Integration based on the parametrisation of conics

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1. *Introduction*

Integration is an important topic discussed in first year calculus. Many subsequent topics, such as solving first-order or second-order differential equations, requires the evaluation of at least one integral, and sometimes several.

The most challenging material in integration is arguably the techniques of integration. Unfortunately, due to time constraints, only basic techniques are commonly taught, while in reality, many integrals having special forms are solvable only by using more sophisticated techniques. For instance, the reader may be tempted to try solving

$$\int \frac{dx}{x^2\sqrt{x^2 - 2x}}, \int \frac{x + 2}{x(x + 1)\sqrt{x^2 + 1}}\,dx, \text{ and } \int \frac{x^2}{\sqrt{-x^2 + 1} + x^2 + 1}\,dx.$$

Each of these integrals has the form

$$\int R\left(x, \sqrt{ax^2 + bx + c}\right)dx, \tag{1}$$

which we will look into. Here a, b, c are constants with $a \neq 0$, and R is a rational function of x and $\sqrt{ax^2 + bx + c}$, i.e. a function composed of a polynomial divided by a polynomial, where each polynomial is in terms of x and $\sqrt{ax^2 + bx + c}$. We assume that each of the square root terms appears with an odd exponent, and that $ax^2 + bx + c$ is not a perfect square, i.e., $D = b^2 - 4ac \neq 0$, since otherwise (1) contains no square root.

In the literature there have been some proposed techniques to solve integrals of the form (1) systematically. Section 5.3 of [1], for instance, discusses several techniques quite exhaustively. In Section 2.25 to 2.28 of [2], as well as in Section 5.3 and 5.4 of [3], several techniques are also found, including Euler's famous substitution. Some other techniques, which are particularly useful when R has certain forms, are also listed in Chapter 3 of [4]. However, many of these techniques seem to appear without any background or intuitive explanation on where they were developed from.

This article demonstrates how integrals of the form (1) can be solved by using a substitution technique conveniently derived from pre-calculus, namely the parametrisation of ellipses and hyperbolas, together with straightforward trigonometric relations. This derivation mentioned will be the starting point of our discussion. Once we have formulated the substitutions, we will finish by applying them to solve the three integrals given above.

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2. *The derivation*

Firstly, one would agree that the difficulty of evaluating (1) is primarily caused by the existence of the square root, $\sqrt{ax^2 + bx + c}$. We therefore seek to eliminate it. One way is by letting

$$y = \sqrt{ax^2 + bx + c}.$$

Squaring both sides, completing the square, and rearranging, we obtain

$$\left(x + \frac{b}{2a}\right)^2 - \frac{y^2}{a} = \frac{D}{4a^2} \tag{2}$$

where $D = b^2 - 4ac$. This should look familiar to most of us. Yes, it looks like the equation of a conic!

At this point we start bringing conics into the scene. One aspect of conics which is useful for simplification is their parametrisation. The parametrisation, of course, depends on the type of the conic. Looking at (2), we can see that the type of the conic is determined by the signs of a and D; $a > 0$ and $D > 0$ gives an east-west opening hyperbola, $a > 0$ and $D < 0$ gives a north-south opening hyperbola, and $a < 0$ and $D > 0$ gives an ellipse. The case $a < 0$ and $D < 0$ makes $\sqrt{ax^2 + bx + c}$ undefined, and is therefore omitted.

For each of the three feasible cases, we write the usual parametric equations $x = f(t)$, $y = g(t)$ involving trigonometric functions, and then perform a substitution $z = \tan\frac{1}{2}t$ which transforms all the trigonometric functions into rational expressions of z, namely

$$\sin t = \frac{2z}{1 + z^2}, \cos t = \frac{1 - z^2}{1 + z^2}, \sec t = \frac{1 + z^2}{1 - z^2} \text{ and } \tan t = \frac{2z}{1 - z^2}.$$

This converts (1) into an integral of a rational function of z, i.e. an integral of the form $\int \frac{P(z)}{Q(z)}\,dz$, where P and Q are polynomials, which then can be solved using known techniques.

Case 1: $a > 0, D > 0$

We write $a = |a|$ and $D = |D|$, so that (2) becomes

$$\left(x + \frac{b}{2|a|}\right)^2 - \frac{y^2}{|a|} = \frac{|D|}{4a^2}.$$

In this case, the parametric equations

$$x = -\frac{b}{2|a|} + \frac{\sqrt{|D|}}{2|a|}\sec t, \; y = \frac{\sqrt{|D|}}{2\sqrt{|a|}}\tan t$$

work. Applying the substitution mentioned above, we obtain

$$x = -\frac{b}{2|a|} + \frac{\sqrt{|D|}}{2|a|}\cdot\frac{1 + z^2}{1 - z^2}, \; y = \frac{\sqrt{|D|}}{\sqrt{|a|}}\cdot\frac{z}{1 - z^2} \tag{3}$$

