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VARIABILITY REGIONS FOR THE *n*th DERIVATIVE OF BOUNDED ANALYTIC FUNCTIONS

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Abstract

Let $\mathcal{H}$ be the class of all analytic self-maps of the open unit disk $\mathbb{D}$. Denote by $H^n f(z)$ the nth-order hyperbolic derivative of $f \in \mathcal{H}$ at $z \in \mathbb{D}$. We develop a method allowing us to calculate higher-order hyperbolic derivatives in an expeditious manner. We also generalise certain classical results for variability regions of the nth derivative of bounded analytic functions. For $z_0 \in \mathbb{D}$ and $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{n-1}) \in \mathbb{D}^n$, let $\mathcal{H}(\gamma) = \{f \in \mathcal{H} : f(z_0) = \gamma_0, H^1 f(z_0) = \gamma_1, \dots, H^{n-1} f(z_0) = \gamma_{n-1}\}$. We determine the variability region $\{f^{(n)}(z_0) : f \in \mathcal{H}(\gamma)\}$ to prove a Schwarz–Pick lemma for the nth derivative. We apply this result to establish an nth-order Dieudonné lemma, which provides an explicit description of the variability region $\{h^{(n)}(z_0) : h \in \mathcal{H}, h(0) = 0, h(z_0) = w_0, h'(z_0) = w_1, \dots, h^{(n-1)}(z_0) = w_{n-1}\}$ for given z_0, w_0, $w_1, \dots, w_{n-1}$. Moreover, we determine the form of all extremal functions.

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1. Introduction and preliminaries

We denote by $\mathbb{D}$ and $\overline{\mathbb{D}}$ the open and closed unit disks in the complex plane $\mathbb{C}$. For $c \in \mathbb{C}$ and $\rho > 0$, let $\mathbb{D}(c, \rho) = \{z \in \mathbb{C} : |z - c| < \rho\}$ and $\overline{\mathbb{D}}(c, \rho) = \{z \in \mathbb{C} : |z - c| \le \rho\}$. Let $\mathcal{H}$ be the class of analytic maps of $\mathbb{D}$ into itself and $\mathcal{H}_0 = \{h \in \mathcal{H} : h(0) = 0\}$. In addition, denote by $\mathcal{S}$ the Schur class, that is, the set of analytic functions from $\mathbb{D}$ to $\overline{\mathbb{D}}$.

First, we recall the classical Schwarz lemma, which says that if $f \in \mathcal{H}_0$, then $|f(z_0)| \le |z_0|$ for any nonzero z_0 in $\mathbb{D}$ and $|f'(0)| \le 1$, with equality if and only if $f(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$. A number of sharpened and refined versions of the Schwarz lemma can be found in [4–8].

The classical Schwarz–Pick lemma [17] states that

$$|f'(z)| \le \frac{1 - |f(z)|^2}{1 - |z|^2} \quad \text{for } f \in \mathcal{H} \text{ and } z \in \mathbb{D},$$

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with equality if and only if $f(z) = e^{i\theta}(z-a)/(1-\bar{a}z)$ for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$. If we define T_a by $T_a(z) := (z+a)/(1+\bar{a}z)$ for $z, a \in \mathbb{D}$, then the Schwarz–Pick lemma can be slightly modified as the following lemma.

THEOREM 1.1 (Schwarz–Pick lemma). *Let $z_0, \gamma_0 \in \mathbb{D}$. Consider $f \in \mathcal{H}$ with $f(z_0) = \gamma_0$. Then, the region of values of $f'(z_0)$, that is, $\{f'(z_0) : f \in \mathcal{H}, f(z_0) = \gamma_0\}$, is the closed disk*

$$\overline{\mathbb{D}}\left(0, \frac{1-|\gamma_0|^2}{1-|z_0|^2}\right) = \{f_\alpha'(z_0) : \alpha \in \overline{\mathbb{D}}\} \quad \text{where } f_\alpha(z) = T_{\gamma_0}(\alpha T_{-z_0}(z)),$$

and $f(z)$ can be expressed as $T_{\gamma_0}(T_{-z_0}(z)f^*(z))$ for some $f^* \in \mathcal{S}$. Further, $f'(z_0) \in \partial\mathbb{D}(0, (1-|\gamma_0|^2)/(1-|z_0|^2))$ if and only if $f(z) = f_\alpha(z)$ for some constant $\alpha \in \partial\mathbb{D}$.

