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TEMPORAL LEONTIEF INVERSE

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Interest in structural change over time has created a demand for analytical tools that can assist in exploiting trends and uncover tendencies in individual sectors or parts of sectors within the context of an economywide system of accounts. This paper offers an alternative approach and is designed to be used with annual input-output or social accounting systems. To date, analysts have been faced with the prospect of conducting simple comparative static approaches or the enormity of the task involved in constructing dynamic models with complex lead and lag structures. The temporal Leontief inverse, introduced in this paper, offers a less complex, more tractable method for examining structural change when a time series of input-output tables is available. The method draws upon some earlier work that proposed the notion of a field of influence of change and explored alternative methods of decomposition of change.

Keywords: Input-Output, Temporal Leontief Inverse, Decomposition

1. INTRODUCTION

In the input-output literature, there has been a great deal of debate about the appropriateness of input-output systems for forecasting over periods longer than three to five years. The issue of coefficient stability has attracted a great deal of attention and, for detractors, the problem of constant coefficients is seen as one of the major drawbacks to the use of input-output models. However, some analysts have ventured opinions based on research that suggest that over a short time period, the problem may not be as severe as once thought. For example, Carter (1970) has ventured the proposition that structural change is a moving-average process, with new firms entering and older ones leaving in any given time interval, such that the technology changes slowly and, perhaps, predictably. However, when the forecast horizon extends over a decade or more, alternative approaches are required.

The predictability issue, as well as the time horizon for structural change, has generated some very different responses. The best-known response centers on Leontief's (1970) dynamic system and the associated dynamic inverse matrix. However, few models of this type have been built; the data requirements and estimating procedures have proven to be formidable. An alternative, and more

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data-parsimonious model, was proposed by Miernyk et al. (1970); one of the attractive features of their approach was the identification of *best-practice* firms whose technology it was thought would become the average technology in 10–15 years. Few have taken up Miernyk's challenge.

New methods of analysis, focusing on price- and quantity-clearing computable general equilibrium models have directed attention once again to the issue of structural change. In the case of a quantity-adjustment general equilibrium model developed for Chicago [see Israilevich et al. (1997)], it has proven possible to extract forecasted input-output tables from the economywide model and to use the same methodology to backcast the input-output tables for earlier years. The result is a set of input-output tables in which coefficients have been adjusted (in lieu of prices) to ensure market clearing. Almost 40 tables are available for the period 1970–2018 at the level of 36 sectors. The major economic content of this paper is to provide a vehicle to transfer from the static description of economic interdependencies to a discrete time version that handles the dynamics of these interdependencies. The central issue of this paper is the sensitivity analysis or comparative statics of the Leontief inverse matrix so that it will be possible to interpret the structure of annual changes in a quantity-adjusted equilibrium model of the kind developed for Chicago.

The challenge for exploring the comparative statics presents a problem for constructing a new perturbation theory of matrix inversion under conditions of all possible changes in the direct coefficients matrix. Inverse matrix perturbation theories based on small changes in the eigenvalues of the matrices are not applicable here. The initial explorations in this field were provided by Sherman and Morrison (1949, 1950); the mathematical content of the present paper provides the elaboration of this initial theory specifically tailored to the economic content of the structural changes observed in the temporal Leontief inverse. The main components of this perturbation theory are the matrix fields of influence of change and the decomposition of the perturbed Leontief inverse into the sum of the nonperturbed Leontief inverse and the superimposition of the fields of influence of change weighted by the products of the perturbations in direct inputs.

In Section 2, a brief review of time-based Leontief systems is provided. Thereafter, in Section 3, the notion of temporal multipliers and temporal increments are illustrated. Section 4 describes the multiplicative and additive forms of the temporal Leontief inverse; its fine structure is discussed in Sections 5 and 6. The fine structure of temporal multipliers occupies Section 7, and applications to structural-path analysis are revealed in Sections 8 and 9. A further application, to temporal changes in the Miyazawa partitioned matrix multipliers, is the subject of Section 10. The paper concludes with some observations and remarks about the methodology.

2. THE LEGACY: COMPARATIVE STATICS AND LEONTIEF DYNAMICS

Some of the earlier approaches to problems of structural change can be divided into two categories: those who use a comparative static approach like Tiebout

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(1969), and others, who attempt to form a discrete-time, dynamic system, such as Leontief (1970) and Miernyk et al. (1970). Because Miernyk's scheme is a derivative (although with some important differences) from Leontief's, only the latter are presented. Tiebout's approach was ingenious, involving comparison of the structure of a regional economy A at time $t + n$ with another economy B at the present time t. He assumed that the regional economy A essentially would look like economy B in n years, and thus borrowed the structure from B as a first estimator of the future structure of region A's economy. The method suffers most from a dearth of comparative data and has even less information about the evolution of the structures of economies.

