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ORDERING CONSECUTIVE *k*-OUT-OF-*n*:F SYSTEMS

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Stochastic comparisons of linear (circular) consecutive *k*-out-of-*n*:F systems with independent components are studied. A sufficient condition is given under which the lifetime of a circular consecutive *k*-out-of-*n*:F system with independent and nonidentically distributed components is stochastically decreasing in *n* for fixed *k*. Furthermore, the likelihood ratio orderings of the lifetimes of linear (circular) consecutive *k*-out-of-*n*:F systems with independent and identically distributed components are also established, and some counterexamples are given to show that these orderings are not true in general.

1. INTRODUCTION

The concept of the consecutive *k*-out-of-*n* was introduced by Chiang and Niu [5] and has attracted a lot of attention in the field of reliability. A linear (circular) consecutive *k*-out-of-*n*:F system consists of *n* linearly (circularly) ordered components such that the system fails if and only if at least *k* consecutive components fail. For given *k* and *n*, one should note the difference between the linear and circular consecutive *k*-out-of-*n*:F systems. In the linear case, components 1 and *n* are not consecutive (or adjacent), while in the circular case they are consecutive. Consecutive *k*-out-of-*n*:F systems have been used to model some systems in industrial engineering including oil pipeline systems, telecommunication systems, vacuum systems in accelerators, etc. For more applications in industrial engineering, one may refer to Kuo and Zuo [12].

In recent decades, there have been a huge literature of research on reliability of consecutive *k*-out-of-*n* systems and related systems, under various assumptions. See Sfakianakis and Papastavridis [15], Fu and Koutras [11], Chao, Fu, and Koutras [4], Aki and Hirano [1], Eryilmaz [6–9], Eryilmaz, Kan, and Akici [10], and references

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therein. Stochastic comparisons of the lifetimes of consecutive k-out-of-n:F systems can be found in Boland and Samaniego [3], Eryilmaz [6], Navarro and Eryilmaz [13] and Salehi, Asadi, and Eryilmaz [14].

Let $T_{\mathrm{L},k|n}$ [$T_{\mathrm{C},k|n}$] denote the lifetime of a linear (circular) consecutive k-out-of-n:F system with n components having lifetimes $X_1, \ldots, X_n$, and let $T_{\mathrm{L},k|n+1}$ [$T_{\mathrm{C},k|n+1}$] denote the lifetime of a linear [circular] consecutive k-out-of-$(n+1)$:F system with $n+1$ components having lifetimes $X_1, \ldots, X_{n+1}$, where $X_1, \ldots, X_n, X_{n+1}$ are assumed to be independent. Boland and Samaniego [3] proved that

$$T_{\mathrm{L},k|n} \leq_{\mathrm{st}} T_{\mathrm{L},k+1|n} \quad \text{and} \quad T_{\mathrm{C},k|n} \leq_{\mathrm{st}} T_{\mathrm{C},k+1|n} \quad \text{for } k = 1, \ldots, n-1, \tag{1.1}$$

and

$$T_{\mathrm{L},k|n+1} \leq_{\mathrm{st}} T_{\mathrm{L},k|n} \quad \text{for } k = 1, \ldots, n. \tag{1.2}$$

Moreover, under the additional assumption that all X_i are identically distributed, they obtained that

$$T_{\mathrm{C},k|n+1} \leq_{\mathrm{st}} T_{\mathrm{C},k|n} \quad \text{for } k = 1, \ldots, n, \tag{1.3}$$

and pointed out that (1.3) is not true in general. Here, $\leq_{\mathrm{st}}$ denotes the *usual stochastic order*. The other two stochastic orders used in this paper are the *hazard rate order* ($\leq_{\mathrm{hr}}$) and the *likelihood ratio order* ($\leq_{\mathrm{lr}}$). The formal definitions of these orders and their properties can be found in Shaked and Shanthikumar [16]. In fact, (1.1) and (1.2) are trivial because, for a fixed set of n failure times, the time until there are k consecutive components failed is always less than or equal to the time until there are $k+1$ consecutive failures. Observe that having $k+1$ consecutive failures in $X_1, \ldots, X_n, X_{n+1}$ implies that one must have k consecutive failures in $X_1, \ldots, X_n$. It thus follows that

$$T_{\mathrm{L},k|n} \leq_{\mathrm{st}} T_{\mathrm{L},k+1|n+1} \quad \text{and} \quad T_{\mathrm{C},k|n} \leq_{\mathrm{st}} T_{\mathrm{C},k+1|n+1} \quad \text{for } k = 1, \ldots, n. \tag{1.4}$$

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For independent and identically distributed (i.i.d.) component lifetimes, Navarro and Eryilmaz [13] established that

$$T_{\mathrm{L},k|n+1} \leq_{\mathrm{hr}} T_{\mathrm{L},k|n} \quad \text{for } n \leq 2k \tag{1.5}$$

and

$$T_{\mathrm{L},k|n} \leq_{\mathrm{lr}} T_{\mathrm{L},k+1|n} \quad \text{for } k+1 < n \leq 2k, \tag{1.6}$$

strengthening (1.1) and (1.2) for linear consecutive k-out-of-n:F systems.