There are many results on regions of variability (see, for example, [1, 11, 12, 15, 20, 23]). Among these is the Schwarz–Pick lemma of the second derivative, which describes the region of values of $f''(z_0)$ for $f \in \mathcal{H}$ and $z_0 \in \mathbb{D}$ with $f(z_0) = \gamma_0$.

THEOREM 1.2 (Schwarz–Pick lemma of the second derivative; [6, Theorem 2.2]). *Let $z_0, \gamma_0 \in \mathbb{D}$ and $\gamma_1 \in \overline{\mathbb{D}}$. Then,*

$$\left\{f''(z_0) : f \in \mathcal{H}, f(z_0) = \gamma_0, f'(z_0) = \frac{1-|\gamma_0|^2}{1-|z_0|^2}\gamma_1\right\} = \overline{\mathbb{D}}(c_2, \rho_2),$$

where

$$\begin{cases} c_2 = c_2(z_0, \gamma_0, \gamma_1) = \dfrac{2(1-|\gamma_0|^2)}{(1-|z_0|^2)^2}(\overline{z_0} - \overline{\gamma_0}\gamma_1)\gamma_1, \\ \rho_2 = \rho_2(z_0, \gamma_0, \gamma_1) = \dfrac{2(1-|\gamma_0|^2)(1-|\gamma_1|^2)}{(1-|z_0|^2)^2}. \end{cases} \tag{1.1}$$

Further, for $|\gamma_1| < 1$, $f''(z_0) \in \partial\mathbb{D}(c_2, \rho_2)$ if and only if $f(z) = T_{\gamma_0}(T_{-z_0}(z)T_{\gamma_1}(\alpha T_{-z_0}(z)))$ for some constant $\alpha \in \partial\mathbb{D}$.

The following interesting question arises from Theorems 1.1 and 1.2 for $n \geq 1$: is it possible to describe the variability region of $f^{(n)}(z_0)$ for $z_0 \in \mathbb{D}$ and $f \in \mathcal{H}$? The answer is yes and the nth-order result is given in Theorem 2.5.

The nth derivative $f^{(n)}(z_0)$ is related to the nth-order hyperbolic derivative $H^n f(z_0)$, which was first introduced by Rivard [18]. For $z, w \in \overline{\mathbb{D}}$, we define the complex pseudo-hyperbolic distance

$$[z, w] := \begin{cases} \dfrac{z-w}{1-\overline{w}z} & \text{if } z \neq w, \\ 0 & \text{if } z = w. \end{cases}$$

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Thus, we can define an operator Δ_{z_0} by

$$\Delta_{z_0} f(z) = \begin{cases} \dfrac{[f(z),\ f(z_0)]}{[z,\ z_0]} & \text{for } z \neq z_0, \\ \dfrac{(1-|z_0|^2)f'(z_0)}{1-|f(z_0)|^2} & \text{for } z = z_0. \end{cases}$$

Following [2], we can iterate the process and construct the hyperbolic difference quotient of order j of the function f for distinct parameters $z_1, \dots, z_j$ by

$$\Delta^j f(z; z_j, \dots, z_1) = (\Delta_{z_j} \circ \cdots \circ \Delta_{z_1}) f(z).$$

We can rewrite this recursive definition as

$$\Delta^j f(z; z_j, \dots, z_1) = \frac{[\Delta^{j-1} f(z; z_{j-1}, \dots, z_1), \Delta^{j-1} f(z_j; z_{j-1}, \dots, z_1)]}{[z, z_j]}.$$

Using a limiting process, we can define the hyperbolic divided quotient of order n of f with parameter z by

$$\Delta_z^n f(\zeta) = \Delta^n f(\zeta; z, \dots, z) := \lim_{z_n \to z} \cdots \lim_{z_1 \to z} \Delta^n f(\zeta; z_n, \dots, z_1) \quad (\zeta \neq z).$$

Therefore, we define the nth-order hyperbolic derivative of $f \in \mathcal{H}$ at $z \in \mathbb{D}$ by

$$H^n f(z) := \Delta_z^n f(z) = \Delta^n f(z; z, \dots, z) = \lim_{\zeta \to z} \Delta^n f(\zeta; z, \dots, z).$$

The usual hyperbolic derivative (see [3]) coincides with the first-order hyperbolic derivative,

$$f^h(z) = \frac{(1-|z|^2)f'(z)}{1-|f(z)|^2} = H^1 f(z).$$

To generalise Theorems 1.1 and 1.2, we consider the following problem.