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and so

$$dx = \frac{\sqrt{|D|}}{|a|}.\frac{2z}{(1-z^2)^2}\,dz. \tag{4}$$

After substituting these into (1) and integrating, we need to rewrite the result in terms of the original variable x. For this purpose, we give the following expression for z in terms of x, which one could easily verify:

$$z = \sqrt{\frac{\frac{2|a|}{\sqrt{|D|}}\left(x+\frac{b}{2|a|}\right)-1}{\frac{2|a|}{\sqrt{|D|}}\left(x+\frac{b}{2|a|}\right)+1}}. \tag{5}$$

Case 2: $a > 0, D < 0$

We write $a = |a|$ and $D = -|D|$, so that (2) becomes

$$\frac{y^2}{|a|} - \left(x + \frac{b}{2|a|}\right)^2 = \frac{|D|}{4a^2}$$

In this case, the parametric equations

$$x = -\frac{b}{2|a|} + \frac{\sqrt{|D|}}{2|a|}\tan t,\ y = \frac{\sqrt{|D|}}{2\sqrt{|a|}}\sec t$$

work. Applying the substitution mentioned above, we obtain

$$x = -\frac{b}{2|a|} + \frac{\sqrt{|D|}}{|a|}.\frac{z}{1-z^2},\ y = \frac{\sqrt{|D|}}{2\sqrt{|a|}}.\frac{1+z^2}{1-z^2} \tag{6}$$

and so

$$dx = \frac{\sqrt{|D|}}{|a|}.\frac{1+z^2}{(1-z^2)^2}\,dz. \tag{7}$$

The expression for z in terms of x is

$$z = \frac{\sqrt{\frac{4a^2}{|D|}\left(x+\frac{b}{2|a|}\right)^2+1}-1}{\frac{2|a|}{\sqrt{|D|}}\left(x+\frac{b}{2|a|}\right)}. \tag{8}$$

Case 3: $a < 0, D > 0$

We write $a = -|a|$ and $D = |D|$, so that (2) becomes

$$\left(x - \frac{b}{2|a|}\right)^2 + \frac{y^2}{|a|} = \frac{|D|}{4a^2}.$$

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In this case, the parametric equations

$$x = \frac{b}{2|a|} + \frac{\sqrt{|D|}}{2|a|}\sin t,\ y = \frac{\sqrt{|D|}}{2\sqrt{|a|}}\cos t$$

work. Applying the substitution mentioned above, we obtain

$$x = \frac{b}{2|a|} + \frac{\sqrt{|D|}}{|a|}\cdot\frac{z}{1+z^2},\ y = \frac{\sqrt{|D|}}{2\sqrt{|a|}}\cdot\frac{1-z^2}{1+z^2} \tag{9}$$

and so

$$dx = \frac{\sqrt{|D|}}{|a|}\cdot\frac{1-z^2}{(1+z^2)^2}\,dz. \tag{10}$$

The expression for z in terms of x is

$$z = \frac{\frac{2|a|}{\sqrt{|D|}}\left(x - \frac{b}{2|a|}\right)}{\sqrt{1 - \frac{4a^2}{|D|}\left(x - \frac{b}{2|a|}\right)^2 + 1}}. \tag{11}$$

3. *Some examples*

As examples, we illustrate the applications of the technique in evaluating the three integrals given in the beginning of this article.

Example 1:

$$\int \frac{dx}{x^2\sqrt{x^2-2x}}.$$

Let

$$y = \sqrt{x^2-2x}.$$

Here we have $a = 1, b = -2, c = 0$, and thus

$$D = b^2 - 4ac = (-2)^2 - 4\times 1\times 0 = 4.$$

Since $a > 0$ and $D > 0$, this integral belongs to Case 1. The substitution (3) in this particular case is

$$x = 1 + \frac{1+z^2}{1-z^2} = \frac{2}{1-z^2},\ y = \frac{2z}{1-z^2}.$$

Meanwhile, (4) becomes

$$dx = \frac{4z}{(1-z^2)^2}\,dz,$$

and (5) becomes

$$z = \sqrt{\frac{x-2}{x}}.$$

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Substituting these into the integral gives

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2 - 2x}} &= \int \frac{\frac{4z}{(1-z^2)^2}}{\left(\frac{2}{1-z^2}\right)^2 \frac{2z}{1-z^2}}\, dz \\ &= \frac{1}{2}\int (1 - z^2)\, dz \\ &= \frac{1}{2}\left(z - \frac{1}{3}z^3\right) + c \\ &= \frac{1}{2}\sqrt{\frac{x-2}{x}} - \frac{1}{6}\left(\sqrt{\frac{x-2}{x}}\right)^3 + c.\end{aligned}$$