Leontief's (1970) dynamic approach adopted the following procedure:

$$X_t - A_t X_t - C_{t+1}(X_{t+1} - X_t) = f_t, \tag{1}$$

where A and C are the matrices of current and capital inputs and $\boldsymbol{X}$ and $\boldsymbol{f}$ are the vectors of output and final demand, all subscripted by time. This summary then sets off a chain of demands from present production for capital that is used in productive activity the following year. Through a process of substitution, the solution can be summarized as

$$\begin{bmatrix} X_{-m} \\ \vdots \\ X_{-2} \\ X_{-1} \\ X_0 \end{bmatrix} = \begin{bmatrix} G_{-m}^{-1} \cdots R_{-m} \cdots R_{-3} R_{-2} G_{-1}^{-1} & R_{-m} \cdots R_{-3} R_{-2} R_{-1} G_{-0}^{-1} \\ \vdots & \vdots \\ R_{-2} G_{-1}^{-1} & R_{-2} R_{-1} G_{-0}^{-1} \\ G_{-1}^{-1} & R_{-1} G_{-0}^{-1} \\ & G_{-0}^{-1} \end{bmatrix} \begin{bmatrix} f_{-m} \\ \vdots \\ f_{-2} \\ f_{-1} \\ f_0 \end{bmatrix}, \tag{2}$$

where $R_t = G_t^{-1} C_{t+1} = (I - A_t + C_{t+1})^{-1} C_{t+1}$.

Two major problems surround this formulation: The first centers on the estimation of the capital coefficients matrix C and the attendant problems of distinguishing between expansion and replacement capital. The second problem addresses the changes in the A matrix. In Leontief's system, there is no mechanism for change.

In this paper, an alternative vision is illustrated. The procedure includes consideration only of the sequence $A_0, A_1, \ldots, A_t, A_{t+1}, \ldots$ of direct input matrices for different time periods, exploits the notions of discrete time changes and corresponding temporal multipliers, and proposes temporal Leontief inverse discrete dynamics in lieu of the complexities underlying the formal structure of dynamic inverses described in (1) and (2). As noted above, the time series of input-output matrices have been derived from a comprehensive general equilibrium system that operates to change the input-output coefficients as part of the market-clearing process. Because computable general equilibrium models are becoming more widespread in their use, it would appear that, in comparison to the usual dynamic inverse, the temporal inverse does offer some enhanced prospect for empirical implementation. The proposed temporal Leontief inverse formulation is comprehensive because it is possible to trace the impact of each change in each time period to the final state of the economy.

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3. TEMPORAL MULTIPLIERS AND TEMPORAL INCREMENTS

Let $A = \|a_{ij}\|$ be an $n \times n$ matrix of direct input coefficients, $E = \|e_{ij}\|$ be a matrix of the incremental changes in the direct input coefficients between time t and t_0, $B = (I - A)^{-1} = \|b_{ij}\|$ and $B(E) = (I - A - E)^{-1} = \|b_{ij}(E)\|$ be the Leontief inverse matrices before and after changes. Hence, it is easy to see that

$$\begin{aligned} B(E) &= (I - A - E)^{-1} = [(I - A)(I - BE)]^{-1} = (I - BE)^{-1}B \\ B(E) &= (I - A - E)^{-1} = [(I - EB)(I - A)]^{-1} = B(I - EB)^{-1}. \end{aligned} \tag{3}$$

The matrices $M_L = M_L(B, E) = (I - BE)^{-1}$, $M_R = M_R(B, E) = (I - EB)^{-1}$ are called the *left* and *right* temporal multipliers. Obviously,

$$B(E) = M_L B = BM_R; \qquad M_L = B(E)(I - A); \qquad M_R = (I - A)B(E). \tag{4}$$

The multiplicative representations (4) of the Leontief inverse $B(E)$ can be converted into the following additive decompositions:

$$\begin{aligned} B(E) &= M_L B = B + (M_L - I)B \\ B(E) &= BM_R = B + B(M_R - I). \end{aligned} \tag{5}$$

Using the relations

$$M_L(I - BE) = (I - BE)M_L = I; \qquad M_R(I - EB) = (I - EB)M_R = I,$$

one obtains

$$M_L - I = M_L BE = BEM_L; \qquad M_R - I = M_R EB = EBM_R. \tag{6}$$

Therefore,

$$B(E) = B + D(B, E), \tag{7}$$

where

$$D(B, E) = B(E) - B = (M_L - I)B = M_L BEB = BEBM_R. \tag{8}$$

The term $D(B, E)$ can be referred to as the temporal increment.

4. MULTIPLICATIVE AND ADDITIVE FORMS OF THE TEMPORAL LEONTIEF INVERSE

A definition of the temporal Leontief inverse can be introduced in the following form: Consider a sequence of time periods, $t_0, t_1, \ldots, t_T$ such that, in the initial period t_0, there exists a matrix $A_0 = \|a_{ij}^0\|$ of direct input coefficients, and an associated Leontief inverse matrix $B_0 = (I - A_0)^{-1}$. In each time period t_s, there are matrices of changes in direct coefficients, $E_s = \|e_{ij}^s\|$, such that the matrices $A_s = \|a_{ij}^s\|$ of direct inputs and Leontief inverses, $B_s = (I - A_s)^{-1} = \|b_{ij}^s\|$, will have the form

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$$\begin{aligned} A_s &= A_{s-1} + E_s = A_0 + E_1 + E_2 + \cdots + E_s, \qquad s = 1, 2, \ldots, T, \\ B_s &= (I - A_{s-1} - E_s)^{-1} = (I - A_0 - E_1 - E_2 - \cdots - E_s)^{-1}. \end{aligned} \tag{9}$$

