}) < 2^{-m}\}$ under the topology of G^ω. For $x, y \in G^\omega$, we have $x \in \overline{V_m y} \iff xy^{-1} \in \overline{V_m}$, so

$$d_u(xy^{-1}, 1_{G^\omega}) < 2^{-m} \Rightarrow x \in \overline{V_m y} \Rightarrow d_u(xy^{-1}, 1_{G^\omega}) \leq 2^{-m}.$$

($\Rightarrow$). Let $x \leftrightsquigarrow y$. Then for each $m \in \omega$, there exist $\gamma_m^x, \gamma_m^y \in c(G)$ such that $\gamma_m^x x \in \overline{V_m(\gamma_m^y y)}$. So $d_u(\gamma_m^x x y^{-1} (\gamma_m^y)^{-1}, 1_{G^\omega}) \leq 2^{-m}$. Let $\gamma_m = \gamma_m^x$ and $\eta_m = (\gamma_m^y)^{-1}$, then $\gamma_m x y^{-1} \eta_m \rightrightarrows 1_{G^\omega}$.

($\Leftarrow$). Let $(\gamma_m), (\eta_m) \in c(G)^\omega$ with $\gamma_m x y^{-1} \eta_m \rightrightarrows 1_{G^\omega}$. It is easy to check that $\eta_m^{-1} y x^{-1} \gamma_m^{-1} \rightrightarrows 1_{G^\omega}$. For any open $V \ni 1_{c(G)}$ and open set U of G^ω with $U \cap ([x] \cup [y]) \neq \emptyset$, there exists $k \in \omega$ with $V_k \subseteq V$. Without loss of generality, we may assume that $U \cap [x] \neq \emptyset$ and $U = U_0 \times \cdots \times U_l \times G^\omega$ with $U_0, \ldots, U_l$ open in G. So there exists $\gamma \in c(G)$ with $\gamma x \in U$, i.e., $\gamma(n)x(n) \in U_n$ for each $n \leq l$.

Fix a large enough m such that

$$d_u(\gamma_m x y^{-1} \eta_m, 1_{G^\omega}) < 2^{-k}, \quad d_u(\eta_m^{-1} y x^{-1} \gamma_m^{-1}, 1_{G^\omega}) < 2^{-k}.$$

Now we define

$$\gamma^x(n) = \begin{cases} \gamma(n), & n \leq l, \\ \gamma_m(n), & n > l, \end{cases} \qquad \gamma^y(n) = \begin{cases} \gamma(n)x(n)y(n)^{-1}, & n \leq l, \\ \eta_m(n)^{-1}, & n > l. \end{cases}$$

Then we have $\gamma^x, \gamma^y \in c(G)$ and $\gamma^x x, \gamma^y y \in U$. Note that

$$\gamma^x(n)x(n)y(n)^{-1}(\gamma^y(n))^{-1} = \begin{cases} 1_G, & n \leq l, \\ \gamma_m(n)x(n)y(n)^{-1}\eta_m(n), & n > l. \end{cases}$$

It is easy to see that

$$d_u(\gamma^x x y^{-1} (\gamma^y)^{-1}, 1_{G^\omega}) \leq d_u(\gamma_m x y^{-1} \eta_m, 1_{G^\omega}) < 2^{-k},$$

$$d_u(\gamma^y y x^{-1} (\gamma^x)^{-1}, 1_{G^\omega}) \leq d_u(\eta_m^{-1} y x^{-1} \gamma_m^{-1}, 1_{G^\omega}) < 2^{-k}.$$

It gives that $\gamma^x x \in \overline{V_k(\gamma^y y)} \subseteq \overline{V(\gamma^y y)}$ and $\gamma^y y \in \overline{V_k(\gamma^x x)} \subseteq \overline{V(\gamma^x x)}$. Thus we have $x \leftrightsquigarrow y$. ⊣

DEFINITION 5.3. Let G be a Polish group and $c(G) \curvearrowright G^\omega$ the action of left multiplication. We say that G is *l.m.-unbalanced* if there exist $x, y \in G^\omega$ such that $x \leftrightsquigarrow y$ and $[x] \neq [y]$, where l.m. stands for left multiplication.

THEOREM 5.4. *Any TSI Polish group is not l.m.-unbalanced.*

PROOF. Let G be a Polish group and $c(G) \curvearrowright G^\omega$ the action of left multiplication. Assume that $x \leftrightsquigarrow y$ for some $x, y \in G^\omega$. From Lemma 5.2, there exist $(\gamma_m), (\eta_m) \in$

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$c(G)^\omega$ with $\gamma_m x y^{-1} \eta_m \rightrightarrows 1_{G^\omega}$. Let d be a two-sided invariant compatible metric on G and d_u the supremum metric on G^ω. Then d_u is also two-sided invariant, so

$$d_u(\eta_m \gamma_m x y^{-1}, 1_{G^\omega}) = d_u(\gamma_m x y^{-1} \eta_m, 1_{G^\omega}) \to 0,$$

thus we have $\eta_m \gamma_m x y^{-1} \rightrightarrows 1_{G^\omega}$. It follows that $\eta_m \gamma_m \rightrightarrows y x^{-1}$. Since $c(G)$ is closed under uniform convergence, $y x^{-1} \in c(G)$. So $[x] = [y]$. ⊣

As an application of the notion of l.m.-unbalanced groups, we present the following theorem. Since it is a special case of Theorem 5.15, we omit the proof at this moment.

THEOREM 5.5. *Let G, H be Polish groups. If G is l.m.-unbalanced but H is not, then $E(G) \not\leq_B E(H)$.*

COROLLARY 5.6. *Let G be a Polish group. If G is l.m.-unbalanced, then $E_0^\omega <_B E(G)$.*

PROOF. It follows from Theorems 3.5(2) and 5.5. ⊣

5.2. α-unbalanced relations.

DEFINITION 5.7 (Allison–Panagiotopoulos [1, Definition 7.1]). Let X be a Polish G-space, V an open neighborhood of 1_G, and let $\alpha < \omega_1$. We define $\leftrightsquigarrow_V^\alpha$ by induction. We say:

(1) $x \leftrightsquigarrow_V^0 y$, if $y \in \overline{Vx}$ and $x \in \overline{Vy}$;
(2) $x \leftrightsquigarrow_V^\alpha y$ for $\alpha > 0$, if for any open neighborhood $W \ni 1_G$ and any open neighborhood $U \subseteq X$ of x or y, there exist $g^x, g^y \in V$ with $g^x x, g^y y \in U$, so that $g^x x \leftrightsquigarrow_W^\beta g^y y$ for all $\beta < \alpha$.

REMARK 5.8. This definition appeared in an old version of [1]. It is slightly different with which in new version of [1] that the requirement of U has been modified to $U \cap ([x] \cup [y]) \neq \emptyset$. It should be noted that the definitions of these two versions are not equivalent for $V \neq G$ or $\alpha > 1$.

It is trivial that, if $x \leftrightsquigarrow_V^\alpha y$, then we have $x \leftrightsquigarrow_V^{\alpha'} y$ for $0 < \alpha' \leq \alpha$. Note that $g^x x \in U \cap Vx$ for any open neighborhood U of y, so $U \cap Vx \neq \emptyset$, and hence $y \in \overline{Vx}$. Similarly, $x \in \overline{Vy}$. Therefore, $x \leftrightsquigarrow_V^\alpha y$ implies $x \leftrightsquigarrow_V^0 y$ too.

LEMMA 5.9. *Let V and V' be two open neighborhoods of 1_G and $g, h \in G$ with $(gVg^{-1} \cup gVh^{-1} \cup hVh^{-1} \cup hVg^{-1}) \subseteq V'$, and let $\alpha < \omega_1, x, y \in X$. If $x \leftrightsquigarrow_V^\alpha y$, then $gx \leftrightsquigarrow_{V'}^\alpha hy$. In particular, we have $gx \leftrightsquigarrow_{gVg^{-1}}^\alpha gy$.*

PROOF. We prove by induction on α. If $\alpha = 0$, since $x \leftrightsquigarrow_V^0 y$ means $x \in \overline{Vy}$ and $y \in \overline{Vx}$, we have $gx \in \overline{gVy} = \overline{(gVh^{-1})(hy)} \subseteq \overline{V'(hy)}$ and $hy \in \overline{hVx} = \overline{(hVg^{-1})(gx)} \subseteq \overline{V'(gx)}$, so $gx \leftrightsquigarrow_{V'}^0 hy$.

Assume that the result holds for all $\beta < \alpha$. If $x \leftrightsquigarrow_V^\alpha y$, then for any open $W \ni 1_G$ and any open set $U \subseteq X$ such that U contains gx or hy. Without loss of generality, suppose $gx \in U$, i.e., $x \in g^{-1}U$. Then there exist $g^x, g^y \in V$ with $g^x x, g^y y \in g^{-1}U$, so that $g^x x \leftrightsquigarrow_{g^{-1}Wg}^\beta g^y y$ for all $\beta < \alpha$. By the inductive hypothesis, $(gg^x g^{-1})(gx) = gg^x x \leftrightsquigarrow_W^\beta gg^y y = (gg^y h^{-1})(hy)$ for all $\beta < \alpha$. It is trivial to check that $gg^x g^{-1}$ and $gg^y h^{-1}$ witnesses that $gx \leftrightsquigarrow_{V'}^\alpha hy$. ⊣

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LEMMA 5.10. *For all $\alpha < \omega_1$, the relation $\leftrightsquigarrow^{\alpha}_{G}$ is G-invariant. Moreover, $x \leftrightsquigarrow y$ iff $x \leftrightsquigarrow^{1}_{G} y$ for $x, y \in X$.*

PROOF. For any $x, y \in X$ and $g, h \in G$, by Lemma 5.9, $x \leftrightsquigarrow^{\alpha}_{G} y$ implies $gx \leftrightsquigarrow^{\alpha}_{G} hy$. So $\leftrightsquigarrow^{\alpha}_{G}$ is G-invariant.

By their definitions, it is trivial that $x \leftrightsquigarrow y$ implies $x \leftrightsquigarrow^{1}_{G} y$.

On the other hand, if $x \leftrightsquigarrow^{1}_{G} y$, we will show $x \leftrightsquigarrow y$. For any open $V \ni 1_G$ and any open set $U \subseteq X$ with $U \cap ([x] \cup [y]) \neq \emptyset$, without loss of generality, we can assume that $U \cap [x] \neq \emptyset$. There exists $g \in G$ with $gx \in U$, i.e., $x \in g^{-1}U$. So there are $g^x, g^y \in G$ such that $g^x x, g^y y \in g^{-1}U$, $g^x x \in \overline{(g^{-1}Vg)g^y y}$, and $g^y y \in \overline{(g^{-1}Vg)g^x x}$. Then gg^x and gg^y witness that $x \leftrightsquigarrow y$. ⊣

LEMMA 5.11. *Let $0 < \alpha < \omega_1$, V an open neighborhood of 1_G. If $x \leftrightsquigarrow^{\alpha}_{V} y$, then for any open set $W \ni 1_G$ and any open set U of X such that U contains x or y, there exist nonempty open sets $V^x, V^y \subseteq V$ such that, whenever $g^x \in V^x$ and $g^y \in V^y$, we have $g^x x, g^y y \in U$ and $g^x x \leftrightsquigarrow^{\beta}_{W} g^y y$ for all $\beta < \alpha$.*

PROOF. Given an open set $W \ni 1_G$ and an open set U of X with $x \in U$ or $y \in U$. We can find an open $W_0 \ni 1_G$ with $W_0^{-1} = W_0$ and $W_0^3 \subseteq W$. Then there exist $h^x, h^y \in V$ such that $h^x x, h^y y \in U$ and $h^x x \leftrightsquigarrow^{\beta}_{W_0} h^y y$ for all $\beta < \alpha$. We can find an open $V_0 \ni 1_G$ such that $(V_0 h^x x \cup V_0 h^y y) \subseteq U$. By Lemma 5.9, $w^x h^x x \leftrightsquigarrow^{\beta}_{W} w^y h^y y$ holds for all $w^x, w^y \in W_0$. Then we can put $V^x = W_0 h^x \cap V_0 h^x \cap V$ and $V^y = W_0 h^y \cap V_0 h^y \cap V$ as desired. ⊣

THEOREM 5.12. *Let G, H be two Polish groups, X a Polish G-space, and Y a Polish H-space, and let $f : X \to Y$ be a Baire measurable homomorphism, i.e., $xE^X_G y$ implies $f(x)E^Y_H f(y)$ for any $x, y \in X$. Then there exists a dense G_δ subset $C \subseteq X$ such that $x \leftrightsquigarrow^{\alpha}_{G} y$ implies $f(x) \leftrightsquigarrow^{\alpha}_{H} f(y)$ for any $x, y \in C$ and $0 < \alpha < \omega_1$.*

PROOF. Let C be the dense G_δ subset of X in [25, Lemma 2.5].

CLAIM. *Let $0 < \alpha < \omega_1$, $x_0, y_0 \in C$, and let $V \ni 1_G$ open in G, $W \ni 1_H$ open in H, and U open in X with $x_0, y_0 \in U$. Suppose*:

(1) $\forall x \in (U \cap C)\, \forall^* g \in V\, (f(gx) \in Wf(x))$;
(2) $x_0 \leftrightsquigarrow^{\alpha}_{V} y_0$.

Then we have $f(x_0) \leftrightsquigarrow^{\alpha}_{W} f(y_0)$.

PROOF OF CLAIM. By [25, Lemma 2.5], the following conditions hold:

(i) $f \restriction C$ is continuous;
(ii) $\forall x \in C\, \forall^* g \in G\, (gx \in C)$;
(iii) for any open $W' \ni 1_H$, there exist an open set $V' \ni 1_G$ and an open neighborhood U' of x_0 such that

$$\forall x \in (U' \cap C)\, \forall^* g \in V'\, (f(gx) \in W' f(x)).$$

Let $W' \ni 1_H$ be any open subset of H. Then we find two open sets $V' \ni 1_G$ and $U' \ni x_0$ satisfying (iii). Given any open set $O \subseteq Y$ with $f(x_0) \in O$ or $f(y_0) \in O$, without loss of generality, we can assume that $f(x_0) \in O$. Since f is continuous

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on C and $x_0 \in U' \cap C$, by shrinking, we can assume that $f(U' \cap C) \subseteq O$. Since $x_0 \leftrightsquigarrow^{\alpha}_{V} y_0$, by Lemma 5.11, we have

$$\exists^* g^x \in V\, \exists^* g^y \in V\, (g^x x_0, g^y y_0 \in U' \wedge \forall \beta < \alpha\, (g^x x_0 \leftrightsquigarrow^{\beta}_{V'} g^y y_0)).$$

Moveover, since $x_0, y_0 \in U \cap C$, by (1) and (ii), we can find $g^x, g^y \in V$ such that $g^x x_0, g^y y_0 \in (U' \cap C)$, $g^x x_0 \leftrightsquigarrow^{\beta}_{V'} g^y y_0$ $(\forall \beta < \alpha)$, $f(g^x x_0) \in Wf(x_0)$ and $f(g^y y_0) \in Wf(y_0)$. Thus there exist $h^x, h^y \in W$ so that $f(g^x x_0) = h^x f(x_0)$ and $f(g^y y_0) = h^y f(y_0)$.

Now we prove $f(x_0) \leftrightsquigarrow^{\alpha}_{W} f(y_0)$ by induction on α. Since $g^x x_0, g^y y_0 \in (U' \cap C)$, we have $h^x f(x_0), h^y f(y_0) \in O$. Thus we only need to show that $h^x f(x_0) \leftrightsquigarrow^{\beta}_{W'} h^y f(y_0)$ for $\beta < \alpha$.

For any open neighborhood N of $h^x f(x_0) = f(g^x x_0)$, we have $g^x x_0 \in f^{-1}(N)$. By the continuity of $f \restriction C$, there is an open set $V_0 \ni 1_G$ such that $(V_0 g^x x_0 \cap C) \subseteq f^{-1}(N)$. Since $g^y y_0 \in (U' \cap C)$, we can find $v \in V_0 \cap V'$ such that $v g^y y_0 \in C$ and $f(v g^y y_0) \in \overline{W' f(g^y y_0)} = \overline{W' h^y f(y_0)}$. So $N \cap (W' h^y f(y_0)) \neq \emptyset$. It follows that $h^x f(x_0) \in \overline{W' h^y f(y_0)}$. Similarly, we have $h^y f(y_0) \in \overline{W' h^x f(x_0)}$. Therefore, $h^x f(x_0) \leftrightsquigarrow^{0}_{W'} h^y f(y_0)$.

If $\alpha = 1$, we have already proved $f(x_0) \leftrightsquigarrow^{1}_{W} f(y_0)$.

