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# Non-uniqueness of even order permutations with maximal entropy

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*Abstract.* In this paper we prove that permutations of even length with maximal entropy are not unique. There are in fact two distinct permutations, $\theta_n$ and $\widetilde{\theta_n}$, with maximal entropy, both of which are self-dual and $\widetilde{\theta_n}$ is the reverse of $\theta_n$.

## 1. *Introduction and preliminaries*

A finite, fully invariant set of a continuous map of the interval induces a permutation of the invariant set. By calculating the entropy of a permutation $\theta$, we obtain a lower bound for the topological entropy of an interval map $f$ which exhibits a permutation of type $\theta$. It has been shown by Block and Coppel [**2**] that any periodic orbit which is strongly simple has minimal entropy. In contrast to this situation, Geller and Weiss [**4**] have shown that, if $n$ is odd, the maximal entropy orbit types are unique (up to duality). In this paper we use the methods of Geller and Weiss [**4**] to show that for $n$ even there are exactly two distinct (up to duality) $n$-permutations with maximal entropy in $P_n$.

Let $f$ be a continuous map of a compact interval $I$ into itself. The *orbit* of a point $x \in I$ is a fully invariant set for $f$ which is defined as the sequence $\mathrm{Orb}(x) = \{x, f(x), f^2(x), \dots\}$. If $x = f^m(x)$ for some $m \in \mathbb{N}$, then $\mathrm{Orb}(x)$ is finite and the orbit is *periodic*. Recall that a *permutation* on $n$ letters is a bijective map $\theta : \{1, \dots, n\} \to \{1, \dots, n\}$. If $\theta$ has the property that for $1 \leq k \leq n$, $\theta^k(1) = 1$ if and only if $k = n$, then $\theta$ is a *cycle*. If $x$ has a periodic orbit of period $n$ we can write $\mathrm{Orb}(x) = \{p_1, \dots, p_n\}$ with $p_1 < p_2 < \cdots < p_n$. This induces a cycle $\theta$ of length $n$ in the following way: $\theta(i) = j$ if and only if $f(p_i) = p_j$. The cycle $\theta$ is called the *orbit type* of $\mathrm{Orb}(x)$. More generally, if $S = \{p_1, \dots, p_n\} \subseteq I$ with $p_1 < p_2 < \cdots < p_n$ is any finite fully invariant set for $f$, we can define the *type* of $S$ to be the permutation $\theta$ of length $n$ with $\theta(i) = j$ if and only if $f(p_i) = p_j$.

We define $P_n$ to be the set of all permutations on $n$ letters and $C_n$ to be the set of all cycles on $n$ letters. We also let $P = \cup_{n \geq 1} P_n$ and $C = \cup_{n \geq 1} C_n$. The *dual* of a permutation $\theta \in P_n$ is the permutation $\overline{\theta} \in P_n$ where $\overline{\theta}(i) = n + 1 - \theta(n + 1 - i)$ for $i = 1, \dots, n$. The *reverse* of a permutation $\theta \in P_n$ is the permutation $\tilde{\theta} \in P_n$ where $\tilde{\theta}(i) = \theta(n + 1 - i)$ for $i = 1, \dots, n$.

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*Definition 1.* The *entropy of a permutation* $\theta \in P$ is

$$h(\theta) := \inf\{h(f) : f \text{ is a map with an invariant set of type } \theta\},$$

where $h(f)$ is the topological entropy of the map $f$.

We note that for each $\theta \in P$, $h(\overline{\theta}) = h(\theta)$.

*Notation 2.* For $a, b \in \mathbb{R}$ with $a \leq b$

$$O[a, b] = \{m \in \mathbb{N} : m \text{ is odd and } a \leq m \leq b\};$$
$$E[a, b] = \{m \in \mathbb{N} : m \text{ is even and } a \leq m \leq b\}.$$

If $S$ is an invariant set for $f$ of type $\theta \in P_n$, then there is a unique map $f_\theta : [1, n] \to [1, n]$ satisfying

(i) $f_\theta(i) = \theta(i)$ for $i \in \{1, \ldots, n\}$,

(ii) $f_\theta|_{[i,i+1]}$ is linear for $i \in \{1, \ldots, n-1\}$.