PROBLEM 1.3. For $z_0 \in \mathbb{D}$ and $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{n-1}) \in \mathbb{D}^n$, let

$$\mathcal{H}(\gamma) = \{f \in \mathcal{H} : f(z_0) = \gamma_0, H^1 f(z_0) = \gamma_1, \dots, H^{n-1} f(z_0) = \gamma_{n-1}\}.$$

Determine the variability region $\{f^{(n)}(z_0) : f \in \mathcal{H}(\gamma)\}$.

Based on the Schwarz–Pick lemma, Dieudonné [9] gave a description of the variability region of $h'(z_0)$ for $h \in \mathcal{H}_0$ and $z_0 \in \mathbb{D}$ (see also [3, 10]).

THEOREM 1.4 (Dieudonné's lemma). *Let $z_0, w_0 \in \mathbb{D}$ with $|w_0| < |z_0|$. Suppose that $h \in \mathcal{H}_0$ and $h(z_0) = w_0$. Set $\gamma_0 = w_0/z_0$, $h_\alpha(z) = zT_{\gamma_0}(\alpha T_{-z_0}(z))$. Then, the region of values of $h'(z_0)$, that is, $\{h'(z_0) : h \in \mathcal{H}_0, h(z_0) = w_0\}$, is the closed disk*

$$\overline{\mathbb{D}}(c_1'(z_0, w_0), \rho_1'(z_0, w_0)) = \{h_\alpha'(z_0) : \alpha \in \overline{\mathbb{D}}\},$$

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where

$$c_1'(z_0,w_0)=\frac{w_0}{z_0},\quad \rho_1'(z_0,w_0)=\frac{|z_0|^2-|w_0|^2}{|z_0|(1-|w_0|^2)},$$

and $h(z)$ can be expressed as $zT_{\gamma_0}(T_{-z_0}(z)h^*(z))$ for some $h^*\in\mathcal{S}$. Further, $h'(z_0)\in\partial\mathbb{D}(c_1'(z_0,w_0),\rho_1'(z_0,w_0))$ if and only if $h(z)=h_\alpha(z)$ for some constant $\alpha\in\partial\mathbb{D}$.

Rivard [19] extended Dieudonné's lemma to the second order.

THEOREM 1.5 (Dieudonné's lemma of the second derivative). *Let $z_0,w_0\in\mathbb{D}$ with $|w_0|<|z_0|$, $\gamma_1\in\overline{\mathbb{D}}$ and*

$$w_1=\frac{w_0}{z_0}+\frac{|z_0|^2-|w_0|^2}{\overline{z_0}(1-|w_0|^2)}\gamma_1.$$

Then, $\{h''(z_0):h\in\mathcal{H}_0,h(z_0)=w_0,h'(z_0)=w_1\}=\overline{\mathbb{D}}(c_2',\rho_2')$, where

$$\begin{cases}c_2'=c_2'(z_0,w_0,\gamma_1)=\dfrac{2(|z_0|^2-|w_0|^2)}{|z_0|^2(1-|z_0|^2)^2}\gamma_1\Big(1-\dfrac{z_0\overline{w_0}}{\overline{z_0}}\gamma_1\Big),\\ \rho_2'=\rho_2'(z_0,w_0,\gamma_1)=\dfrac{2(|z_0|^2-|w_0|^2)}{|z_0|(1-|z_0|^2)^2}(1-|\gamma_1|^2).\end{cases}$$

Further, for $|\gamma_1|<1$, $h''(z_0)\in\partial\mathbb{D}(c_2',\rho_2')$ if and only if $h(z)=zT_{\gamma_0}(T_{-z_0}(z)T_{\gamma_1}(\alpha T_{-z_0}(z)))$ for some constant $\alpha\in\partial\mathbb{D}$, where $\gamma_0=w_0/z_0$.

Theorems 1.4 and 1.5 suggest an interesting question for $n\geq 1$: can we obtain an nth-order Dieudonné lemma for the region of values of $h^{(n)}(z_0)$ for $h\in\mathcal{H}_0$ and $z_0\in\mathbb{D}$? We give the nth-order result in Theorem 3.1.