If $\alpha > 1$, then for all $0 < \beta < \alpha$, by inductive hypothesis, $g^x x_0 \leftrightsquigarrow^{\beta}_{V'} g^y y_0$ implies $h^x f(x_0) = f(g^x x_0) \leftrightsquigarrow^{\beta}_{W'} f(g^y y_0) = h^y f(y_0)$. This completes the proof. ⊣

Put $V = G$, $W = H$, and $U = X$, then the proceeding Claim gives that, $x \leftrightsquigarrow^{\alpha}_{G} y$ implies $f(x) \leftrightsquigarrow^{\alpha}_{H} f(y)$ for $x, y \in C$ and $0 < \alpha < \omega_1$. ⊣

Now we return to $E(G)$ and the left multiplication action $c(G) \curvearrowright G^\omega$.

LEMMA 5.13. *Let G be a Polish group, $V \subseteq c(G)$ an open neighborhood of $1_{c(G)}$, $\alpha < \omega_1$, and $x, y, z \in G^\omega$. If $x \leftrightsquigarrow^{\alpha}_{V} y$, then $xz \leftrightsquigarrow^{\alpha}_{V} yz$.*

PROOF. We prove by induction on α.

If $\alpha = 0$, then we have $x \in \overline{Vy}$ and $y \in \overline{Vx}$. So $xz \in \overline{Vyz}$ and $yz \in \overline{Vxz}$, i.e., $xz \leftrightsquigarrow^{0}_{V} yz$.

If $\alpha > 0$, then for any open $W \ni 1_{c(G)}$ and any open $U \subseteq G^\omega$ with $xz \in U$ or $yz \in U$, we have $x \in Uz^{-1}$ or $y \in Uz^{-1}$, so there exist $g^x, g^y \in V$ such that $g^x x, g^y y \in Uz^{-1}$ and $g^x x \leftrightsquigarrow^{\beta}_{W} g^y y$ for $\beta < \alpha$. Then we have $g^x xz, g^y yz \in U$, and by induction hypothesis, $g^x xz \leftrightsquigarrow^{\beta}_{W} g^y yz$ for $\beta < \alpha$. Therefore, $xz \leftrightsquigarrow^{\alpha}_{V} yz$. ⊣

DEFINITION 5.14. Let G be a Polish group, $0 < \alpha < \omega_1$. We say that G is *α-l.m.-unbalanced* if there exist $x, y \in G^\omega$ such that $x \leftrightsquigarrow^{\alpha}_{c(G)} y$ and $[x] \neq [y]$, or equivalently, there exists $z \in G^\omega \setminus c(G)$ such that $z \leftrightsquigarrow^{\alpha}_{c(G)} 1_{G^\omega}$.

It is clear that G is 0-l.m.-unbalanced if G contains at least two elements. And Lemma 5.10 gives that G is l.m.-unbalanced iff it is 1-l.m.-unbalanced.

THEOREM 5.15. *Let G, H be Polish groups, $0 < \alpha < \omega_1$. If G is α-l.m.-unbalanced but H is not, then $E(G) \not\leq_B E(H)$.*

PROOF. Assume for contradiction that there is a Borel reduction θ of $E(G)$ to $E(H)$. From Theorem 5.12, there is a comeager set $C \subseteq G^\omega$ such that for any

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$x, y \in C$, if $x \leftrightsquigarrow^{\alpha}_{c(G)} y$, then $\theta(x) \leftrightsquigarrow^{\alpha}_{c(H)} \theta(y)$. Since G is α-l.m.-unbalanced, there exists $z \in G^\omega \setminus c(G)$ such that $z \leftrightsquigarrow^{\alpha}_{c(G)} 1_{G^\omega}$. Since $C \cap zC \neq \emptyset$, we can find an $x \in (C \cap zC)$ and put $x = zy$. Then $x, y \in C$ and $xy^{-1} = z \notin c(G)$, so $[x] \neq [y]$. By Lemma 5.13, $x \leftrightsquigarrow^{\alpha}_{c(G)} y$. It follows that $\theta(x) \leftrightsquigarrow^{\alpha}_{c(H)} \theta(y)$ and $[\theta(x)] \neq [\theta(y)]$, contradicting that H is not α-l.m.-unbalanced. ⊣

5.3. Examples of α-l.m.-unbalanced groups. Let G be a topological group, recall that G is said to be *distal* provided that $1_G \notin \overline{\{ghg^{-1} : g \in G\}}$ for all $h \neq 1_G \in G$ (cf. [27]). It is trivial that all TSI Polish groups are distal. Moveover, if H is a closed normal nilpotent subgroup of G and G/H is compact, then G is distal (cf. [27, Section 3. Proposition]). In particular, all nilpotent topological groups are distal.

THEOREM 5.16. *Let G be a Polish group. If G is not distal, then G is l.m.-unbalanced.*

PROOF. Since G is not distal, there exist $h \neq 1_G$ in G and a sequence (g_m) in G with $g_m h g_m^{-1} \to 1_G$. For each $m, n \in \omega$, let $\gamma_m(n) = g_m$ and $\eta_m(n) = g_m^{-1}$. Then $\gamma_m, \eta_m \in c(G)$. We define $z \in G^\omega$ as

$$z(n) = \begin{cases} 1_G, & n = 2k, \\ h, & n = 2k+1. \end{cases}$$

It is clear that $\gamma_m z \eta_m \rightrightarrows 1_{G^\omega}$. From Lemma 5.2, for any $x, y \in G^\omega$, if $xy^{-1} = z$, we have $x \leftrightsquigarrow y$. But it is trivial that $[x] \neq [y]$. ⊣

THEOREM 5.17. *Let G be a locally compact Polish group. Then G is not distal iff it is l.m.-unbalanced.*

PROOF. From Theorem 5.16, we only need to prove that if G is l.m.-unbalanced then it is not distal.

Let $x, y \in G^\omega$ with $x \leftrightsquigarrow y$ but $[x] \neq [y]$. By Lemma 5.2, there exist two sequences $(\gamma_m), (\eta_m)$ in $c(G)$ such that $\gamma_m x y^{-1} \eta_m \rightrightarrows 1_{G^\omega}$. Since $\gamma_m, \eta_m \in c(G)$ for each m, we can find $g_m, v_m \in G$ such that $g_m = \lim_n \gamma_m(n)$ and $v_m = \lim_n \eta_m(n)$.

Let $B \subseteq G$ be an open neighborhood of 1_G with $\overline{B}$ compact. There is a large enough $k \in \omega$ such that $\gamma_k(n)x(n)y(n)^{-1}\eta_k(n) \in B \subseteq \overline{B}$ for all $n \in \omega$. Thus there exist a $p \in \overline{B}$ and a strictly increasing natural numbers $n_0 < n_1 < \cdots$ such that $\lim_i \gamma_k(n_i)x(n_i)y(n_i)^{-1}\eta_k(n_i) = p$. By $[x] \neq [y]$, we have $xy^{-1} \notin c(G)$, thus $\gamma_k x y^{-1} \eta_k \notin c(G)$. So there exist a $q \in \overline{B}$ with $q \neq p$ and another strictly increasing natural numbers $l_0 < l_1 < \cdots$ such that $\lim_i \gamma_k(l_i)x(l_i)y(l_i)^{-1}\eta_k(l_i) = q$. Therefore, we have

$$\lim_i x(n_i)y(n_i)^{-1} = g_k^{-1} p v_k^{-1}, \quad \lim_i x(l_i)y(l_i)^{-1} = g_k^{-1} q v_k^{-1}.$$

By the property of uniformly convergence of $\gamma_m x y^{-1} \eta_m \rightrightarrows 1_{G^\omega}$, we have

$$\begin{aligned} \lim_m g_m g_k^{-1} p v_k^{-1} v_m &= \lim_m \lim_i \gamma_m(n_i)x(n_i)y(n_i)^{-1}\eta_m(n_i) \\ &= \lim_i \lim_m \gamma_m(n_i)x(n_i)y(n_i)^{-1}\eta_m(n_i) \\ &= \lim_i 1_G = 1_G. \end{aligned}$$

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Similarly, we have $\lim_m g_m g_k^{-1} q v_k^{-1} v_m = 1_G$. In the end, we set $h = g_k^{-1} p q^{-1} g_k \neq 1_G$. It follows that

$$\lim_m g_m h g_m^{-1} = \lim_m (g_m g_k^{-1} p v_k^{-1} v_m)(g_m g_k^{-1} q v_k^{-1} v_m)^{-1} = 1_G.$$

So G is not distal. ⊣

COROLLARY 5.18. *Let G, H be two Polish groups. If G is not distal, and H is TSI, or locally compact and distal, then $E(G) \not\leq_B E(H)$.*

PROOF. It follows from Theorems 5.4, 5.5, 5.16, and 5.17. ⊣

EXAMPLE 5.19. Let G be the group of all 2×2 real upper triangular matrices whose determinant $= 1$, equipped with the usual topology. Then $h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $g_m = \begin{pmatrix} 1/m & 0 \\ 0 & m \end{pmatrix}$ witness that G is not distal.

EXAMPLE 5.20. Let Λ be an infinite countable discrete group, G a Polish group containing at least two elements, then the wreath product $\Lambda \wr G$ is not distal. To see this, fix an $a \in G$ with $a \neq 1_G$. We define $\chi : \Lambda \to G$ as $\chi(\lambda) = \begin{cases} a, & \lambda = 1_\Lambda, \\ 1_G, & \lambda \neq 1_\Lambda. \end{cases}$ and let $\Lambda = \{\lambda_m : m \in \omega\}$. Then $h = (1_\Lambda, \chi)$ and $g_m = (\lambda_m, 1_{G^\Lambda})$ witness that $\Lambda \wr G$ is not distal.

THEOREM 5.21. *Let G be a locally compact Polish group. Then G is not 2-l.m.-unbalanced.*

PROOF. Let d be a compatible metric on G and d_u the supremum metric on G^ω. Since G is locally compact, we can find an $r > 0$ with $\overline{B(1_G, r)}$ is compact. Let $V = \{\gamma \in c(G) : d_u(1_{c(G)}, \gamma) < r\}$.

Assume for contradiction that there exist $x, y \in G^\omega$ with $x \leftrightsquigarrow^2_{c(G)} y$ but $[x] \neq [y]$, then there exist $g^x, g^y \in c(G)$ such that $g^x x \leftrightsquigarrow^1_V g^y y$. Following the arguments of $\Rightarrow$ part in the proof of Lemma 5.2, we can find two sequences (γ_m) and (η_m) in V such that $\gamma_m g^x x y^{-1} (g^y)^{-1} \eta_m \rightrightarrows 1_{G^\omega}$. Let $g_m = \lim_n \gamma_m(n)$. Following the arguments in the proof of Theorem 5.17, there exists $h \neq 1_G$ such that $g_m h g_m^{-1} \to 1_G$.

Since $\gamma_m \in V$, we have $g_m \in \overline{B(1_G, r)}$. There exists a subsequence of (g_m) converging to some $g \in G$, so $ghg^{-1} = 1_G$, i.e., $h = 1_G$. A contradiction! ⊣

For $c(G) \curvearrowright G^\omega$, Lemma 5.2 can simplify arguments concerning $\leftrightsquigarrow$. Following the same spirit, we define a notion $\leftrightarrow^\alpha_V$ to simplify arguments concerning $\leftrightsquigarrow^\alpha_V$.

DEFINITION 5.22. Let X be a Polish G-space, V an open neighborhood of 1_G, and let $\alpha < \omega_1$. We define $\leftrightarrow^\alpha_V$ by induction. We say:

(1) $x \leftrightarrow^0_V y$, if $y \in \overline{Vx}$ and $x \in \overline{Vy}$;
(2) $x \leftrightarrow^\alpha_V y$ for $\alpha > 0$, if for any open $W \ni 1_G$, there exist $g^x, g^y \in V$ such that $g^x x \leftrightarrow^\beta_W g^y y$ for all $\beta < \alpha$.

From their definitions, it is trivial to see that $x \leftrightsquigarrow^\alpha_V y$ implies $x \leftrightarrow^\alpha_V y$. Moreover, if $x \leftrightarrow^\alpha_V y$, then we have $x \leftrightarrow^{\alpha'}_V y$ for $0 < \alpha' \leq \alpha$, but $x \leftrightarrow^0_V y$ may fail.

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LEMMA 5.23. *Let G be a Polish group, $c(G) \curvearrowright G^\omega$ the left multiplication action. Then $x \leftrightarrow^\alpha_{c(G)} y$ implies $x \leftrightsquigarrow^\alpha_{c(G)} y$ for all $x, y \in G^\omega$ and $\alpha < \omega_1$.*

PROOF. Let $d \leq 1$ be a compatible metric on G and d_u the supremum metric on G^ω. For $\varepsilon > 0$, we define $V_\varepsilon = \{\gamma \in c(G) : d_u(1_{c(G)}, \gamma) < \varepsilon\}$.

Let $x, y, x', y' \in G^\omega$, $m \in \omega$, and $\alpha < \omega_1$. If

(1) $x \leftrightarrow^\alpha_{V_\varepsilon} y$,
(2) $x(n) = x'(n), y(n) = y'(n)$ for $n \geq m$, and
(3) $x'(y')^{-1}, y'(x')^{-1} \in \overline{V_{\varepsilon/2}}$,

then we claim that $x' \leftrightsquigarrow^\alpha_{V_\varepsilon} y'$.

To see this, we prove by induction on α. It is trivial that (3) implies $x' \leftrightsquigarrow^0_{V_\varepsilon} y'$. For any $\delta > 0$ and any open $U \subseteq G^\omega$ with $x' \in U$ or $y' \in U$, without loss of generality, we can assume that $x' \in U$, and $U = U_0 \times \cdots \times U_l \times G^\omega$ with $l \geq m$ and $U_0, \ldots, U_l$ open in G. By $x \leftrightarrow^\alpha_{V_\varepsilon} y$, there exist $\gamma^x, \gamma^y \in V_\varepsilon$ such that $\gamma^x x \leftrightarrow^\beta_{V_{\delta/2}} \gamma^y y$ for $\beta < \alpha$. We put

$$(\gamma^x)'(n) = \begin{cases} 1_G, & n \leq l, \\ \gamma^x(n), & n > l, \end{cases} \qquad (\gamma^y)'(n) = \begin{cases} x'(n)y'(n)^{-1}, & n \leq l, \\ \gamma^y(n), & n > l. \end{cases}$$

Then we have $(\gamma^x)', (\gamma^y)' \in V_\varepsilon$ and $(\gamma^x)'x', (\gamma^y)'y' \in U$. Note that $\gamma^x x \leftrightarrow^0_{V_{\delta/2}} \gamma^y y$ gives $\gamma^x x(\gamma^y y)^{-1}, \gamma^y y(\gamma^x x)^{-1} \in \overline{V_{\delta/2}}$. Since

$$(\gamma^x)'(n)x'(n)((\gamma^y)'(n)y'(n))^{-1} = \begin{cases} 1_G, & n \leq l, \\ \gamma^x(n)x(n)(\gamma^y(n)y(n))^{-1}, & n > l, \end{cases}$$

we have $(\gamma^x)'x'((\gamma^y)'y')^{-1} \in \overline{V_{\delta/2}}$. Similarly, $(\gamma^y)'y'((\gamma^x)'x')^{-1} \in \overline{V_{\delta/2}}$. Since $\gamma^x x \leftrightarrow^\beta_{V_{\delta/2}} \gamma^y y$ implies $\gamma^x x \leftrightarrow^\beta_{V_\delta} \gamma^y y$ for $\beta < \alpha$. By the induction hypothesis, $(\gamma^x)'x' \leftrightsquigarrow^\beta_{V_\delta} (\gamma^y)'y'$ for $\beta < \alpha$. Thus $(\gamma^x)'$ and $(\gamma^y)'$ witness that $x' \leftrightsquigarrow^\alpha_{V_\varepsilon} y'$.

In the end, by setting $V_{\varepsilon/2} = c(G)$, it follows that $x \leftrightarrow^\alpha_{c(G)} y$ implies $x \leftrightsquigarrow^\alpha_{c(G)} y$. ⊣

Given two sets X, Y and a map $f : X \to Y$, recall that the map $f^\omega : X^\omega \to Y^\omega$ is defined as: $f^\omega(x)(n) = f(x(n))$ for $x \in X^\omega$ and $n \in \omega$.