The sequential application of (4) provides the following *multiplicative* decomposition of the temporal Leontief inverse:

$$\begin{aligned} B_s &= M_L^s B_{s-1} \\ &= M_L^s M_L^{s-1} B_{s-2} \\ &\vdots \\ &= M_L^s M_L^{s-1} \cdots M_L^2 M_L^1 B_0, \end{aligned} \tag{10}$$

where temporal multipliers M_L^s are defined as

$$M_L^s = (I - B_{s-1} E_s)^{-1}, \qquad s = 1, 2, \ldots, T. \tag{11}$$

In turn, the sequential application of (6) provides the *additive* decomposition of the temporal Leontief inverse:

$$\begin{aligned} B_s &= B_{s-1} + D_s \\ &= B_{s-2} + D_{s-1} + D_s \\ &\vdots \\ &= B_1 + D_2 + \cdots + D_{s-1} + D_s \\ &= B_0 + D_1 + D_2 + \cdots + D_{s-1} + D_s, \end{aligned} \tag{12}$$

where the temporal increments D_s are defined as

$$D_s = D(B_{s-1}, E_s) = B_s - B_{s-1} = \left(M_L^s - I\right) B_{s-1}, \qquad s = 1, 2, \ldots, T. \tag{13}$$

Using (11) one can obtain from (12) the following:

$$\begin{aligned} B_s = {} & \underbrace{I + (B_0 - I)} + \left(M_L^1 - I\right) B_0 + \left(M_L^2 - I\right) M_L^1 B_0 + \cdots \\ & + \left(M_L^s - I\right) M_L^{s-1} \cdots M_L^2 M_L^1 B_0 \\ = {} & \underbrace{B_0 + \left(M_L^1 - I\right) B_0} + \left(M_L^2 - I\right) M_L^1 B_0 + \cdots \\ & + \left(M_L^s - I\right) M_L^{s-1} \cdots M_L^2 M_L^1 B_0 \\ = {} & \underbrace{\begin{array}{l} B_1 + \left(M_L^2 - I\right) M_L^1 B_0 + \cdots \\ \qquad\qquad \vdots \end{array}} \\ & + \left(M_L^s - I\right) M_L^{s-1} \cdots M_L^2 M_L^1 B_0 \\ = {} & B_{s-1} + \left(M_L^s - I\right) M_L^{s-1} \cdots M_L^2 M_L^1 B_0. \end{aligned} \tag{14}$$

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This representation provides for an interpretation of the temporal Leontief inverse that shares a common feature with its dynamic cousin: the inverse depends on its *evolutionary tail* of changes and this dependence is highly nonlinear. Equations (11) and (13) can serve as the bases for temporal analysis of an evolving input-output system. For example, if $\boldsymbol{f}_s$ is the final demand vector in the sth time period, then the corresponding gross output vector $\boldsymbol{x}_s = B_s f_s$ can be decomposed into a sum of the effects of the first time period, the second time period through to the sth time-period increments

$$\begin{aligned} x_s = B_s f_s = {} & f_s \\ & + (B_0 - I) f_s \\ & + \left(M_L^1 - I\right) B_0 f_s \\ & + \left(M_L^2 - I\right) M_L^1 B_0 f_s \\ & \qquad \vdots \\ & + \left(M_L^s - I\right) M_L^{s-1} \cdots M_L^2 M_L^1 B_0 f_s. \end{aligned} \tag{15}$$

5. FINE STRUCTURE OF TEMPORAL INCREMENTS $D(B, E)$

This section addresses the possibilities of tracing the impact of each change in the individual direct inputs in each time period through to the final state of the economy. The analytical technique that is used uncovers the fine structure of the temporal multipliers and temporal increments. This analytical technique represents the basics of the perturbation theory for matrix inversion. The fine structure of the increment, $D(B, E)$, is revealed through the following decomposition (see Sonis and Hewings, 1989):

$$D(B, E) = \frac{1}{Q(E)} \left[\sum_{k=1}^{n} \sum_{\substack{i_r < i_{r+1} \\ j_r \neq j_s}}{}' F(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k) e_{j_1 i_1} e_{j_2 i_2} \cdots e_{j_k i_k} \right], \tag{16}$$

where

$$\begin{aligned} Q(E) = \frac{\det B}{\det B(E)} = 1 - \sum_{i_1 j_1} b_{j_1 i_1} e_{i_1 j_1} \\ + \sum_{k=2}^{n} (-1)^k \sum_{\substack{i_r < i_{r+1} \\ j_r \neq j_s}}{}' B_{\mathrm{or}}(j_1, j_2, \ldots, j_k; i_1, i_2, \ldots, i_k) e_{i_1 j_1} e_{i_2 j_2} \cdots e_{i_k j_k} \end{aligned} \tag{17}$$

and where $B_{\mathrm{or}}(j_1, j_2, \ldots, j_k; i_1, i_2, \ldots, i_k)$ is a determinant of order k that includes the components of the Leontief inverse B from the ordered set of columns $i_1, i_2, \ldots, i_k$ and rows $j_1, j_2, \ldots, j_k$; further, in the sum $\sum'$, the products of