In this paper, we continue to investigate the stochastic ordering properties for linear (circular) consecutive k-out-of-n:F systems with independent components. In Section 2, a sufficient condition for (1.3) is given when the systems have independent and nonidentical components. In Section 3, the usual stochastic orderings in (1.1) and (1.3)–(1.5) are extended to the likelihood ratio orderings under some restrictions on (k, n) and the assumption that the lifetimes of components of the systems are i.i.d. Some counterexamples are given to illustrate that these orderings may not be true in general.

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2. USUAL STOCHASTIC ORDERINGS

Let $R(p_1,\ldots,p_n)$ denote the reliability of any coherent system with n components having respective survival probabilities p_i, $i=1,\ldots,n$. According to Barlow and Proschan [2], the reliability may be expressed as follows by making use of the path and/or cut sets of the system:

$$R(p_1,\ldots,p_n)=\sum_{P\in\mathcal{P}}\prod_{i\in P}p_i\prod_{j\notin P}q_j=1-\sum_{K\in\mathcal{K}}\prod_{i\notin K}p_i\prod_{j\in K}q_j \tag{2.1}$$

with $q_i=1-p_i$ for $i=1,\ldots,n$, where $\mathcal{P}$ and $\mathcal{K}$ denote the path sets and cut sets of the system, respectively.

As mentioned in Section 1, Boland and Samaniego [3] pointed out that (1.3) is not true when the distribution functions F_i of the components are not identical. Also they built a sufficient and necessary condition for (1.3) in this case. Denote by F_i the distribution function of X_i for each i. The sufficient and necessary condition can be written as follows:

$$T_{\mathrm{C},k|n+1}\le_{\mathrm{st}}T_{\mathrm{C},k|n}\iff\overline{F}_{n+1}(t)\le\theta_{k|n}(\overline{F}_1(t),\ldots,\overline{F}_n(t)),\qquad\forall\,t \tag{2.2}$$

where $\theta_{k|n}(p_1,\ldots,p_n)$ is a threshold parameter which takes the form

$$\theta_{k|n}(p_1,\ldots,p_n)=\frac{R_{\mathrm{C},k|n}(p_1,\ldots,p_n)-R_{\mathrm{C},k|n+1}(p_1,\ldots,p_n,0)}{R_{\mathrm{L},k|n}(p_1,\ldots,p_n)-R_{\mathrm{C},k|n+1}(p_1,\ldots,p_n,0)}. \tag{2.3}$$

Although the appearance of $\theta_{k|n}(p_1,\ldots,p_n)$ nicely establishes an equivalent characterization for (1.3), the threshold parameter is not intuitive and not easy to compute. A counterexample can be given to illustrate that $\theta_{k|n}(p_1,\ldots,p_n)$ is not monotone in each p_i.

We now present an intuitive sufficient condition for (1.3), and from which we can obtain an effective lower bound for the threshold parameter $\theta_{k|n}$.

PROPOSITION 2.1: *If* $p_{n+1}\le\min\{p_1,\ldots,p_n\}$, *then*

$$R_{\mathrm{C},k|n}(p_1,\ldots,p_n)\ge R_{\mathrm{C},k|n+1}(p_1,\ldots,p_n,p_{n+1}).$$

PROOF: Denote by $\mathcal{K}_1$ and $\mathcal{K}_2$ the classes of cut sets of the systems with reliability functions $R_{\mathrm{C},k|n}(p_1,\ldots,p_n)$ and $R_{\mathrm{C},k|n+1}(p_1,\ldots,p_{n+1})$, respectively. From (2.1), it follows that

$$R_{\mathrm{C},k|n}(p_1,\ldots,p_n)=1-\sum_{K\in\mathcal{K}_1}\prod_{i\in\Lambda_n\setminus K}p_i\prod_{j\in K}q_j, \tag{2.4}$$

$$R_{\mathrm{C},k|n+1}(p_1,\ldots,p_{n+1})=1-\sum_{K\in\mathcal{K}_2}\prod_{i\in\Lambda_{n+1}\setminus K}p_i\prod_{j\in K}q_j, \tag{2.5}$$

with $\Lambda_n=\{1,2,\ldots,n\}$ and $q_i=1-p_i$ for each i. Now, we consider the cut sets of the system with reliability function $R_{\mathrm{C},k|n}(p_1,\ldots,p_n)$. For each cut set K in $\mathcal{K}_1$, there

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exists a cut set $K_0 = \{j_1, j_2, \ldots, j_\tau\} \subseteq K$, where $j_1 < j_\tau$, and τ is large enough such that components $j_1, j_2, \ldots, j_\tau$ are consecutive and K_0 is the longest sequence of failures. Clearly, $\tau \geq k$. Two cases arise.