LEMMA 5.24. *Let G, H be two Polish groups, $\phi : G \to H$ a continuous homomorphism. Then for any $x, y \in G^\omega$, $\alpha < \omega_1$, open neighborhood $W \subseteq c(H)$ of $1_{c(H)}$, and any open neighborhood $V \subseteq c(G)$ of $1_{c(G)}$ with $\phi^\omega(V) \subseteq W$, we have $x \leftrightarrow^\alpha_V y$ implies $\phi^\omega(x) \leftrightarrow^\alpha_W \phi^\omega(y)$.*

PROOF. Let d be a compatible metric on G, and let d_u be the supremum metric on G^ω. By the continuity of ϕ, we can see that $\phi^\omega : G^\omega \to H^\omega$ is continuous, and $\phi^\omega \restriction c(G)$ is also continuous from $c(G)$ to $c(H)$.

We prove by induction on α. If $\alpha = 0$, note that $x \leftrightarrow^0_V y$ iff $xy^{-1}, yx^{-1} \in \overline{V}$. It follows from the continuity of ϕ^ω that $\phi^\omega(x)\phi^\omega(y)^{-1} = \phi^\omega(xy^{-1}) \in \overline{W}$ and $\phi^\omega(y)\phi^\omega(x)^{-1} = \phi^\omega(yx^{-1}) \in \overline{W}$, and hence $\phi^\omega(x) \in \overline{W\phi^\omega(y)}$ and $\phi^\omega(y) \in \overline{W\phi^\omega(x)}$, i.e., $\phi^\omega(x) \leftrightarrow^0_W \phi^\omega(y)$.

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For $\alpha > 0$, we let $W' \subseteq c(H)$ be an open neighborhood of $1_{c(H)}$. By the continuity of $\phi^\omega \restriction c(G)$, we can find an open neighborhood V' of $1_{c(G)}$ such that $\phi^\omega(V') \subseteq W'$. Note that $x \leftrightarrow_V^\alpha y$, so there exist $\gamma^x, \gamma^y \in V$ such that $\gamma^x x \leftrightarrow_{V'}^\beta \gamma^y y$ for $\beta < \alpha$. By induction hypothesis, we have $\phi^\omega(\gamma^x)\phi^\omega(x) = \phi^\omega(\gamma^x x) \leftrightarrow_{W'}^\beta \phi^\omega(\gamma^y y) = \phi^\omega(\gamma^y)\phi^\omega(y)$. Therefore, $\phi^\omega(\gamma^x)$ and $\phi^\omega(\gamma^y)$ witness that $\phi^\omega(x) \leftrightarrow_W^\alpha \phi^\omega(y)$. ⊣

THEOREM 5.25. *Let G be a Polish group, Λ an infinite countable discrete group, and $\alpha < \omega_1$. Then G is α-l.m.-unbalanced iff $\Lambda \wr G$ is $(\alpha+1)$-l.m.-unbalanced.*

PROOF. Let ρ be a compatible metric on $\Lambda \wr G$, ρ_u the supremum metric on $(\Lambda \wr G)^\omega$. For any $\varepsilon > 0$, we define

$$V_\varepsilon = \{\gamma \in c(\Lambda \wr G) : \rho_u(1_{c(\Lambda \wr G)}, \gamma) < \varepsilon\}.$$

($\Rightarrow$). Suppose G is α-l.m.-unbalanced. Then there exists an $a \in G^\omega \setminus c(G)$ such that $a \not\leftrightarrow_{c(G)}^\alpha 1_{G^\omega}$, and hence $a \leftrightarrow_{c(G)}^\alpha 1_{G^\omega}$. For any $n \in \omega$, let

$$\chi_n(\lambda) = \begin{cases} a(n), & \lambda = 1_\Lambda, \\ 1_G, & \lambda \neq 1_\Lambda. \end{cases}$$

We define $z \in (\Lambda \wr G)^\omega$ as $z(n) = (1_\Lambda, \chi_n)$ for $n \in \omega$. It is trivial that $z \notin c(\Lambda \wr G)$. To see that $\Lambda \wr G$ is $(\alpha+1)$-l.m.-unbalanced, by Lemma 5.23, we only need to show that $z \leftrightarrow_{c(\Lambda \wr G)}^{\alpha+1} 1_{(\Lambda \wr G)^\omega}$.

Note that the underlying topology of $\Lambda \wr G$ is the product topology on $\Lambda \times G^\Lambda$. Thus for any given $\varepsilon > 0$, there exists a $\lambda_0 \in \Lambda$ such that, for any $\chi \in G^\Lambda$, if $\chi(\lambda) = 1_G$ for all $\lambda \neq \lambda_0$, then $\rho(1_{\Lambda \wr G}, (1_\Lambda, \chi)) < \varepsilon/2$. Define

$$H = \{(1_\Lambda, \chi) \in \Lambda \wr G : \forall \lambda \in \Lambda\, (\lambda \neq \lambda_0 \Rightarrow \chi(\lambda) = 1_G)\}.$$

It is clear that H is a closed subgroup of $\lambda \wr G$. Because $\rho_u(1_{(\Lambda \wr G)^\omega}, x) \leq \varepsilon/2 < \varepsilon$ for all $x \in H^\omega$, we have

$$c(H) = (c(\Lambda \wr G) \cap H^\omega) \subseteq V_\varepsilon.$$

We define $f : G \to H$ as: for $g \in G$, $f(g) = (1_\Lambda, \chi^g)$ with

$$\chi^g(\lambda) = \begin{cases} g, & \lambda = \lambda_0, \\ 1_G, & \lambda \neq \lambda_0. \end{cases}$$

It is easy to see that f is a topological isomorphism from G to H, and hence a continuous homomorphism from G to $\Lambda \wr G$. Note that $f^\omega(c(G)) \subseteq c(H) \subseteq V_\varepsilon$. Then Lemma 5.24 gives $f^\omega(a) \leftrightarrow_{V_\varepsilon}^\alpha 1_{(\Lambda \wr G)^\omega}$.

In $\Lambda \wr G$, for $n \in \omega$, we have

$$(\lambda_0, 1_{G^\Lambda}) z(n) (\lambda_0, 1_{G^\Lambda})^{-1} = (1_\Lambda, \chi^{a(n)}) = f(a(n)).$$

Define $\gamma_0 \in c(\Lambda \wr G)$ as $\gamma_0(n) = (\lambda_0, 1_{G^\Lambda})$ for $n \in \omega$, then $\gamma_0 z \gamma_0^{-1} = f^\omega(a)$. It follows that $\gamma_0 z = f^\omega(a)\gamma_0 \leftrightarrow_{V_\varepsilon}^\alpha \gamma_0$. From Definition 5.22, we can see that $\gamma_0 z \leftrightarrow_{V_\varepsilon}^\beta \gamma_0$ for $0 < \beta \leq \alpha$. To prove for $\beta = 0$, because $f^\omega(a), f^\omega(a)^{-1} \in H^\omega$, we have $\rho_u(1_{(\Lambda \wr G)^\omega}, f^\omega(a)) < \varepsilon$ and $\rho_u(1_{(\Lambda \wr G)^\omega}, f^\omega(a)^{-1}) < \varepsilon$. So $f^\omega(a), f^\omega(a)^{-1} \in \overline{V_\varepsilon}$, then we have $f^\omega(a)\gamma_0 \in \overline{V_\varepsilon \gamma_0}$ and $\gamma_0 \in \overline{V_\varepsilon f^\omega(a)\gamma_0}$, and hence $\gamma_0 z = f^\omega(a)\gamma_0 \leftrightarrow_{V_\varepsilon}^0 \gamma_0$. It follows that $z \leftrightarrow_{c(\Lambda \wr G)}^{\alpha+1} 1_{(\Lambda \wr G)^\omega}$.

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($\Leftarrow$). On the other hand, suppose $\Lambda \wr G$ is $(\alpha+1)$-l.m.-unbalanced. Then there exists $z \in (\Lambda \wr G)^\omega \setminus c(\Lambda \wr G)$ such that $z \leftrightarrow^{\alpha+1}_{c(\Lambda\wr G)} 1_{(\Lambda\wr G)^\omega}$. Thus for any $\varepsilon > 0$, there exist $\gamma^z_\varepsilon, \gamma_\varepsilon \in c(\Lambda \wr G)$ such that $\gamma^z_\varepsilon z \leftrightarrow^{\alpha}_{V_\varepsilon} \gamma_\varepsilon$ and $\gamma^z_\varepsilon z \leftrightarrow^{0}_{V_\varepsilon} \gamma_\varepsilon$. Then we have $\gamma^z_\varepsilon z\gamma_\varepsilon^{-1} \leftrightarrow^{\alpha}_{V_\varepsilon} 1_{(\Lambda\wr G)^\omega}$ and $\gamma^z_\varepsilon z\gamma_\varepsilon^{-1} \in \overline{V_\varepsilon}$.

Fix a small enough $\varepsilon_0 > 0$ such that, for any $\gamma \in \overline{V_{\varepsilon_0}}$ and any $n \in \omega$, $\gamma(n) = (1_\Lambda, \chi)$ for some $\chi \in G^\Lambda$, i.e., $\overline{V_{\varepsilon_0}}$ is a subset of the subgroup $(\{1_\Lambda\} \times G^\Lambda)^\omega$ of $(\Lambda \wr G)^\omega$. So we can write $\gamma^z_{\varepsilon_0}(n)z(n)\gamma_{\varepsilon_0}(n)^{-1} = (1_\Lambda, \xi_n)$ with $\xi_n \in G^\Lambda$ for each $n \in \omega$. Note that $\gamma^z_{\varepsilon_0} z\gamma^{-1}_{\varepsilon_0} \notin c(\Lambda \wr G)$, we can find a $\lambda_0 \in \Lambda$ such that $\lim_n \xi_n(\lambda_0)$ diverges.

Now we define $\pi : (\{1_\Lambda\} \times G^\Lambda) \to G$ as $\pi(1_\Lambda, \chi) = \chi(\lambda_0)$ for $\chi \in G^\Lambda$. It is clear that π is a continuous homomorphism. Note that $\gamma^z_{\varepsilon_0} z\gamma^{-1}_{\varepsilon_0} \in \overline{V_{\varepsilon_0}} \subseteq (\{1_\Lambda\} \times G^\Lambda)^\omega$. Then $\pi((\gamma^z_{\varepsilon_0} z\gamma^{-1}_{\varepsilon_0})(n)) = \xi_n(\lambda_0)$ for all $n \in \omega$, so $\pi^\omega(\gamma^z_{\varepsilon_0} z\gamma^{-1}_{\varepsilon_0}) \notin c(G)$. By the continuity of $\pi : (\{1_\Lambda\} \times G^\Lambda) \to G$, we have $\pi^\omega(c(\{1_\Lambda\} \times G^\Lambda)) \subseteq c(G)$, and hence $\pi^\omega(V_{\varepsilon_0}) \subseteq c(G)$. Then Lemma 5.24 gives $\pi^\omega(\gamma^z_{\varepsilon_0} z\gamma^{-1}_{\varepsilon_0}) \leftrightarrow^{\alpha}_{c(G)} 1_{G^\omega}$. Therefore, G is α-l.m.-unbalanced. ⊣

COROLLARY 5.26. *Let G be a Polish group, Λ an infinite countable discrete group, and $\alpha < \omega_1$. If G is α-l.m.-unbalanced but not $(\alpha+1)$-l.m.-unbalanced, then $E(G) <_B E(\Lambda \wr G)$.*

PROOF. It is trivial that G is topologically isomorphic to a closed subgroup of $\Lambda \wr G$, so $E(G) \leq_B E(\Lambda \wr G)$. It follows from Theorems 5.15 and 5.25 that $E(\Lambda \wr G) \not\leq_B E(G)$. ⊣

Let G_0 be a TSI Polish group containing at least two elements, and let Λ be an infinite countable discrete group. We define $G_{n+1} = \Lambda \wr G_n$ for $n < \omega$. Then G_n is n-l.m.-unbalanced but not $(n+1)$-l.m.-unbalanced. Therefore,

$$E(G_0) <_B \cdots <_B E(G_n) <_B E(G_{n+1}) <_B \ldots.$$

From [14, Theorem 2.2.11], all these G_n are CLI. So far, we have not found any example of ω-l.m.-unbalanced CLI Polish group.

On the other hand, if G is a non-CLI Polish group, let $z \in G^\omega$ and $(\gamma_m) \in c(G)^\omega$ be which defined in the proof of Theorem 4.3. We can inductively prove that, for any $V \ni 1_{c(G)}$, there exists m large enough, such that $\gamma_m^{-1} z \leftrightarrow^{\alpha}_{V} 1_{G^\omega}$. So $z \leftrightarrow^{\alpha}_{c(G)} 1_{G^\omega}$, and hence G is α-l.m.-unbalanced for all ordinal α (even if $\alpha \geq \omega_1$).

§6. On TSI Polish groups.

DEFINITION 6.1. Let G be a Polish group. We define equivalence relation $E_*(G)$ on G^ω as: for $x, y \in G^\omega$,

$$xE_*(G)y \iff \lim_n x(0)x(1)\ldots x(n)y(n)^{-1}\ldots y(1)^{-1}y(0)^{-1} \text{ converges.}$$

It is clear that $E(G) \sim_B E_*(G)$ (cf. [9, Proposition 2.2]).

In this section, we focus on TSI Polish groups. It turns out that, for TSI Polish groups, $E_*(G)$ is a more convenient research object than $E(G)$.

LEMMA 6.2. *Let G be a TSI Polish group, d a complete compatible two-sided invariant metric on G.*

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(1) *For* $g_0, \dots, g_n, h_0, \dots, h_n \in G$, *we have*

$$d(g_0 \dots g_n, h_0 \dots h_n) = d(g_0 \dots g_n h_n^{-1} \dots h_0^{-1}, 1_G) \leq \sum_{k=0}^{n} d(g_k, h_k).$$

(2) *For* $x, y \in G^\omega$, *we have*

$$xE_*(G)y \iff \lim_n \sup_{n \leq m} d(x(n) \dots x(m), y(n) \dots y(m)) = 0.$$

(3) *For* $x, y \in G^\omega$, *if* $xE_*(G)y$, *then* $\lim_n d(x(n), y(n)) = 0$.

PROOF. (1) Since d is two-sided invariant, we have

$$\begin{aligned} d(g_0 g_1, h_0 h_1) &= d(g_0 g_1 h_1^{-1} h_0^{-1}, 1_G) = d(g_1 h_1^{-1}, g_0^{-1} h_0) \\ &\leq d(g_1 h_1^{-1}, 1_G) + d(1_G, g_0^{-1} h_0) = d(g_0, h_0) + d(g_1, h_1). \end{aligned}$$

Then we can easily complete the proof of (1) by induction on n.

(2) By Cauchy criterion, $\lim_n x(0)x(1) \dots x(n)y(n)^{-1} \dots y(1)^{-1}y(0)^{-1}$ converges iff for any $\varepsilon > 0$, there exists $N \in \omega$ such that, for $m > n > N$, we have

$$\begin{aligned} \varepsilon &> d(x(0) \dots x(n)y(n)^{-1} \dots y(0)^{-1}, x(0) \dots x(m)y(m)^{-1} \dots y(0)^{-1}) \\ &= d(1_G, x(n+1) \dots x(m)y(m)^{-1} \dots y(n+1)^{-1}) \\ &= d(x(n+1) \dots x(m), y(n+1) \dots y(m)). \end{aligned}$$

It follows that

$$xE_*y \iff \lim_n \sup_{n \leq m} d(x(n) \dots x(m), y(n) \dots y(m)) = 0.$$

(3) It is an easy corollary of (2). ⊣

For the sake of brevity, we write

$$d(x, y)|_{[n,m+1)} = d(x, y)|_{[n,m]} = d(x(n) \dots x(m), y(n) \dots y(m)).$$

6.1. Borel reducibility. For any given metric space (M, d), recall that $E(M; 0)$ is an equivalence relation on M^ω (cf. [7, Definition 3.2]) defined as

$$xE(M; 0)y \iff \lim_n d(x(n), y(n)) = 0$$

for $x, y \in M^\omega$. If G is a Polish group, then $E(G; 0)$ is independent of the choice of left-invariant compatible metric d on G, since $d(x(n), y(n)) \to 0$ iff $x(n)^{-1}y(n) \to 1_G$.