The organisation of this paper is as follows. In Section 2, we obtain a useful formula for directly calculating the hyperbolic derivatives and use it to solve Problem 1.3. In Section 3, we apply the Schwarz–Pick lemma of the nth derivative to establish a Dieudonné lemma of the nth derivative. The connection between $\mathcal{H}_0$ and $\mathcal{H}$ is as follows. If $h\in\mathcal{H}_0$ is not a rotation about the origin and $f(z)=h(z)/z$, then $f\in\mathcal{H}$. Since $h^{(n)}(z)=zf^{(n)}(z)+nf^{(n-1)}(z)$, we just need to determine the variability region of $f^{(n)}(z_0)$ for $f\in\mathcal{H}$.

2. Formula for hyperbolic derivatives and the Schwarz–Pick lemma of the nth derivative

Since the recursive formula established by Rivard [19] does not lend itself easily to simple calculations for hyperbolic derivatives, we will develop a method that enables us to calculate higher-order hyperbolic derivatives more efficiently.

For $n=1,2,3,\ldots$, we define a sequence of functions F_n of the n complex variables $\gamma_1,\gamma_2,\ldots,\gamma_n$ inductively by

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$$F_1(\gamma_1) = \gamma_1, \quad F_2(\gamma_1, \gamma_2) = (1 - |\gamma_1|^2)\gamma_2,$$
$$F_n(\gamma_1, \ldots, \gamma_n) = (1 - |\gamma_1|^2)F_{n-1}(\gamma_2, \ldots, \gamma_n) - \bar{\gamma}_1 \sum_{k=2}^{n-1} F_{n-k}(\gamma_2, \ldots, \gamma_{n-k+1})F_k(\gamma_1, \ldots, \gamma_k), \quad n \geq 3. \tag{2.1}$$

By construction, the function $F_n(\gamma_1, \ldots, \gamma_n)$ is a polynomial in $\gamma_1, \bar{\gamma}_1, \ldots, \gamma_{n-1}, \bar{\gamma}_{n-1}, \gamma_n$ with integer coefficients. From (2.1),

$$F_3 = (1 - |\gamma_1|^2)(1 - |\gamma_2|^2)\gamma_3 - (1 - |\gamma_1|^2)\bar{\gamma}_1\gamma_2^2,$$
$$F_4 = (1 - |\gamma_1|^2)(1 - |\gamma_2|^2)(1 - |\gamma_3|^2)\gamma_4 - (1 - |\gamma_1|^2)(1 - |\gamma_2|^2)\gamma_3(2\bar{\gamma}_1\gamma_2 + \bar{\gamma}_2\gamma_3) + (1 - |\gamma_1|^2)\bar{\gamma}_1^2\gamma_2^3.$$

By induction, it is easy to verify Schur's recurrence relation.

LEMMA 2.1 [22]. *For each $n \geq 2$, there exists a function $G_n(\gamma_1, \ldots, \gamma_{n-1})$ of $n-1$ complex variables $\gamma_1, \ldots, \gamma_{n-1}$ such that*

$$F_n(\gamma_1, \ldots, \gamma_n) = (1 - |\gamma_1|^2) \cdots (1 - |\gamma_{n-1}|^2)\gamma_n + G_n(\gamma_1, \ldots, \gamma_{n-1}).$$

LEMMA 2.2 [14]. *Let $g(z) = \sum_{n=1}^{\infty} a_n z^n$ be a function in $\mathcal{H}_0$ with its hyperbolic derivatives $H^j g(0) = \gamma_j (j = 1, 2, \ldots)$. Then, $a_n = F_n(\gamma_1, \gamma_2, \ldots, \gamma_n)$ for $n = 1, 2, 3, \ldots$.*

For $f \in \mathcal{H}$, Peschl's invariant derivatives $D^n f(z)$ [16] (see also [13, 21]) are defined by the Taylor series expansion

$$z \mapsto g(z) := \frac{f(\frac{z+z_0}{1+\bar{z}_0 z}) - f(z_0)}{1 - \overline{f(z_0)} f(\frac{z+z_0}{1+\bar{z}_0 z})} = \sum_{n=1}^{\infty} \frac{D^n f(z_0)}{n!} z^n, \quad z, z_0 \in \mathbb{D}. \tag{2.2}$$