If $\{1, n\} \subseteq K_0$, define

$$K_1 = K \cup \{n+1\} \quad \text{and} \quad K_2 = (K \backslash \{j_\tau\}) \cup \{n+1\}.$$

Then, K_1 and K_2 are cut sets since $K_0 \cup \{n+1\}$ and $(K_0 \backslash \{j_\tau\}) \cup \{n+1\}$ are cut sets. Also, for different K, the K_2 are also different. For such a K,

$$\begin{aligned}
\prod_{i \in \Lambda_n \backslash K} p_i \prod_{j \in K} q_j &= \prod_{i \in \Lambda_{n+1} \backslash K_1} p_i \prod_{j \in K_1} q_j + \left(\prod_{i \in \Lambda_n \backslash K} p_i \prod_{j \in K} q_j \right) p_{n+1} \\
&= \prod_{i \in \Lambda_{n+1} \backslash K_1} p_i \prod_{j \in K_1} q_j + \left(\prod_{i \in \Lambda_{n+1} \backslash K_2} p_i \prod_{j \in K_2} q_j \right) \frac{p_{n+1}}{p_{j_\tau}} \frac{q_{j_\tau}}{q_{n+1}} \\
&\leq \prod_{i \in \Lambda_{n+1} \backslash K_1} p_i \prod_{j \in K_1} q_j + \prod_{i \in \Lambda_{n+1} \backslash K_2} p_i \prod_{j \in K_2} q_j, \qquad \textbf{(2.6)}
\end{aligned}$$

since $p_{n+1} \leq p_{j_\tau}$.

If $1 \notin K_0$ or $n \notin K_0$, define

$$K_1 = \{i_1, \ldots, i_m\} \quad \text{and} \quad K_2 = \{i_1, \ldots, i_m, n+1\}.$$

Then, $K_1, K_2 \in \mathcal{K}_2$ and

$$\prod_{i \in \Lambda_n \backslash K} p_i \prod_{j \in K} q_j = \prod_{i \in \Lambda_{n+1} \backslash K_1} p_i \prod_{j \in K_1} q_j + \prod_{i \in \Lambda_{n+1} \backslash K_2} p_i \prod_{j \in K_2} q_j. \qquad \textbf{(2.7)}$$

From (2.6) and (2.7), it follows that for each cut set $K \in \mathcal{K}_1$, there exist two cut sets $K_1, K_2 \in \mathcal{K}_2$ such that the probability corresponding to cut set K in (2.4) is less than or equal to the sum of probabilities corresponding to two cut sets K_1 and K_2 in (2.5). This means that

$$\sum_{K \in \mathcal{K}_1} \prod_{i \in \Lambda_n \backslash K} p_i \prod_{j \in K} q_j \leq \sum_{K \in \mathcal{K}_2} \prod_{i \in \Lambda_{n+1} \backslash K} p_i \prod_{j \in K} q_j,$$

implying $R_{C,k|n}(p_1, \ldots, p_n) \geq R_{C,k|n+1}(p_1, \ldots, p_{n+1})$ by (2.4) and (2.5). This completes the proof of the proposition. ■

Combining Proposition 2.1 with (2.2), the following corollary presents an effective lower bound for the threshold parameter $\theta_{k|n}$. Meanwhile, it generates Corollary 1 of Boland and Samaniego [3], which states that $\theta_{k|n}(p, \ldots, p) \geq p$ for all $p \in [0, 1]$.

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COROLLARY 2.2: *For a circular consecutive k-out-of-n:F system consisting of n independent components with respective reliabilities* $p_1, \ldots, p_n$, *it holds*

$$\theta_{k|n}(p_1, \ldots, p_n) \geq \min\{p_1, \ldots, p_n\}$$

for each $p_i \in [0, 1]$, *where* $\theta_{k|n}(p_1, \ldots, p_n)$ *is defined as in* (2.3).

Proposition 2.1 can be restated as the following theorem.

THEOREM 2.3: *Let* $T_{\mathrm{C},k|n}$ *denote the lifetime of a circular consecutive k-out-of-n:F system with component lifetimes* $X_1, \ldots, X_n$, *and let* $T_{\mathrm{C},k|n+1}$ *denote the lifetime of a circular consecutive k-out-of-*$(n+1)$*:F system with component lifetimes* $X_1, \ldots, X_{n+1}$, *where* $X_1, \ldots, X_{n+1}$ *are independent but not necessarily identically distributed. If* $X_{n+1} \leq_{\mathrm{st}} X_i$ *for* $i = 1, \ldots, n$, *then*

$$T_{\mathrm{C},k|n+1} \leq_{\mathrm{st}} T_{\mathrm{C},k|n} \quad \text{for } k = 1, \ldots, n.$$