The map $f_\theta$ is entirely determined by $\theta$ and is called the *linearisation* of $f$ with respect to its invariant set of type $\theta$.

It is known that for $\theta \in P$, $h(\theta) = h(f_\theta) = \log \rho(M(\theta))$ (see [**8**] and [**1**] for proofs) where $\rho(M(\theta))$ is the spectral radius of the *induced matrix* of $\theta$, which is defined as follows.

The *induced matrix* $M(\theta)$ of $\theta \in P_n$ is the $(n-1) \times (n-1)$ matrix with $ij$th entry given by

$$a_{ij} = \begin{cases} 1, & \text{if } f_\theta([i, i+1]) \supset [j, j+1]; \\ 0, & \text{otherwise.} \end{cases}$$

It was shown in [**6**] that, for $n$ even, the family of self-dual permutations $\theta_n$, defined below, achieves maximal entropy amongst permutations of period $n$. We now define a new family of self-dual permutations, $\widetilde{\theta}_n$, which also achieves maximal entropy in $P_n$.

*Definition 3.* Let $n$ be even and $k = n/2$. We define the permutation $\theta_n$ of period $n$ by

$$\theta_n : j \mapsto \begin{cases} k - j + 1, & \text{if } j \in O[1, k], \\ j - k, & \text{if } j \in O[k+1, n], \\ k + j, & \text{if } j \in E[1, k], \\ 3k - j + 1, & \text{if } j \in E[k+1, n]. \end{cases}$$

*Definition 4.* Let $n$ be even and $k = n/2$. The permutation $\widetilde{\theta}_n$ of period $n$ is defined by

$$\widetilde{\theta}_n : j \mapsto \begin{cases} k + j, & \text{if } j \in O[1, k], \\ 3k - j + 1, & \text{if } j \in O[k+1, n], \\ k - j + 1, & \text{if } j \in E[1, k], \\ j - k, & \text{if } j \in E[k+1, n]. \end{cases}$$

It is easy to show that $\widetilde{\theta}_n$ is the reverse of $\theta_n$. The map $f_{\theta_n}$ has a local minimum at one whilst the map $f_{\widetilde{\theta}_n}$ has a local maximum at one.

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For any permutation $\theta$, we can define its induced matrix $M(\theta)$ by specifying each column of $M(\theta)$. To this end we introduce the following notation:

Given $a, b \in \mathbb{R}$ with $1 \leq a \leq b \leq n-1$ we define $\langle a, b\rangle$ to be a column of an $(n-1) \times (n-1)$ matrix with $i$th element $(1 \leq i \leq n-1)$

$$\langle a, b\rangle_i = \begin{cases} 1, & \text{if } a \leq i \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

If column $j$ of matrix $M$ is equal to $\langle a, b\rangle$ we call $\langle a, b\rangle$ the *column support* of $M^{(j)}$ and write $\langle M^{(j)}\rangle = \langle a, b\rangle$ or simply $M^{(j)} = \langle a, b\rangle$.

We can now determine the induced matrices $A = M(\theta_n)$ and $\tilde{A} = M(\widetilde{\theta_n})$.

PROPOSITION 5. *For $n \equiv 0 \pmod 4$ the column $A^{(j)}$ is given by*

$$\begin{array}{ll} \langle k-j+1, k+j\rangle, & \text{when } j \in O[1, k-1], \\ \langle k-j, k+j-1\rangle, & \text{when } j \in E[1, k-1], \\ \langle 1, 2k-1\rangle, & \text{when } j = k, \\ \langle j-k, 3k-j-1\rangle, & \text{when } j \in O[k+1, n-1], \\ \langle j-k+1, 3k-j\rangle, & \text{when } j \in E[k+1, n-1]. \end{array}$$

*When $n \equiv 2 \pmod 4$ the column $A^{(j)}$ is given by*

$$\begin{array}{ll} \langle k-j, k+j-1\rangle, & \text{when } j \in O[1, k-1], \\ \langle k-j+1, k+j\rangle, & \text{when } j \in E[1, k-1], \\ \langle 1, 2k-1\rangle, & \text{when } j = k, \\ \langle j-k+1, 3k-j\rangle, & \text{when } j \in O[k+1, n-1], \\ \langle j-k, 3k-j-1\rangle, & \text{when } j \in E[k+1, n-1]. \end{array}$$