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changes $e_{i_1 j_1} e_{i_2 j_2} \cdots e_{i_k j_k}$ that differ only by the order of multiplication are counted only once. Furthermore, F, the matrix field of influence of the incremental changes $e_{i_1 j_1} e_{i_2 j_2} \cdots e_{i_k j_k}$ include the components

$$\begin{aligned} f_{ij}(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k) &= (-1)^k [B_{\text{or}}(i_1, i_2, \ldots, i_k, i; j_1, j_2, \ldots, j_k, j) \\ &\quad - b_{ij} B_{\text{or}}(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k)] \\ &= (-1)^k \begin{vmatrix} b_{i_1 j_1} & b_{i_1 j_2} & \ldots & b_{i_1 j_k} & b_{i_1 j} \\ b_{i_2 j_1} & b_{i_2 j_2} & \ldots & b_{i_2 j_k} & b_{i_2 j} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ b_{i_k j_1} & b_{i_k j_2} & \ldots & b_{i_k j_k} & b_{i_k j} \\ b_{i j_1} & b_{i j_2} & \ldots & b_{i j_k} & 0 \end{vmatrix}, \qquad i, j = 1, 2, \ldots, n. \end{aligned} \tag{18}$$

Remark 1. It is easy to see, from (18), that the fields of influence and the determinant $B_{\text{or}}(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k)$ are identical with the accuracy of sign for all possible products $e_{i_1 j_1} e_{i_2 j_2} \cdots e_{i_k j_k}$ or $e_{j_1 i_1} e_{j_2 i_2} \cdots e_{j_k i_k}$. These elements belong to the same determinant,

$$\begin{vmatrix} e_{i_1 j_1} & e_{i_1 j_2} & \cdots & e_{i_1 j_k} \\ e_{i_2 j_1} & e_{i_2 j_2} & \cdots & e_{i_2 j_k} \\ \vdots & \vdots & \cdots & \vdots \\ e_{i_k j_1} & e_{i_k j_2} & \cdots & e_{i_k j_k} \end{vmatrix}. \tag{19}$$

Therefore, it is possible to write (16) and (17) in the form

$$D(B, E) = \frac{1}{Q(E)} \times \left[\sum_{k=1}^{n} \sum_{\substack{i_r < i_{r+1} \\ j_r \neq j_s}}{}' F(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k) \begin{vmatrix} e_{i_1 j_1} & e_{i_1 j_2} & \cdots & e_{i_1 j_k} \\ e_{i_2 j_1} & e_{i_2 j_2} & \cdots & e_{i_2 j_k} \\ \vdots & \vdots & \cdots & \vdots \\ e_{i_k j_1} & e_{i_k j_2} & \cdots & e_{i_k j_k} \end{vmatrix} \right], \tag{20}$$

where

$$\begin{aligned} Q(E) &= 1 - \sum_{i_1 j_1} b_{j_1 i_1} e_{i_1 j_1} \\ &\quad + \sum_{k=2}^{n} (-1)^k \sum_{\substack{i_r < i_{r+1} \\ j_r \neq j_s}}{}' B_{\text{or}}(j_1, j_2, \ldots, j_k; i_1, i_2, \ldots, i_k) \begin{vmatrix} e_{i_1 j_1} & e_{i_1 j_2} & \cdots & e_{i_1 j_k} \\ e_{i_2 j_1} & e_{i_2 j_2} & \cdots & e_{i_2 j_k} \\ \vdots & \vdots & \cdots & \vdots \\ e_{i_k j_1} & e_{i_k j_2} & \cdots & e_{i_k j_k} \end{vmatrix}. \end{aligned} \tag{21}$$

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Further, equation (18) provides for the following description of the structure of the fields of influence:

(1) The first-order field of influence, $F(i_1, j_1)$, associated with the increment, $e_{j_1 i_1}$ is the matrix generated by the multiplication of the j_1th column of the Leontief inverse B with its i_1th row:

$$F[i_1, j_1] = \begin{pmatrix} b_{1 j_1} \\ b_{2 j_2} \\ \vdots \\ b_{n j_1} \end{pmatrix} \begin{pmatrix} b_{i_1 1} & b_{i_1 2} & \cdots & b_{i_1 n} \end{pmatrix}. \tag{22}$$

This means that if the change e occurs in only one place, (i_1, j_1), then the Leontief inverse $B(e)$ has the form

$$B(e) = B + \frac{e}{1 - b_{j_1 i_1} e} F[i_1, j_1] \tag{23}$$

or, in the original Sherman and Morrison (1949, 1950) coordinate form,

$$b_{ij}(e) = b_{ij} + \frac{b_{i j_1} b_{i_1 j} e}{1 - b_{j_1 i_1} e}. \tag{24}$$

(2) In the case in which changes occur in the i_1th row of the A matrix, $(e_{i_1 1}, e_{i_1 2}, \ldots, e_{i_1 n})$, then

$$B(E) = B + \frac{\sum_{s=1}^{n} e_{i_1 s} F[s; i_1]}{1 - \sum_{s=1}^{n} b_{s i_1} e_{i_1 s}}. \tag{25}$$

or, in the original Sherman and Morrison (1949) coordinate form,

$$b_{ij}(E) = b_{ij} + \frac{b_{i i_1} \sum_{s=1}^{n} b_{sj} e_{i_1 s}}{1 - \sum_{j=1}^{n} b_{s i_1} e_{i_1 s}}. \tag{26}$$

(3) The second-order synergism between two incremental changes, $e_{j_1 i_1}$ and $e_{j_2 i_2}$, is reduced to the following linear combination of four first-order fields of influence:

$$F(i_1, i_2; j_1, j_2) = -b_{i_1 j_1} F(i_2, j_2) + b_{i_1 j_2} F(i_2, j_1) + b_{i_2 j_1} F(i_1, j_2) - b_{i_2 j_2} F(i_1, j_1). \tag{27}$$

(4) The following recurrent formula provides the possibility of presenting the field of influence of order k through the use of fields of influence of lesser orders, 1, 2, $\ldots, k-1$:

$$F(i_1, \ldots, i_k; j_1, \ldots, j_k) = \frac{1}{k-1} \sum_{s=1}^{k} \sum_{r=1}^{k} (-1)^{s+r+1} b_{i_s j_r} \times F(i_1, \ldots, i_{s-1}, i_{s+1}, \ldots, i_k; j_1, \ldots, j_{r-1}, j_{r+1}, \ldots, j_k). \tag{28}$$

The proofs are provided in Section 6.

