Ergod. Th. & Dynam. Sys. (2011), **31**, 1081–1094 © *Cambridge University Press, 2010*
doi:10.1017/S0143385710000350

On quasi-compact Markov nets

WOJCIECH BARTOSZEK† and NAZIFE ERKURŞUN‡

† *Department of Mathematics, Gdańsk University of Technology, ul. Narutowicza 11/12, 80-233 Gdańsk, Poland*
(e-mail: bartowk@mifgate.mif.pg.gda.pl)

‡ *Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey*
(e-mail: erkursun@metu.edu.tr)

(*Received* 5 *August* 2009 *and accepted in revised form* 21 *April* 2010)

Abstract. We extend a theorem of Lotz, which says that any Markov operator T acting on $C(X)$ such that T^* is mean ergodic and all invariant measures have non-meager supports must be quasi-compact, to Lotz–Räbiger nets.

1. *Introduction*

Of great interest in the ergodic theory of operators (or one-parameter operator semigroups) is the asymptotic behavior of iterates. Limits of positive operators acting on Banach lattices have attracted special attention. The interesting properties and regular asymptotic behavior of iterates hold also for quasi-compact Markov operators (or operator semigroups). We refer the reader to [**1–3, 15–18**] for more details; here we merely mention that the Cesàro means $T_n = (I + P + \cdots + P^{n-1})/n$ of a single Markov operator P converge in the operator norm to a finite-dimensional (Markovian) projection if and only if P is quasi-compact [**14–16**]. The main purpose of this paper is to study conditions that would imply quasi-compactness of Markov nets. Before proceeding to Markov nets, let us briefly outline the notion of abstract ergodic nets. Given a Banach space $(\mathfrak{X}, \|\cdot\|)$, let $\mathcal{L}(\mathfrak{X})$ be the Banach algebra of all bounded linear operators on $\mathfrak{X}$ (where $I = I_{\mathfrak{X}}$ stands for the identity operator). A family $\Theta = (T_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{L}(\mathfrak{X})$, indexed by a directed set $\Lambda = (\Lambda, \preceq)$, is called an *operator net* (see [**19, 20, 22**]).

Definition 1. A net $\Theta = (T_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{L}(\mathfrak{X})$ is called a *Lotz–Räbiger net*, or $(\mathcal{L}, \mathcal{R})$ net for short, if the following conditions hold:

(LR1) Θ is equi-continuous;

(LR2) $\lim_{\lambda\to\infty} \|T_\lambda \circ (T_\tau - I)x\| = 0$ for all $x \in \mathfrak{X}$ and every fixed $\tau \in \Lambda$;

(LR3) $\lim_{\lambda\to\infty} \|(T_\tau - I) \circ T_\lambda x\| = 0$ for all $x \in \mathfrak{X}$ and every fixed $\tau \in \Lambda$.

Remark 1. With a slight abuse of notation (in the above definition and elsewhere in this paper), $\lim_{\lambda\to\infty}$ is understood to mean the limit along a net. In particular, the statement

$$\lim_{\lambda\to\infty} \|T_\lambda \circ (T_\tau - I)\, x\| = 0$$

https://doi.org/10.1017/S0143385710000350 Published online by Cambridge University Press

means that for every $\varepsilon > 0$ there exists $\lambda_\varepsilon \in \Lambda$ such that for all $\lambda \succeq \lambda_\varepsilon$ we have

$$\|T_\lambda \circ (T_\tau - I)\, x\| < \varepsilon.$$

Standard examples of $(\mathcal{L}, \mathcal{R})$ nets are the Cesàro means

$$T_n = \frac{1}{n} \sum_{k=0}^{n-1} P^k$$

of a linear contraction P, where $\Lambda = \mathbb{N}$, or

$$T_t = \frac{1}{t} \int_0^t P(s)\, ds$$

for a one-parameter semigroup of contractions $\{P(s)\}_{s\in\mathbb{R}_+}$, where $\Lambda = \mathbb{R}_+$. The ergodic properties of generalized averages such as $(\mathcal{L}, \mathcal{R})$ nets have been studied in many papers (for example, [**5, 6, 8, 22, 24**]). Obviously, the notion of $(\mathcal{L}, \mathcal{R})$ nets is much broader than Cesàro averaging and, in particular, encompasses Cesàro means of higher orders for both discrete and continuous semigroups. Some non-standard examples of $(\mathcal{L}, \mathcal{R})$ nets (including commutative convex semigroups of contractions with suitable Λ) and their unified theory are treated in several recent papers by Emel'yanov (see [**6, 8**]). Without loss of generality, we shall assume that the nets considered contain the identity operator I. Let us recall the following definition.

Definition 2. Let X be a topological Hausdorff compact space. A linear operator $T : C(X) \to C(X)$ is said to be *Markov* if $T\mathbf{1} = \mathbf{1}$ and $Tf \geq 0$ for all non-negative $f \in C(X)$.

This paper is devoted to ergodic properties of $(\mathcal{L}, \mathcal{R})$ nets of Markov operators. We now give two examples of $(\mathcal{L}, \mathcal{R})$ Markov nets whose asymptotic structure differs from that of classical Cesàro sequences.

Example 1. Let $\ell^2(\kappa)$ be the standard Hilbert space (see [**4**]), where κ is a limit ordinal (of course, $\ell^2(\kappa)$ is separable if and only if $\kappa \prec \omega_1$). The vectors $e_\iota(j) = \delta_{\iota,j}$, for $\iota \prec \kappa$, form a complete orthonormal basis for $\ell^2(\kappa)$. Let Λ be the set of all finite subsets (including the empty set) of κ with the natural ordering given by $\lambda_1 \preceq \lambda_2$ if and only if $\lambda_1 \subseteq \lambda_2$. Given $\lambda = \{\xi_1, \ldots, \xi_n\} \in \Lambda$, define T_λ to be the orthogonal projection onto $\mathrm{lin}\{e_{\xi_1}, \ldots, e_{\xi_n}\}^\perp$. Obviously, the T_λ form an $(\mathcal{L}, \mathcal{R})$ net. Notice that for each $\eta \in \ell^2(\kappa)$, we have $\lim_{\lambda\to\infty} \|T_\lambda \eta\|_2 = 0$. Moreover, given η, there exists an increasing sequence $\lambda_1 \preceq \lambda_2 \preceq \cdots$ (which depends on η) such that the ordinary limit $\lim_{n\to\infty} \|T_{\lambda_n} \eta\|_2$ is equal to zero. However, if κ is uncountable, then there exists no subsequence $\lambda_1 \preceq \lambda_2 \preceq \cdots$ that is universally good for all $\eta \in \ell^2(\kappa)$. Clearly, the net under consideration can be lifted up to the Banach lattice $C(X)$ of all (real-valued) continuous functions, where X stands for the unit (closed) ball of $\ell^2(\kappa)$ endowed with weak* (or weak) topology. Specifically, define $\mathbf{T}_\lambda f(\eta) = f(T_\lambda \eta)$; it then follows that $\lim_{\lambda\to\infty} \mathbf{T}_\lambda f(\eta) = f(0)$ pointwise (a close look at the structure of weakly continuous functions gives that the convergence $\lim_{\lambda\to\infty} \mathbf{T}_\lambda f = f(0)\mathbf{1}$ is uniform on X, i.e. in the sup norm).