Then, $D^n f(z_0) = g^{(n)}(0)$ and $H^n f(z_0) = H^n g(0)$ for $n = 1, 2, \ldots$ (see [8, Proposition 3.3]). The first few $D^n f(z)$ are given by

$$D^1 f(z) = \frac{(1 - |z|^2) f'(z)}{1 - |f(z)|^2},$$
$$D^2 f(z) = \frac{(1 - |z|^2)^2}{1 - |f(z)|^2}\left[f''(z) - \frac{2\bar{z} f'(z)}{1 - |z|^2} + \frac{2\overline{f(z)} f'(z)^2}{1 - |f(z)|^2}\right],$$
$$D^3 f(z) = \frac{(1 - |z|^2)^3}{1 - |f(z)|^2}\left[f'''(z) - \frac{6\bar{z} f''(z)}{1 - |z|^2} + \frac{6\overline{f(z)} f'(z) f''(z)}{1 - |f(z)|^2} + \frac{6\bar{z}^2 f'(z)}{(1 - |z|^2)^2} - \frac{12\overline{f(z)} f'(z)^2}{(1 - |z|^2)(1 - |f(z)|^2)} + \frac{6\overline{f(z)}^2 f'(z)^3}{(1 - |f(z)|^2)^2}\right].$$

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When we express g in (2.2) by the series expansion $g(z) = \sum_{n=1}^{\infty} a_n z^n$, we can obtain the formula for $D^n f(z_0)$,

$$\frac{D^n f(z_0)}{n!} = (1 - |\gamma_1|^2) \cdots (1 - |\gamma_{n-1}|^2)\gamma_n + G_n(\gamma_1, \ldots, \gamma_{n-1}). \tag{2.3}$$

Next, we obtain a formula for $H^n f$ in terms of $D^n f$ and the lower-order hyperbolic derivatives $H^1 f, \ldots, H^{n-1} f$, allowing us to calculate the hyperbolic derivatives in a more convenient way. From the preceding definitions,

$$H^1 f(z) = D^1 f(z) = f^h(z) = \frac{(1 - |z|^2) f'(z)}{1 - |f(z)|^2}.$$

A Blaschke product of degree $n \in \mathbb{N}$ is a function of the form $B(z) = e^{i\theta} \prod_{j=1}^{n} [z, z_j]$, with $z, z_j \in \mathbb{D}$, $\theta \in \mathbb{R}$. The class of Blaschke products of degree n is denoted by $\mathcal{B}_n$.

THEOREM 2.3. *Let $n \geq 2$, $f \in \mathcal{H} \setminus \bigcup_{j=0}^{j=n-1} \mathcal{B}_j$ and $z_0 \in \mathbb{D}$. Then,*

$$H^n f(z_0) = \frac{D^n f(z_0) - n!\, G_n(\gamma_1, \ldots, \gamma_{n-1})}{n!\,(1 - |\gamma_1|^2) \cdots (1 - |\gamma_{n-1}|^2)} \quad \text{where } \gamma_j = H^j f(z_0). \tag{2.4}$$

Using (2.4), one can easily obtain an explicit form for hyperbolic derivatives. We give the first few cases. Noting that $G_2(\gamma_1) = 0$ and $G_3(\gamma_1, \gamma_2) = -(1 - |\gamma_1|^2)\overline{\gamma_1}\gamma_2^2$, we immediately obtain

$$H^2 f(z_0) = \frac{D^2 f(z_0)}{2(1 - |H^1 f(z_0)|^2)},$$
$$H^3 f(z_0) = \frac{D^3 f(z_0) + 6(1 - |H^1 f(z_0)|^2)\overline{H^1 f(z_0)}(H^2 f(z_0))^2}{6(1 - |H^1 f(z_0)|^2)(1 - |H^2 f(z_0)|^2)}.$$

It is possible to obtain expressions for $D^n f$ in terms of $f^{(n)}$ and its lower-order derivatives, as well as the derivatives $D^1 f, \ldots, D^n f$.

LEMMA 2.4 [13, Corollary 7.5]. *Let $f \in \mathcal{H}$. Then,*

$$D^n f(z) = \sum_{k=1}^{n} \alpha_{n,k} \frac{(\overline{z})^{n-k}(1 - |z|^2)^k f^{(k)}(z)}{1 - |f(z)|^2} - \sum_{k=2}^{n} k!\, (-\overline{f(z)})^{k-1} A_{n,k}(D^1 f, \ldots, D^{n-k+1} f),$$

where

$$\alpha_{n,k} := \begin{cases} (-1)^{n-k} \dfrac{n!\,(n-1)!}{k!\,(k-1)!\,(n-k)!} & \text{if } 1 \leq k \leq n, \\ 0 & \text{otherwise}, \end{cases}$$