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Remark 2. The Sherman-Morrison formula (23) can be generalized to the case of the multiregional input-output system using the block matrix of direct inputs,

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{n1} & A_{nn} & \cdots & A_{nn} \end{pmatrix}, \tag{30}$$

and a block matrix, E, of incremental changes represented, in this case, in the ith row:

$$E = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ E_{i_0 1} & E_{1_0 j} & \cdots & \vdots & E_{i_0 n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{31}$$

Matrix E represents changes in the forward linkages for the ith region; the case for backward linkage changes can be treated in a similar fashion. The generalized Sherman-Morrison formula has a form (Sonis and Hewings, 1995),

$$B(E) = B + \begin{bmatrix} B_{1i_0} \\ B_{2i_0} \\ \vdots \\ B_{ni_0} \end{bmatrix} \Delta(E) \sum_{s=1}^{n} E_{i_0 s} [B_{s1}, B_{s2}, \ldots, B_{sn}], \tag{32}$$

where

$$\begin{bmatrix} B_{1i_0} \\ B_{2i_0} \\ \vdots \\ B_{ni_0} \end{bmatrix} \quad \text{and} \quad [B_{s1}, B_{s2}, \ldots, B_{sn}], \qquad s = 1, 2, \ldots, n$$

are the rectangular matrices representing the i_0th block column and block rows of the Leontief inverse, B, and

$$\Delta(E) = \left(I - \sum_{s=1}^{n} E_{i_0 s} B_{s i_0} \right)^{-1}. \tag{33}$$

Comparison of the Sherman-Morrison form (25) with its generalized block form (32) indicates that the transfer from the single-region to the multiregional economy leads to the separation of the column and row blocks in the block fields of influence.

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6. PROOFS

6.1. Auxiliary Propositions

For the proofs, we need three general properties for an arbitrary matrix $C = (c_{ij})$ and its determinant $\det C$. The first property is:

$$\frac{\partial \det C}{\partial c_{ij}} = C_{ij}, \tag{34}$$

where C_{ij} are the cofactors of the elements c_{ij}. This property follows immediately from the well-known Laplace decomposition for the determinant of C,

$$\det C = c_{i1}C_{i1} + \cdots + c_{ij}C_{ij} + \cdots + c_{in}C_{in}$$

by differentiating $\det C$ by c_{ij}.

The property (34) may be generalized by induction:

$$\frac{\partial^k \det C}{\partial c_{i_1 j_1} \partial c_{i_2 j_2} \cdots \partial c_{i_k j_k}}$$

$$= \begin{cases} (-1)^z M[(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)] & i_r \neq i_s,\ j_r \neq j_s \\ 0 & \text{otherwise} \end{cases}, \tag{35}$$

where $z = i_1 + i_2 + \cdots + i_k + j_1 + j_2 + \cdots + j_k + d(i_1 i_2 \cdots i_k) + d(j_1 j_2 \cdots j_k)$ and where $M[(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)]$ is a minor derived from $\det C$ by removal of rows $(i_1 i_2 \cdots i_k)$ and columns $(j_1 j_2 \cdots j_k)$ and $d(l_1 l_2 \cdots l_k)(l_s = i_s \text{or} l_s = j_s, s = 1, 2, \ldots, k)$ is the disorder of the sequence $(l_1, l_2, \ldots, l_k)$ which is equal to the number of inversions, i.e., to the number of pairwise transpositions of integers, $(l_1, l_2, \ldots, l_k)$, that converts this sequence into a totally ordered sequence $(l'_1 < l'_2 < \cdots < l'_k)$.

The conditions (34) and (35) can be simplified in the case of an invertible matrix C by using the following relationships [see, for example, Gantmacher (1966)]:

$$(-1)^{i_1+i_2+\cdots+i_k+j_1+j_2+\cdots+j_k} M[(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)]$$

$$= \det C \bullet B[(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)], \tag{36}$$

where $B[(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)]$ are the minors of the matrix $B = C^{-1}$, including components from the columns $(i_1 i_2 \cdots i_k)$ and rows $(j_1 j_2 \cdots j_k)$ of matrix B.

Then, the property (34) can be written in the following form:

$$\frac{\partial \det C}{\partial c_{ij}} = \det C \bullet b_{ji}, \tag{37}$$

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and the property (35) can be presented in the form

$$\frac{\partial^k \det C}{\partial c_{i_1 j_1}\partial c_{i_2 j_2}\cdots\partial c_{i_k j_k}} = \begin{cases} (-1)^d \det C \bullet B[(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)] & i_r \neq i_s,\ j_r \neq j_s \\ 0 & \text{otherwise} \end{cases}, \quad \textbf{(38)}$$

where $d = d(i_1 i_2 \cdots i_k) + d(j_1 j_2 \cdots j_k)$.