Example 2. Consider the order interval $X = \{\xi : 0 \preceq \xi \preceq \omega_1\}$, where ω_1 is the first uncountable ordinal. Clearly (see, e.g., [**9**]), X equipped with the topology generated by the basis $\{(\iota_1, \iota_2] : 0 \preceq \iota_1 \prec \iota_2 \preceq \omega_1\}$ is a Hausdorff compact space. As usual, $\mathfrak{X} = C(X)$ denotes the Banach space of all continuous functions on X with the sup norm. Let us again

https://doi.org/10.1017/S0143385710000350 Published online by Cambridge University Press

take Λ to be the family of all finite (non-empty) subsets of $[0, \omega) = \mathbb{N}$ and define

$$T_\lambda f(\iota) = \begin{cases} \dfrac{1}{\mathrm{card}(\lambda)} \displaystyle\sum_{j\in\lambda} f(\iota + j) & \text{if } \iota \text{ is not a limit ordinal,} \\ f(\iota) & \text{if } \iota \text{ is a limit ordinal.} \end{cases} \qquad (\bullet)$$

The T_λ are positive linear contractions on $C(X)$ satisfying $T_\lambda \mathbf{1} = \mathbf{1}$; they form an $(\mathcal{L}, \mathcal{R})$ net. For each $f \in \mathfrak{X}$, $\lim_{\lambda\to\infty} T_\lambda f = \overline{f}$ exists in the sup norm. (For ι that are not limit ordinals, we have $\overline{f}(\iota) = f(\xi_\iota)$ where $\xi_\iota = \min\{\xi \succ \iota : \xi \text{ is a limit ordinal}\}$.) We now modify $(\bullet)$ by introducing

$$S_\lambda f(\iota) = \frac{1}{\mathrm{card}(\lambda)} \sum_{j\in\lambda} f(\iota - j) \qquad (\diamond)$$

(suppose $\iota = \eta_\iota + k_\iota$ with $k_\iota \in \mathbb{N}$ such that $k_\iota \le j$; then set $\iota - j = \eta_\iota$ if η_ι is a limit ordinal supporting ι from below, or set $\iota - j = 0$ if $\eta_\iota = 0$). Clearly,

$$\lim_{\lambda\to\infty} S_\lambda f(\iota) = \overline{f}(\iota) = f(\eta_\iota)$$

exists pointwise, but the limit function is not continuous in general.

Recall that a power-bounded $T \in \mathcal{L}(\mathfrak{X})$ is said to be *mean ergodic* if its Cesàro means T_n converge in the strong operator topology. In general, one might ask when (given an $(\mathcal{L}, \mathcal{R})$ net) the limit $\lim_{\lambda\to\infty} T_\lambda x$ exists (in the norm) for any $x \in \mathfrak{X}$. It was proved in [**8**, Proposition 1.2] that the limit $\lim_{\lambda\to\infty} T_\lambda x$ exists if and only if there is a decomposition

$$\mathfrak{X} = \mathfrak{X}_\Theta \oplus \overline{\mathrm{lin} \bigcup_{\lambda\in\Lambda} (I - T_\lambda)\mathfrak{X}},$$

where $\mathfrak{X}_\Theta = \{x \in \mathfrak{X} : T_\lambda x = x \ \forall \lambda \in \Lambda\}$ denotes the manifold of all Θ-invariant elements. An Eberlein-like theorem (cf. [**8**, Theorem 1.2] and [**22**]) is valid, and the convergence holds if the net $(T_\lambda x)_{\lambda\in\Lambda}$ has a weak cluster point as $\lambda \to \infty$. Finally (see [**8**, Theorem 3.1]), the net version of Sine's criterion in [**25**] holds; so we have that an $(\mathcal{L}, \mathcal{R})$ net Θ is strong operator convergent if and only if $\mathfrak{X}_\Theta$ separates $\mathfrak{X}^*_{\Theta^*}$.

Definition 3. An $(\mathcal{L}, \mathcal{R})$ net $\Theta = (T_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{L}(\mathfrak{X})$ is said to be *ergodic* if for each $x \in \mathfrak{X}$ we have $\lim_{\lambda\to\infty} \|T_\lambda x - Qx\| = 0$ (of course, Q is a linear projection on $\mathfrak{X}_\Theta$).

We have already observed that the nets of Markov operators defined in Example 1 and $(\bullet)$ of Example 2 are ergodic, while the Markov net defined by $(\diamond)$ in Example 2 is not ergodic. It should be pointed out that ergodicity of Θ does not imply ergodicity of Θ^*. Even worse, Θ^* is not necessarily an $(\mathcal{L}, \mathcal{R})$ net in general. Indeed, consider the shift operator $\sigma : \ell^2 \to \ell^2$ defined by $\sigma(\eta_1, \eta_2, \ldots) = (\eta_2, \eta_3, \ldots)$. Clearly, σ is a contraction, and for each $\eta \in \ell^2$ one has $\lim_{n\to\infty} \|\sigma^n(\eta)\| = 0$. It follows that $\Theta = (\sigma^n)_{n\ge0}$ forms an ergodic $(\mathcal{L}, \mathcal{R})$ net. On the other hand, the adjoint family $\Theta^* = (\sigma^{*n})_{n\ge0}$, where $\sigma^*(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots)$, does not satisfy condition (LR2), since

$$\|\sigma^{*n}(I - \sigma^{*k})\xi\|^2 = \|\xi - \sigma^{*k}\xi\|^2 > 0$$

for all $n \ge 0$ and $\xi \ne 0$. The family Θ^* is therefore not an $(\mathcal{L}, \mathcal{R})$ net.

This paper is devoted to studying the uniform ergodicity of Markov $(\mathcal{L}, \mathcal{R})$ nets. Before going into details, we recall the following definition (see [**22**]).

https://doi.org/10.1017/S0143385710000350 Published online by Cambridge University Press

Definition 4. A net $\Theta = (T_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{L}(\mathfrak{X})$ is called a *uniform Lotz–Räbiger net* if it satisfies the following conditions:

(ULR1) Θ is equi-continuous;

(ULR2) $\lim_{\lambda\to\infty} \|T_\lambda \circ (T_\tau - I)\| = 0$ for every $\tau \in \Lambda$;

(ULR3) $\lim_{\lambda\to\infty} \|(T_\tau - I) \circ T_\lambda\| = 0$ for every $\tau \in \Lambda$.

Example 3. Given a linear contraction $T \in \mathcal{L}(\mathfrak{X})$, the Cesàro means T_n satisfy

$$\|(I - T)T_n\| = \|T_n(I - T)\| \leq \frac{2}{n}.$$

A standard argument gives that for each fixed $\tau \in \mathbb{N}$,

$$\|(I - T_\tau)T_n\| = \|T_n(I - T_\tau)\| \leq \frac{2\tau}{n} \to 0.$$

Hence, the family of Cesàro means $\{T_n : n = 1, 2, \ldots\}$ is a uniform $(\mathcal{L}, \mathcal{R})$ net.