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and $A_{n,k}$ is the Bell polynomial given by

$$A_{n,k}(x_1,\dots,x_{n-k+1}) := \sum_{j\in I_k} \frac{n!}{j_1!j_2!\cdots j_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{j_1}\left(\frac{x_2}{2!}\right)^{j_2}\cdots\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}$$
$$= n!\sum_{j\in I_k}\prod_{i=1}^{n-k+1}\frac{x_i^{j_i}}{(i!)^{j_i}j_i!},$$

where I_k consists of all multi-indexes $(j_1,\dots,j_{n-k+1})$ such that

$$j_1,\dots,j_{n-k+1}\geq 0,\quad j_1+j_2+\cdots+j_{n-k+1}=k,\quad j_1+2j_2+\cdots+(n-k+1)j_{n-k+1}=n.$$

We note here that the ordinary derivative $f^{(n)}$ can also be expressed in terms of the invariant derivatives $D^k f(k=1,2,\dots,n)$ by

$$\frac{(1-|z|^2)^n}{1-|f(z)|^2}\frac{f^{(n)}(z)}{n!} = \sum_{k=1}^{n}\binom{n-1}{k-1}(\bar z)^{n-k}\cdot b_k,$$

where

$$b_k = \sum_{l=1}^{k}\frac{l!}{k!}(-\overline{f(z)})^{l-1}A_{k,l}(D^1f,\dots,D^{k-l+1}f).$$

Since $\alpha_{n,n}=1$, we have

$$D^n f(z) = \frac{(1-|z|^2)^n f^{(n)}(z)}{1-|f(z)|^2} + s_{n-1}(z),$$

where

$$s_{n-1}(z) = \sum_{k=1}^{n-1}\alpha_{n,k}\frac{(\bar z)^{n-k}(1-|z|^2)^k f^{(k)}(z)}{1-|f(z)|^2}$$
$$-\sum_{k=2}^{n}k!\,(-\overline{f(z)})^{k-1}A_{n,k}(D^1f(z),\dots,D^{n-k+1}f(z)).$$

Note that $s_{n-1}(z)$ depends only on $D^1f(z),\dots,D^{n-1}f(z)$ together with z and $f(z)$. By (2.4), it is easy to verify that $s_{n-1}(z)$ depends only on $H^1f(z),\dots,H^{n-1}f(z)$ together with z and $f(z)$. For instance, $s_1(z) = -2H^1f(z)(\bar z - \overline{f(z)}H^1f(z))$.

Together with (2.3), we have $f^{(n)}(z_0) = c_n + \rho_n\gamma_n$, where $\gamma_n = H^nf(z_0)\in\overline{\mathbb{D}}$ and

$$\begin{cases} c_n = \dfrac{(1-|\gamma_0|^2)}{(1-|z_0|^2)^n}[n!\,G_n(\gamma_1,\dots,\gamma_{n-1}) - s_{n-1}(z_0)], \\ \rho_n = \dfrac{n!\prod_{k=0}^{n-1}(1-|\gamma_k|^2)}{(1-|z_0|^2)^n}. \end{cases} \tag{2.5}$$

In addition, we set $G_1=0$ and $s_0(z_0)=0$.

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THEOREM 2.5 (Schwarz–Pick lemma of the nth derivative). *Let $n \in \mathbb{N}$, $z_0 \in \mathbb{D}$ and $\gamma = (\gamma_0, \dots, \gamma_{n-1}) \in \overline{\mathbb{D}}^n$. Suppose that $f \in \mathcal{H}$, $f(z_0) = \gamma_0$, $H^1 f(z_0) = \gamma_1, \dots, H^{n-1} f(z_0) = \gamma_{n-1}$.*

(1) *If $|\gamma_0| < 1, \dots, |\gamma_{j-1}| < 1$ and $|\gamma_j| = 1$ for some $j = 1, \dots, n-1$, then $\gamma_{j+1} = \cdots = \gamma_{n-1} = 0$ and $f^{(n)}(z_0) = c_n$. Furthermore,*

$$f(z) = T_{\gamma_0}(T_{-z_0}(z)T_{\gamma_1}(\cdots T_{-z_0}(z)T_{\gamma_{j-1}}(\gamma_j T_{-z_0}(z))\cdots)),$$

which is a Blaschke product of degree j.

