A search for sums of powers

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The following problem appeared as one of Chris Maslanka's puzzles in the *Saturday Guardian* during July 2014:

Most people know that $3^2 + 4^2 = 5^2$.

Fewer people know that $10^2 + 11^2 + 12^2 = 13^2 + 14^2$.

Still fewer people know that $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$.

What is the next term in this sequence of sums of sequential squares?

Some people may recognise the sequence beginning 3, 10, 21, … and know what follows and spot the pattern. Please feel free to solve the problem before reading further.

And then you may ask if the sequence of equations continues for ever. To answer that we need some algebra. Each equation is of the form

$$p^2 + (p+1)^2 + \ \ldots\ + (p+n)^2 = (p+n+1)^2 + \ \ldots\ + (p+2n)^2 \qquad (1)$$

where p is the first number and n is the number of terms following the first. Expanding the brackets and collecting the terms in powers of p, we have:

$$(n+1)p^2 + 2p\sum_{r=1}^{n} r + \sum_{r=1}^{n} r^2 = np^2 + 2p\sum_{r=n+1}^{2n} r + \sum_{r=n+1}^{2n} r^2$$

and so

$$p^2 - 2p\left(\sum_{r=1}^{2n} r - 2\sum_{r=1}^{n} r\right) - \left(\sum_{r=1}^{2n} r^2 - 2\sum_{r=1}^{n} r^2\right) = 0$$

and using standard formulae for sums of terms and squares of terms we have the quadratic equation in p given by $p^2 + 2n^2p - n^2(2n+1) = 0$ which has the solution $p = n(2n+1)$, and since $n(2n+1)$ is a sequence composed of every second triangular number (which are 1, 3, 6, 10, …). So equation (1) is a formula for our sequence of sums of squares when p is every second triangular number and $n = 1, 2, 3, \ldots$.

Another question is whether this extends to higher powers. For example, starting with the cubic, we have the equivalent of equation (1):

$$p^3 + (p+1)^3 + \ \ldots\ + (p+n)^3 = (p+n+1)^3 + \ \ldots\ + (p+2n)^3.$$

If we follow the same method we used for squares we obtain equation (2):

$$p^3 + 3p^2\left(2\sum_{r=1}^{n} r - \sum_{r=1}^{2n} r\right) + 3p\left(2\sum_{r=1}^{n} r^2 - \sum_{r=1}^{2n} r^2\right) + \left(2\sum_{r=1}^{n} r^3 - \sum_{r=1}^{2n} r^3\right) = 0. \qquad (2)$$

If we are to develop this we need a general method for generating formulae for sums of powers. That is, we need to find formulae for $U_n^{(k)} = \sum_{r=1}^{n} r^k$, where $k = 1, 2, 3, \ldots$.

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This turns out to be easier than you might think, by making use of Pascal's Triangle. The $(k + 1)$th differences of the sequence are equal, so $U_n^{(k)}$ can be written as a polynomial of degree $k + 1$.

So $U_n^{(k)} = a_1 n + a_2 n^2 + \ldots + a_{k+1} n^{k+1}$ and we need to find the coefficients $a_1, a_2, \ldots, a_{k+1}$. (For any k, the null sequence $U_0^{(k)} = a_0$ would be a sequence of no terms. So a_0 is zero.)

Let $PT^{(k)}$ be the $(k + 1) \times (k + 1)$ triangular matrix extracted from the first $k + 1$ rows of Pascal's Triangle and $p^{(k)}$ be the column vector extracted from the kth row of Pascal's Triangle. For example,

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array} \qquad PT^{(3)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 4 \end{pmatrix}, p^{(3)} = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}$$

This shows Pascal's Triangle, the matrix $PT^{(3)}$ and the vector $p^{(3)}$.

Let $\left(a^{(k)}\right)^T = (a_1, a_2, \ldots, a_{k+1})$ be the vector of coefficients of $U_n^{(k)}$. Then

$$a^{(k)} = \left(PT^k\right)^{-1} p^{(k)}. \tag{3}$$

And if $\left(n^{(k)}\right)^T = (n, n^2, \ldots, n^{k+1})$ then

$$U_n^{(k)} = \left(n^{(k)}\right)^T \left(PT^{(k)}\right)^{-1} p^{(k)}. \tag{4}$$