Example 2:

$$\int \frac{x + 2}{x(x + 1)\sqrt{x^2 + 1}}\, dx.$$

Let

$$y = \sqrt{x^2 + 1}.$$

Here we have $a = 1, b = 0, c = 1$, and thus

$$D = b^2 - 4ac = 0^2 - 4 \times 1 \times 1 = -4.$$

Since $a > 0$ and $D < 0$, this integral belongs to Case 2. The substitution (6) in this particular case is

$$x = \frac{2z}{1 - z^2}, y = \frac{1 + z^2}{1 - z^2}.$$

Meanwhile, (7) becomes

$$dx = \frac{2(1 + z^2)}{(1 - z^2)^2}\, dz,$$

and (8) becomes

$$z = \frac{\sqrt{x^2 + 1} - 1}{x},$$

Substituting these into the integral gives

$$\begin{aligned}\int \frac{x+2}{x(x+1)\sqrt{x^2+1}}\, dx &= \int \frac{\frac{2z}{1-z^2} + 2}{\frac{2z}{1-z^2}\left(\frac{2z}{1-z^2} + 1\right)\frac{1+z^2}{1-z^2}} \cdot \frac{2(1+z^2)}{(1-z^2)^2}\, dz \\ &= 2\int \frac{z^2 - z - 1}{z(z - 1 - \sqrt{2})(z - 1 + \sqrt{2})}\, dz \\ &= 2\int \left(\frac{1}{z} + \frac{\frac{\sqrt{2}}{4}}{z - 1 - \sqrt{2}} - \frac{\frac{\sqrt{2}}{4}}{z - 1 + \sqrt{2}}\right) dz\end{aligned}$$

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$$= 2\int \frac{dz}{z} + \frac{\sqrt{2}}{2}\int \frac{dz}{z-1-\sqrt{2}} - \frac{\sqrt{2}}{2}\int \frac{dz}{z-1+\sqrt{2}}$$

$$= 2\ln|z| + \frac{\sqrt{2}}{2}\ln|z-1-\sqrt{2}| - \frac{\sqrt{2}}{2}\ln|z-1+\sqrt{2}| + c$$

$$= 2\ln\left|\frac{\sqrt{x^2+1}-1}{x}\right| + \frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{x} - 1 - \sqrt{2}\right|$$

$$- \frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{x} - 1 + \sqrt{2}\right| + c.$$

Example 3:

$$\int \frac{x^2}{\sqrt{-x^2+1}+x^2+1}\,dx.$$

Let

$$y = \sqrt{-x^2+1}.$$

Here we have $a = -1, b = 0, c = 1$, and thus

$$D = b^2 - 4ac = 0^2 - 4\times(-1)\times 1 = 4.$$

Since $a < 0$ and $D > 0$, this integral belongs to Case 3. The substitution (9) in this particular case is

$$x = \frac{2z}{1+z^2}, y = \frac{1-z^2}{1+z^2}.$$

Meanwhile, (10) becomes

$$dx = \frac{2(1-z^2)}{(1+z^2)^2}\,dz,$$

and (11) becomes

$$z = \frac{x}{\sqrt{1-x^2+1}}.$$

Substituting these into the integral gives

$$\int \frac{x^2}{\sqrt{-x^2+1}+x^2+1}\,dx$$

$$= \int \frac{\left(\frac{2z}{1+z^2}\right)^2}{\frac{1-z^2}{1+z^2} + \left(\frac{2z}{1+z^2}\right)^2 + 1}\cdot\frac{2(1-z^2)}{(1+z^2)^2}\,dz$$

$$= 4\int \frac{z^2(1-z^2)}{(1+z^2)^2(1+3z^2)}\,dz$$

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$$= 4\int\left[\frac{1}{(1+z^2)^2} - \frac{1}{1+3z^2}\right]dz$$

$$= 4\int\frac{1}{(1+z^2)^2}\,dz - 4\int\frac{1}{1+3z^2}\,dz$$

$$= 4\int\frac{\frac{1}{2}(1+z^2)+\frac{1}{2}(1-z^2)}{(1+z^2)^2}\,dz - 4\int\frac{1}{1+3z^2}\,dz$$

$$= 2\int\frac{dz}{1+z^2} + 2\int\frac{1-z^2}{(1+z^2)^2}\,dz - 4\int\frac{1}{1+(\sqrt{3}z)^2}\,dz$$

$$= 2\int\frac{dz}{1+z^2} + 2\int\frac{\frac{1}{z^2}-1}{(\frac{1}{z}+z)^2}\,dz - 4\int\frac{1}{1+(\sqrt{3}z)^2}\,dz$$

$$= 2\int\frac{dz}{1+z^2} - 2\int\frac{d(\frac{1}{z}+z)}{(\frac{1}{z}+z)^2} - \frac{4}{\sqrt{3}}\int\frac{d(\sqrt{3}z)}{1+(\sqrt{3}z)^2}$$

$$= 2\arctan z + \frac{2}{\frac{1}{z}+z} - \frac{4}{\sqrt{3}}\arctan(\sqrt{3}z) + c$$

$$= 2\arctan\left(\frac{x}{\sqrt{1-x^2}+1}\right) + \frac{2\left(\frac{x}{\sqrt{1-x^2}+1}\right)}{1+\left(\frac{x}{\sqrt{1-x^2}+1}\right)^2} - \frac{4}{\sqrt{3}}\arctan\left(\frac{\sqrt{3}x}{\sqrt{1-x^2}+1}\right) + c$$

$$= 2\arctan\left(\frac{x}{\sqrt{1-x^2}+1}\right) + x - \frac{4}{\sqrt{3}}\arctan\left(\frac{\sqrt{3}x}{\sqrt{1-x^2}+1}\right) + c.$$

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