LEMMA 6.3. *Let* G *be a TSI Polish group, then we have*

$$E(G) \times E(G; 0) \leq_B E(G).$$

PROOF. We define $\theta : G^\omega \times G^\omega \to G^\omega$ as

$$\theta(x, x')(n) = \begin{cases} x(k), & n = 2k, \\ x(k)x'(k), & n = 2k+1. \end{cases}$$

Then θ witnesses that $E(G) \times E(G; 0) \leq_B E(G)$. ⊣

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LEMMA 6.4. *Let G, H, K be three TSI Polish groups. Suppose $\psi : G \to H$ and $\varphi : H \to K$ are continuous homomorphisms with $\varphi(\psi(G)) = K$ such that* $\ker(\varphi \circ \psi)$ *is non-Archimedean. Then $E(G) \leq_B E(H) \times E(G;0)$.*

PROOF. By [14, Corollary 2.3.4], $K \cong G/\ker(\varphi \circ \psi)$. Without loss of generality we may assume that $K = G/\ker(\varphi \circ \psi)$. Let $\phi = \varphi \circ \psi$. Then $\phi : G \to G/\ker(\varphi \circ \psi)$ is a continuous surjective homomorphism. Let d_G, d_H be two-sided invariant complete compatible metrics on G and H, respectively. Let

$$\begin{aligned} d_\phi(\phi(g), \phi(g')) &= \inf\{d_G(hg, h'g') : h, h' \in \ker(\varphi \circ \psi)\} \\ &= \inf\{d_G(hg, g') : h \in \ker(\varphi \circ \psi)\}. \end{aligned}$$

Then d_ϕ is a two-sided invariant complete compatible metric on $G/\ker(\varphi \circ \psi)$ (c.f. [14, Exercise 2.2.7]). It is clear that $d_\phi(\phi(g), \phi(g')) \leq d_G(g, g')$ for $g, g' \in G$.

We only need to prove that $E_*(G) \leq_B E_*(H) \times E(G;0)$. Define $\vartheta : G^\omega \to H^\omega \times G^\omega$ as, for $x \in G^\omega$ and $n \in \omega$,

$$\vartheta(x)(n) = (\psi(x(n)), x(n)).$$

For $x, y \in G^\omega$, if $xE_*(G)y$, then $\lim_k \sup_{k \leq m} d_G(x, y)|_{[k,m]} = 0$. So $\lim_n d_G(x(n), y(n)) = 0$. Since d_G, d_H are two-sided invariant and ψ is a continuous homomorphism, we have

$$\lim_k \sup_{k \leq m} d_H(\psi(x(\cdot)), \psi(y(\cdot)))|_{[k,m]} = 0.$$

Thus we have $(\vartheta(x), \vartheta(y)) \in E_*(H) \times E(G;0)$.

On the other hand, if $(\vartheta(x), \vartheta(y)) \in E_*(H) \times E(G;0)$, then we have $\lim_n d_G(x(n), y(n)) = 0$, and there exists $h_0 \in H$ such that

$$\begin{aligned} &\lim_n \psi(x(0) \dots x(n) y(n)^{-1} \dots y(0)^{-1}) \\ &= \lim_n \psi(x(0)) \dots \psi(x(n)) \psi(y(n))^{-1} \dots \psi(y(0))^{-1} \\ &= h_0. \end{aligned}$$

From $\varphi(h_0) \in K = \varphi(\psi(G)) = \phi(G)$, there exists $g_0 \in G$ with $\varphi(h_0) = \phi(g_0)$. By continuity of φ and $\phi = \varphi \circ \psi$, we have

$$\lim_n \phi(x(0) \dots x(n) y(n)^{-1} \dots y(0)^{-1}) = \varphi(h_0) = \phi(g_0).$$

Since $\ker(\varphi \circ \psi)$ is non-Archimedean, there exists a sequence of open subgroups (M_l) of $\ker(\varphi \circ \psi)$ which forms a neighborhood base of 1_G in $\ker(\varphi \circ \psi)$. For any $\varepsilon > 0$, there exist $l \in \omega$ and $0 < \varepsilon' < \varepsilon$ such that

$$\{h \in \ker(\varphi \circ \psi) : d_G(1_G, h) < 3\varepsilon'\} \subseteq M_l \subseteq \{h \in \ker(\varphi \circ \psi) : d_G(1_G, h) < \varepsilon\}.$$

Then there is an $N \in \omega$ such that

$$d_G(x(n), y(n)) < \varepsilon',$$

$$d_\phi(\phi(x(0) \dots x(n) y(n)^{-1} \dots y(0)^{-1}), \phi(g_0)) < \varepsilon'$$

for $n > N$. By the definition of d_ϕ, there exist $h_n \in \ker(\varphi \circ \psi)$ for each $n > N$ such that

$$d_G(x(0) \dots x(n) y(n)^{-1} \dots y(0)^{-1}, h_n g_0) < \varepsilon'.$$

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Note that

$$\begin{aligned} &d_G(x(0)\dots x(n)x(n+1)y(n+1)^{-1}y(n)^{-1}\dots y(0)^{-1},\\ &\quad x(0)\dots x(n)y(n)^{-1}\dots y(0)^{-1})\\ &= d_G(x(n+1),y(n+1)) < \varepsilon'. \end{aligned}$$

Then $d_G(1_G, h_n^{-1}h_{n+1}) = d_G(h_ng_0, h_{n+1}g_0) < 3\varepsilon'$. So $h_n^{-1}h_{n+1} \in M_l$. It follows that $h_n^{-1}h_m$ is in the open subgroup M_l for $m \geq n > N$. Thus $d_G(h_n, h_m) < \varepsilon$, and hence

$$\begin{aligned} &d_G(x(0)\dots x(n)y(n)^{-1}\dots y(0)^{-1}, x(0)\dots x(m)y(m)^{-1}\dots y(0)^{-1})\\ &< 2\varepsilon' + d_G(h_ng_0, h_mg_0) < 3\varepsilon. \end{aligned}$$

This gives $xE_*(G)y$.

Therefore, ϑ witnesses that $E_*(G) \leq_B E_*(H) \times E(G;0)$. ⊣

THEOREM 6.5. *Let G, H, K be three TSI Polish groups. Suppose $\psi : G \to H$ and $\varphi : H \to K$ are continuous homomorphisms with $\varphi(\psi(G)) = K$ such that $\ker(\varphi \circ \psi)$ is non-Archimedean. If the interval $[0,1]$ embeds into H, then $E(G) \leq_B E(H)$.*

PROOF. By [7, Theorem 3.4(ii)], we have $E(G;0) \leq_B E([0,1];0)$.

Let $f : [0,1] \to H$ be an embedding. By the uniformly continuity of f and $f^{-1} : f([0,1]) \to [0,1]$, it is trivial that $E([0,1];0) \sim_B E(f([0,1]);0) \leq_B E(H;0)$.

Then Lemmas 6.3 and 6.4 give $E(G) \leq_B E(H)$. ⊣

LEMMA 6.6. *Let G, H be two TSI Polish groups, G_c and H_c open normal subgroups of G and H, respectively, and let T_G meets each coset of G_c at exact one point. Let $\phi : G_c \to H_c$ be a topological group isomorphism, and $\theta : G \to H$ such that, for $u, v \in T_G$ and $g \in G_c$,*

(1) $\phi(ugu^{-1}) = \theta(u)\phi(g)\theta(u)^{-1}$;
(2) $\theta(ug) = \theta(u)\phi(g)$;
(3) *if $u \neq v$, then $\theta(u)H_c \neq \theta(v)H_c$.*

Then θ^ω is a continuous reduction of $E_*(G)$ to $E_*(H)$.

PROOF. From (2), it is trivial to see that $\theta : G \to H$ is continuous, so θ^ω is also continuous. we only need to check that θ^ω is a reduction.

Let $x, y \in G^\omega$. We write, for brevity, $x(n) = u_ng_n$ and $y(n) = v_nh_n$ such that $u_n, v_n \in T_G$ and $g_n, h_n \in G_c$ for each n. Let d_G and d_H be two-sided invariant complete compatible metrics on G and H, respectively.

If $xE_*(G)y$, then we have $\lim_k \sup_{k\leq m} d_G(x,y)|_{[k,m]} = 0$. In particular, we have $\lim_n d_G(x(n), y(n)) = 0$. Since G_c is open, there exists an $N \in \omega$ such that, for $n > N$, $x(n)$ and $y(n)$ are in the same coset of G_c, i.e., $u_n = v_n$. Now let $m \geq k > N$, we have

$$\begin{aligned} &x(k)x(k+1)\dots x(m)\\ &= u_kg_ku_{k+1}g_{k+1}\dots u_mg_m\\ &= (u_kg_ku_k^{-1})(u_ku_{k+1}g_{k+1}u_{k+1}^{-1}u_k^{-1})\dots(u_k\dots u_mg_mu_m^{-1}\dots u_k^{-1})u_k\dots u_m\\ &\overset{\text{Def}}{=} g'_kg'_{k+1}\dots g'_mu_k\dots u_m,\\ &y(k)y(k+1)\dots y(m)\\ &= u_kh_ku_{k+1}h_{k+1}\dots u_mh_m\\ &= (u_kh_ku_k^{-1})(u_ku_{k+1}h_{k+1}u_{k+1}^{-1}u_k^{-1})\dots(u_k\dots u_mh_mu_m^{-1}\dots u_k^{-1})u_k\dots u_m\\ &\overset{\text{Def}}{=} h'_kh'_{k+1}\dots h'_mu_k\dots u_m. \end{aligned}$$

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Note that

$$\begin{aligned}
\phi(g'_k) &= \phi(u_k g_k u_k^{-1}) = \theta(u_k)\phi(g_k)\theta(u_k)^{-1},\\
\phi(g'_{k+1}) &= \phi(u_k u_{k+1} g_{k+1} u_{k+1}^{-1} u_k^{-1})\\
&= \theta(u_k)\phi(u_{k+1} g_{k+1} u_{k+1}^{-1})\theta(u_k^{-1})\\
&= \theta(u_k)\theta(u_{k+1})\phi(g_{k+1})\theta(u_{k+1})^{-1}\theta(u_k)^{-1},\\
\phi(g'_m) &= \phi(u_k \dots u_m g_m u_m^{-1} \dots u_k^{-1})\\
&= \theta(u_k)\phi(u_{k+1} \dots u_m g_m u_m^{-1} \dots u_{k+1}^{-1})\theta(u_k^{-1})\\
&= \dots\\
&= \theta(u_k) \dots \theta(u_m)\phi(g_m)\theta(u_m)^{-1} \dots \theta(u_k)^{-1}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\theta(x(k))\theta(x(k+1)) \dots \theta(x(m))\\
&= \theta(u_k)\phi(g_k)\theta(u_{k+1})\phi(g_{k+1}) \dots \theta(u_m)\phi(g_m)\\
&= \phi(g'_k)\phi(g'_{k+1}) \dots \phi(g'_m)\theta(u_k) \dots \theta(u_m).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\theta(y(k))\theta(y(k+1)) \dots \theta(y(m))\\
&= \theta(u_k)\phi(h_k)\theta(u_{k+1})\phi(h_{k+1}) \dots \theta(u_m)\phi(h_m)\\
&= \phi(h'_k)\phi(h'_{k+1}) \dots \phi(h'_m)\theta(u_k) \dots \theta(u_m).
\end{aligned}$$

It follows that from $\lim_k \sup_{k\le m} d_G(x,y)|_{[k,m]} = 0$ that

$$d_G(g'_k \dots g'_m, h'_k \dots h'_m) = d_G(x(k) \dots x(m), y(k) \dots y(m)) \to 0.$$

Since ϕ is a topological group isomorphism, we have

$$d_H(\theta^\omega(x), \theta^\omega(y))|_{[k,m]} = d_H(\phi(g'_k) \dots \phi(g'_m), \phi(h'_k) \dots \phi(h'_m)) \to 0.$$

And hence $\theta^\omega(x) E_*(H) \theta^\omega(y)$.

On the other hand, if $\theta^\omega(x) E_*(H) \theta^\omega(y)$, we have $\lim_n d_H(\theta(x(n)), \theta(y(n))) = 0$. By $\theta(x(n)) = \theta(u_n)\phi(g_n)$ and $\theta(y(n)) = \theta(v_n)\phi(h_n)$, we have $\theta(u_n)H_c = \theta(v_n)H_c$, i.e., $u_n = v_n$, for n large enough. Then the similar arguments give $xE_*(G)y$. ⊣

For $u \in G$, define a topological group automorphism $\iota_u : G_c \to G_c$ as $\iota_u(g) = ugu^{-1}$. We define

$$\mathrm{Inn}_G(G_c) = \{\iota_u : u \in G\}.$$

THEOREM 6.7. *Let G, H be two TSI Polish groups containing at least two elements, G_c and H_c open normal subgroups of G and H, respectively. If there exists a topological group isomorphism $\phi : G_c \to H_c$ such that $\phi \mathrm{Inn}_G(G_c)\phi^{-1} \subseteq \mathrm{Inn}_H(H_c)$, then $E_*(G) \le_B E_*(H)$.*

PROOF. Corollary 3.7 gives $E_*(H \times \mathbb{Z}) \sim_B E_*(H)$, so we only need to show that $E_*(G) \le_B E_*(H \times \mathbb{Z})$. Define a topological group isomorphism ϕ_0 from G_c to the open normal subgroup $H_c \times \{0\}$ of $H \times \mathbb{Z}$ as $\phi_0(g) = (\phi(g), 0)$ for $g \in G_c$.

Fix a $T_G \subseteq G$ such that T_G meats each coset of G_c at exact one point. For each $u \in T_G$, since $\phi\iota_u\phi^{-1} \in \mathrm{Inn}_H(H_c)$, there exists some $w \in H$ such that $\phi\iota_u\phi^{-1} = \iota_w$.

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Thus $\phi_0 \imath_u \phi_0^{-1} = \imath_{(w,n)}$ for all $n \in \mathbb{Z}$. Therefore, we can find $\theta : T_G \to (H \times \mathbb{Z})$ such that, for $u, v \in T_G$,

(i) $\phi_0 \imath_u \phi_0^{-1} = \imath_{\theta(u)}$;
(ii) if $u \neq v$, then $\theta(u)(H_c \times \{0\}) \neq \theta(v)(H_c \times \{0\})$.

Then we can extend θ to a map $G \to H$ satisfying clauses (1)–(3) in the proceeding lemma. ⊣

6.2. Borel irreducibility.

DEFINITION 6.8 (Farah [13]).

(1) A map $\psi : \prod_n X_n \to \prod_n X'_n$ is *additive* if there exist $0 = l_0 < l_1 < \cdots < l_j < \cdots$ and maps $T_j : X_j \to \prod_{n \in [l_j, l_{j+1})} X'_n$ such that, for $x \in \prod_n X_n$,

$$\psi(x) = T_0(x(0)) ^\frown T_1(x(1)) ^\frown T_2(x(2)) ^\frown \dots .$$

(2) Let E and F be equivalence relations on $\prod_n X_n$ and $\prod_n X'_n$, respectively, we say that E is *additive reducible* to F, denoted by $E \leq_A F$, if there exists an additive reduction of E to F.

Let E be an equivalence relation on $\prod_n X_n$, and let $I \subseteq \omega$ be infinite. Fix an element $w \in \prod_{n \notin I} X_n$. For $x \in \prod_{n \in I} X_n$, define $x \oplus w \in \prod_n X_n$ as: $(x \oplus w)(n) = x(n)$ for $n \in I$ and $(x \oplus w)(n) = w(n)$ for $n \notin I$. We define $E|_I^w$ on $\prod_{n \in I} X_n$ as: for $x, y \in \prod_{n \in I} X_n$,

$$x E|_I^w y \iff (x \oplus w) E (y \oplus w).$$

Let (F_n) be a sequence of finite sets. A special equivalence relation $E_0(\prod_n F_n)$ defined as: for $x, y \in \prod_n F_n$,

$$x E_0(\prod_n F_n) y \iff \exists m \, \forall n > m \, (x(n) = y(n)).$$

The following lemma converts a Borel reduction to an additive reduction. This turns out to be a powerful tool to prove Borel irreducibility.

LEMMA 6.9. *Suppose G is a TSI Polish group. Let (F_n) be a sequence of finite sets, E a Borel equivalence relation on $\prod_n F_n$ with $E_0(\prod_n F_n) \subseteq E$. If $E \leq_B E_*(G)$, then there exist an infinite $I \subseteq \omega$ and a $w \in \prod_{n \notin I} F_n$ such that $E|_I^w \leq_A E_*(G)$.*

PROOF. The following proof is a modification of the proof of [12, Theorem 2.2], claims (i)–(iii). We omit some similar arguments.