Definition 5. An $(\mathcal{L}, \mathcal{R})$ net $\Theta = (T_\lambda)_{\lambda\in\Lambda} \subseteq \mathcal{L}(\mathfrak{X})$ is said to be *uniformly ergodic* if there exists a linear projection $Q \in \mathcal{L}(\mathfrak{X})$ such that

$$\lim_{\lambda\to\infty} \|T_\lambda - Q\| = 0.$$

It is clear that uniform ergodicity implies ergodicity. However, the reverse implication does not, in general, hold.

LEMMA 1. *Each uniformly ergodic* $(\mathcal{L}, \mathcal{R})$ *net* Θ *is a uniform* $(\mathcal{L}, \mathcal{R})$ *net.*

Proof. It is enough to verify conditions (ULR2) and (ULR3). Since Θ converges uniformly to a linear projection Q and the range of Q is $\mathfrak{X}_\Theta$, it follows that for every $x \in \mathfrak{X}$ one has

$$(I - T_\tau)Qx = Q(I - T_\tau)x = 0 \quad \text{for all } \tau \in \Lambda.$$

Now we obtain (ULR2) and (ULR3) as follows:

$$\begin{aligned}\lim_{\lambda\to\infty} \|T_\lambda \circ (T_\tau - I)\| &= \lim_{\lambda\to\infty} \|T_\lambda \circ (T_\tau - I) - Q \circ (T_\tau - I)\| \\ &= \lim_{\lambda\to\infty} \|(T_\lambda - Q) \circ (T_\tau - I)\| \\ &\leq \lim_{\lambda\to\infty} \|T_\lambda - Q\|\|T_\tau - I\| = 0\end{aligned}$$

and

$$\begin{aligned}\lim_{\lambda\to\infty} \|(T_\tau - I) \circ T_\lambda\| &= \lim_{\lambda\to\infty} \|(T_\tau - I) \circ T_\lambda - (T_\tau - I) \circ Q\| \\ &= \lim_{\lambda\to\infty} \|(T_\tau - I) \circ (T_\lambda - Q)\| \\ &\leq \lim_{\lambda\to\infty} \|T_\tau - I\|\|T_\lambda - Q\| = 0.\end{aligned}$$

Thus Θ is a uniform $(\mathcal{L}, \mathcal{R})$ net. □

We focus on the question of which uniformly ergodic nets tend to finite-dimensional projections.

Definition 6. A power-bounded linear operator $T \in \mathcal{L}(\mathfrak{X})$ is said to be *quasi-compact* if there exists a compact operator $K \in \mathcal{L}(\mathfrak{X})$ such that $\lim_{n\to\infty} \|T^n - K\| < 1$ for some $n \geq 1$.

https://doi.org/10.1017/S0143385710000350 Published online by Cambridge University Press

Note that for any Banach space $\mathfrak{X}$, quasi-compactness can be equivalently defined as $\lim_{n\to\infty} \operatorname{dist}(T^n, \mathcal{K}) = 0$, where $\mathcal{K}$ stands for the ideal of compact operators on $\mathfrak{X}$. Quasi-compact operators (or one-parameter operator semigroups) and their asymptotic properties have been thoroughly studied.

For positive linear contractions, it is well known that quasi-compactness is equivalent to norm asymptotic periodicity [**3**]. Here, much stronger asymptotic behavior holds, namely: for some natural d the iterates T^{nd} (rather than the Cesàro averages T_n) converge to a finite-dimensional projection.

In his recent paper [**6**], Emel'yanov proposed defining quasi-compact operator nets as those which are asymptotically uniformly equivalent to an operator net consisting of compact operators. (As far as the authors are aware, Emel'yanov's manuscript and his development of the notion of abstract $(\mathcal{L}, \mathcal{R})$ quasi-compactness had not reached a final stage at the time of this writing.)

Combining Yosida and Kakutani's result from [**27**] with Lin's characterization in [**16, 17**], we have that the Cesàro means T_n of a Markov operator T converge in the operator norm to a finite-dimensional projection if and only if T is quasi-compact. It is worth adding that in the case of continuous-time Markov semigroups, the convergence of Cesàro averages

$$(T_t)_{t\geq 0} = \left(\frac{1}{t}\int_0^t \mathbf{P}(s)\, ds\right)_{t\geq 0}$$

to a finite-dimensional projection does not, in general, imply that there exist a compact operator K and an $s \geq 0$ such that $\|\mathbf{P}(s) - K\| < 1$. To see this, consider $\mathbf{P}(s) : C(\mathbb{T}) \to C(\mathbb{T})$ where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is a one-dimensional torus and $\mathbf{P}(s) f(z) := f(z \cdot \exp(2\pi i s))$. Clearly, for each s the operator $\mathbf{P}(s)$ is a Markov invertible isometry (in particular, $\mathbf{P}(s)$ is far from $\mathcal{K}$). Nevertheless, the Cesàro averages T_t (which form an $(\mathcal{L}, \mathcal{R})$ net) are norm-convergent, with $\lim_{t\to\infty} T_t = E$ where

$$E(f) = \int_{\mathbb{T}} f(z)\, d\lambda(z)$$

is a one-dimensional projection on constants (here λ denotes the normalized Haar measure on $\mathbb{T}$). Here we distinguish a semigroup $\{\mathbf{P}(s)\}_{s\geq 0}$, which is not quasi-compact and does not form an $(\mathcal{L}, \mathcal{R})$ net, from its Cesàro means $(T_t)_{t\geq 0}$, which are quasi-compact. Similar examples also arise in more general settings. However, in the case of abstract $(\mathcal{L}, \mathcal{R})$ nets, which are not formed as averages over any particular semigroup, we need not worry about ambiguity. Following a characterization of quasi-compactness due to Lin (see [**17**]), we adopt the definition below.

Definition 7. An $(\mathcal{L}, \mathcal{R})$ net $\Theta = (T_\lambda)_{\lambda\in\Lambda}$ of Markov operators on $C(X)$ is said to be *quasi-compact* if it is uniformly ergodic and the limit projection $Q = \lim_{\lambda\to\infty} T_\lambda$ is finite-dimensional.

In the special case where T is a Markov operator on $C(X)$, it appears that T is quasi-compact if the adjoint Cesàro means T_n^* are ergodic and all invariant measures have non-meager supports (see [**1, 2, 18**]). The main goal of this paper is to extend the above characterization of quasi-compactness to $(\mathcal{L}, \mathcal{R})$ Markovian nets.