(2) *If $|\gamma_0| < 1, \dots, |\gamma_{n-1}| < 1$, then the region of values of $f^{(n)}(z_0)$ is the closed disk*

$$\overline{\mathbb{D}}(c_n, \rho_n) = \{f^{(n)}_{\gamma,\varepsilon}(z_0) : \varepsilon \in \overline{\mathbb{D}}\},$$

where $f_{\gamma,\varepsilon}(z) = T_{\gamma_0}(T_{-z_0}(z)T_{\gamma_1}(\cdots T_{-z_0}(z)T_{\gamma_{n-1}}(\varepsilon T_{-z_0}(z))\cdots))$, and $f(z)$ can be expressed as

$$f(z) = T_{\gamma_0}(T_{-z_0}(z)T_{\gamma_1}(\cdots T_{-z_0}(z)T_{\gamma_{n-1}}(T_{-z_0}(z)f^*(z))\cdots))$$

for some $f^* \in \mathcal{S}$. Further, $f^{(n)}(z_0) \in \partial\mathbb{D}(c_n, \rho_n)$ if and only if $f(z) = f_{\gamma,\varepsilon}(z)$ for some constant $\varepsilon \in \partial\mathbb{D}$.

Theorems 1.1 and 1.2 are both simple corollaries of Theorem 2.5.

3. Dieudonné's lemma of the nth derivative

In this section, we will apply Theorem 2.5 to formulate a Dieudonné lemma of the nth derivative $h^{(n)}$ for $h \in \mathcal{H}_0$.

Let $n \in \mathbb{N}$, $z_0 \in \mathbb{D}$ and $\gamma = (\gamma_0, \dots, \gamma_{n-1}) \in \mathbb{D}^n$. Suppose that $f \in \mathcal{H}$, $f(z_0) = \gamma_0$ and $H^1 f(z_0) = \gamma_1, \dots, H^{n-1} f(z_0) = \gamma_{n-1}$. Then, we have $h(z) = zf(z) \in \mathcal{H}_0$ and $h^{(k)}(z) = kf^{(k-1)}(z) + zf^{(k)}(z)$ for $k = 1, \dots, n$. In particular, $h(z_0) = z_0\gamma_0$,

$$h'(z_0) = f(z_0) + z_0 f'(z_0) = \gamma_0 + \frac{1 - |\gamma_0|^2}{1 - |z_0|^2} z_0\gamma_0 = \gamma_0 + |z_0|\rho_1 \frac{|z_0|}{\overline{z_0}}\gamma_0,$$

and, for $k = 2, \dots, n$,

$$\begin{aligned} h^{(k)}(z_0) &= kf^{(k-1)}(z_0) + z_0 f^{(k)}(z_0) \\ &= k(c_{k-1} + \rho_{k-1}\gamma_{k-1}) + z_0(c_k + \rho_k\gamma_k) \\ &= [k(c_{k-1} + \rho_{k-1}\gamma_{k-1}) + z_0 c_k] + |z_0|\rho_k \frac{|z_0|}{\overline{z_0}}\gamma_k, \end{aligned}$$

where $\gamma_n = H^n f(z_0) \in \overline{\mathbb{D}}$. Note that c_k and ρ_k are defined by (2.5). Therefore, we conclude that the region of values of $h^{(n)}(z_0)$ is the closed disk $\overline{\mathbb{D}}(c_n', \rho_n')$, where

$$\begin{cases} c_n' = n(c_{n-1} + \rho_{n-1}\gamma_{n-1}) + z_0 c_n, \\ \rho_n' = |z_0|\rho_n. \end{cases}$$

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THEOREM 3.1 (Dieudonné's lemma of the nth derivative). *Let $z_0, w_0 \in \mathbb{D}$ with $|w_0| < |z_0|$ and let $\gamma_1, \dots, \gamma_{n-1} \in \overline{\mathbb{D}}$. For $k = 1, \dots, n-1$, let $w_k \in \overline{\mathbb{D}}(c_k', \rho_k')$ be given by $w_k = c_k' + \rho_k' \gamma_k |z_0|/\overline{z_0}$, where*

$$c_1' = \frac{w_0}{z_0}, \quad \rho_1' = \frac{|z_0|^2 - |w_0|^2}{|z_0|(1 - |w_0|^2)}$$

and, for $k = 2, \dots, n-1$,

$$c_k' = k(c_{k-1} + \rho_{k-1}\gamma_{k-1}) + z_0 c_k, \quad \rho_k' = |z_0|\rho_k.$$

Suppose that $h \in \mathcal{H}_0$, $h(z_0) = w_0, h'(z_0) = w_1, \dots, h^{(n-1)}(z_0) = w_{n-1}$. Set $\gamma_0 = w_0/z_0$.