3. LIKELIHOOD RATIO ORDERINGS

Computing the reliability of a linear (circular) consecutive k-out-of-n system has been widely studied in the literature, and most of the methods are basically related to recursive relationships. For $n \leq 2k$ $[n \leq 2k+1]$, the reliability of a linear (circular) consecutive k-out-of-n:F system can be explicitly expressed. Recall the notations in the previous section. Let $R_{\mathrm{L},k|n}(p_1, \ldots, p_n)$ and $R_{\mathrm{C},k|n}(p_1, \ldots, p_n)$ denote the reliabilities of the linear consecutive k-out-of-n:F system and the circular consecutive k-out-of-n:F system with lifetimes $T_{\mathrm{L},k|n}$ and $T_{\mathrm{C},k|n}$, respectively, where $p_i = \overline{F}_i(t)$, $i = 1, \ldots, n$. According to Sfakianakis and Papastavridis [15], we have

$$R_{\mathrm{L},k|n}(p_1, \ldots, p_n) = 1 - \sum_{i=1}^{n-k+1} \prod_{j=1}^{k} q_{i+j-1} + \sum_{i=1}^{n-k} \prod_{j=1}^{k+1} q_{i+j-1}, \quad k+1 \leq n \leq 2k, \tag{3.1}$$

$$R_{\mathrm{C},k|n}(p_1, \ldots, p_n) = 1 - \sum_{i=1}^{n} p_i \prod_{j=1}^{k} q_{[i+j]} - \prod_{i=1}^{n} q_i, \quad k+1 \leq n \leq 2k+1, \tag{3.2}$$

where $q_i = 1 - p_i$, $i = 1, \ldots, n$, and $[j+i] = j + i - n$ if $j + i > n$, otherwise $j + i$. It should be noted that the systems are actually parallel systems when $k = n$.

As a special case, if the lifetimes X_i of the components are i.i.d. with a common distribution function F, then

$$R_{\mathrm{L},k|n}(p, \ldots, p) = 1 - (n-k+1)q^k + (n-k)q^{k+1}, \qquad k+1 \leq n \leq 2k, \tag{3.3}$$

$$R_{\mathrm{C},k|n}(p, \ldots, p) = 1 - nq^k + nq^{k+1} - q^n, \qquad k+1 \leq n \leq 2k+1, \tag{3.4}$$

with $q = 1 - p = \overline{F}(t)$. Assume that the common distribution F is absolutely continuous with density function f. Then, the density functions of $T_{\mathrm{L},k|n}$ and $T_{\mathrm{C},k|n}$ are,

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respectively, given by

$$f_{\mathrm{L},k|n}(t) = f(t)\left[k(n-k+1)F^{k-1}(t) - (k+1)(n-k)F^{k}(t)\right], \qquad k+1 \le n \le 2k, \tag{3.5}$$

$$f_{\mathrm{C},k|n}(t) = f(t)\left[nkF^{k-1}(t) - n(k+1)F^{k}(t) + nF^{n-1}(t)\right], \qquad k+1 \le n \le 2k+1. \tag{3.6}$$

Under some restrictions on (k,n) and the assumption that the lifetimes of components of the systems are i.i.d., the following theorem partially extends the usual stochastic orderings in (1.1) and (1.3)–(1.5) to the likelihood ratio orderings. All the proofs are given in the Appendix.

THEOREM 3.1: *Let $T_{\mathrm{L},k|n}$ [$T_{\mathrm{C},k|n}$] denote the lifetime of a linear (circular) consecutive k-out-of-n system with i.i.d. component lifetimes. Then,*

$$\begin{aligned}
&T_{\mathrm{L},k|n+1} \le_{\mathrm{lr}} T_{\mathrm{L},k|n} \quad \text{for } k+1 \le n \le 2k-1,\\
&T_{\mathrm{L},k|n} \le_{\mathrm{lr}} T_{\mathrm{L},k+1|n+1} \quad \text{for } k+1 \le n \le 2k,\\
&T_{\mathrm{C},k|n} \le_{\mathrm{lr}} T_{\mathrm{C},k+1|n} \quad \text{for } k+2 \le n \le 2k+1,\\
&T_{\mathrm{C},k|n+1} \le_{\mathrm{lr}} T_{\mathrm{C},k|n} \quad \text{for } k+1 \le n \le 2k,\\
&T_{\mathrm{C},k|n} \le_{\mathrm{lr}} T_{\mathrm{C},k+1|n+1} \quad \text{for } k+1 \le n \le 2k+1.
\end{aligned}$$

The following example illustrates that hazard rate orderings (and hence likelihood ratio orderings) in Theorem 3.1 may not hold when the systems consist of heterogenous components.

Example 3.2: Let us consider some special consecutive k-out-of-n:F systems.