Let us denote by $M(X)$ the dual space $C(X)^*$, which may be identified with the Banach lattice of all bounded signed and Radon (regular) measures on X with the total

https://doi.org/10.1017/S0143385710000350 Published online by Cambridge University Press

variation norm $\|\cdot\|$. The convex and weak*-compact set of probabilistic Radon measures on X is denoted by $P(X)$. Given an $(\mathcal{L}, \mathcal{R})$ net Θ of Markov operators on $C(X)$, a probability measure $\mu \in M(X)$ is said to be Θ-invariant if $T_\lambda^* \mu = \mu$ for every $\lambda \in \Lambda$. Clearly, the set of all invariant probabilities (denoted by $P_\Theta(X)$) is a non-empty weak*-compact and convex subset of $P(X)$. To see this, note that even though the $(\mathcal{L}, \mathcal{R})$ nets are not necessarily abelian (and hence the Kakutani theorem is not directly applicable), it follows from property (LR2) that the limit points $\overline{\mu}$ of nets $\{T_\lambda^* \mu\}_{\lambda \in \Lambda}$ are Θ-invariant. As usual, we say that a non-empty subset A of X is absorbing (with respect to a family $\Theta = (T_\lambda)_{\lambda \in \Lambda}$ of Markov operators) if for all $x \in A$ and $\lambda \in \Lambda$ we have $T_\lambda^* \delta_x(A) = 1$. By the Kuratowski–Zorn lemma, each closed absorbing set contains a (non-empty and closed) minimal absorbing set. If X is minimal (absorbing), then we say that the *dynamical system* (X, Θ) is *irreducible*. Clearly (see [**26**] for the case of a single Markov operator), the topological support $\operatorname{supp}(\mu)$ of a Θ-invariant (where $\Theta = (T_\lambda)_{\lambda \in \Lambda}$) probability measure $\mu \in P_\Theta(X)$ is absorbing. The whole space X is minimal if and only if $\operatorname{supp}(\mu) = X$ for every $\mu \in P_\Theta(X)$. An $(\mathcal{L}, \mathcal{R})$ net is said to be *uniquely ergodic* if $P_\Theta(X) = \{\mu\}$ is a singleton. If, moreover, for this unique invariant probability we have $\operatorname{supp}(\mu) = X$, then (X, Θ) is said to be *strictly ergodic*.

Definition 8. We say that a pointwise convergent $(\mathcal{L}, \mathcal{R})$ net $\Theta = (T_\lambda)_{\lambda \in \Lambda}$ of Markov operators on $C(X)$ (such that for each $x \in X$ and $f \in C(X)$ the limit $\lim_{\lambda \to \infty} T_\lambda f(x) = \widetilde{Q} f(x)$ exists) is *asymptotically individually controlled by subsequences* (*aics* for short) if for each $f \in C(X)$ there exists an increasing sequence $\lambda_1 \preceq \lambda_2 \preceq \cdots$ such that for every $x \in X$,

$$\lim_{k \to \infty} \sup_{\lambda \succeq \lambda_k} T_\lambda f(x) = \lim_{k \to \infty} \inf_{\lambda \succeq \lambda_k} T_\lambda f(x) = \widetilde{Q} f(x). \tag{1}$$

The Cesàro means T_n of a weak* ergodic Markov operator T on $C(X)$ (such that for each continuous $f \in C(X)$ the Cesàro averages $T_n f(x)$ converge pointwise to $\widetilde{Q} f(x)$, where the limit function $\widetilde{Q} f$ is Borel but not necessarily continuous) form an $(\mathcal{L}, \mathcal{R})$ net which is asymptotically individually controlled by the identity sequence $n_k = k$ (see [**11–13**]). Notice that the $(\mathcal{L}, \mathcal{R})$ nets of the Markov operators in Examples 1 and 2 satisfy the *aics* condition as well.

Definition 9. Let $\Theta = (T_\lambda)_{\lambda \in \Lambda}$ be an $(\mathcal{L}, \mathcal{R})$ net on $C(X)$ such that the adjoint operators $\Theta^* = (T_\lambda^*)_{\lambda \in \Lambda}$ form an ergodic $(\mathcal{L}, \mathcal{R})$ net. We say that Θ is *presmoothing* if for every $x \in X$ there exist an open set U containing x and a sequence $\lambda_1 \preceq \lambda_2 \preceq \cdots$ such that for every $u \in U$,

$$\left\| T_{\lambda_{k_u}}^* \delta_u - \lim_{\lambda \to \infty} T_\lambda^* \delta_u \right\| < 2 \tag{2}$$

for some natural number k_u.

The following lemma shows, as mentioned earlier, that the *aics* and presmoothing conditions are naturally satisfied when we deal with classical Cesàro means. In particular, we have the following lemma.

LEMMA 2. *Let T be a Markov operator on $C(X)$ such that its adjoint T^* is mean ergodic (as a linear contraction on $M(X)$). Then the Cesàro means T_n form an $(\mathcal{L}, \mathcal{R})$ net which is presmoothing and satisfies the* aics *condition.*

https://doi.org/10.1017/S0143385710000350 Published online by Cambridge University Press

Proof. By the assumptions, $\|T_n^*(\mu) - \mathfrak{Q}(\mu)\| \to 0$. Taking $U = X$ and $\lambda_k = k$ gives that Θ is presmoothing. Now, given a function $f \in C(X)$ and $x \in X$, we have

$$\lim_{k\to\infty} \sup_{\lambda\succeq\lambda_k} T_\lambda f(x) - \lim_{k\to\infty} \inf_{\lambda\succeq\lambda_k} T_\lambda f(x) = \langle f, \mathfrak{Q}(\delta_x)\rangle - \langle f, \mathfrak{Q}(\delta_x)\rangle = 0.$$

Hence $(T_n)_{n\in\mathbb{N}}$ satisfies the *aics* condition with $\widetilde{Q} f(x) = \langle f, \mathfrak{Q}(\delta_x)\rangle$. □

In what follows, we will find that for general directed sets Λ, the *aics* property holds whenever an $(\mathcal{L}, \mathcal{R})$ net is ergodic. Specifically, we have the following result.

LEMMA 3. *Let $\Theta = (T_\lambda)_{\lambda\in\Lambda}$ be an ergodic $(\mathcal{L}, \mathcal{R})$ net of Markov operators on $C(X)$. Then the* aics *condition holds.*

Proof. Let us write $\lim_{\lambda\to\infty} T_\lambda = Q$ in the strong operator topology on $C(X)$. For each $k \in \mathbb{N}$, choose $\widetilde{\lambda_k} \in \Lambda$ such that

$$\sup_{\lambda\succeq\widetilde{\lambda_k}} \|T_\lambda f - Qf\| \le \frac{1}{k}.$$

As $(\Lambda, \preceq)$ is a directed set, we may select (by induction) an increasing sequence $\lambda_1 \preceq \lambda_2 \preceq \cdots$ such that $\lambda_n \succeq \widetilde{\lambda_j}$ for $j = 1, 2, \ldots, n$. It follows that

$$\sup_{\lambda\succeq\lambda_k} \|T_\lambda f - Qf\| \le \frac{1}{k}.$$

We have thus obtained that, uniformly for $x \in X$,

$$Qf(x) - \frac{1}{k} \le \inf_{\lambda\succeq\lambda_k} T_\lambda f(x) \le Qf(x) \le \sup_{\lambda\succeq\lambda_k} T_\lambda f(x) \le Qf(x) + \frac{1}{k}.$$

So, finally,

$$\lim_{k\to\infty} \inf_{\lambda\succeq\lambda_k} T_\lambda f(x) = \lim_{k\to\infty} \sup_{\lambda\succeq\lambda_k} T_\lambda f(x) = Qf(x).$$ □

LEMMA 4. *Let $\Theta = (T_\lambda)_{\lambda\in\Lambda}$ be an $(\mathcal{L}, \mathcal{R})$ net such that Θ converges to Q uniformly. Then $\Theta = (T_\lambda)_{\lambda\in\Lambda}$ is presmoothing.*