Assume that θ is a Borel reduction of E to $E_*(G)$. Let d be a two-sided invariant complete compatible metric on G. Following claims (i), (ii), and the arguments after Claim (ii) in the proof of [12, Theorem 2.2], we construct two sequences of natural numbers $0 = n_0 < n_1 < n_2 < \cdots$ and $0 = l_0 < l_1 < l_2 < \cdots$, a sequence (s_j) with $s_j \in \prod_{n_j < n < n_{j+1}} F_n$ for each $j \in \omega$. Put $I = \{n_j : j \in \omega\}$, and put $w = \bigcup_j s_j \in \prod_{n \notin I} F_n$. Our construction confirms that, for any $x, y \in \prod_{n \in I} F_n$, we have:

(a) if $x(n) = y(n)$ for $n > n_j$, then for $l_{j+1} \leq k \leq m$,

$$d(\theta(x \oplus w), \theta(y \oplus w))|_{[k,m]} < 2^{-j};$$

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(b) if $x(n) = y(n)$ for $n \leq n_j$, then for $k \leq m < l_{j+1}$,

$$d(\theta(x \oplus w), \theta(y \oplus w))|_{[k,m]} < 2^{-j}.$$

For each $n \in I$, fix an $a_n^{\#} \in F_n$. Define $p_j : F_{n_j} \to \prod_{n\in I} F_n$ for each $j \in \omega$ as $p_j(a)(n) = \begin{cases} a, & n = n_j, \\ a_n^{\#}, & n \neq n_j \end{cases}$ for $a \in F_{n_j}$ and $n \in \omega$. Then we define $T_{n_j} : F_{n_j} \to G^{l_{j+1}-l_j}$ as: for $a \in F_{n_j}$ and $i < l_{j+1} - l_j$,

$$T_{n_j}(a)(i) = \theta(p_j(a) \oplus w)(l_j + i).$$

The additive mapping $\psi : \prod_{n\in I} F_n \to G^{\omega}$ is defined as: for $x \in \prod_{n\in I} F_n$,

$$\psi(x) = T_{n_0}(x(n_0))^\frown T_{n_1}(x(n_1))^\frown T_{n_2}(x(n_2))^\frown \dots .$$

Now we will show that ψ is a reduction of $E|_I^w$ to $E_*(G)$.

For any $x \in \prod_{n\in I} F_n$ and $j \in \omega$, define $e_j(x), e'_j(x) \in \prod_{n\in I} F_n$ as

$$e_j(x)(n) = \begin{cases} x(n), & n = n_j, \\ a_n^{\#}, & n \neq n_j, \end{cases} \qquad e'_j(x)(n) = \begin{cases} x(n), & n \leq n_j, \\ a_n^{\#}, & n > n_j. \end{cases}$$

Applying (a) for $l_j \leq k \leq m$, we have

$$d(\theta(e_j(x) \oplus w), \theta(e'_j(x) \oplus w))|_{[k,m]} < 2^{-(j-1)}$$

and applying (b) for $k \leq m < l_{j+1}$, we have

$$d(\theta(x \oplus w)), \theta(e'_j(x) \oplus w))|_{[k,m]} < 2^{-j}.$$

Claim. $\theta(x \oplus w) E_*(G) \psi(x)$.

Proof of Claim. Since d is complete and two-sided invariant, by Cauchy criterion, we only need to show that

$$\lim_k \sup_{k\leq m} d(\theta(x \oplus w), \psi(x))|_{[k,m]} \to 0.$$

For any $k \leq m$, there exists $j \leq j'$ with $l_j \leq k < l_{j+1}$ and $l_{j'} \leq m < l_{j'+1}$. If $j = j'$, note that $p_j(x(n_j)) = e_j(x)$, so we have

$$\psi(x) \restriction [l_j, l_{j+1}) = \theta(e_j(x) \oplus w) \restriction [l_j, l_{j+1}),$$

and hence

$$\begin{aligned} & d(\theta(x \oplus w), \psi(x))|_{[k,m]} \\ \leq\ & d(\theta(x \oplus w), \theta(e'_j(x) \oplus w))|_{[k,m]} + d(\theta(e'_j(x) \oplus w), \theta(e_j(x) \oplus w))|_{[k,m]} \\ <\ & 2^{-j} + 2^{-(j-1)} = 3 \cdot 2^{-j}; \end{aligned}$$

otherwise, $j < j'$, we have

$$\begin{aligned} & d(\theta(x \oplus w), \psi(x))|_{[k,m]} \\ \leq\ & d(\theta(x \oplus w), \psi(x))|_{[k,l_{j+1})} + \textstyle\sum_{i=j+1}^{j'-1} d(\theta(x \oplus w), \psi(x))|_{[l_i,l_{i+1})} \\ & + d(\theta(x \oplus w), \psi(x))|_{[l_{j'},m]} \\ <\ & 3 \cdot \textstyle\sum_{i=j}^{j'} 2^{-i} < 3 \cdot 2^{-(j-1)}. \end{aligned}$$

This gives $\theta(x \oplus w) E_*(G) \psi(x)$. ⊣

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In the end, for $x, y \in \prod_{n\in I} F_n$, $xE|_I^w y$ means $(x \oplus w)E(y \oplus w)$, so

$$xE|_I^w y \iff \theta(x \oplus w)E_*(G)\theta(y \oplus w) \iff \psi(x)E_*(G)\psi(y).$$

This competes the proof. ⊣

Let G, H be two TSI Polish groups, and let d_G and d_H be complete compatible two-sided invariant metrics on G and H, respectively. Assume that θ is a Borel reduction of $E_*(G)$ to $E_*(H)$. Let (F_n) be a sequence of finite subsets of G such that:

(i) $1_G \in F_n = F_n^{-1}$,
(ii) $F_{n-1}^{3n+2} \subseteq F_n$, and
(iii) $\bigcup_n F_n$ is dense in G.

Let E be the restriction of $E_*(G)$ on $\prod_n F_n$. There exist an infinite $I \subseteq \omega$ and a $w \in \prod_{n\notin I} F_n$ such that $E|_I^w \leq_A E_*(H)$. So there are natural numbers $0 = n_0 < n_1 < n_2 < \cdots$ with $I = \{n_j : j \in \omega\}$, $0 = l_0 < l_1 < l_2 < \cdots$, $T_{n_j} : F_{n_j} \to H^{l_{j+1}-l_j}$, and $\psi : \prod_{n\in I} F_n \to H^\omega$ with

$$\psi(x) = T_{n_0}(x(n_0))^\frown T_{n_1}(x(n_1))^\frown T_{n_2}(x(n_2))^\frown \dots,$$

such that ψ is an additive reduction of $E|_I^w$ to $E_*(H)$.

For $s = (h_0, \dots, h_{l-1})$ and $t = (h'_0, \dots, h'_{l-1})$ in H^l, let

$$d_H^\infty(s,t) = \max_{0\leq k\leq m<l} d_H(h_k \dots h_m, h'_k \dots h'_m).$$

Lemma 6.10. *For any $q \in \omega$, there exists a $\delta_q > 0$ such that*

$$\forall^\infty n \in I\, \forall g, g' \in F_n\, (d_G(g,g') < \delta_q \Rightarrow d_H^\infty(T_n(g), T_n(g')) < 2^{-q}).$$

Proof. If not, then there exist an $\varepsilon_0 > 0$, a strictly increasing sequence (j_p), and $g_p, g'_p \in F_{n_{j_p}}$ for each p, such that

$$d_G(g_p, g'_p) < 2^{-p}, \quad d_H^\infty(T_{n_{j_p}}(g_p), T_{n_{j_p}}(g'_p)) \geq \varepsilon_0.$$

For each $n \in I$, put

$$x(n) = \begin{cases} g_p, & n = n_{j_p}, \\ 1_G, & \text{otherwise}, \end{cases} \qquad y(n) = \begin{cases} g'_p, & n = n_{j_p}, \\ 1_G, & \text{otherwise}. \end{cases}$$

Then for any $k \leq m$, letting $n_{j_p} \leq k < n_{j_{p+1}}$, we have

$$\begin{aligned} &d_G(x \oplus w, y \oplus w)|_{[k,m]} \\ &\leq \textstyle\sum_{k\leq n\leq m} d_G((x \oplus w)(n), (y \oplus w)(n)) \\ &\leq \textstyle\sum_{r\geq p} d_G(g_r, g'_r) < \sum_{r\geq p} 2^{-r} = 2^{-(p-1)}. \end{aligned}$$

It follows that $(x \oplus w)E_*(G)(y \oplus w)$. But for any $p \in \omega$,

$$\max_{l_{j_p}\leq k\leq m<l_{j_{p+1}}} d_H(\psi(x), \psi(y))|_{[k,m]} = d_H^\infty(T_{n_{j_p}}(g_p), T_{n_{j_p}}(g'_p)) \geq \varepsilon_0.$$

So $(\psi(x), \psi(y)) \notin E_*(H)$, contradicting that ψ is a reduction. ⊣

In the rest of this section, we assume that H is locally compact.

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Then there exists a $q_c \in \omega$ such that the closure of $W_c = \{h \in H : d_H(1_H, h) < 2^{-q_c}\}$ is compact. Note that $W_c = W_c^{-1}$ and $hW_ch^{-1} = W_c$ for all $h \in H$. Define $H_c = \bigcup_m W_c^m$. Then H_c is an open normal subgroup of H. Let $V_c = \{g \in G : d_G(1_G, g) < \delta_{q_c}\}$ and $G_c = \bigcup_m V_c^m$. Note that $V_c = V_c^{-1}$ and $gV_cg^{-1} = V_c$ for all $g \in G$. So G_c is an open normal subgroup of G.

For $j > 0$, we define

$$u_{n_j} = w(n_{j-1} + 1) \dots w(n_j - 1),$$

and $u_{n_0} = u_0 = 1_G$. For any $g \in \bigcup_n F_n$, if $g \in F_n$ for some $n < n_j$, we have $u_{n_j}^{-1} g \in F_{n_{j-1}}^{n_j} \subseteq F_{n_j}$. We define

$$S_{n_j}(g) = T_{n_j}(u_{n_j}^{-1}g)(0) \dots T_{n_j}(u_{n_j}^{-1}g)(l_{j+1} - l_j - 1).$$

Note that $d_G(u_{n_j}^{-1}g, u_{n_j}^{-1}g') = d_G(g, g')$. From the preceding lemma, we have

$$\forall q \, \forall^\infty j \, \forall g, g' \in F_{n_j} \, (d_G(g, g') < \delta_q \Rightarrow d_H(S_{n_j}(g), S_{n_j}(g')) < 2^{-q}).$$

Fix a coset C of G_c, note that C is open, we can fix a $v \in C \cap \bigcup_n F_n$. For $g \in G_c \cap \bigcup_n F_n$, there exists an $m > 0$ such that $g \in V_c^m$. So we can find $1_G = g_0', g_1', \dots, g_m' = g \in G_c$ such that $(g_i')^{-1} g_{i+1}' \in V_c$. Put $g_0 = g_0'$ and $g_m = g_m'$. For $1 \leq i < m$, we can also find g_i in some small enough neighborhood of g_i' such that $g_i \in G_c \cap \bigcup_n F_n$ and $g_i^{-1} g_{i+1} \in V_c$. Then $d_G(g_i, g_{i+1}) = d_G(1_G, g_i g_{i+1}^{-1}) < \delta_{q_c}$ for $i < m$. Moveover, $d_G(v^{-j} g_i v^{j+1}, v^{-j} g_{i+1} v^{j+1}) = d_G(g_i, g_{i+1}) < \delta_{q_c}$ for all $j \in \omega$. For j large enough, we have $v, g_i \in F_{n_j - 1}$, and hence $v^{-j} g_i v^{j+1} \in F_{n_j-1}^{2j+2} \subseteq F_{n_j}$. So, for $i < m$,

$$d_H(S_{n_j}(v^{-j} g_i v^{j+1}), S_{n_j}(v^{-j} g_{i+1} v^{j+1})) < 2^{-q_c}.$$

Thus $S_{n_j}(v^{-j} g v^{j+1}) S_{n_j}(v)^{-1} \in W_c^m \subseteq \overline{W_c^m}$. We define

$$S_{n_j}^C(g) = S_{n_0}(v) \dots S_{n_{j-1}}(v) S_{n_j}(v^{-j} g v^{j+1}) S_{n_j}(v)^{-1} S_{n_{j-1}}(v)^{-1} \dots S_{n_0}(v)^{-1}.$$

Then $S_{n_j}^C(g) \in W_c^m \subseteq \overline{W_c^m}$ and $S_{n_j}^C(1_G) = 1_H$. Fix an ultrafilter $\mathfrak{A}$ on ω such that $\mathfrak{A}$ does not contain any finite set. Note that $\overline{W_c^m} = \overline{W}_c^m$ is compact. We now define

$$S^C(g) = \lim_{j \in \mathfrak{A}} S_{n_j}^C(g) \in \overline{W_c^m}.$$

For the definition and the existence of the ultrafilter limit $\lim_{j \in \mathfrak{A}}$, one can see [5, Section 1.5]. Since $\overline{W_c^m} \subseteq W_c^{m+1} \subseteq H_c$ for each $m \in \omega$, we have $S^C(g) \in H_c$. Note that

$$\forall q \, \forall g, g' \in G_c \cap \bigcup_n F_n \, (d_G(g, g') < \delta_q \Rightarrow d_H(S^C(g), S^C(g')) \leq 2^{-q}).$$

So S^C is uniformly continuous on $G_c \cap \bigcup_n F_n$. Therefore, we can extend S^C to a uniformly continuous map from G_c to H_c, which is still denoted as S^C for brevity.

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It is worth noting that $S^C(1_G) = 1_H$.
For $x \in G_c^\omega$, recall that $(S^C)^\omega(x) \in H_c^\omega$ is defined as

$$(S^C)^\omega(x)(p) = S^C(x(p)) \quad (\forall p \in \omega).$$

LEMMA 6.11. *For $x, y \in G_c^\omega$, if $\lim_p d_G(x(p), y(p)) = 0$, then*

$$xE_*(G)y \iff (S^C)^\omega(x)E_*(H)(S^C)^\omega(y).$$

PROOF. For each $p \in \omega$, we can find $g_p, g'_p \in G_c \cap \bigcup_n F_n$ such that

$$d_G(g_p, x(p)) < 2^{-p}, \quad d_G(g'_p, y(p)) < 2^{-p},$$

$$d_H(S^C(g_p), S^C(x(p))) < 2^{-p}, \quad d_H(S^C(g'_p), S^C(y(p))) < 2^{-p}.$$

We can find a large enough $j(p) \in \omega$ for $p \in \omega$ such that:

(0) $j(0) < j(1) < j(2) < \cdots$;
(1) $v \in F_n$ for some $n < n_{j(0)}$;
(2) $g_p, g'_p \in F_n$ for some $n < n_{j(p)}$;
(3) for brevity, we write $n(p) = n_{j(p)}$, then

$$d_H(S^C(g_p), S^C_{n(p)}(g_p)) < 2^{-p}, \quad d_H(S^C(g'_p), S^C_{n(p)}(g'_p)) < 2^{-p}.$$

For $n \in I$, we define

$$\hat{x}(n) = \begin{cases} u_{n(p)}^{-1} v^{-j(p)} g_p v^{j(p)+1}, & n = n(p), \\ u_n^{-1} v, & \text{otherwise}, \end{cases}$$

$$\hat{y}(n) = \begin{cases} u_{n(p)}^{-1} v^{-j(p)} g'_p v^{j(p)+1}, & n = n(p), \\ u_n^{-1} v, & \text{otherwise}. \end{cases}$$

Note that $\hat{x}(n), \hat{y}(n) \in F_{n-1}^{3n+2} \subseteq F_n$ for each $n \in I$.
Then for $k \le m$, if $n(q-1) < k \le m \le n(q)$, we have

$$\begin{aligned} & d_G(\hat{x} \oplus w, \hat{y} \oplus w)|_{[k,m]} \\ \le\ & d_G(u_{n(q)}^{-1} v^{-j(q)} g_q v^{j(q)+1}, u_{n(q)}^{-1} v^{-j(q)} g'_q v^{j(q)+1}) \\ =\ & d_G(g_q, g'_q) \to 0 \quad (q \to \infty). \end{aligned}$$

If $n(q-1) < k \le n(q)$ and $n(r) \le m < n(r+1)$ $(q \le r)$, we have

$$\begin{aligned} & d_G(\hat{x} \oplus w, \hat{y} \oplus w)|_{[k,m]} \\ =\ & d_G(u_{n(q)}^{-1} v^{-j(q)} g_q \dots g_r v^{j(r)+1}, u_{n(q)}^{-1} v^{-j(q)} g'_q \dots g'_r v^{j(r)+1}) \\ =\ & d_G(g_q \dots g_r, g'_q \dots g'_r). \end{aligned}$$

Thus we have

$$\begin{aligned} & \left| d_G(\hat{x} \oplus w, \hat{y} \oplus w)|_{[k,m]} - d_G(x, y)|_{[q,r]} \right| \\ =\ & \left| d_G(g_q \dots g_r, g'_q \dots g'_r) - d_G(x(q) \dots x(r), y(q) \dots y(r)) \right| \\ \le\ & d_G(g_q \dots g_r, x(q) \dots x(r)) + d_G(g'_q \dots g'_r, y(q) \dots y(r)) \\ \le\ & \textstyle\sum_{q \le p \le r} (d_G(g_p, x(p)) + d_G(g'_p, y(p))) \\ <\ & 2 \cdot \textstyle\sum_{q \le p} 2^{-p} = 2^{-(q-2)} \to 0 \quad (q \to \infty). \end{aligned}$$

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This implies that

$$xE_*(G)y \iff (\hat{x} \oplus w)E_*(G)(\hat{y} \oplus w).$$

Since ψ is an additive reduction, we have

$$(\hat{x} \oplus w)E_*(G)(\hat{y} \oplus w) \iff \hat{x}E|_I^w \hat{y} \iff \psi(\hat{x})E_*(H)\psi(\hat{y}).$$

On the other hand, for $k \leq m$, if $l_j \leq k < l_{j+1}$ and $l_s \leq m < l_{s+1}$, then $j \leq s$, so there exist q and r such that $n(q) \leq n_j < n(q+1)$ and $n(r) < n_s \leq n(r+1)$. Then

$$\begin{aligned} &d_H(\psi(\hat{x}), \psi(\hat{y}))|_{[k,m]} \\ &\leq d_H(\psi(\hat{x}), \psi(\hat{y}))|_{[k,l_{j+1})} + d_H(\psi(\hat{x}), \psi(\hat{y}))|_{[l_{j+1},l_s)} + d_H(\psi(\hat{x}), \psi(\hat{y}))|_{[l_s,m]}. \end{aligned}$$

Note that

$$d_H(\psi(\hat{x}), \psi(\hat{y}))|_{[k,l_{j+1})} \leq d_H^\infty(T_{n_j}(\hat{x}(n_j)), T_{n_j}(\hat{y}(n_j))).$$

If $n_j \neq n(q)$, then $\hat{x}(n_j) = \hat{y}(n_j)$; if $n_j = n(q)$, since

$$\begin{aligned} d_G(\hat{x}(n(q)), \hat{y}(n(q))) &= d_G(g_q, g'_q) \\ &\leq d_G(x(q), y(q)) + 2^{-q} + 2^{-q} \to 0 \quad (q \to \infty), \end{aligned}$$

we have

$$d_H^\infty(T_{n(q)}(\hat{x}(n(q))), T_{n(q)}(\hat{y}(n(q)))) \to 0 \quad (q \to \infty).$$

Therefore, $d_H(\psi(\hat{x}), \psi(\hat{y}))|_{[k,l_{j+1})} \to 0$ as $k \to \infty$.