In (38), the minor $B[(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)]$ includes the components of columns $i_1, i_2, \ldots, i_k$ and rows $j_1, j_2, \ldots, j_k$ in their natural order as in matrix B. The change in this natural order into the order corresponding to the order of ∂c_{ij} in the derivative

$$\frac{\partial^k \det C}{\partial c_{i_1 j_1}\partial c_{i_2 j_2}\cdots\partial c_{i_k j_k}}$$

implies the disappearance of the multiplier $(-1)^d$. Therefore, instead of the expression $(-1)^d B[(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)]$, one can use the determinant of $B_{\text{or}}[(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)]$ that includes the rows and columns from $B[(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)]$ in their order in the derivative

$$\frac{\partial^k \det C}{\partial c_{i_1 j_1}\partial c_{i_2 j_2}\cdots\partial c_{i_k j_k}}.$$

Moreover, if the determinant $B_{\text{or}}[(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)]$ includes two identical rows and columns, $j_r = j_s$ or $i_r = i_s$, then it is equal to zero. Therefore, the property (33) can be rewritten in the form

$$\frac{\partial^k \det C}{\partial c_{i_1 j_1}\partial c_{i_2 j_2}\cdots\partial c_{i_k j_k}} = \det C \bullet B_{\text{or}}[(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)]. \quad \textbf{(39)}$$

Further, each component b_{ij} of the inverse matrix $B = C^{-1}$ can be considered as a scalar function of the matrix $C = (c_{pq})$, where the matrix C is considered as a point in n^2-dimensional space. Therefore, the gradient of this function, $\text{grad}[b_{ij}(C)]$, is an n^2-dimensional vector with components

$$\frac{\partial b_{ij}(C)}{\partial c_{pq}}.$$

The properties (34) and (38) imply that

$$\begin{aligned} \frac{\partial b_{ij}(C)}{\partial c_{pq}} &= \frac{\partial}{\partial c_{pq}}\left[\frac{\partial \det C}{\partial c_{ij}} \Big/ \det C\right] \\ &= \frac{\partial^2 \det C}{\partial c_{pq}\partial c_{ji}} \Big/ \det C - \frac{\partial \det C}{\partial c_{ji}} \bullet \frac{\partial \det C}{\partial c_{pq}} \Big/ (\det C)^2 \\ &= B_{\text{or}}\begin{pmatrix} q & i \\ p & j \end{pmatrix} - b_{ij} b_{pq} = \begin{vmatrix} b_{qp} & b_{qj} \\ b_{ip} & b_{ij} \end{vmatrix} - b_{ij} b_{pq} = -b_{ip} b_{qj}. \end{aligned}$$

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Let us construct the matrix, including all components of the vector grad$[b_{ij}(C)]$, according to the following rule:

For the pq-coordinate of the gradient,

$$[\operatorname{grad} b_{ij}(C)]_{pq} = \frac{\partial b_{ij}(C)}{\partial c_{pq}} \qquad p, q = 1, 2, \ldots, n.$$

Then,

$$\operatorname{grad} b_{ij}(C) = -\begin{pmatrix} b_{i1}b_{1j} & b_{i2}b_{1j} & \ldots & b_{in}b_{1j} \\ b_{i1}b_{2j} & b_{i2}b_{2j} & \ldots & b_{in}b_{2j} \\ \vdots & \vdots & \cdots & \vdots \\ b_{i1}b_{nj} & b_{i2}b_{nj} & \ldots & b_{in}b_{nj} \end{pmatrix}. \tag{40}$$

It is easy to see that the grad$[b_{ij}(C)]$ is the product of the jth column of the matrix B and its ith row:

$$\operatorname{grad}\ b_{ij}(C) = -\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}(b_{i1} \quad b_{i2} \quad \cdots \quad b_{in}). \tag{41}$$

For $C = I - A$, the gradient of b_{ij}, considered to be scalar function of A, has the components

$$\frac{\partial b_{ij}(A)}{\partial a_{pq}} = \frac{\partial b_{ij}}{\partial c_{pq}} \frac{\partial c_{pq}}{\partial a_{pq}} = b_{ip}b_{qj}$$

and

$$\operatorname{grad} b_{ij}(A) = -\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}(b_{i1} \quad b_{i2} \quad \cdots \quad b_{in}). \tag{42}$$

6.2. Proof of equation (17)

It is possible to see, from the definition of the determinant, that the $\det(I - A - E) = \det C(E) = D(E)$ is a polynomial of degree n in the increments $e_{ij}(i, j = 1, 2, \ldots, n)$:

$$D(E) = \alpha_0 + \sum_{i_1 j_1} \alpha_{i_1 j_1} e_{i_1 j_1} + \sum_{k=2}^{n} \sum_{\substack{i_r < i_{r+1} \\ j_r \neq j_s}}{}' \alpha_{i_1 j_1 i_2 j_2 \cdots i_k j_k} e_{i_1 j_1} e_{i_2 j_2} \cdots e_{i_k j_k}, \tag{43}$$

where, in $\sum'$, the products $e_{i_1 j_1} e_{i_2 j_2} \cdots e_{i_k j_k}$ appear only once.