Proof. We choose $\lambda_1 \preceq \lambda_2 \preceq \cdots$ such that

$$\sup_{\lambda\succeq\lambda_k} \|T_\lambda - Q\| < \frac{1}{k}.$$

It follows that

$$\sup_{x\in X} \sup_{\lambda\succeq\lambda_k} \|T_\lambda^*\delta_x - Q^*\delta_x\| < \frac{1}{k}.$$

Clearly, $Q^*\delta_x = \lim_{\lambda\to\infty} T_\lambda^*\delta_x$. In particular, the presmoothing condition is satisfied if we set $k_x \equiv 1$ and $U = X$. □

2. *Unique ergodicity of minimal Markov $(\mathcal{L}, \mathcal{R})$ nets*

In the following, we will need a condition that guarantees (under a minimality assumption) strict ergodicity. For classical dynamical systems, this topic was discussed by, among others, Oxtoby [**21**], Furstenberg [**10**], Lotz [**18**] and Raimi [**23**]. Then, in a series of papers [**11–13**], Iwanik characterized (Cesàro) strictly ergodic Markov operators on $C(X)$. Specifically, he proved that a minimal Markov operator $T : C(X) \to C(X)$ is uniquely ergodic if and only if it is weak* ergodic. The main result of this section shows that

https://doi.org/10.1017/S0143385710000350 Published online by Cambridge University Press

Iwanik's characterization can be extended to $(\mathcal{L}, \mathcal{R})$ Markov nets as long as the *aics* condition is satisfied.

THEOREM 1. *Let $\Theta = (T_\lambda)_{\lambda\in\Lambda}$ be an irreducible $(\mathcal{L}, \mathcal{R})$ net of Markov operators on $C(X)$ such that for every $f \in C(X)$ and each $x \in X$, the limit $\lim_{\lambda\to\infty} T_\lambda f(x) = \widetilde{Q} f(x)$ exists. Then Θ is strictly ergodic if and only if it satisfies the* aics *condition.*

Proof. The proof of necessity follows directly from applying Lemma 3. For the opposite implication, it is enough to show that $P_\Theta(X) = \{\mu\}$ is a singleton. If it is not, then there exists a continuous function $f \in C(X)$ such that the affine weak*-continuous function $\widehat{f} : P_\Theta(X) \to \mathbb{R}$ defined by

$$\widehat{f}(\nu) = \int_X f \, d\nu$$

is non-constant. It follows from the weak* compactness of $P_\Theta(X)$ that $\widehat{f}$ attains its minimum and maximum on extreme measures, say on ν_1, $\nu_2 \in \operatorname{ex} P_\Theta(X)$, respectively. By the minimality assumption, invariant measures are supported on the whole space, i.e. the topological supports satisfy $\operatorname{supp}(\nu_1) = \operatorname{supp}(\nu_2) = X$. Clearly, $L^2(\nu_i) \cap C(X)$ is dense in $L^2(\nu_i)$ with respect to the $\|\cdot\|_{L^2(\nu_i)}$ norm. Now, for each $g \in L^2(\nu_i) \cap C(X)$ and any $\lambda_0 \in \Lambda$,

$$\lim_{\lambda\to\infty} \|T_\lambda(I - T_{\lambda_0})g\|_2 \le \lim_{\lambda\to\infty} \|T_\lambda(I - T_{\lambda_0})g\|_{\sup} = 0$$

and

$$\lim_{\lambda\to\infty} \|(I - T_{\lambda_0})T_\lambda g\|_2 \le \lim_{\lambda\to\infty} \|(I - T_{\lambda_0})T_\lambda g\|_{\sup} = 0.$$

Therefore, for $i = 1, 2$, the $T_\lambda : L^2(\nu_i) \to L^2(\nu_i)$ form $(\mathcal{L}, \mathcal{R})$ nets. It follows from the invariance of the measures ν_i that for each $\lambda \in \Lambda$, the operator T_λ is doubly stochastic on $L^2(\nu_i)$, i.e. $T_\lambda \ge 0$, $T_\lambda \mathbf{1} = \mathbf{1}$ and $T'_\lambda \mathbf{1} = \mathbf{1}$, where T'_λ stands for the adjoint operator. In particular, the T_λ are positive linear contractions on $L^2(\nu_i)$.

From [**8**] we get that $\Theta = (T_\lambda)_{\lambda\in\Lambda}$ is strongly convergent on $L^2(\nu_i)$. We will show that

$$\lim_{\lambda\to\infty} \left\| T_\lambda f - \left(\int_X f \, d\nu_i \right) \mathbf{1} \right\|_{L^2(\nu_i)} = 0$$

for each $f \in L^2(\nu_i)$. Since the strong limit is a projection onto fixed space, it is enough to show that $L^2(\nu_i)_\Theta = \{c\mathbf{1} : c \in \mathbb{R}\}$. Suppose, to the contrary, that there exists a non-constant and invariant function $f \in L^2(\nu_1)_\Theta$. It is well known (see, e.g., [**14**]) that $L^2(\nu_1)_\Theta$ is a sublattice of $L^2(\nu_1)$. In particular, there exists $c \in \mathbb{R}$ such that $(c - f)^+$ and $(c - f)^-$ are non-zero, non-negative, invariant and lattice orthogonal functions in $L^2(\nu_1)$. Let us denote by A_+ and A_- measure supports of $(c - f)^+$ and $(c - f)^-$, respectively. It is easy to see that $T'_\lambda 1_{A_+} \ge 1_{A_+}$, $T'_\lambda 1_{A_-} \ge 1_{A_-}$ and $\nu_1(A_+ \cap A_-) = 0$. Because doubly stochastic operators are conservative, it follows that for each $\lambda \in \Lambda$, we have $T'_\lambda 1_{A_+} = 1_{A_+}\nu_1$ a.e. and $0 < \nu_1(A_+) \le 1 - \nu_1(A_-) < 1$. Now, for each continuous function $g \in C(X)$ and each $\lambda \in \Lambda$,

$$\int_X g 1_{A_+} \, d\nu_1 = \int_X g T'_\lambda 1_{A_+} \, d\nu_1 = \int_X T_\lambda g 1_{A_+} \, d\nu_1$$

and

$$\int_X g(1 - 1_{A_+}) \, d\nu_1 = \int_X g(T'_\lambda 1 - T'_\lambda 1_{A_+}) \, d\nu_1 = \int_X T_\lambda g(1 - 1_{A_+}) \, d\nu_1.$$

https://doi.org/10.1017/S0143385710000350 Published online by Cambridge University Press

We get

$$\nu_1 = \nu_1(A_+)\frac{1}{\nu_1(A_+)}1_{A_+}\nu_1 + (1-\nu_1(A_+))\frac{1}{(1-\nu_1(A_+))}(1-1_{A_+})\nu_1;$$

hence ν_1 is a convex combination of invariant probability measures $\{1_{A_+}/\nu_1(A_+)\}\nu_1$ and $\{(1-1_{A_+})/(1-\nu_1(A_+))\}\nu_1$, which contradicts the fact that ν_1 is an extreme point of $P_\Theta(X)$.