Similarly, we also have $d_H(\psi(\hat{x}), \psi(\hat{y}))|_{[l_s,m]} \to 0$ as $m \to \infty$.

In the end, we have

$$\psi(\hat{x})(l_{j+1}) \dots \psi(\hat{x})(l_s - 1) = S_{n_{j+1}}(u_{n_{j+1}}\hat{x}(n_{j+1})) \dots S_{n_{s-1}}(u_{n_{s-1}}\hat{x}(n_{s-1})).$$

For $n_i \in I$ with $j+1 \leq i \leq s-1$, if $n_i = n(p)$ for some $q+1 \leq p \leq r$, then

$$S_{n_i}(u_{n_i}\hat{x}(n_i)) = S_{n_{i-1}}(v)^{-1} \dots S_{n_0}(v)^{-1} S^C_{n(p)}(g_p) S_{n_0}(v) \dots S_{n_i}(v);$$

and if $n_i \neq n(p)$ for all $q+1 \leq p \leq r$, then

$$S_{n_i}(u_{n_i}\hat{x}(n_i)) = S_{n_i}(v).$$

This gives that

$$\begin{aligned} &\psi(\hat{x})(l_{j+1}) \dots \psi(\hat{x})(l_s - 1) \\ &= S_{n_j}(v)^{-1} \dots S_{n_0}(v)^{-1} S^C_{n(q+1)}(g_{q+1}) \dots S^C_{n(r)}(g_r) S_{n_0}(v) \dots S_{n_{s-1}}(v). \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\psi(\hat{y})(l_{j+1}) \dots \psi(\hat{y})(l_s - 1) \\ &= S_{n_j}(v)^{-1} \dots S_{n_0}(v)^{-1} S^C_{n(q+1)}(g'_{q+1}) \dots S^C_{n(r)}(g'_r) S_{n_0}(v) \dots S_{n_{s-1}}(v). \end{aligned}$$

Thus we have

$$\begin{aligned} &d_H(\psi(\hat{x}), \psi(\hat{y}))|_{[l_{j+1},l_s)} \\ &= d_H(S^C_{n(q+1)}(g_{q+1}) \dots S^C_{n(r)}(g_r), S^C_{n(q+1)}(g'_{q+1}) \dots S^C_{n(r)}(g'_r)). \end{aligned}$$

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Note that

$$\begin{aligned}&d_H(S^C_{n(q+1)}(g_{q+1})\dots S^C_{n(r)}(g_r), S^C(x(q+1))\dots S^C(x(r)))\\ \le &\sum_{q+1\le p\le r} d_H(S^C_{n(p)}(g_p), S^C(x(p)))\\ \le &\sum_{q+1\le p\le r}(d_H(S^C_{n(p)}(g_p), S^C(g_p)) + d_H(S^C(g_p), S^C(x(p))))\\ < &\, 2\cdot\sum_{q+1\le p\le r} 2^{-p} < 2^{-(q-1)} \to 0 \quad (q\to\infty).\end{aligned}$$

Similarly, we also have

$$d_H(S^C_{n(q+1)}(g'_{q+1})\dots S^C_{n(r)}(g'_r), S^C(y(q+1))\dots S^C(y(r))) \to 0 \quad (q\to\infty).$$

To sum up, we have

$$\begin{aligned}&\lim_k d_H(\psi(\hat{x}), \psi(\hat{y}))|_{[k,m]} = 0\\ \iff &\lim_j d_H(\psi(\hat{x}), \psi(\hat{y}))|_{[l_{j+1}, l_s)} = 0\\ \iff &\lim_q d_H((S^C)^\omega(x), (S^C)^\omega(y))|_{[q+1,r]} = 0.\end{aligned}$$

It follows that

$$\psi(\hat{x})E_*(H)\psi(\hat{y}) \iff (S^C)^\omega(x)E_*(H)(S^C)^\omega(y),$$

and hence $xE_*(G)y \iff (S^C)^\omega(x)E_*(H)(S^C)^\omega(y)$. ⊣

Recall that a topological group G is said to *have no small subgroups* if there exists an open set $U \ni 1_G$ such that $\{1_G\}$ is the only subgroup of G contained in U (cf. [18, Definition 2.16]).

LEMMA 6.12. *Suppose G has no small subgroups. Then there is a $D > 0$ such that, for all $1_G \neq h \in G_c$, if $d_G(1_G, h) \le D$, then*

$$\inf_{g\in G_c} d_H(S^C(g), S^C(hg)) > 0.$$

In particular, for $g, g' \in G_c$, if $0 < d_G(g, g') \le D$, then $S^C(g) \neq S^C(g')$.

PROOF. If not, there exists a sequence (h_q) in G_c such that $h_q \neq 1_G$ and $\inf_{g\in G_c} d_H(S^C(g), S^C(h_q g)) = 0$ for each $q \in \omega$, and $\lim_q d_G(1_G, h_q) = 0$.

Since G has no small subgroups, there exists some $D_0 > 0$ such that, for each $q \in \omega$, we can find an $m_q > 0$ such that $d_G(1_G, h_q^{m_q}) \ge D_0$. And for $i < m_q$, we can find an element $g_{q,i} \in G_c$ such that

$$d_H(S^C(g_{q,i}), S^C(h_q g_{q,i})) < 2^{-(q+i+3)}.$$

Let $M_{-1} = 0$ and $M_q = m_0 + \cdots m_q$ for $q \in \omega$.

Now for $p \in \omega$, define

$$x(p) = \begin{cases} g_{q,i}, & p = 2(M_{q-1}+i), i < m_q,\\ g_{q,i}^{-1}, & p = 2(M_{q-1}+i)+1, i < m_q,\end{cases}$$
$$y(p) = \begin{cases} h_q g_{q,i}, & p = 2(M_{q-1}+i), i < m_q,\\ g_{q,i}^{-1}, & p = 2(M_{q-1}+i)+1, i < m_q.\end{cases}$$

By $\lim_q d_G(1_G, h_q) = 0$, we have $\lim_p d_G(x(p), y(p)) = 0$, then it follows from Lemma 6.11 that

$$xE_*(G)y \iff (S^C)^\omega(x)E_*(H)(S^C)^\omega(y).$$

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Note that $d_G(x,y)|_{[2M_{q-1},2M_q)} = d_G(1_G, h_q^{m_q}) \geq D_0$, so $xE_*(G)y$ fails.

On the other hand, for $k \leq k'$, if $2M_{r-1} \leq k < 2M_r$ and $2M_{r'-1} \leq k' < 2M_{r'}$, then

$$\begin{aligned}
&d_H((S^C)^\omega(x), (S^C)^\omega(y))|_{[k,k']} \\
&\leq \sum_{p=2M_{r-1}}^{2M_{r'}-1} d_H(S^C(x(p)), S^C(y(p))) \\
&= \sum_{q=r-1}^{r'-1} \sum_{i=0}^{m_q-1} d_H(S^C(g_{q,i}), S^C(h_q g_{q,i})) \\
&< \sum_{q=r-1}^{r'-1} \sum_{i=0}^{m_q-1} 2^{-(q+i+3)} < \sum_{q=r-1}^{r'-1} 2^{-(q+2)} < 2^{-r}.
\end{aligned}$$

This gives $(S^C)^\omega(x) E_*(H) (S^C)^\omega(y)$. A contradiction! ⊣

We conclude this section by the following theorem.

THEOREM 6.13 (Pre-rigid Theorem). *Let G, H be two TSI Polish groups such that H is locally compact. If $E(G) \leq_B E(H)$, then there exists an open normal subgroups G_c of G and a continuous map $S : G_c \to H$ with $S(1_G) = 1_H$ such that, for $x, y \in G_c^\omega$, if $\lim_n d_G(x(n), y(n)) = 0$, then*

$$xE_*(G_c)y \iff S^\omega(x) E_*(H) S^\omega(y).$$

In particular, if $G = G_c$ and the interval $[0,1]$ embeds in H, then the converse is also true.

PROOF. Suppose $E(G) \leq_B E(H)$, then Lemma 6.11 gives the desired results.

Conversely, if $G = G_c$, from such an $S : G \to H$ we can define a continuous map $\theta : G^\omega \to H^\omega \times G^\omega$ as: for $x \in G^\omega$ and $n \in \omega$,

$$\theta(x(n)) = (S(x(n)), x(n)).$$

It is clear that S is a Borel reduction of $E_*(G)$ to $E_*(H) \times E(G;0)$. Since the interval $[0,1]$ embeds in H, following the arguments in the proof of Theorem 6.5, we can see that $E(G) \leq_B E(H)$. ⊣

Under the assumption in Pre-rigid Theorem, the existence of Borel reduction is transformed into the existence of a well-behaved continuous mapping between topological groups.

6.3. Applications on Lie groups. A *Lie group* is a group which is also a smooth manifold such that the group operations are smooth functions. A Lie group is Polish iff it is separable iff it has only countably many connected components. Let G be a Lie group, we denote by G_0 the connected component of 1_G, then G_0 is an open normal subgroup of G. For more details on Lie groups, we refer to [24, 30].

THEOREM 6.14. *Let G and H be two separable TSI Lie groups. If $E(G) \leq_B E(H)$, then there exists a continuous locally injective map $S : G_0 \to H_0$.*

PROOF. If $E(G) \leq_B E(H)$, then $E_*(G) \leq_B E_*(H)$. The open normal subgroup G_c of G defined in the arguments before Lemma 6.11 is clopen, so G_c contains the connected component G_0. By continuity of S, we have $S(G_0) \subseteq H_0$. Let S^C be which appeared in Lemma 6.12. Note that any Lie group is locally compact and has no small subgroups (cf. [18, Proposition 2.17]). So the restriction of S^C on G_0 is the desired continuous locally injective map. ⊣

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Let $f : \tilde{X} \to X$ be a continuous surjection between two topological spaces. We say that f is a *covering map* provided that, for any $x \in X$, there exists an open $V \ni x$ such that $f^{-1}(V)$ is the disjoint union of open subsets of $\tilde{X}$ each of which is mapped homeomorphically onto V under f. Then $\tilde{X}$ is said to be a *covering space* of X.

THEOREM 6.15. *Let G and H be two separable TSI Lie groups with $E(G) \leq_B E(H)$. Then we have*:

(1) $\dim(G) \leq \dim(H)$;
(2) *if* $\dim(G) = \dim(H) \geq 2$ *and G_0 is compact, then H_0 is also compact and G_0 is a covering space of H_0.*

PROOF. Clause (1) follows from Theorem 6.14 trivially.

If $\dim(G) = \dim(H) \geq 2$ and G_0 is compact, by Theorem 6.14, there exists a continuous locally injective map $S : G_0 \to H_0$. It follows from [26, Lemma 1] that S is an open map, so $S(G_0)$ is an open subset of H_0. Since G_0 is compact, we have $S(G_0)$ is closed, and hence clopen in H_0. So $S(G_0) = H_0$, thus H_0 is compact. Moreover, since S is an open continuous locally injective map, it is a locally homeomorphism. Then by [6, Proposition 1], S is a covering map. ⊣

Clause (2) is also valid for $\dim(G) = \dim(H) = 1$. We will show it later as a special case of that G_0 and H_0 are abelian.

REMARK 6.16. Lemma 5.8 of [8] concerning finite-dimensional Banach spaces is a special case of the preceding theorem. Indeed, it was this lemma from [8] that served as the inspiration for our definition of the equivalence relations induced by Polish groups.

It is well known that all abelian Polish groups and compact Polish groups are TSI (cf. [14, Exercise 2.1.5]). On the other hand, a connected locally compact Polish group is TSI iff it is the product of a compact Polish group and a group $\mathbb{R}^n$ (cf. [14, Exercise 2.1.4] and [11, Theorem 16.4.6]). Let G be a connected TSI Lie group. Since any closed subgroup of a Lie group is also a Lie group (cf. [2, Theorem 1.4]), we can write $G = \mathbb{R}^n \times G_K$, where G_K is a compact connected Lie group.

Now we focus on the case that G_0 and H_0 are abelian. It is well known that every connected abelian Lie group is of the form $\mathbb{R}^c \times \mathbb{T}^e$ (cf. [2, Proposition 1.12]). We can write

$$G_0 = \mathbb{R}^{c_0} \times \mathbb{T}^{e_0}, \quad H_0 = \mathbb{R}^{c_1} \times \mathbb{T}^{e_1}.$$

Suppose $E(G) \leq_B E(H)$, we will show that $S^C : G_0 \to H_0$ is a group homomorphism for each coset C of G_0. Note that $\mathbb{T} = \{\exp(\mathrm{i}t) : t \in \mathbb{R}\}$.

Given $g_0, g_1 \in G_0$ and fix their expressions as

$$g_0 = (\tau_0^0, \dots, \tau_0^{c_0-1}, \exp(\mathrm{i}\tau_0^{c_0}), \dots, \exp(\mathrm{i}\tau_0^{c_0+e_0-1})),$$

$$g_1 = (\tau_1^0, \dots, \tau_1^{c_0-1}, \exp(\mathrm{i}\tau_1^{c_0}), \dots, \exp(\mathrm{i}\tau_1^{c_0+e_0-1})).$$

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For $t \in [0,1]$, define $g(t) \in G_0$ such that

$$\mathrm{proj}_i(g(t)) = \begin{cases} \tau_0^i + t(\tau_1^i - \tau_0^i), & 0 \le i < c_0, \\ \exp(\mathrm{i}(\tau_0^i + t(\tau_1^i - \tau_0^i))), & c_0 \le i < c_0 + e_0. \end{cases}$$

Then $g(0) = g_0$ and $g(1) = g_1$.

Fix an $i_0 < c_1 + e_1$. If $i_0 < c_1$, we define $F_{i_0} : [0,1] \to \mathbb{R}$ as $F_{i_0}(t) = \mathrm{proj}_{i_0}(S^C(g(t)))$. Note that $[0,1]$ is simply connected. If $c_1 \le i_0 < c_1 + e_1$, by [19, Definition A2.6], we can also find a continuous function $F_{i_0} : [0,1] \to \mathbb{R}$ such that $\exp(\mathrm{i}F_{i_0}(t)) = \mathrm{proj}_{i_0}(S^C(g(t)))$.

LEMMA 6.17. *$F_{i_0}(t) = F_{i_0}(0) + t(F_{i_0}(1) - F_{i_0}(0))$ for $t \in [0,1]$.*

PROOF. Define $f : [0,1] \to \mathbb{R}$ as $f(t) = F_{i_0}(t) - F_{i_0}(0) - t(F_{i_0}(1) - F_{i_0}(0))$. Note that f is continuous and $f(0) = f(1) = 0$. We only need to prove that $f(t) = 0$ for all $t \in (0,1)$.