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Consider the polynomial

$$Q(E) = \frac{D(E)}{D} = \beta_0 + \sum_{i_1 j_1} \beta_{i_1 j_1} e_{i_1 j_1} + \sum_{k=2}^{n} \sum_{\substack{i_r < i_{r+1} \\ j_r \neq j_s}}{}' \beta_{i_1 j_1 i_2 j_2 \cdots i_k j_k} e_{i_1 j_1} e_{i_2 j_2} \cdots e_{i_k j_k}, \tag{44}$$

where $D = \det(I - A)$ and $\beta_{i_1 j_1 i_2 j_2 \cdots i_k j_k} = \alpha_{i_1 j_1 i_2 j_2 \cdots i_k j_k}/D$. Obviously, for $E = 0$, $\alpha_0 = D(0) = D$ and $\beta_0 = 1$. For the evaluation of the coefficient $\alpha_{i_1 j_1 i_2 j_2 \cdots i_k j_k}$, one can take the partial derivative of the order k from $D(E)$ by $\partial e_{i_1 j_1} \partial e_{i_2 j_2} \cdots \partial e_{i_k j_k}$ and put $E = 0$:

$$\alpha_{i_1 j_1 i_2 j_2 \cdots i_k j_k} = \left. \frac{\partial D(E)}{\partial e_{i_1 j_1} \partial e_{i_2 j_2} \cdots \partial e_{i_k j_k}} \right|_{E=0}. \tag{45}$$

Then, (43) implies that

$$\begin{aligned} \beta_{i_1 j_1 i_2 j_2 \cdots i_k j_k} &= \frac{1}{D} \frac{\partial D(E)}{\partial c_{i_1 j_1}(E) \cdots \partial c_{i_k j_k}(E)} \bullet \frac{\partial c_{i_1 j_1}(E)}{\partial e_{i_1 j_1}} \cdots \frac{\partial c_{i_k j_k}(E)}{\partial e_{i_k j_k}} \Bigg|_{E=0} \\ &= \frac{(-1)^k}{D} \bullet \frac{\partial D}{\partial c_{i_1 j_1} \cdots \partial c_{i_k j_k}} \\ &= (-1)^k B_{\text{or}}[(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)]. \end{aligned} \tag{46}$$

Therefore, (41) implies that

$$\begin{aligned} Q(E) &= \frac{D(E)}{D} \\ &= 1 - \sum_{i_1 j_1} b_{j_1 i_1} e_{i_1 j_1} \\ &\quad + \sum_{k=2}^{n} (-1)^k \sum_{\substack{i_r < i_{r+1} \\ i_s \neq j_s}}{}' B_{\text{or}}[(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)] e_{i_1 j_1} \cdots e_{i_k j_k}. \end{aligned} \tag{47}$$

6.3. Proof of (16)

The property (34) provides that

$$\frac{\partial D(E)}{\partial e_{ji}} = \frac{\partial D(E)}{\partial c_{ji}(E)} \frac{\partial c_{ji}(E)}{\partial e_{ji}} = -C_{ji}(E), \tag{48}$$

where $C_{ji}(E)$ is the cofactor of the element $c_{ji}(E) = c_{ji} - e_{ji}$. Therefore, from (48) and (44),

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$$\frac{C_{ji}(E)}{D} = \frac{1}{D}\frac{\partial D(E)}{\partial e_{ji}} = \frac{\partial Q(E)}{\partial e_{ji}}$$
$$= b_{ij} - \sum_{i_1 j_1} B_{\mathrm{or}}[(i_1, j_1), (i, j)]e_{j_1 i_1}$$
$$+ \sum_{k=2}^{n-1}(-1)^k \sum_{\substack{i_r \neq j_r \\ j_r \neq j_{r+1}}}{}' B_{\mathrm{or}}[(i_1, j_1), \ldots, (i_k, j_k)(i, j)]e_{j_1 i_1} \cdots e_{j_k i_k}$$
$$= b_{ij} + R_{ij}(E),$$

where

$$R_{ij} = -\sum_{i_1 j_1} B_{\mathrm{or}}[(i_1, j_1), (i, j)]e_{j_1 i_1}$$
$$+ \sum_{k=2}^{n-1}(-1)^k \sum_{\substack{i_r \neq j_r \\ j_r \neq j_{r+1}}}{}' B_{\mathrm{or}}[(i_1, j_1), \ldots, (i_k, j_k)(i, j)]e_{j_1 i_1} \cdots e_{j_k i_k}. \tag{49}$$

Further,

$$b_{ij}(E) = \frac{C_{ji}(E)}{D(E)} = \frac{C_{ji}(E)}{D} \Big/ \frac{D(E)}{D} = \frac{b_{ij} + R_{ij}(E)}{Q(E)}$$

and

$$b_{ij}(E) - b_{ij} = \frac{1}{Q(E)}[b_{ij} + R_{ij}(E) - b_{ij}Q(E)] = \frac{P_{ij}(E)}{Q(E)},$$

where, from (49),

$$P_{ij}(E) = b_{ij} + R_{ij}(E) - b_{ij}Q(E)$$
$$= b_{ij} - \sum_{i_1 j_1} B_{\mathrm{or}}[(i_1, j_1), (i, j)]e_{j_1 i_1}$$
$$+ \sum_{k=2}^{n-1}(-1)^k \sum_{\substack{i_r \neq j_r \\ j_r < j_{r+1}}}{}' B_{\mathrm{or}}[(i_1, j_1), \ldots, (i_k, j_k)(i, j)]e_{j_1 i_1} \cdots e_{j_k i_k}$$
$$- b_{ij}\left\{1 - \sum_{i_1 j_1} b_{i_1 j_1} e_{j_1 i_1} + \sum_{k=2}^{n-1}(-1)^k \right.$$
$$\left. \times \sum_{\substack{i_r \neq j_r \\ j_r < j_{r+1}}}{}' B_{\mathrm{or}}[(i_1, j_1), \ldots, (i_k, j_k)(i, j)]e_{j_1 i_1} \cdots e_{j_k i_k} + (-1)^n \det B \det E \right\}$$