It follows that

$$\lim_{\lambda\to\infty}\left\|T_\lambda f - \left(\int_X f\,d\nu_1\right)\mathbf{1}\right\|_{L^2(\nu_1)} = 0$$

for each $f \in L^2(\nu_1)$ and, analogously,

$$\lim_{\lambda\to\infty}\left\|T_\lambda f - \left(\int_X f\,d\nu_2\right)\mathbf{1}\right\|_{L^2(\nu_2)} = 0$$

for all $f \in L^2(\nu_2)$.

Now, given a continuous function $f \in C(X)$, we choose a sequence $\widetilde{\lambda_1} \preceq \widetilde{\lambda_2} \preceq \cdots$ such that

$$\sup_{\lambda\succeq\widetilde{\lambda_k}}\left\|T_\lambda f - \left(\int_X f\,d\nu_i\right)\mathbf{1}\right\|_{L^2(\nu_i)} < \frac{1}{k}.$$

Since the *aics* condition holds, we can choose a sequence $\lambda_1 \preceq \lambda_2 \preceq \cdots$ such that for each $x \in X$ we have

$$\lim_{k\to\infty}\sup_{\lambda\succeq\lambda_k} T_\lambda f(x) = \lim_{k\to\infty}\inf_{\lambda\succeq\lambda_k} T_\lambda f(x) = \widetilde{Q}f(x).$$

Then, we find an increasing sequence $\widetilde{\beta_1} \preceq \widetilde{\beta_2} \preceq \cdots$ such that $\widetilde{\lambda_k} \preceq \widetilde{\beta_k}$ and $\lambda_k \preceq \widetilde{\beta_k}$ for each $k \in \mathbb{N}$. It follows from the Riesz convergence theorem that there exists another subsequence $\beta_1, \beta_2, \ldots$ (of the sequence $\widetilde{\beta_1}, \widetilde{\beta_2}, \ldots$) such that

$$\lim_{k\to\infty} T_{\beta_k} f(x) = \int_X f\,d\nu_i$$

for ν_i-a.e. $x \in X$ (for $i = 1, 2$). Clearly,

$$\lim_{k\to\infty}\sup_{\lambda\succeq\beta_k} T_\lambda f(x) = \lim_{k\to\infty}\inf_{\lambda\succeq\beta_k} T_\lambda f(x) = \widetilde{Q}f(x).$$

It follows from the minimality assumption that the sets

$$E_1 = \left\{x \in X : \lim_{k\to\infty} T_{\beta_k} f(x) = \int_X f\,d\nu_1\right\}$$

and

$$E_2 = \left\{x \in X : \lim_{k\to\infty} T_{\beta_k} f(x) = \int_X f\,d\nu_2\right\}$$

are dense in X. On the other hand,

$$G_1 = \bigcap_{K=1}\bigcup_{k\geq K}\left\{x \in X : T_{\beta_k} f(x) < \frac{2}{3}\int_X f\,d\nu_1 + \frac{1}{3}\int_X f\,d\nu_2 - \frac{1}{k}\right\}$$

https://doi.org/10.1017/S0143385710000350 Published online by Cambridge University Press

and

$$G_2 = \bigcap_{K=1} \bigcup_{k \geq K} \left\{ x \in X : T_{\beta_k} f(x) > \frac{1}{3} \int_X f \, d\nu_1 + \frac{2}{3} \int_X f \, d\nu_2 + \frac{1}{k} \right\}$$

are G_δ sets, and they are dense as $E_1 \subseteq G_1$ and $E_2 \subseteq G_2$. The Baire category theorem then tells us that the intersection $G_1 \cap G_2$ is non-empty. For each $x \in G_1 \cap G_2$ we have

$$\begin{aligned} \widetilde{Q} f(x) &= \lim_{k \to \infty} \sup_{\lambda \geq \beta_k} T_\lambda f(x) \geq \limsup_{k \to \infty} T_{\beta_k} f(x) \geq \frac{1}{3} \int_X f \, d\nu_1 + \frac{2}{3} \int_X f \, d\nu_2 \\ &> \frac{2}{3} \int_X f \, d\nu_1 + \frac{1}{3} \int_X f \, d\nu_2 \geq \liminf_{k \to \infty} T_{\beta_k} f(x) \\ &\geq \lim_{k \to \infty} \inf_{\lambda \geq \beta_k} T_\lambda f(x) = \widetilde{Q} f(x), \end{aligned}$$

which contradicts the *aics* condition. □

3. *Quasi-compact Markov* $(\mathcal{L}, \mathcal{R})$ *nets*

It was proved in [**1**] (see also [**2, 18**]) that if a Markov operator T on $C(X)$ is such that topological supports of invariant measures are non-meager subsets of X and T^* is mean ergodic, then T is quasi-compact and hence its Cesàro means T_n converge in the operator norm to a finite-dimensional Markov projection Q. Now we are in a position to prove the $(\mathcal{L}, \mathcal{R})$ version of this result.

THEOREM 2. *Let $\Theta = (T_\lambda)_{\lambda \in \Lambda}$ be a uniform $(\mathcal{L}, \mathcal{R})$ net of Markov operators on $C(X)$ such that the support of each invariant probability is a non-meager subset of X and the adjoint $(\mathcal{L}, \mathcal{R})$ net $(T^*_\lambda)_{\lambda \in \Lambda}$ is mean ergodic. Then Θ is quasi-compact if and only if it is presmoothing and satisfies the* aics *condition.*

Proof. Suppose that Θ is quasi-compact. Then it is uniformly convergent (so that Θ^* is ergodic) and, by Lemmas 3 and 4, the *aics* and presmoothing conditions hold.

In the other direction, the proof is considerably more complex and requires a few steps which imitate the arguments originally used in [**18**] and [**1**]. Given a probability invariant measure $\mu \in P_\Theta(X)$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Lambda$, we write

$$H_{\lambda_1, \ldots, \lambda_n, \mu} = \{ h \in C(X) : 0 \leq h \leq g_{\lambda_1, \ldots, \lambda_n, \mu} \},$$

where

$$g_{\lambda_1, \ldots, \lambda_n, \mu}(x) = \| T^*_{\lambda_n} \circ \cdots \circ T^*_{\lambda_1} \delta_x \wedge \mu \|.$$

From the invariance of μ it follows that

$$g_{\lambda_1, \ldots, \lambda_n, \lambda_{n+1}, \mu}(x) \geq g_{\lambda_1, \ldots, \lambda_n, \mu}(x)$$

for all $x \in X$. Let

$$H = \bigcup_{\substack{\lambda_1, \ldots, \lambda_n \in \Lambda \\ \mu \in P_\Theta(X)}} H_{\lambda_1, \ldots, \lambda_n, \mu} \quad \text{and} \quad A = \{ x \in X : h(x) = 0 \ \forall h \in H \}.$$

If the set A is empty, then by compactness of X there must be finitely many $h_1, \ldots, h_j \in H$ such that

$$h(x) = \sum_{k=1}^{j} h_k(x) > \varepsilon > 0$$

https://doi.org/10.1017/S0143385710000350 Published online by Cambridge University Press

for some $\varepsilon > 0$ and all $x \in X$. We have

$$0 < \frac{\varepsilon}{j} < \frac{h(x)}{j} \leq \left\| \frac{\sum_{k=1}^{j} T^*_{\lambda^k_{n_k}} \circ \cdots \circ T^*_{\lambda^k_{n_1}} \delta_x}{j} \wedge \frac{\mu_1 + \cdots + \mu_j}{j} \right\|.$$

It follows that for all continuous $0 \leq f \leq 1$,

$$\frac{\varepsilon}{j} \leq \frac{1}{j} \sum_{k=1}^{j} T_{\lambda^k_{n_1}} \circ \cdots \circ T_{\lambda^k_{n_k}} (1 - f) + \overline{\mu}(f)$$

or, equivalently,

$$\left(\frac{1}{j} \sum_{k=1}^{j} T_{\lambda^k_{n_1}} \circ \cdots \circ T_{\lambda^k_{n_k}} \right) f - \overline{\mu}(f) \leq 1 - \frac{\varepsilon}{j}$$

where $\overline{\mu} = (\mu_1 + \mu_2 + \cdots + \mu_j)/j$.