As proof of equation (3) consider:

$$\begin{aligned} (n + 1)^k &= \sum_{r=1}^{n+1} r^k - \sum_{r=1}^{n} r^k = U_{n+1}^{(k)} - U_n^{(k)} \\ &= a_1 (n + 1) + a_2 (n + 1)^2 + \ldots + a_{k+1} (n + 1)^{k+1} \\ &\quad - a_1 n - a_2 n^2 - \ldots - a_{k+1} n^{k+1} \\ &= a_1 \left[\binom{1}{0} 1 \right] + a_2 \left[\binom{2}{0} + \binom{2}{1} n \right] + a_3 \left[\binom{3}{0} + \binom{3}{1} n + \binom{3}{2} n^2 \right] + \ldots \\ &\quad + a_{k+1} \left[\binom{k+1}{0} + \binom{k+1}{1} n + \ldots + \binom{k+1}{k} n^k \right]. \end{aligned}$$

Equating powers of n we obtain the following set of $k + 1$ equations:

$$\binom{k}{i} = a_{k+1} \binom{k+1}{i} + a_k \binom{k}{i} + a_{k-1} \binom{k-1}{i} + \ldots + a_{i+1} \binom{1+1}{i}$$

for $i = 0, 1, 2, \ldots, k$.

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In matrix form this system is,

$$\begin{pmatrix} \binom{k}{0} \\ \binom{k}{1} \\ \binom{k}{2} \\ \vdots \\ \binom{k}{k} \end{pmatrix} = \begin{pmatrix} \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \cdots & \binom{k+1}{0} \\ 0 & \binom{2}{1} & \binom{3}{1} & \cdots & \binom{k+1}{1} \\ 0 & 0 & \binom{3}{2} & \cdots & \binom{k+1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{k+1}{k} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_k \end{pmatrix}.$$

So with $(PT)^{(k)}$, $p^{(k)}$, and $a^{(k)}$ as described, we have $(PT)^{(k)}a^{(k)} = p^{(k)}$ and inverting the matrix gives us equation (3).

So our cubic sequence is $U_n^{(3)} = 1^3 + 2^3 + \ldots + n^3$

$$= \begin{pmatrix} n & n^2 & n^3 & n^4 \end{pmatrix}. \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 4 \end{pmatrix}^{-1} . \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix} = \tfrac{1}{4}n^2(n+1)^2.$$

Similarly, $U_n^{(2)} = 1^4 + 2^4 + \ldots + n^4$

$$= \begin{pmatrix} n & n^2 & n^3 & n^4 & n^5 \end{pmatrix}. \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 6 & 10 \\ 0 & 0 & 0 & 4 & 10 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}^{-1} . \begin{pmatrix} 1 \\ 4 \\ 6 \\ 4 \\ 1 \end{pmatrix}$$

$$= \tfrac{1}{30}\left(-n + 10n^3 + 15n^4 + 6n^5\right).$$

So we are ready to return to equation (2) which can be written more succinctly as

$$p^3 + 3p_1p^2 + 3p_2p + p_3 = 0$$

where

$$p_1 = 2\sum_{r=1}^{n} r - \sum_{r=1}^{2n} r,\ p_2 = 2\sum_{r=1}^{n} r^2 - \sum_{r=1}^{2n} r^2 \text{ and } p_3 = 2\sum_{r=1}^{n} r^3 - \sum_{r=1}^{2n} r^3$$

and p_1, p_2 and p_3 are polynomials of n of degree 1, 2 and 3 respectively and are computed from the coefficient vectors $a^{(1)}$, $a^{(2)}$ and $a^{(3)}$ respectively.

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$$a^{(1)} = \left(PT^{(1)}\right)^{-1} p^{(1)} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix},$$

$$a^{(2)} = \left(PT^{(2)}\right)^{-1} p^{(2)} = \frac{1}{6}\begin{pmatrix}1\\3\\2\end{pmatrix},$$

$$a^{(3)} = \left(PT^{(3)}\right)^{-1} p^{(3)} = \frac{1}{4}\begin{pmatrix}0\\1\\2\\1\end{pmatrix}.$$

$$p_1 = 2\begin{pmatrix} n & n^2 \end{pmatrix} a^{(1)} - \begin{pmatrix} 2n & 4n^2 \end{pmatrix} a^{(1)} = \begin{pmatrix} 0 & -2n^2 \end{pmatrix} a^{(1)} = -n^2$$

$$p_2 = 2\begin{pmatrix} n & n^2 & n^3 \end{pmatrix} a^{(2)} - \begin{pmatrix} 2n & 4n^2 & 8n^3 \end{pmatrix} a^{(2)}$$

$$= \begin{pmatrix} 0 & -2n^2 & -6n^3 \end{pmatrix} a^{(2)} = -n^2(1 + 2n)$$

$$p_3 = 2\begin{pmatrix} n & n^2 & n^3 & n^4 \end{pmatrix} a^{(3)} - \begin{pmatrix} 2n & 4n^2 & 8n^3 & 16n^4 \end{pmatrix} a^{(3)}$$

$$= \begin{pmatrix} 0 & -2n^2 & -6n^3 & -14n^4 \end{pmatrix} a^{(3)} = -\frac{1}{2}n^2(1 + 6n + 7n^2).$$