If not, there exists $t_0 \in (0,1)$ such that $f(t_0) \neq 0$. Without loss of generality, assume that $f(t_0) > 0$. Let

$$\xi = \inf\{t \le t_0 : f(t) \ge f(t_0)\}, \quad \zeta = \sup\{t \ge t_0 : f(t) \ge f(t_0)\}.$$

By the continuity, we have $f(\xi) = f(\zeta) = f(t_0)$. Again by the continuity, we can find $0 < \xi_0 < \xi_1 < \xi_2 < \cdots < \xi$ such that $f(\xi_l) = \frac{l+1}{l+2} f(t_0)$ for each $l \in \omega$. Let $\xi' = \lim_l \xi_l$, then $\xi' \le \xi$ and $f(\xi') = \lim_l f(\xi_l) = f(t_0)$. By the definition of ξ, we have $\xi = \xi'$. There exists $L \in \omega$ such that $\xi - \xi_L < 1 - \zeta$. Thus we can find $1 > \zeta_0 > \zeta_1 > \zeta_2 > \cdots > \zeta$ such that, for $l \ge L$, we have

$$\xi - \xi_l > \zeta_l - \zeta > \xi - \xi_{l+1}.$$

This gives $\zeta = \lim_l \zeta_l$.

For $p \in \omega$, we set

$$x(p) = \begin{cases} g(\xi), & p = 2l, \\ g(\zeta), & p = 2l+1, \end{cases} \qquad y(p) = \begin{cases} g(\xi_l), & p = 2l, \\ g(\zeta_l), & p = 2l+1. \end{cases}$$

It is clear that $\lim_p d_G(x(p), y(p)) = 0$, so Lemma 6.11 gives

$$xE_*(G)y \iff (S^C)^\omega(x) E_*(H) (S^C)^\omega(y).$$

From the alternating series test, the following series

$$(\xi - \xi_0) + (\zeta - \zeta_0) + \cdots + (\xi - \xi_l) + (\zeta - \zeta_l) + \cdots$$

is convergent. We also note that

$$\mathrm{proj}_i(g(\xi)g(\xi_l)^{-1}) = \begin{cases} (\xi - \xi_l)(\tau_1^i - \tau_0^i), & 0 \le i < c_0, \\ \exp(\mathrm{i}(\xi - \xi_l)(\tau_1^i - \tau_0^i)), & c_0 \le i < c_0 + e_0, \end{cases}$$

$$\mathrm{proj}_i(g(\zeta)g(\zeta_l)^{-1}) = \begin{cases} (\zeta - \zeta_l)(\tau_1^i - \tau_0^i), & 0 \le i < c_0, \\ \exp(\mathrm{i}(\zeta - \zeta_l)(\tau_1^i - \tau_0^i)), & c_0 \le i < c_0 + e_0. \end{cases}$$

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It follows that, for all $i < c_0 + e_0$, the following are convergent:

$$\sum_p \mathrm{proj}_i(x(p)y(p)^{-1}) = \sum_l \mathrm{proj}_i(g(\xi)g(\zeta)g(\xi_l)^{-1}g(\zeta_l)^{-1}) \quad (0 \le i < c_0),$$
$$\prod_p \mathrm{proj}_i(x(p)y(p)^{-1}) = \prod_l \mathrm{proj}_i(g(\xi)g(\zeta)g(\xi_l)^{-1}g(\zeta_l)^{-1}) \quad (c_0 \le i < c_0 + e_0).$$

So we have $xE_*(G)y$.

On the other hand, we have

$$\sum_l (f(\xi) - f(\xi_l) + f(\zeta) - f(\zeta_l)) \ge \sum_l (f(\xi) - f(\xi_l)) = \sum_l \frac{f(t_0)}{l+2} = \infty.$$

Since

$$\begin{aligned} &F_{i_0}(\xi) - F_{i_0}(\xi_l) + F_{i_0}(\zeta) - F_{i_0}(\zeta_l) \\ &= (f(\xi) - f(\xi_l) + f(\zeta) - f(\zeta_l)) + (\xi - \xi_l + \zeta - \zeta_l)(F_{i_0}(1) - F_{i_0}(0)), \end{aligned}$$

and $\sum_l (\xi - \xi_l + \zeta - \zeta_l)$ converges, we have

$$\sum_l (F_{i_0}(\xi) - F_{i_0}(\xi_l) + F_{i_0}(\zeta) - F_{i_0}(\zeta_l)) = \infty.$$

If $i_0 < c_1$, we have

$$\sum_l \mathrm{proj}_{i_0}(S^C(g(\xi))S^C(g(\xi_l))^{-1}S^C(g(\zeta))S^C(g(\zeta_l))^{-1}) = \infty;$$

if $c_1 \le i_0 < c_1 + e_1$, since $(F_{i_0}(\xi) - F_{i_0}(\xi_l) + F_{i_0}(\zeta) - F_{i_0}(\zeta_l)) \to 0$,

$$\begin{aligned} &\prod_l \mathrm{proj}_{i_0}(S^C(g(\xi))S^C(g(\xi_l))^{-1}S^C(g(\zeta))S^C(g(\zeta_l))^{-1}) \\ =&\prod_l \exp(\mathrm{i}(F_{i_0}(\xi) - F_{i_0}(\xi_l) + F_{i_0}(\zeta) - F_{i_0}(\zeta_l))) \text{ diverges.} \end{aligned}$$

Therefore, $\prod_p S^C(x(p))S^C(y(p))^{-1}$ diverges. In other words, we have $(S^C)^\omega(x)E_*(H)(S^C)^\omega(y)$ fails. A contradiction! ⊣

LEMMA 6.18. *$S^C : G_0 \to H_0$ is a group homomorphism.*

PROOF. For any $i_0 < c_1 + e_1$, Lemma 6.17 gives $F_{i_0}(1/2) = F_{i_0}(0) + (F_{i_0}(1) - F_{i_0}(0))/2$, i.e., $F_{i_0}(0) + F_{i_0}(1) = 2F_{i_0}(1/2)$. It follows that

$$S^C(g(0))S^C(g(1)) = S^C(g(1/2))^2.$$

For $t \in [0,1]$, define $g^*(t) \in G_0$ as

$$\mathrm{proj}_i(g^*(t)) = \begin{cases} t(\tau_0^i + \tau_1^i), & 0 \le i < c_0, \\ \exp(\mathrm{i}(t(\tau_0^i + \tau_1^i))), & c_0 \le i < c_0 + e_0. \end{cases}$$

Similarly, we have

$$S^C(g^*(0))S^C(g^*(1)) = S^C(g^*(1/2))^2.$$

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From the definition of S^C, we have $S^C(1_G) = 1_H$. Note that $g(0) = g_0$ and $g(1) = g_1$, also note that $g^*(0) = 1_G, g^*(1) = g_0g_1$, and $g^*(1/2) = g(1/2)$. So we have

$$S^C(g_0g_1) = S^C(g^*(1)) = S^C(g^*(1/2))^2 = S^C(g(1/2))^2 = S^C(g_0)S^C(g_1).$$

Therefore, S^C is a group homomorphism. ⊣

THEOREM 6.19. *$E(\mathbb{R}^{c_0} \times \mathbb{T}^{e_0}) \leq_B E(\mathbb{R}^{c_1} \times \mathbb{T}^{e_1})$ iff $e_0 \leq e_1$ and $c_0 + e_0 \leq c_1 + e_1$.*

PROOF. The ($\Leftarrow$) part follows form Theorem 6.5 and Proposition 3.4.

Now suppose $E(\mathbb{R}^{c_0} \times \mathbb{T}^{e_0}) \leq_B E(\mathbb{R}^{c_1} \times \mathbb{T}^{e_1})$. Theorem 6.15 gives $c_0 + e_0 \leq c_1 + e_1$. Since $E(\mathbb{T}^{e_0}) \leq_B E(\mathbb{R}^{c_0} \times \mathbb{T}^{e_0}) \leq_B E(\mathbb{R}^{c_1} \times \mathbb{T}^{e_1})$, by Theorem 6.14 and Lemma 6.18, there exists a continuous locally injective homomorphism $S : \mathbb{T}^{e_0} \to \mathbb{R}^{c_1} \times \mathbb{T}^{e_1}$. Then there are continuous homomorphisms $R : \mathbb{T}^{e_0} \to \mathbb{R}^{c_1}$ and $T : \mathbb{T}^{e_0} \to \mathbb{T}^{e_1}$ such that $S(g) = (R(g), T(g))$ for $g \in \mathbb{T}^{e_0}$. It is trivial to see that $R(g) = 0$ for all $g \in \mathbb{T}^{e_0}$. So T should be locally injective, and hence $e_0 \leq e_1$. ⊣

Generalizing the results above, the authors [10] proved a Rigid Theorem concerning TSI Lie groups as following.

THEOREM [10, Theorem 1.8]. *Let G, H be two separable TSI Lie groups such that G is connected. Then $E(G) \leq_B E(H)$ iff there exists a continuous locally injective homomorphism $S : G \to H$.*

Next we provide an example of a disconnected Lie group. Let $\mathbb{Z}_2 = \{0, 1\}$ and denote by $\mathbb{Z}_2 \ltimes \mathbb{R}$ the semi product $\mathbb{Z}_2 \ltimes_\phi \mathbb{R}$ with $\phi(0) = \mathrm{id}_{\mathbb{R}}$ and $\phi(1)(t) = -t$ for $t \in \mathbb{R}$. Then we have $(1, 0)(0, t)(1, 0) = (0, -t)$ for $t \in \mathbb{R}$. In the following, we write $v = (1, 0)$ for brevity. Note that $v^{-1} = v$ and $v(0, t)v^{-1} = (0, -t)$.

LEMMA 6.20. *$E(\mathbb{Z}_2 \ltimes \mathbb{R}) \not\leq_B E(\mathbb{T})$.*

PROOF. We use notions defined in the arguments before Lemma 6.11. We can assume $v \in F_0$. In order to write succinctly, we identify $\{0\} \times \mathbb{R}$ with $\mathbb{R}$.

Assume for contradiction that $E(\mathbb{Z}_2 \ltimes \mathbb{R}) \leq_B E(\mathbb{T})$. Let $G = \mathbb{Z}_2 \ltimes \mathbb{R}$ and $H = \mathbb{T}$. Since $\mathbb{T}$ is compact, we can put $H_c = W_c = \mathbb{T}$. Then we can put $G_c = V_c$ to be either $\mathbb{R}$ or G.

First, for $G_c = \mathbb{R}$, there are two cosets $\mathbb{R}$ and $\mathbb{R}v$. By Lemma 6.18, $S^{\mathbb{R}}, S^{\mathbb{R}v} : \mathbb{R} \to \mathbb{T}$ are continuous locally injective homomorphisms. Thus there exist $a, b \in \mathbb{R} \setminus \{0\}$ such that, for $t \in \mathbb{R}$,

$$S^{\mathbb{R}}((0, t)) = \exp(\mathrm{i}at), \quad S^{\mathbb{R}v}((0, t)) = \exp(\mathrm{i}bt).$$

Secondly, for $G_c = G$, there is only one coset G. Since $H = \mathbb{T}$ is abelian and $v^2 = 1_G$, we have

$$S^{\mathbb{R}v}_{n_{2j}}(g) = S_{n_{2j}}(gv)S_{n_{2j}}(v)^{-1}, \quad S^{G}_{n_{2j}}(h) = S_{n_{2j}}(h)S_{n_{2j}}(1_G)^{-1}$$

for $g = (0, t) \in \mathbb{R} \cap F_{n_{2j}-2}$, $v \in F_{n_{2j}-2}$ and any $h \in F_{n_{2j}-1}$, so

$$S^{\mathbb{R}v}_{n_{2j}}((0, t)) = S_{n_{2j}}((0, t)v)S_{n_{2j}}(v)^{-1} = S^{G}_{n_{2j}}((0, t)v)S^{G}_{n_{2j}}(v)^{-1}.$$

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Assume that the ultrafilter $\mathfrak{A} \ni \{2j : j \in \omega\}$, then we have

$$S^{\mathbb{R}v}((0,t)) = S^G((0,t)v)S^G(v)^{-1}.$$

So $S^G((0,t)v) = \exp(\mathrm{i}bt)S^G(v)$ for $t \in \mathbb{R}$.

Now define $x, y \in G^\omega$ as $x(p) = v$ and $y(p) = (0, \frac{1}{p+1})v$ for $p \in \omega$. Then we have $\lim_p d_G(x(p), y(p)) = 0$, and hence by Lemma 6.11,

$$xE_*(G)y \iff (S^G)^\omega(x)E_*(H)(S^G)^\omega(y).$$

For $p \in \omega$, we have $x(0)\dots x(p) = v^{p+1}$ and

$$\begin{aligned} y(0)\dots y(p) &= (0,1)v\dots(0,\tfrac{1}{p+1})v \\ &= (0,1)(v(0,\tfrac{1}{2})v^{-1})\dots(v^p(0,\tfrac{1}{p+1})v^{-p})v^{p+1} \\ &= (0, 1 + \tfrac{-1}{2} \cdots + \tfrac{(-1)^p}{p+1})v^{p+1}. \end{aligned}$$

Since $\sum_p \frac{(-1)^p}{p+1}$ converges, we have $xE_*(G)y$.

On the other hand, we have $S^G(x(0))\dots S^G(x(p)) = S^G(v)^{p+1}$ and

$$\begin{aligned} S^G(y(0))\dots S^G(y(p)) &= \exp(\mathrm{i}b)S^G(v)\dots\exp(\tfrac{\mathrm{i}b}{p+1})S^G(v) \\ &= \exp(\mathrm{i}b)\exp(\tfrac{\mathrm{i}b}{2})\dots\exp(\tfrac{\mathrm{i}b}{p+1})S^G(v)^{p+1}. \end{aligned}$$

Since $\sum_p \frac{1}{p+1} = \infty$ and $\frac{1}{p+1} \to 0$, we have $\prod_p \exp(\frac{\mathrm{i}b}{p+1})$ diverges, and hence $(S^G)^\omega(x)E_*(H)(S^G)^\omega(y)$ fails. A contradiction! ⊣

We also denote by $\mathbb{Z}_2 \ltimes \mathbb{T}$ the semi product $\mathbb{Z}_2 \ltimes_\phi \mathbb{T}$ with $\phi(0) = \mathrm{id}_\mathbb{T}$ and $\phi(1)(\exp(\mathrm{i}t)) = \exp(-\mathrm{i}t)$ for $t \in [0, 2\pi)$.

COROLLARY 6.21.

(1) $E(\mathbb{T})$ *and* $E(\mathbb{Z}_2 \ltimes \mathbb{R})$ *are Borel incomparable*;
(2) $E(\mathbb{R}) <_B E(\mathbb{T}) <_B E(\mathbb{Z}_2 \ltimes \mathbb{T})$;
(3) $E(\mathbb{R}) <_B E(\mathbb{Z}_2 \ltimes \mathbb{R}) <_B E(\mathbb{Z}_2 \ltimes \mathbb{T})$.

PROOF. Clause (1) follows from Lemmas 6.18 and 6.20. By Proposition 3.4 and Theorem 6.5, we have $E(\mathbb{R}) \leq_B E(\mathbb{T}) \leq_B E(\mathbb{Z}_2 \ltimes \mathbb{T})$ and $E(\mathbb{R}) \leq_B E(\mathbb{Z}_2 \ltimes \mathbb{R}) \leq_B E(\mathbb{Z}_2 \ltimes \mathbb{T})$. Then (2) and (3) follow from (1). ⊣

COROLLARY 6.22. *Let G be a separable TSI Lie group with $G_0 = \mathbb{R}$. Define $\iota_\mathbb{R} : \mathbb{R} \to \mathbb{R}$ as $\iota_\mathbb{R}(t) = -t$. Then we have*:

(1) $E(G) \sim_B E(\mathbb{R}) \iff \mathrm{Inn}_G(G_0) = \{\mathrm{id}_\mathbb{R}\}$;
(2) $E(G) \sim_B E(\mathbb{Z}_2 \ltimes \mathbb{R}) \iff \mathrm{Inn}_G(G_0) = \{\mathrm{id}_\mathbb{R}, \iota_\mathbb{R}\}$.

PROOF. Suppose that there are $u \in G$ and $b \in \mathbb{R}$ with $b \neq \pm 1$ such that $utu^{-1} = bt$ for all $t \in \mathbb{R} = G_0$. Without loss of generality, we can assume that $|b| < 1$. Then $u^n t u^{-n} = b^n t \to 0$, contradicting that G is TSI. So $\mathrm{Inn}_G(G_0) \subseteq \{\mathrm{id}_\mathbb{R}, \iota_\mathbb{R}\}$.