(1) *If $|\gamma_1| < 1, \dots, |\gamma_{j-1}| < 1$, $|\gamma_j| = 1$, $\gamma_{j+1} = \cdots = \gamma_{n-1} = 0$ for some $j = 1, \dots, n-1$, then $h^{(n)}(z_0) = c_n'$ and*

$$h(z) = zT_{\gamma_0}(T_{-z_0}(z)T_{\gamma_1}(\cdots T_{-z_0}(z)T_{\gamma_{j-1}}(\gamma_j T_{-z_0}(z))\cdots)).$$

(2) *If $|\gamma_1| < 1$, $|\gamma_2| < 1, \dots, |\gamma_{n-1}| < 1$, then the region of values of $h^{(n)}(z_0)$ is the closed disk*

$$\overline{\mathbb{D}}(c_n', \rho_n') = \{h_{\gamma,\varepsilon}^{(n)}(z_0) : \varepsilon \in \overline{\mathbb{D}}\},$$

where $h_{\gamma,\varepsilon}(z) = zT_{\gamma_0}(T_{-z_0}(z)T_{\gamma_1}(\cdots T_{-z_0}(z)T_{\gamma_{n-1}}(\varepsilon T_{-z_0}(z))\cdots))$, and $h(z)$ can be expressed as

$$zT_{\gamma_0}(T_{-z_0}(z)T_{\gamma_1}(\cdots T_{-z_0}(z)T_{\gamma_{n-1}}(T_{-z_0}(z)h^*(z))\cdots))$$

for some $h^* \in \mathcal{S}$. Further, $h^{(n)}(z_0) \in \partial\mathbb{D}(c_n', \rho_n')$ if and only if $h(z) = h_{\gamma,\varepsilon}(z)$ for some constant $\varepsilon \in \partial\mathbb{D}$.

PROOF. We just need to consider the case $\gamma_1, \dots, \gamma_{n-1} \in \mathbb{D}$ for $n \geq 2$. Assume that $f(z) = h(z)/z$, with $f \in \mathcal{H}$. It is easily seen that $f(z_0) = \gamma_0$, $H^1 f(z_0) = \gamma_1, \dots, H^{n-1} f(z_0) = \gamma_{n-1}$. From the preceding discussion, $h^{(n)}(z_0) \in \overline{\mathbb{D}}(c_n', \rho_n')$.

Next, we show that $\overline{\mathbb{D}}(c_n', \rho_n')$ is covered. Define the class $\mathcal{H}_0(z_0, w_0, \dots, w_{n-1})$ by

$$\mathcal{H}_0(z_0, w_0, \dots, w_{n-1}) = \{h \in \mathcal{H}_0 : h(z_0) = w_0, h'(z_0) = w_1, \dots, h^{(n-1)}(z_0) = w_{n-1}\}.$$

From the proof of Theorem 2.5 and the relations $h^{(k)}(z) = kf^{(k-1)}(z) + zf^{(k)}(z)$ for $k = 1, \dots, n$, it is easy to see that $h_{\gamma,\varepsilon} \in \mathcal{H}_0(z_0, w_0, \dots, w_{n-1})$ and $h_{\gamma,\varepsilon}^{(n)}(z_0) = c_n' + \rho_n' \varepsilon |z_0|/\overline{z_0}$. The closed disk $\overline{\mathbb{D}}(c_n', \rho_n')$ is covered since $\varepsilon \in \overline{\mathbb{D}}$ is arbitrary.

Next, $h^{(n)}(z_0) \in \partial\mathbb{D}(c_n', \rho_n')$ if and only if $h(z) = zf(z)$, where f is a Blaschke product of degree n satisfying $f(z_0) = \gamma_0$, $H^1 f(z_0) = \gamma_1, \dots, H^{n-1} f(z_0) = \gamma_{n-1}$. From the proof of Theorem 2.5, we can easily check that

$$f(z) = T_{\gamma_0}(T_{-z_0}(z)T_{\gamma_1}(\cdots T_{-z_0}(z)T_{\gamma_{n-1}}(\varepsilon T_{-z_0}(z))\cdots)), \quad \varepsilon \in \partial\mathbb{D}. \qquad \square$$

Theorems 1.4 and 1.5 can be directly derived from Theorem 3.1.

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