PROPOSITION 6. *For $n \equiv 0 \pmod 4$ the column $\tilde{A}^{(j)}$ is given by*

$$\begin{array}{ll} \langle k-j, k+j-1\rangle, & \text{when } j \in O[1, k-1], \\ \langle k-j+1, k+j\rangle, & \text{when } j \in E[1, k-1], \\ \langle 1, 2k-1\rangle, & \text{when } j = k, \\ \langle j-k+1, 3k-j\rangle, & \text{when } j \in O[k+1, n-1], \\ \langle j-k, 3k-j-1\rangle, & \text{when } j \in E[k+1, n-1]. \end{array}$$

*When $n \equiv 2 \pmod 4$ the column $\tilde{A}^{(j)}$ is given by*

$$\begin{array}{ll} \langle k-j+1, k+j\rangle, & \text{when } j \in O[1, k-1], \\ \langle k-j, k+j-1\rangle, & \text{when } j \in E[1, k-1], \\ \langle 1, 2k-1\rangle, & \text{when } j = k, \\ \langle j-k, 3k-j-1\rangle, & \text{when } j \in O[k+1, n-1], \\ \langle j-k+1, 3k-j\rangle, & \text{when } j \in E[k+1, n-1]. \end{array}$$

It is interesting to note that the formulae for $A^{(j)}$ and $\tilde{A}^{(j)}$ when $n \equiv 0 \pmod 4$ are the same as those for $\tilde{A}^{(j)}$ and $A^{(j)}$, respectively, when $n \equiv 2 \pmod 4$.

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2. *Proof of the theorem*

It was shown in [**6**] that, for $n$ even, the permutation $\theta_n$ has maximal entropy among all $n$-permutations. In Theorem 7 we show that the permutation $\widetilde{\theta_n}$ also has maximal entropy in $P_n$ and in Theorem 14 we show these are the only two permutations with this property. In the following proofs we will make use of the *flip* permutation, $\phi$, defined as follows: $\phi : \{1, \ldots, n\} \mapsto \{1, \ldots, n\}$, such that

$$\phi(i) = n + 1 - i, \quad \text{for any } i \in \{1, \ldots, n\}.$$

The induced matrix, $\Phi = M(\phi)$, is the $(n-1)\times(n-1)$ antidiagonal matrix; that is, $\Phi_{ij} = 1$ if and only if $j = n - i$. Note the following easily verifiable equations: $\overline{\theta_n} = \phi \circ \theta_n \circ \phi$, $\widetilde{\theta_n} = \theta_n \circ \phi$, $\Phi^2 = I$, $\overline{A} = \Phi A \Phi$ and $\tilde{A} = \Phi A$. Since $\theta_n$ is self-dual, which is equivalent to $A = \Phi A \Phi$ or $\Phi A = A \Phi$, we see $\tilde{A} = \Phi A = A\Phi$. It immediately follows that $\Phi \tilde{A} = A$ and $\tilde{A}\Phi = A$, which shows $\widetilde{\theta_n}$ is also self-dual. These properties make the following results straightforward to prove.

THEOREM 7. *For $n$ even, the self-dual permutations $\theta_n$ and $\widetilde{\theta_n}$ have maximal entropy among all permutations of length $n$.*

*Proof.* By the comments above it is enough to show that $h(\widetilde{\theta_n}) = h(\theta_n)$. Let $k = n/2$. We denote the vector $\mathbf{u} \in \mathbb{R}^{n-1}$ by

$$\mathbf{u} = (u_1, u_2, \ldots, u_{k-1}, u_k, u_{n-(k-1)}, \ldots, u_{n-2}, u_{n-1})$$

and define the cone $\mathcal{P}$ to be the set of all vectors $\mathbf{u} \in \mathbb{R}^{n-1}$ such that

$$0 \leq u_1 = u_{n-1} \leq u_2 = u_{n-2} \leq \cdots \leq u_{k-1} = u_{n-(k-1)} \leq u_k.$$