By Theorem 6.7 and $E(\mathbb{Z}_2 \ltimes \mathbb{R}) \not\leq_B E(\mathbb{R})$, we complete the proof. ⊣

By similar arguments, we can prove the following corollary.

COROLLARY 6.23. *Let G be a separable TSI Lie group with $G_0 = \mathbb{T}$. Define $\iota_\mathbb{T} : \mathbb{T} \to \mathbb{T}$ as $\iota_\mathbb{T}(\exp(\mathrm{i}t)) = \exp(-\mathrm{i}t)$. Then we have*:

(1) $E(G) \sim_B E(\mathbb{T}) \iff \mathrm{Inn}_G(G_0) = \{\mathrm{id}_\mathbb{T}\}$;
(2) $E(G) \sim_B E(\mathbb{Z}_2 \ltimes \mathbb{T}) \iff \mathrm{Inn}_G(G_0) = \{\mathrm{id}_\mathbb{T}, \iota_\mathbb{T}\}$.

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6.4. Applications on *p*-adic solenoids. Let $p \geq 2$ be a natural number. Recall that the *p-adic solenoid* $\mathbb{T}_p$ is the closed subgroup of $\mathbb{T}^\omega$ as follows:

$$\mathbb{T}_p = \{(g_l) \in \mathbb{T}^\omega : \forall l\ (g_l = g_{l+1}^p)\}.$$

Then $\mathbb{T}_p$ is a compact connected abelian Polish group, but not arcwise connected. The arc component of $1_{\mathbb{T}_p}$ in $\mathbb{T}_p$ is

$$R_p = \{(\exp(\mathrm{i}t/p^l)) : t \in \mathbb{R}\}$$

(cf. [19, Exercise E1.11(iv)]). For more on p-adic solenoids, we refer to [19].

Given $p \geq 2$, let $N(p)$ be the set of all prime divisors of p.

THEOREM 6.24. *Let $p, q \geq 2$ be natural numbers, then we have*:

(1) $E(\mathbb{T}_p) \leq_B E(\mathbb{T}_q) \iff N(p) \supseteq N(q)$;
(2) $E(\mathbb{R}) <_B E(\mathbb{T}_p) <_B E(\mathbb{T})$.

PROOF. For $p \geq 2$, let $\phi_p : \mathbb{T}_p \to \mathbb{T}$ as $\phi_p((g_l)) = g_0$ for $(g_l) \in \mathbb{T}_p$. Then ϕ_p is a closed continuous surjective group homomorphism, so $\mathbb{T} \cong \mathbb{T}_p/\ker(\phi_p)$. Let $\psi_p : \mathbb{R} \to \mathbb{T}_p$ as $\psi_p(t) = (\exp(\mathrm{i}t/p^l))$ for $t \in \mathbb{R}$. Then ψ_p is a continuous injective group homomorphism with $R_p = \psi_p(\mathbb{R})$.

(1) First, if $q|p$, there is $d \in \mathbb{N}$ with $p = qd$. Let $\phi_{p,q} : \mathbb{T}_p \to \mathbb{T}_q$ as $\phi_{p,q}((g_l)) = (g_l^{d^l})$ for $(g_l) \in \mathbb{T}_p$. Then $\phi_{p,q}$ is a closed continuous surjective group homomorphism with

$$\ker(\phi_{p,q}) = \mathbb{T}_p \cap \{(g_l) \in \mathbb{T}^\omega : \forall l\ \exists k < d^l\ (g_l = \exp(2k\pi\mathrm{i}/d^l))\}.$$

It is clear that $\ker(\phi_{p,q})$ is non-Archimedean and $\mathbb{T}_q \cong \mathbb{T}_p/\ker(\phi_{p,q})$. By Theorem 6.5, we have $E(\mathbb{T}_p) \leq_B E(\mathbb{T}_q)$.

Secondly, if $N(p) \supseteq N(q)$, there is $e \in \mathbb{N}$ with $q|p^e$. Let $\phi : \mathbb{T}_p \to \mathbb{T}_{p^e}$ as $\phi((g_l)) = (g_{el})$ for $(g_l) \in \mathbb{T}_p$. Then ϕ is a topological group isomorphism. So $\mathbb{T}_p \cong \mathbb{T}_{p^e}$, and hence $E(\mathbb{T}_p) \sim_B E(\mathbb{T}_{p^e}) \leq_B E(\mathbb{T}_q)$.

On the other hand, if $E(\mathbb{T}_p) \leq_B E(\mathbb{T}_q)$, since $\mathbb{T}_q$ is compact, by Lemma 6.11, there exists a continuous map $S : \mathbb{T}_p \to \mathbb{T}_q$ with $S(1_{\mathbb{T}_p}) = 1_{\mathbb{T}_q}$ such that, for $x, y \in \mathbb{T}_p^\omega$, if $\lim_i d_{\mathbb{T}_p}(x(i), y(i)) = 0$, then

$$xE_*(\mathbb{T}_p)y \iff S^\omega(x)E_*(\mathbb{T}_q)S^\omega(y).$$

Moreover, since ϕ_q is a continuous homomorphism, we have

$$S^\omega(x)E_*(\mathbb{T}_q)S^\omega(y) \Rightarrow \phi_q^\omega(S^\omega(x))E_*(\mathbb{T})\phi_q^\omega(S^\omega(y)).$$

Given $\tau \in \mathbb{R}$, without loss of generality, assume that $\tau > 0$. For $0 \leq t \leq \tau$, $\psi_p(t)$ is in the arc component R_p of $1_{\mathbb{T}_p}$, thus $S(\psi_p(t))$ is in the arc component R_q of $1_{\mathbb{T}_q}$. Let $F : [0, \tau] \to \mathbb{R}$ be a continuous function such that $\exp(\mathrm{i}F(t)) = \phi_q(S(\psi_p(t)))$ with $F(0) = 0$. Following the arguments in the proof of Lemma 6.17, we have $F(t) = tF(\tau)/\tau$. Due to the arbitrariness of τ, there is $a \in \mathbb{R}$ such that $a = F(t)/t$ for all $t \neq 0$, so $S(\psi_p(t)) = \psi_q(at)$. It is clear that $a \neq 0$.

Note that $\lim_m \psi_p(p^m) = 1_{\mathbb{T}_p}$. By the continuity of S, $\lim_m \psi_q(ap^m) = 1_{\mathbb{T}_q}$. For each $l \in \omega$, we have $\lim_m \exp(\mathrm{i}ap^m/q^l) = 1$ and hence $\frac{ap^m}{2\pi q^l} \in \mathbb{Z}$ for m large enough. Thus $N(p) \supseteq N(q)$.

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(2) By (1) we only need to show $E(\mathbb{R}) \leq_B E(\mathbb{T}_p) \leq_B E(\mathbb{T})$. Note that

$$\ker(\phi_p) = \mathbb{T}_p \cap \{(g_l) \in \mathbb{T}^\omega : \forall l\, \exists k < p^l\, (g_l = \exp(2k\pi \mathrm{i}/p^l))\}.$$

It is clear that $\ker(\phi_p)$ is non-Archimedean. Thus by Theorem 6.5, we have $E(\mathbb{T}_p) \leq_B E(\mathbb{T})$.

Note that $\phi_p(\psi_p(\mathbb{R})) = \mathbb{T}$ and $\ker(\phi_p \circ \psi_p) = \{2k\pi : k \in \mathbb{Z}\}$ is discrete. Since $[0, 1]$ embeds into $\mathbb{T}_p$, by Theorem 6.5, we have $E(\mathbb{R}) \leq_B E(\mathbb{T}_p)$. ⊣

REMARK 6.25. The authors also generalized the above theorem to P-adic solenoids, where P is a sequence of natural numbers ≥ 2 (see [9, Theorem 3.2]). Furthermore, the Borel reducibility among $E(G)$'s between $E(\mathbb{R})$ and $E(\mathbb{T})$ are extremely complicated that the partial ordered set $P(\omega)/\mathrm{Fin}$ embeds into them (see [9, Theorem 3.6]).

§7. Further remarks. Although we have already presented many results concerning equivalence relations $E(G)$'s, more interesting questions remaind unanswered. For instance, Theorem 3.5(5) gives $E(S_\infty) \sim_B =^+$. And we already obtained a sequence of CLI non-Archimedean Polish groups (G_n) so that

$$E_0 \sim_B E(G_0) <_B \cdots <_B E(G_n) <_B E(G_{n+1}) <_B \cdots <_B E(S_\infty) \sim_B =^+ .$$

However, we do not know:

QUESTION 7.1. *Is there a sequence of non-Archimedean Polish groups $(G_n)_{n\in\omega}$ such that these $E(G_n)$'s are pairwise Borel incomparable*?

In Section 5, we use the notion of l.m.-unbalanced groups to prove some $E(G)$ are not Borel reducible to $E(H)$ for any TSI Polish group H. As mentioned in the first paragraph of Section 5.3, all nilpotent topological groups are distal. And by Theorem 5.17, we know all nilpotent locally compact Polish groups are not l.m.-unbalanced. For instance, let $T^u(3)$ be the group of all real matrices as $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$. It is clear that $T^u(3)$ is nilpotent locally compact, but not TSI. It follows from [10, Theorem 1.2] that $E(T^u(3)) \not\leq_B E(H)$ for any TSI Polish group H. In particular, we have $E(T^u(3)) \not\leq_B E(\mathbb{R}^3)$. So far, we do not know:

QUESTION 7.2. *Does $E(\mathbb{R}^3) \leq_B E(T^u(3))$*?

As mentioned at the end of Section 5, we do not know any example of ω-l.m.-unbalanced CLI Polish group. So perhaps the most noteworthy problem is:

QUESTION 7.3. *Does α-l.m.-unbalanced CLI Polish groups exist for each $\alpha < \omega_1$? If not, can we find more sophisticated tools to form a hierarchy of length ω_1 for all CLI Polish groups under Borel reducibility among equivalence relations induced by them*?

Some questions asked in earlier versions of this article have been answered in [9, 10]. For instance, Theorem 1.8 of [10] is a positive answer for the following question.

QUESTION 7.4. *Let G, H be two connected TSI Lie groups, does $E(G) \leq_B E(H)$ iff there exists a continuous locally injective homomorphism from G to H*?

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QUESTION 7.5. *Let G, H be two separable TSI Lie groups such that their identity components G_0 and H_0 are topologically isomorphic, does $E(G) \leq_B E(H)$ iff there exists an topological isomorphism $\phi : G_0 \to H_0$ such that $\phi \mathrm{Inn}_G(G_0)\phi^{-1} \subseteq \mathrm{Inn}_H(H_0)$?*

Theorem 6.18 confirms the special case of G, H are abelian for Question 7.4. And Corollaries 6.22 and 6.23 confirm the very special case of G_0 and H_0 are topologically isomorphic to either $\mathbb{R}$ or $\mathbb{T}$ for Question 7.5.

The authors [10] considered equivalence relations induced by separable strongly NSS Fréchet spaces. So far, we know almost nothing about Borel reducibility among equivalence relations induced by non-strongly NSS infinitely dimensional Polish groups.

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REFERENCES

[1] S. ALLISON and A. PANAGIOTOPOULOS, *Dynamical obstructions for classification by (co)homology and other TSI-group invariants*. ***Transactions of the American Mathematical Society***, vol. 374 (2021), pp. 8793–8811.

[2] A. ARVANITOYEORGOS, ***An Introduction to Lie Groups and the Geometry of Homogeneous Spaces***, Student Mathematical Library, 22, American Mathematical Society, Providence, 2003.

[3] H. BECKER, *Polish group actions: Dichotomies and generalized elementary embeddings*. ***Journal of the American Mathematical Society***, vol. 11 (1998), pp. 397–449.

[4] H. BECKER and A. S. KECHRIS, ***The Descriptive Set Theory of Polish Group Actions***, London Mathematical Society Lecture Note Series, 232, Cambridge University Press, Cambridge, 1996.

[5] C. C. CHANG and H. JEROME KEISLER, ***Continuous Model Theory***, Annals of Mathematics Studies, 58, Princeton University Press, New Jersey, 1966.

[6] J. CHARATONIK and P. PELLICER COVARRUBIAS, *On covering mappings on solenoids*. ***Proceedings of the American Mathematical Society***, vol. 130 (2001), pp. 2145–2154.

[7] L. DING, *Borel reducibility and Hölder(α) embeddability between Banach spaces*. ***The Journal of Symbolic Logic***, vol. 77 (2012), pp. 224–244.

[8] ———, *On equivalence relations generated by Schauder bases*. ***The Journal of Symbolic Logic***, vol. 82 (2017), pp. 1459–1481.

[9] L. DING and Y. ZHENG, *On equivalence relations induced by locally compact abelian Polish groups*. ***The Journal of Symbolic Logic***, vol. 90 (2025), pp. 221–236.

[10] ———, *On equivalence relations induced by Polish groups admitting compatible two-sided invariant metrics*. ***The Journal of Symbolic Logic***, (2025), pp. 1–38. https://doi.org/10.1017/jsl.2025.9.

[11] J. DIXMIER, ***C^*-Algebras, Translations of: Les C^*-Algèbres et Leurs représentations***, North-Holland Mathematical Library, 15, North-Holland, New York, 1977.

[12] R. DOUGHERTY and G. HJORTH, *Reducibility and nonreducibility between ℓ^p equivalence relations*. ***Transactions of the American Mathematical Society***, vol. 351 (1999), pp. 1835–1844.

[13] I. FARAH, *Basis problem for turbulent action II: c_0-equalities*. ***Proceedings of the London Mathematical Society***, vol. 82 (2001), pp. 1–30.

https://doi.org/10.1017/jsl.2025.10177 Published online by Cambridge University Press

[14] S. Gao, ***Invariant Descriptive Set Theory***, Monographs and Textbooks in Pure and Applied Mathematics, 293, CRC Press, Boca Raton, 2009.

[15] S. Gao and S. Jackson, *Countable abelian groups actions and hyperfinite equivalence relations*. ***Inventiones Mathematicae***, vol. 201 (2015), pp. 309–383.

[16] S. Gao and M. Xuan, *On non-Archimedean Polish groups with two-sided invariant metrics*. ***Topology and its Applications***, vol. 161 (2014), pp. 343–353.

[17] G. Hjorth, ***Classification and Orbit Equivalence Relations***, Mathematics Surveys and Monographs, 75, American Mathematical Society, Providence, 2000.

[18] K. H. Hofmann and S. A. Morris, ***The Lie Theory of Connected Pro-Lie Groups***, EMS Tracts in Mathematics, 2, European Mathematical Society, Zürich, 2007.

[19] ———, ***The Structure of Compact Groups***, De Gruyter Studies in Mathematics, 25, De Gruyter, Berlin, 2013.

[20] K. H. Hofmann, S. A. Morris, and M. Stroppel, *Varieties of topological groups, Lie groups, and SIN-groups*. ***Colloquium Mathematicum***, vol. 70 (1996), pp. 151–163.

[21] V. Kanovei, ***Borel Equivalence Relations: Structure and Classification***, University Lecture Series, 44, American Mathematical Society, Providence, 2008.

[22] A. S. Kechris, *Countable sections for locally compact group actions*. ***Ergodic Theory and Dynamical Systems***, vol. 12 (1992), pp. 283–295.

[23] ———, ***Classical Descriptive Set Theory***, Graduate Texts in Mathematics, 156, Springer-Verlag, New York, 1995.

[24] A.W. Knapp, ***Lie Groups Beyond an Introduction***, first ed., Progress in Mathematics, 140, Birkhäuser, Boston, 2002.

[25] M. Lupini and A. Panagiotopoulos, *Games orbits play and obstructions to Borel reducibility*. ***Groups, Geometry, and Dynamics***, vol. 12 (2018), pp. 1461–1483.

[26] G. H. Meisters and C. Olech, *Locally one-to-one mappings and a classical theorem on Schlicht functions*. ***Duke Mathematical Journal***, vol. 30 (1963), pp. 63–80.

[27] J. Rosenblatt, *A distal property of groups and the growth of connected locally compact groups*. ***Mathematika***, vol. 26 (1979), pp. 94–98.

[28] S. Solecki, *Actions of non-compact and non-locally compact groups*. ***The Journal of Symbolic Logic***, vol. 65 (2000), pp. 1881–1894.

[29] S. Thomas, *Popa superrigidity and countable Borel equivalence relations*. ***Annals of Pure and Applied Logic***, vol. 158 (2009), pp. 175–189.

[30] V. S. Varadarajan, ***Lie Groups, Lie Algebras, and their Representations***, Graduate Texts in Mathematics, 102, Springer-Verlag, New York, 1974.

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