(i) $T_{\mathrm{L},4|7} \not\le_{\mathrm{hr}} T_{\mathrm{L},5|7}$. By (3.1), we have

$$\begin{aligned}
R_{\mathrm{L},4|7}(p_1,\ldots,p_7) &= 1 - (q_1q_2q_3q_4 + q_2q_3q_4q_5 + q_3q_4q_5q_6 + q_4q_5q_6q_7)\\
&\quad + (q_1q_2q_3q_4q_5 + q_2q_3q_4q_5q_6 + q_3q_4q_5q_6q_7),\\
R_{\mathrm{L},5|7}(p_1,\ldots,p_7) &= 1 - (q_1q_2q_3q_4q_5 + q_2q_3q_4q_5q_6 + q_3q_4q_5q_6q_7)\\
&\quad + (q_1q_2q_3q_4q_5q_6 + q_2q_3q_4q_5q_6q_7).
\end{aligned}$$

We compute the reliabilities for two vectors $(0.67, 0.1, 0.07, 0.5, 0.2, 0.3, 0.05)$ and $(0.67, 0.1, 0.07, 0.5, 0.2, 0.3, 0.95)$ and find that

$$\frac{R_{\mathrm{L},4|7}(0.67, 0.1, 0.07, 0.5, 0.2, 0.3, 0.05)}{R_{\mathrm{L},5|7}(0.67, 0.1, 0.07, 0.5, 0.2, 0.3, 0.05)} = 0.837744$$

and

$$\frac{R_{\mathrm{L},4|7}(0.67, 0.1, 0.07, 0.5, 0.2, 0.3, 0.95)}{R_{\mathrm{L},5|7}(0.67, 0.1, 0.07, 0.5, 0.2, 0.3, 0.95)} = 0.835018.$$

This means $T_{\mathrm{L},4|7} \not\le_{\mathrm{hr}} T_{\mathrm{L},5|7}$.

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(ii) $T_{\mathrm{C},4|7} \not\leq_{\mathrm{hr}} T_{\mathrm{C},5|7}$. By (3.2), we have

$$\begin{aligned} R_{\mathrm{C},4|7}(p_1,\ldots,p_7) &= 1-(p_1q_2q_3q_4q_5+p_2q_3q_4q_5q_6+p_3q_4q_5q_6q_7 \\ &\quad +p_4q_5q_6q_7q_1+p_5q_6q_7q_1q_2+p_6q_7q_1q_2q_3 \\ &\quad +p_7q_1q_2q_3q_4+q_1q_2q_3q_4q_5q_6q_7), \\ R_{\mathrm{C},5|7}(p_1,\ldots,p_7) &= 1-(p_1q_2q_3q_4q_5q_6+p_2q_3q_4q_5q_6q_7+p_3q_4q_5q_6q_7q_1 \\ &\quad +p_4q_5q_6q_7q_1q_2+p_5q_6q_7q_1q_2q_3+p_6q_7q_1q_2q_3q_4 \\ &\quad +p_7q_1q_2q_3q_4q_5+q_1q_2q_3q_4q_5q_6q_7). \end{aligned}$$

We compute the reliabilities for two vectors $(0.05, 0.9, 0.3, 0.1, 0.2, 0.3, 0.9)$ and $(0.9, 0.9, 0.3, 0.1, 0.2, 0.3, 0.9)$ and find that

$$\frac{R_{\mathrm{C},4|7}(0.05,0.9,0.3,0.1,0.2,0.3,0.9)}{R_{\mathrm{C},5|7}(0.05,0.9,0.3,0.1,0.2,0.3,0.9)} = 0.663748$$

and

$$\frac{R_{\mathrm{C},4|7}(0.9,0.9,0.3,0.1,0.2,0.3,0.9)}{R_{\mathrm{C},5|7}(0.9,0.9,0.3,0.1,0.2,0.3,0.9)} = 0.661539.$$

This means $T_{\mathrm{C},4|7} \not\leq_{\mathrm{hr}} T_{\mathrm{C},5|7}$.

(iii) $T_{\mathrm{L},4|7} \not\leq_{\mathrm{hr}} T_{\mathrm{L},4|6}$. By (3.1), we have

$$\begin{aligned} R_{\mathrm{L},4|6}(p_1,\ldots,p_6) &= 1-(q_1q_2q_3q_4+q_2q_3q_4q_5+q_3q_4q_5q_6) \\ &\quad +(q_1q_2q_3q_4q_5+q_2q_3q_4q_5q_6). \end{aligned}$$

We compute the reliabilities for two vectors $(0.7, 0.1, 0.1, 0.5, 0.2, 0.3, 0.07)$ and $(0.7, 0.1, 0.9, 0.5, 0.2, 0.3, 0.07)$ and find that

$$\frac{R_{\mathrm{L},4|7}(0.7,0.1,0.1,0.5,0.2,0.3,0.07)}{R_{\mathrm{L},4|6}(0.7,0.1,0.1,0.5,0.2,0.3)} = 0.958436$$

and

$$\frac{R_{\mathrm{L},4|7}(0.7,0.1,0.9,0.5,0.2,0.3,0.07)}{R_{\mathrm{L},4|6}(0.7,0.1,0.9,0.5,0.2,0.3)} = 0.755493.$$

This implies $T_{\mathrm{L},4|7} \not\leq_{\mathrm{hr}} T_{\mathrm{L},4|6}$.