By the proof of Theorem 7 in [**6**], we need to show that for all $\mathbf{u} \in \mathcal{P}$, $\mathbf{u}\tilde{A} = \mathbf{u}A$ (thus $\mathcal{P}$ is invariant under $\tilde{A}$). Noting that $\mathbf{u}\Phi = \mathbf{u}$ for any $\mathbf{u} \in \mathcal{P}$ we have

$$\mathbf{u}\tilde{A} = \mathbf{u}\Phi A \quad (\text{since } \tilde{A} = \Phi A) = \mathbf{u}A$$

as required. □

We now have that the cone $\mathcal{P}$ is invariant under both $A$ and $\tilde{A}$. However, for the proofs of Propositions 12 and 13 we need to define a new cone $\mathcal{Q}$, which is a subcone of $\mathcal{P}$, with strict inequalities.

*Definition 8.* Let $n$ be even and $k = n/2$. We define the cone $\mathcal{Q}$ to be the set of all vectors $\mathbf{u} \in \mathbb{R}^{n-1}$ such that

$$0 < u_1 = u_{n-1} < u_2 = u_{n-2} < \cdots < u_{k-1} = u_{n-(k-1)} < u_k.$$

PROPOSITION 9. *The cone $\mathcal{Q}$ is invariant under $A$ and $\tilde{A}$.*

*Proof.* For $M = A$ or $M = \tilde{A}$ we will show that if $\mathbf{u} \in \mathcal{Q}$, then $\mathbf{x} = \mathbf{u}M \in \mathcal{Q}$. Firstly, we note that for each $j < k$, $\langle M^{(j)} \rangle \subset \langle M^{(j+1)} \rangle$, $M^{(1)}$ is non-zero and $\mathbf{u}$ is positive. Thus $\mathbf{x} = \mathbf{u}M$ satisfies $0 < x_1 < x_2 < \cdots < x_k$. Now for any $\mathbf{u} \in \mathcal{Q}$, $\mathbf{u}\Phi = \mathbf{u}$, hence

$$\mathbf{x}\Phi = \mathbf{u}M\Phi = \mathbf{u}\Phi M (\text{since } M\Phi = \Phi M) = \mathbf{u}M (\text{since } \mathbf{u}\Phi = \mathbf{u}) = \mathbf{x},$$

and so for each $j < k$, $x_j = x_{n-j}$. Thus $\mathbf{x} \in \mathcal{Q}$ as required. □

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Let $\lambda$ be the spectral radius of $A$ and $\mathbf{r}$ a left Perron–Frobenius eigenvector for $A$ such that $\mathbf{r}A = \lambda\mathbf{r}$. Then $h(\theta_n) = \log\lambda$.

An $m \times m$ square matrix $D$ is said to be *reducible* if there exist two complementary subsets $\{i_1, i_2, \ldots, i_\mu\}$ and $\{j_1, j_2, \ldots, j_\nu\}$ of $\{1, 2, \ldots, m\}$ such that $\mu + \nu = m$, $i_s \neq j_t$ for any $s \in U = \{1, \ldots, \mu\}$ and $t \in V = \{1, \ldots, \nu\}$ and $d_{i_s j_t} = 0$ for $s \in U$ and $t \in V$. Otherwise the matrix is said to be *irreducible*. Therefore, $A$ is irreducible since $a_{ik} = a_{ki} = 1$ for all $i \in \{1, \ldots, n-1\}$ and $k = n/2$. By the Perron–Frobenius theorem, this implies $\lambda$ is positive and $\mathbf{r}$ is positive ($r_j > 0$ for all $j$). It is easy to show by direct calculation that, for $n = 4$, $\lambda = \sqrt{2} + 1$. Thus $\lambda > 1$ for all $n$ as $\mathcal{H}(P_n) := \max\{h(\theta) : \theta \in P_n\}$ is increasing with $n$ (see [**7**, 11.7]).

PROPOSITION 10. *The eigenvector* $\mathbf{r} \in \mathcal{Q}$.