Hence the Markov operator

$$S = \frac{1}{j} \sum_{k=1}^{j} T_{\lambda^k_{n_1}} \circ \cdots \circ T_{\lambda^k_{n_k}}$$

satisfies the Doeblin condition and is quasi-compact (see [**18**, Proposition 3]). By property (ULR2),

$$\left\| \left(T_\lambda - T_\lambda \frac{1}{j} \sum_{k=1}^{j} T_{\lambda^k_{n_1}} \circ \cdots \circ T_{\lambda^k_{n_k}} \right) f \right\| < \frac{\epsilon}{2j}$$

for all functions $0 \leq f \leq \mathbf{1}$ whenever $\lambda \in \Lambda$ is large enough. In particular, there exists $\lambda \in \Lambda$ such that

$$\|T_\lambda f - \overline{\mu}(f)\| \leq 1 - (\epsilon/2j) \quad \text{for all } 0 \leq f \leq \mathbf{1}.$$

Therefore T_λ is a quasi-compact Markov operator.

It follows from [**3**] that there exists a finite-dimensional Markov projection R such that for some $d \geq 1$ the convergence

$$\lim_{n \to \infty} \|T^{nd}_\lambda - R\| = 0$$

holds. In particular, given $\alpha > 0$, we have

$$\|T^n_\lambda - R\| < \frac{\alpha}{2}$$

for suitable $n \in \mathbb{N}$. Now, using the conditions (ULR2) and (ULR3), there exists $\gamma_0 = \gamma_0(n, \alpha) \in \Lambda$ such that $\|T^n_\lambda T_\gamma - T_\gamma\| < \alpha/2$ and $\|T_\gamma T^n_\lambda - T_\gamma\| < \alpha/2$ for all $\gamma \succeq \gamma_0$. Hence

$$\|T_\gamma - T_\gamma R\| \leq \|T_\gamma - T_\gamma T^n_\lambda\| + \|T_\gamma T^n_\lambda - T_\gamma R\| < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$$

and, similarly,

$$\|T_\gamma - R T_\gamma\| \leq \alpha.$$

In other words,

$$\lim_{\gamma \to \infty} \|T_\gamma - T_\gamma R\| = \lim_{\gamma \to \infty} \|T_\gamma - R T_\gamma\| = 0.$$

https://doi.org/10.1017/S0143385710000350 Published online by Cambridge University Press

Let us denote by Q^* the strong operator limit of the net T_λ^*. Then Q^* is a positive linear contraction on $M(X)$. Upon applying the above estimations, we find that for all $\varrho \in M(X)$,

$$\|Q^*\varrho - Q^*R^*\varrho\| = \|Q^*\varrho - R^*Q^*\varrho\| = 0$$

or, more simply,

$$Q^* = Q^*R^* = R^*Q^*.$$

It follows that the projection Q^* is finite-dimensional.

Now

$$\lim_{\gamma\to\infty} \|T_\gamma^* - Q^*\| \le \lim_{\gamma\to\infty} \|T_\gamma^* - T_\gamma^*R^*\| + \lim_{\gamma\to\infty} \|T_\gamma^*R^* - Q^*R^*\|.$$

Obviously, both terms on the right-hand side converge to zero (as regards the second term, note that the range of R^* is finite-dimensional and therefore the T_γ^* tend to Q^* uniformly on the unit ball of $R^*(M(X))$).

In summary, we have proved that if $A = \emptyset$, then

$$\lim_{\lambda\to\infty} \|T_\lambda - Q\| = \lim_{\lambda\to\infty} \|T_\lambda^* - Q^*\| = 0,$$

where Q is a finite-dimensional Markov projection (i.e. $(T_\lambda)_{\lambda\in\Lambda}$ is quasi-compact).

Now suppose that A is non-empty. Clearly, A is closed. We claim that A is absorbing, i.e. $T_\lambda^*\delta_a(A) = 1$ for every $a \in A$ and all $\lambda \in \Lambda$. If not, then there exist an open set V with $\overline{V} \cap A = \emptyset$, an $a \in A$ and a $\lambda \in \Lambda$ such that $T_\lambda^*\delta_a(V) = \sigma_1 > 0$. It follows from the construction of the set A that there exist an open subset $W \subseteq V \subseteq A^c$, an invariant measure μ and $\lambda_1, \dots, \lambda_n \in \Lambda$ such that

$$\|T_{\lambda_1}^* \circ \cdots \circ T_{\lambda_n}^* \delta_w \wedge \mu\| \ge h(w) > \eta > 0 \quad \text{for all } w \in W, \text{ for some } \eta > 0,$$

and still we have $T_\lambda^*\delta_a(W) = \sigma_2 > 0$ for some $\sigma_2 \le \sigma_1$. In particular, if $w \in W$, then

$$\inf_{0\le f\le 1} (T_{\lambda_n} \circ \cdots \circ T_{\lambda_1}(1-f)(w) + \mu(f)) \ge h(w) > \eta.$$

Applying the Feller property gives that there is an open set U containing the point a such that $T_\lambda^*\delta_u(W) > \sigma$ for all $u \in U$ and some $0 < \sigma \le \sigma_2$. It follows from the Chapman–Kolmogorov rule that

$$\begin{aligned} T_{\lambda_1}^* \circ \cdots \circ T_{\lambda_n}^* \circ T_\lambda^*\delta_u &= \int T_{\lambda_1}^* \circ \cdots \circ T_{\lambda_n}^*\delta_w \, dT_\lambda^*\delta_u(w) \\ &\ge \int_W T_{\lambda_1}^* \circ \cdots \circ T_{\lambda_n}^*\delta_w \, dT_\lambda^*\delta_u(w). \end{aligned}$$

Now, for a continuous function $0 \le f \le 1$ and any $u \in U$, we get

$$\begin{aligned} &T_\lambda \circ T_{\lambda_n} \circ \cdots \circ T_{\lambda_1}(1-f)(u) + \mu(f) \\ &= \int (1-f)(x)\, d\left(\int T_{\lambda_1}^* \circ \cdots \circ T_{\lambda_n}^*\delta_w \, dT_\lambda^*\delta_u(w) \right)(x) + \mu(f) \\ &= \int [(T_{\lambda_n} \circ \cdots \circ T_{\lambda_1}(1-f)(w) + \mu(f))] \, dT_\lambda^*\delta_u(w) \\ &\ge \int_W (T_{\lambda_n} \circ \cdots \circ T_{\lambda_1}(1-f)(w) + \mu(f)) \, dT_\lambda^*\delta_u(w) \\ &= \int_W \eta \, dT_\lambda^*\delta_u(w) \\ &= \eta T_\lambda^*\delta_u(W) > \eta\sigma > 0. \end{aligned}$$

https://doi.org/10.1017/S0143385710000350 Published online by Cambridge University Press

Thus, for all u in some open neighborhood U of a point $a \in A$ we have

$$\|T^*_{\lambda_1} \circ \cdots \circ T^*_{\lambda_n} \circ T_\lambda \delta_u \wedge \mu\| > \eta\sigma > 0,$$

which is a contradiction.