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(iv) $T_{C,4|7} \not\leq_{hr} T_{C,4|6}$. By (3.2), we have

$$\begin{aligned} R_{C,4|6}(p_1,\ldots,p_6) = 1 - (&p_1q_2q_3q_4q_5 + p_2q_3q_4q_5q_6 + p_3q_4q_5q_6q_1 \\ &+ p_4q_5q_6q_1q_2 + p_5q_6q_1q_2q_3 + p_6q_1q_2q_3q_4 \\ &+ q_1q_2q_3q_4q_5q_6). \end{aligned}$$

We compute the reliabilities for two vectors $(0.1, 0.1, 0.3, 0.5, 0.2, 0.3, 0.9)$ and $(0.9, 0.1, 0.3, 0.5, 0.2, 0.3, 0.9)$ and find that

$$\frac{R_{C,4|7}(0.1, 0.1, 0.3, 0.5, 0.2, 0.3, 0.9)}{R_{C,4|6}(0.1, 0.1, 0.3, 0.5, 0.2, 0.3)} = 1.88776$$

and

$$\frac{R_{C,4|7}(0.9, 0.1, 0.3, 0.5, 0.2, 0.3, 0.9)}{R_{C,4|6}(0.9, 0.1, 0.3, 0.5, 0.2, 0.3)} = 1.03661.$$

This implies $T_{C,4|7} \not\leq_{hr} T_{C,4|6}$.

(v) $T_{L,4|6} \not\leq_{hr} T_{L,5|7}$. Using the expressions above, we compute the reliabilities for two vectors $(0.7, 0.8, 0.7, 0.1, 0.5, 0.4, 0.1)$ and $(0.7, 0.8, 0.7, 0.1, 0.5, 0.4, 0.9)$ and find that

$$\frac{R_{L,4|6}(0.7, 0.8, 0.7, 0.1, 0.5, 0.4, 0.1)}{R_{L,5|7}(0.7, 0.8, 0.7, 0.1, 0.5, 0.4, 0.1)} = 0.97599$$

and

$$\frac{R_{L,4|6}(0.7, 0.8, 0.7, 0.1, 0.5, 0.4, 0.9)}{R_{L,5|7}(0.7, 0.8, 0.7, 0.1, 0.5, 0.4, 0.9)} = 0.92405.$$

implying $T_{L,4|6} \not\leq_{hr} T_{L,5|7}$.

(vi) $T_{C,4|6} \not\leq_{hr} T_{C,5|7}$. Using the expressions above, we compute the reliabilities for two vectors $(0.7, 0.8, 0.7, 0.1, 0.5, 0.4, 0.1)$ and $(0.7, 0.8, 0.7, 0.1, 0.5, 0.4, 0.9)$ and find that

$$\frac{R_{C,4|6}(0.7, 0.8, 0.7, 0.1, 0.5, 0.4, 0.1)}{R_{C,5|7}(0.7, 0.8, 0.7, 0.1, 0.5, 0.4, 0.1)} = 0.97592$$

and

$$\frac{R_{C,4|6}(0.7, 0.8, 0.7, 0.1, 0.5, 0.4, 0.9)}{R_{C,5|7}(0.7, 0.8, 0.7, 0.1, 0.5, 0.4, 0.9)} = 0.86943.$$

implying $T_{C,4|6} \not\leq_{hr} T_{C,5|7}$.

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References

1. Aki, S. & Hirano, K. (1997). Lifetime distributions of consecutive k-out-of-n:F systems. *Nonlinear Analysis, Theorey, Methods and Applications* 30: 555–562.
2. Barlow, R.E. & Proschan, F. (1981). *Statistical theorey of reliability and life testing: probability models.* Silver Spring, MD: To Begin With.
3. Boland, P.J. & Samaniego, F.J. (2004). Stochastic ordering results for consecutive k-out-of-n systems. *IEEE Transactions on Reliability* 53: 7–10.
4. Chao, M.T., Fu, J.C., & Koutras, M.V. (1995). Survey of reliability studies of consecutive k-out- of-n:F and related systems. *IEEE Transactions on Reliability* 44: 120–127.
5. Chiang, D.T. & Niu, S.C. (1981). Reliability of consecutive k-out-of-n:F systems. *IEEE Transactions on Reliability* R30: 87–89.
6. Eryilmaz, S. (2007). On the lifetime distribution of consecutive k-out-of-n:F system. *IEEE Transactions on Reliability* 56: 35–39.
7. Eryilmaz, S. (2009). Reliability properties of consecutive k-out-of-n systems of arbitrily dependent components. *Reliability Engineering and System Safety* 94: 350–356.
8. Eryilmaz, S. (2011). Circular consecutive k-out-of-n systems with exchangeable dependent components. *Journal of Statistical Planning and Inference* 141: 725–733.
9. Eryilmaz, S. (2011). Review of recent advances in reliability of consecutive k-out-of-n and related systems. *Journal of Risk and Reliability* 224: 225–237.
10. Eryilmaz, S., Kan, C., & Akici, F. (2009). Consecutive k-within-m-out-of-n:F system with exchangeable components. *Naval Research Logistics* 56: 503–510.
11. Fu, J.C. & Koutras, M. (1994). Distribution theory of runs: A Markov chain approach. *Journal of American Statistical Association* 89: 1050–1058.
12. Kuo, W. & Zuo, M.J. (2003). *Optimal Reliability Modeling, Principles and Applications*. New York: Wiley.
13. Navarro, J. & Eryilmaz, S. (2007). Mean residual lifetimes of consecutive-k-out-of-n systems. *Journal of Applied Probability* 44: 82–98.
14. Salehi, E.T., Asadi, M., & Eryilmaz, S. (2011). On the mean residual lifetime of consecutive k-out-of-n systems. *Test* 21: 93–115.
15. Sfakianakis, M.E. & Papastavridis, S.G. (1993). Reliability of a general consecutive k-out-of-n:F system. *IEEE Transactions Reliability* 42: 491–496.
16. Shaked, M. & Shanthikumar, J.G. (2007). *Stochastic orders*. New York: Springer.