*Proof.* For each $i \in \{1, \ldots, n-1\}$, $\lambda r_i = \mathbf{r}A^{(i)}$. Now, as in Proposition 9, for each $j < k$, $\langle A^{(j)}\rangle \subset \langle A^{(j+1)}\rangle$, $A^{(1)}$ is non-zero and $\mathbf{r}$ is positive, so $0 < r_1 < r_2 < \cdots < r_k$. Now as $\lambda\mathbf{r} = \mathbf{r}A$,

$$\lambda\mathbf{r}\Phi = \mathbf{r}A\Phi = \mathbf{r}\Phi A,$$

since $A\Phi = \Phi A$. As $\lambda$ has algebraic multiplicity one, we conclude that $\mathbf{r}\Phi$ is a scalar multiple of $\mathbf{r}$. Since $\mathbf{r}\Phi^{(i)} = r_{n-i}$, this is only possible if $\mathbf{r}\Phi = \mathbf{r}$. Thus $\mathbf{r} \in \mathcal{Q}$ as required. □

*Remark 11.* If we replace $\tilde{A}$ for $A$ in the discussion above Proposition 10, so that $\lambda$ is the spectral radius of $\tilde{A}$ and $\mathbf{r}$ is a left Perron–Frobenius eigenvector for $\tilde{A}$, then $\tilde{A}$ is irreducible and Proposition 10 is still true.

Let $\eta$ be an $n$-permutation such that $h(\eta) = h(\theta_n)$ and let $B = M(\eta)$ so that $\lambda$ is also the spectral radius of $B$. We may assume that $f_\eta$ is maximodal and has all maximum values above all minimum values (see [**7**]) and hence $B^{(k)} = \langle 1, n-1\rangle$.

If $\theta$ is any maximodal permutation such that $f_\theta$ assumes a minimum at one and $a_{ij}$ ($i, j \in \{1, \ldots, n-1\}$) are the entries of $M(\theta)$, then

$$a_{ij} = 1 \iff \begin{cases} \theta(i) \le j \le \theta(i+1) - 1, & \text{if } i \text{ is odd,} \\ \theta(i+1) \le j \le \theta(i) - 1, & \text{if } i \text{ is even.} \end{cases}$$

If $f_\theta$ has a maximum at one, then

$$a_{ij} = 1 \iff \begin{cases} \theta(i+1) \le j \le \theta(i) - 1, & \text{if } i \text{ is odd,} \\ \theta(i) \le j \le \theta(i+1) - 1, & \text{if } i \text{ is even.} \end{cases}$$

From this we can deduce that $\Phi A^{(j)}$ cannot be a column of $B$.

Let $\varepsilon > 0$ and define $B^\varepsilon$ as follows:

$$b^\varepsilon_{ij} = \begin{cases} b_{ij}, & \text{if } B^{(j)} = A^{(j)}, \\ b_{ij} + \varepsilon, & \text{if } B^{(j)} \neq A^{(j)}. \end{cases}$$

Then $B^\varepsilon \ge B$; that is, $b^\varepsilon_{ij} \ge b_{ij}$ for all $i, j \in \{1, \ldots, n-1\}$, and $B^\varepsilon$ is irreducible as $b^\varepsilon_{ik} > 0$ and $b^\varepsilon_{ki} > 0$ for all $i \in \{1, \ldots, n-1\}$ and $k = n/2$.

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PROPOSITION 12. *Let* $\mathbf{r}$, $\eta$, $B$, *and* $B^{\varepsilon}$ *be as above such that* $f_{\eta}$ *has a minimum at one. Then* $\mathbf{r}A \geq \mathbf{r}B^{\varepsilon}$.

*Proof.* For any $j \in \{1, \ldots n-1\}$, $\mathbf{r}A^{(j)} \geq \mathbf{r}B^{(j)}$ (see [**6**, Proposition 16]). Clearly, if $A^{(j)} = B^{(j)}$, $\mathbf{r}A^{(j)} = \mathbf{r}B^{(j)}$. If $A^{(j)} \neq B^{(j)}$, then $\mathbf{r}A^{(j)} > \mathbf{r}B^{(j)}$. To see this we note that $B^{(j)}$ has at most the same number of non-zero entries as $A^{(j)}$ (see [**6**, Corollaries 10 and 12]) $\langle A^{(j)} \rangle = \langle x, n-x \pm 1 \rangle$ and $\mathbf{r} \in \mathcal{Q}$. Thus, $\mathbf{r}A^{(j)} = \mathbf{r}B^{(j)}$ if and only if $A^{(j)} = B^{(j)}$ or $\Phi A^{(j)} = B^{(j)}$. However, $f_{\eta}$ has a minimum at one and so $\Phi A^{(j)} \neq B^{(j)}$. By definition:

(i) if $B^{\varepsilon(j)} = B^{(j)}$ then $B^{(j)} = A^{(j)}$ and hence $\mathbf{r}A^{(j)} = \mathbf{r}B^{\varepsilon(j)}$;

(ii) if $B^{\varepsilon(j)} \neq B^{(j)}$, then $B^{(j)} \neq A^{(j)}$ and $b^{\varepsilon}{}_{ij} = b_{ij} + \varepsilon$. Hence $\mathbf{r}B^{\varepsilon(j)} = \mathbf{r}B^{(j)} + \varepsilon\mathbf{r}$. As $\mathbf{r}A^{(j)} > \mathbf{r}B^{(j)}$, for sufficiently small $\varepsilon$, $\mathbf{r}A^{(j)} > \mathbf{r}B^{(j)} + \varepsilon\mathbf{r} = \mathbf{r}B^{\varepsilon(j)}$.

Thus $\mathbf{r}A \geq \mathbf{r}B^{\varepsilon}$ as required. □

PROPOSITION 13. *Let* $\eta$, $B$ *and* $B^{\varepsilon}$ *be as above such that* $f_{\eta}$ *has a maximum at one. Then* $\mathbf{r}\tilde{A} \geq \mathbf{r}B^{\varepsilon}$.

*Proof.* The proof is analogous to the proof of Proposition 12 noting that $\tilde{a}_{ij} = a_{n-ij}$ for all $i, j \in \{1, \ldots, n-1\}$ and that $\Phi\tilde{A}^{(j)}$ cannot be $B^{(j)}$. □

THEOREM 14. *For* $n$ *even, if* $\eta \in P_n$ *and* $h(\eta) = \max\{h(\theta) : \theta \in P_n\}$ *then* $\eta = \theta_n$ *if* $f_{\eta}$ *has a minimum at one or* $\eta = \widetilde{\theta_n}$ *if* $f_{\eta}$ *has a maximum at one.*

*Proof.* Let $\eta \in P_n$ be such that $h(\eta) = \max\{h(\theta) : \theta \in P_n\}$ and $f_{\eta}$ has a minimum at one. Let $\lambda_{\varepsilon}$ be the spectral radius of $B^{\varepsilon}$ and $\mathbf{s}_{\varepsilon}$ a right Perron–Frobenius eigenvector for $B^{\varepsilon}$; that is,

$$B^{\varepsilon}\mathbf{s}_{\varepsilon} = \lambda_{\varepsilon}\mathbf{s}_{\varepsilon}.$$

Then

$$\lambda(\mathbf{r}, \mathbf{s}_{\varepsilon}) = (\lambda\mathbf{r}, \mathbf{s}_{\varepsilon}) = (\mathbf{r}A, \mathbf{s}_{\varepsilon}) \geq (\mathbf{r}B^{\varepsilon}, \mathbf{s}_{\varepsilon}) = (\mathbf{r}, B^{\varepsilon}\mathbf{s}_{\varepsilon}) = \lambda_{\varepsilon}(\mathbf{r}, \mathbf{s}_{\varepsilon}).$$

This implies that $\lambda \geq \lambda_{\varepsilon}$ as $A$ and $B^{\varepsilon}$ are irreducible and therefore $\mathbf{r}$ and $\mathbf{s}_{\varepsilon}$ are positive by the Perron–Frobenius theorem. But as $B^{\varepsilon} \geq B$ and $B^{\varepsilon}$ is irreducible we have $\lambda_{\varepsilon} > \lambda$ unless $B^{\varepsilon} = B$ (see [**3**, Lemma 2, p. 57]), and hence $B = A$. Thus $\eta = \theta_n$.

If $f_{\eta}$ has a maximum at one, then we can replace $A$ by $\tilde{A}$ above and conclude that $\eta = \widetilde{\theta_n}$. □

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*Note added in proof.* We would like to note that since we first submitted this paper, we have become aware of independent work by W. Geller and Z. Zhang [**5**] dealing with this question.

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