At this point let us remind ourselves what we are doing. We are trying to find integer sequences of the form

$$p^k + (p + 1)^k + \ldots + (p + n)^k = (p + n + 1)^k + \ldots + (p + 2n)^k.$$

This is equivalent to finding integer roots to a polynomial given, in vector form, by

$$\begin{pmatrix} p^k & p_1 p^{k-1} & p_2 p^{k-2} & \ldots & p_k \end{pmatrix} p^{(k)} = 0$$

where the coefficients $p_1, p_2, \ldots, p_k$ are as given above and $p^{(k)}$ is the kth Pascal Row vector.

When $k = 1$ we have $(p - n^2)\begin{pmatrix}1\\1\end{pmatrix} = 0$. Solving gives $p = n^2$, which gives us the nice sequence of equations:

$$1 + 2 = 3$$

$$4 + 5 + 6 = 7 + 8$$

$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

When $k = 2$ we have $\left(p^2 - n^2 p - n^2(1 + 2n)\right)\begin{pmatrix}1\\2\\1\end{pmatrix} = 0$ which solves with $p = n(1 + 2n)$ and creates the sequence of sums of squares that we started with.

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When $k = 3$ we have

$$\left(p^3 - n^2p^2 - n^2(1 + 2n)p - \frac{1}{2}n^2\left(1 + 6n + 7n^2\right)\right)\begin{pmatrix}1\\3\\3\\1\end{pmatrix} = 0$$

so

$$p^3 - 3n^2p^2 - 3n^2(1 + 2n)p - \frac{1}{2}n^2\left(1 + 6n + 7n^2\right) = 0.$$

At this point I enlisted the help of an on-line computation application from Stephen Wolfram which combines his Mathematica computation engine with a quantitative search engine. It is freely available on the web and can search for integer solutions of any polynomial, like the one above, across a range of values for n and p. This I did, but finding no integer solutions in range I decided to relax requirements and widen the search. If we do not require that the first term of the sequence on the right-hand side follows directly after the last term of the sequence on the left-hand side then we can look for sequences of the form

$$p^k + (p + 1)^k + \ldots + (p + n)^k = q^k + (q + 1)^k + \ldots + (q + m)^k.$$

We have introduced new variables q and m where previously $q = p + n + 1$ and $m = 2n - 1$. In this case we choose $q > p + n$ so that we do not overlap sequences.

Using our methods from before, this equation can be written in vector form as

$$\left((n+1)p^k \quad U_n^{(1)}p^{k-1} \quad \ldots \quad U_n^{(k)} \right)p^{(k)} = \left((m+1)p^k \quad U_m^{(1)}p^{k-1} \quad \ldots \quad U_m^{(k)} \right)p^{(k)}$$

where $U_n^{(k)}$ and $U_m^{(k)}$ are the 'sum of powers' formulae for each n and m and $p^{(k)}$ is the k th Pascal row vector. This is a useful form for WolframAlpha which accepts matrices and matrix computations easily. This time I had some success, finding solutions which generated the following sequences:

$$3^3 + 4^3 + 5^3 = 6^3$$

and

$$11^3 + 12^3 + 13^3 + 14^3 = 20^3$$

and

$$61^3 + 62^3 + \ldots + 71^3 = 101^3 + 102^3 + 103^3.$$

It is simple to extend the search to higher degree polynomials generating sequences of powers of 4 or 5 or 6, and to higher values of p and q that determine the length of the sequence. I invite any interested reader to have a go.

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1. But it was a jubilant uncoiling power that made the orange ball soar; and the clarity of its flight through the white sky, the parabolic symmetry of its rise and fall, the certainty that the catch would not be fumbled were almost beautiful, an unforced subversion of the surroundings – the concrete, the double fence and its functional Y-shaped posts, the cold.

2. Under the parabola of a ball,
   a child turning into a man,
   I looked into the air too long.

3. Still at the Egg-life –
   Chafing the Shell –
   When you troubled the Ellipse –
   And the Bird fell –

4. And, once in his long year,
   With praeternuptial ecstasy and fear,
   By the delicious law of that ellipse
   Wherein all citizens of ether move,
   With hastening pace to come
   Nearer, though never near,
   His Love
   And always inaccessible sweet Home

5. Not thus HYPERBOLA – with subtlest art
   The blue-eyed wanton plays her changeful part;
   Quick as her conjugated axes move
   Through every posture of luxurious love.

6. Indeed, he seemed to approach the grave as an hyperbolic curve approaches a line less directly as he got nearer, till it was doubtful if he would ever reach it at all.

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