APPENDIX

PROOF OF THEOREM 3.1: According to (3.5) and (3.6), we need to prove that each of the functions

$$\frac{f_{\mathrm{L},k|n+1}(t)}{f_{\mathrm{L},k|n}(t)} = \frac{k(n-k+2)F^{k-1}(t) - (k+1)(n-k+1)F^{k}(t)}{k(n-k+1)F^{k-1}(t) - (k+1)(n-k)F^{k}(t)},$$
$$k+1 \le n \le 2k-1,$$

$$\frac{f_{\mathrm{L},k|n}(t)}{f_{\mathrm{L},k+1|n+1}(t)} = \frac{k(n-k+1)F^{k-1}(t) - (k+1)(n-k)F^{k}(t)}{(k+1)(n-k+1)F^{k}(t) - (k+2)(n-k)F^{k+1}(t)},$$
$$k+1 \le n \le 2k,$$

$$\frac{f_{\mathrm{C},k|n}(t)}{f_{\mathrm{C},k+1|n}(t)} = \frac{nkF^{k-1}(t) - n(k+1)F^{k}(t) + nF^{n-1}(t)}{n(k+1)F^{k}(t) - n(k+2)F^{k+1}(t) + nF^{n-1}(t)},$$
$$k+2 \le n \le 2k+1,$$

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$$\frac{f_{C,k|n+1}(t)}{f_{C,k|n}(t)} = \frac{(n+1)kF^{k-1}(t) - (n+1)(k+1)F^k(t) + (n+1)F^n(t)}{nkF^{k-1}(t) - n(k+1)F^k(t) + nF^{n-1}(t)},$$
$$k+1 \le n \le 2k,$$
$$\frac{f_{C,k|n}(t)}{f_{C,k+1|n+1}(t)} = \frac{nkF^{k-1}(t) - n(k+1)F^k(t) + nF^{n-1}(t)}{(n+1)(k+1)F^k(t) - (n+1)(k+2)F^{k+1}(t) + (n+1)F^n(t)},$$
$$k+1 \le n \le 2k+1$$

is decreasing in $t \in \Re_+$. Clearly, it is equivalent to prove that each of

$$\Delta_1(q) = \frac{k(n-k+2) - (k+1)(n-k+1)q}{k(n-k+1) - (k+1)(n-k)q}, \qquad k+1 \le n \le 2k-1,$$
$$\Delta_2(q) = \frac{k(n-k+1) - (k+1)(n-k)q}{(k+1)(n-k+1)q - (k+2)(n-k)q^2}, \qquad k+1 \le n \le 2k,$$
$$\Delta_3(q) = \frac{k - (k+1)q + q^{n-k}}{(k+1)q - (k+2)q^2 + q^{n-k}}, \qquad k+2 \le n \le 2k+1,$$
$$\Delta_4(q) = \frac{k - (k+1)q + q^{n-k+1}}{k - (k+1)q + q^{n-k}}, \qquad k+1 \le n \le 2k,$$
$$\Delta_5(q) = \frac{k - (k+1)q + q^{n-k}}{(k+1)q - (k+2)q^2 + q^{n-k+1}}, \qquad k+1 \le n \le 2k+1$$

is decreasing in $q \in [0,1]$.

(i),(ii) The proofs are very similar to that of Lemma 3.4 of Navarro and Eryilmaz [13].