It has already been mentioned that a non-empty, closed and absorbing set A contains a minimal closed and absorbing subset A_0. Applying Theorem 1, we obtain that there exists a unique invariant probability measure μ concentrated on A_0. By our assumption, the interior of A_0 is non-empty. By the presmoothing assumption, we can find an open set $U \subseteq \operatorname{Int} A_0$ (without loss of generality we may assume that $\overline{U} \subseteq \operatorname{Int} A_0$) and a sequence $\lambda_1 \preceq \lambda_2 \preceq \cdots$ such that

$$\|T^*_{\lambda_n} \delta_u - \lim_{\lambda \to \infty} T^*_\lambda \delta_u\| = \|T^*_{\lambda_n} \delta_u - \mu\| < 2$$

for all $u \in U$. Clearly, $\lim_{\lambda\to\infty} T^*_\lambda \delta_u = \mu$ (on minimal A_0 the net $\Theta = (T_\lambda)_{\lambda\in\Lambda}$ is uniquely ergodic). In particular, $\sup_k \|T^*_{\lambda_k} \delta_u \wedge \mu\| > 0$. On the other hand (by the definition of A), the sets

$$D_{k,l} = \left\{ u \in U : \|T^*_{\lambda_k} \delta_u \wedge \mu\| < \frac{1}{l} \right\}$$

are open and dense in U (and hence in $\overline{U}$). It follows from the Baire category theorem that $\bigcap_{k,l\geq 1} D_{k,l}$ in a dense G_δ subset of $\overline{U}$. We note that if $a \in \bigcap_{k,l\geq 1} D_{k,l}$, then $\|T^*_{\lambda_k} \wedge \mu\| = 0$ for each k, which is a contradiction. Therefore, we conclude that the set A must be empty. □

Acknowledgements. This paper originated from discussions during a visit of the second author to the Gdańsk University of Technology (from 1 October 2008 to 28 February 2009), which was supported by the Öğretim Üyesi Yetiştirme Program.

References

[1] W. Bartoszek. On a theorem of H. P. Lotz on quasi-compactness of Markov operators. *Colloq. Math.* **52**(2) (1987), 281–284.

[2] W. Bartoszek. On quasi-compactness and invariant measures of Markov operators on $C(X)$. *Bull. Acad. Polon. Sci.* **34** (1986), 69–72.

[3] W. Bartoszek. Asymptotic periodicity of the iterates of positive contractions on Banach lattices. *Studia Math.* **XCI** (1988), 759–188.

[4] C. Bessaga and A. Pełczyński. *Selected Topics in Infinite-Dimensional Topology (Mathematical Monographs, 58)*. PWN, Warsaw, 1975.

[5] W. F. Eberlein. Abstract ergodic theorems and weakly almost periodic functions. *Trans. Amer. Math. Soc.* **67** (1949), 217–240.

[6] E. Yu. Emel'yanov. Lotz–Räbiger nets. *Conference Materials, Bilkent–METU Joint Analysis Seminar*, 2008.

[7] E. Yu. Emel'yanov. *Non-spectral Asymptotic Analysis of One-Parameter Operator Semigroups (Operator Theory: Advances and Applications, 173)*. Birkhäuser, Basel, 2007.

[8] E. Yu. Emel'yanov and N. Erkursun. Generalization of Eberlein's and Sine's ergodic theorems to $(\mathcal{L}, \mathcal{R})$-nets. *Vladikavkaz. Mat. Zh.* **9**(3) (2007), 22–26.

[9] R. Engelking. *General Topology*. Heldermann, Berlin, 1985.

[10] H. Furstenberg. Strict ergodicity and transformations of the torus. *Amer. J. Math.* **83** (1961), 573–601.

[11] A. Iwanik. On pointwise convergence of Cesàro means and separation properties for Markov operators on $C(X)$. *Bull. Acad. Polon. Sci.* **29**(9–10) (1981), 515–520.

https://doi.org/10.1017/S0143385710000350 Published online by Cambridge University Press

[12] A. Iwanik. Unique ergodicity of irreducible Markov operators on $C(X)$. *Studia Math.* **77**(1) (1983), 81–86.
[13] A. Iwanik. Non-uniquely ergodic minimal systems. *Bull. Acad. Polon. Sci.* **32**(7–8) (1984), 471–478.
[14] U. Krengel. *Ergodic Theorems*. De Gruyter, Berlin, 1985.
[15] M. Lin. On the uniform ergodic theorem. *Proc. Amer. Math. Soc.* **43** (1974), 334–340.
[16] M. Lin. Quasi-compactness and uniform ergodicity of Markov operators. *Ann. Inst. H. Poincaré Sect. B* **11** (1975), 345–354.
[17] M. Lin. Quasi-compactness and uniform ergodicity of positive operators. *Israel J. Math.* **29** (1978), 309–311.
[18] H. P. Lotz. Uniform ergodic theorems for Markov operators on $C(X)$. *Math. Z.* **178** (1981), 145–156.
[19] H. P. Lotz. Tauberian theorems for operators on Banach spaces. *Semesterbericht Funktionalanalysis, WS 1983/1984*, Mathematics Institute, Eberhard-Karls-Universität Tübingen, 1984, pp. 1–15.
[20] H. P. Lotz. Tauberian theorems for operators on L^∞ and similar spaces. *Functional Analysis: Surveys and Recent Results*. Eds. K.-D. Biersted and B. Fuchssteiner. North-Holland, Amsterdam, 1984, pp. 117–133.
[21] J. Oxtoby. Ergodic sets. *Bull. Amer. Math. Soc.* **58** (1952), 116–136.
[22] F. Räbiger. Stability and ergodicity of dominated semigroups. *Math. Ann.* **297** (1993), 103–116.
[23] R. A. Raimi. Minimal sets and ergodic measures in $\beta\mathbb{N}\backslash\mathbb{N}$. *Bull. Amer. Math. Soc.* **70** (1964), 711–712.
[24] S. Romanelli. Ergodic subspaces related to operator semigroups. *Bull. Acad. Polon. Sci.* **38**(3–4) (1991), 191–198.
[25] R. Sine. Constricted systems. *Rocky Mountain J. Math.* **21** (1991), 1373–1383.
[26] R. Sine. Geometric theory of a single Markov operator. *Pacific J. Math.* **27** (1968), 155–166.
[27] K. Yosida and S. Kakutani. Operator-theoretical treatment of Markov process and mean ergodic theorem. *Ann. of Math.* (2) **42**(2) (1941), 188–228.

https://doi.org/10.1017/S0143385710000350 Published online by Cambridge University Press