(iii) It suffices to prove that

$$\Delta_{31}(q) = \frac{k - (k+1)q + q^{n-k}}{(k+1) - (k+2)q + q^{n-k-1}}$$

is decreasing in $q \in [0,1]$. Taking derivative of $\Delta_{31}(q)$ with respect to q and after some simplifications, we have

$$\begin{aligned}
\Delta'_{31}(q) &\overset{\text{sgn}}{=} \left(-(k+1) + (n-k)q^{n-k-1}\right)\left((k+1) - (k+2)q + q^{n-k-1}\right) \\
&\quad - \left(-(k+2) + (n-k-1)q^{n-k-2}\right)\left(k - (k+1)q + q^{n-k}\right) \\
&= -1 + 2(k+1)(n-k-1)q^{n-k-1} - (k+2)(n-k-1)q^{n-k} \\
&\quad - k(n-k-1)q^{n-k-2} + q^{2n-2k-2} \\
&\overset{\text{def}}{=} -\Delta_{32}(q),
\end{aligned}$$

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where $\stackrel{\text{sgn}}{=}$ denotes equality in sign. We now prove $\Delta_{32}(q) \geq 0$. Since $\Delta_{32}(q)$ can be rewritten as

$$\begin{aligned}\Delta_{32}(q) &= 1 - 2(k+1)(n-k-1)q^{n-k-1} + (k+1)(n-k-1)q^{n-k} \\ &\quad + (k+1)(n-k-1)q^{n-k-2} \\ &\quad + (n-k-1)q^{n-k} - (n-k-1)q^{n-k-2} - q^{2n-2k-2} \\ &= 1 + (k+1)(n-k-1)q^{n-k-2}(1-q)^2 + (n-k-1)q^{n-k} \\ &\quad - (n-k-1)q^{n-k-2} - q^{2n-2k-2},\end{aligned}$$

and $(k+1)(n-k-1)q^{n-k-2}(1-q)^2 \geq 0$, we have

$$\Delta_{32}(q) \geq 1 + (n-k-1)q^{n-k} - (n-k-1)q^{n-k-2} - q^{2n-2k-2} \stackrel{\text{def}}{=} \Delta_{33}(q).$$

Observe that $\Delta_{33}(0) = 1$ and $\Delta_{33}(1) = 0$, so $\Delta_{33}(q) \geq 0$ if

$$\Delta'_{33}(q) = (n-k-1)q^{n-k-3}\left[(n-k)q^2 - (n-k-2) - 2q^{n-k}\right] \leq 0$$

for $n-k \geq 3$ ($n-k=2$ is trivial), which can be obtained by repeated application of this method. Thus, $\Delta_{33}(q) \geq 0$ and hence $\Delta_{32}(q) \geq 0$.

(iv) It suffices to prove that

$$\Delta_{41}(q) = \frac{q^{n-k} - q^{n-k+1}}{k - (k+1)q + q^{n-k}}$$

is increasing in $q \in [0,1]$, which is true if

$$\Delta_{42}(q) = \frac{1-q}{k - (k+1)q + q^{n-k}}$$

is increasing in $q \in [0,1]$. Taking derivative with respect to q, we have

$$\begin{aligned}\Delta'_{42}(q) &\stackrel{\text{sgn}}{=} -\left(k - (k+1)q + q^{n-k}\right) - (1-q)\left(-(k+1) + (n-k)q^{n-k-1}\right) \\ &= 1 - (n-k)q^{n-k-1} + (n-k-1)q^{n-k} \\ &\stackrel{\text{def}}{=} \Delta_{43}(q).\end{aligned}$$

Since $\Delta_{43}(0) = 1$, $\Delta_{43}(1) = 0$, and

$$\Delta'_{43}(q) = (n-k)(n-k-1)q^{n-k-1} - (n-k)(n-k-1)q^{n-k-2} \leq 0$$

for $n-k \geq 2$ ($n-k=1$ is trivial), we have $\Delta_{43}(q) \geq 0$ and hence $\Delta_{42}(q)$ is increasing in $q \in [0,1]$.

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(v) It suffices to prove that

$$\Delta_{51}(q) = \frac{k-(k+1)q+q^{n-k}}{(k+1)-(k+2)q+q^{n-k}}$$

is decreasing in $q \in [0,1]$. Taking derivative with respect to q and after some simplifications, we have

$$\begin{aligned}\Delta'_{51}(q) &\overset{\text{sgn}}{=} \left(-(k+1)+(n-k)q^{n-k-1}\right)\left((k+1)-(k+2)q+q^{n-k}\right)\\ &\quad -\left(-(k+2)+(n-k)q^{n-k-1}\right)\left(k-(k+1)q+q^{n-k}\right)\\ &= -1-(n-k-1)q^{n-k}+(n-k)q^{n-k-1}\\ &\overset{\text{def}}{=} \Delta_{52}(q).\end{aligned}$$

Since $\Delta_{52}(0) = -1$, $\Delta_{52}(1) = 0$, and

$$\Delta'_{52}(q) = (n-k)(n-k-1)q^{n-k-2} - (n-k)(n-k-1)q^{n-k-1} \geq 0$$

for $n-k \geq 2$ ($n-k=1$ is trivial), then $\Delta_{52}(q) \leq 0$. This completes the proof of the theorem. ■

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