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$$= \sum_{i_1 j_1} \{b_{ij} b_{i_1 j_1} - B_{\text{or}}[(i_1, j_1), (i, j)]\} e_{j_1 i_1} + \sum_{k=2}^{n-1} (-1)^k$$
$$\times \sum_{\substack{i_r \neq j_r \\ j_r < j_{r+1}}}{}' \{B_{\text{or}}[(i_1, j_1), \ldots, (i_k, j_k)(i, j)]$$
$$- b_{ij} B_{\text{or}}[(i_1, j_1), \ldots, (i_k, j_k)]\} e_{j_1 i_1} \cdots e_{j_k i_k} + (-1)^{n+1} b_{ij} \det B \det E. \tag{50}$$

Let us denote for each i, j and $k = 1, 2, \ldots, n$,

$$f_{ij}[(i_1, j_1), \ldots, (i_k, j_k)]$$
$$= (-1)^k \{B_{\text{or}}[(i_1, j_1), \ldots, (i_k, j_k)(i, j)] - b_{ij} B_{\text{or}}[(i_1, j_1), \ldots, (i_k, j_k)]\}. \tag{51}$$

Then, (50) becomes

$$P_{ij}(E) = \sum_{i_1 j_1} f_{ij}[(j_1, i_1)] e_{j_1 i_1} + \sum_{k=2}^{n-1} \sum_{\substack{i_r \neq j_r \\ j_r < j_{r+1}}}{}' f_{ij}[(i_1, j_1), \ldots, (i_k, j_k)] e_{j_1 i_1} \cdots e_{j_k i_k} \tag{52}$$

and

$$b_{ij}(E) = b_{ij} + \frac{P_{ij}(E)}{Q(E)}. \tag{53}$$

The condition (53) provides the coordinate form of the matrix condition (16); if the matrix fields of influence of changes, $e_{j_1 i_1}, \ldots, e_{j_k i_k}$, are constructed,

$$F[(i_1, j_1), \ldots, (i_k, j_k)] = \| f_{ij}[(i_1, j_1), \ldots, (i_k, j_k)] \|, \qquad k = 1, 2, \ldots, n, \tag{54}$$

then the fine structure of the fields of influence will be revealed.

6.4. Proofs for (22), (27), and (28)

From (51), the coefficient of the increment $e_{j_1 i_1}$ is

$$f_{ij}[j_1, i_1] = b_{ij} b_{i_1 j_1} - \begin{vmatrix} b_{i_1 j_1} & b_{i_1 j} \\ b_{i j_1} & b_{ij} \end{vmatrix} = b_{i j_1} b_{i_1 j}. \tag{55}$$

In matrix form,

$$F[j_1, i_1] = \begin{pmatrix} b_{1 j_1} \\ b_{2 j_1} \\ \vdots \\ b_{n j_1} \end{pmatrix} (b_{i_1 1} \quad b_{i_1 2} \quad \cdots \quad b_{i_1 n}) = b_{i j_1} b_{i_1 j}. \tag{56}$$

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Analogously, the coefficient of the product $e_{j_1 i_1} e_{j_2 i_2}$ in (51) is of the form

$$\begin{aligned} f_{ij}[(j_1, i_1), (j_2, i_2)] &= B_{\text{or}}[(j_1, i_1), (j_2, i_2), (i, j)] - b_{ij} B_{\text{or}}[(j_1, i_1), (j_2, i_2)] \\ &= \begin{vmatrix} b_{i_1 j_1} & b_{i_1 j_2} & b_{i_1 j} \\ b_{i_2 j_1} & b_{i_2 j_2} & b_{i_2 j} \\ b_{i j_1} & b_{i j_2} & 0 \end{vmatrix} \\ &= b_{i_1 j_2} b_{i j_1} b_{i_2 j} + b_{i_2 j_1} b_{i j_2} b_{i_1 j} - b_{i_2 j_2} b_{i j_1} b_{i_1 j} - b_{i_1 j_1} b_{i j_2} b_{i_2 j} \\ &= b_{i_1 j_2} f_{ij}[(j_1, i_2)] + b_{i_2 j_1} f_{ij}[(j_2, i_1)] - b_{i_2 j_2} f_{ij}[(j_1, i_1)] \\ &\quad - b_{i_1 j_1} f_{ij}[(j_2, i_2)], \end{aligned}$$

which is the coordinate form of (35). Further, the difference of the determinants in the definition of the components f_{ji} from (43) can be presented in an alternative form:

$$\begin{aligned} \text{Det} &= B_{\text{or}}[(j_1, i_1), \ldots, (j_k, i_k), (j, i)] - b_{ij} B_{\text{or}}[(j_1, i_1), \ldots, (j_k, i_k)] \\ &= \begin{vmatrix} b_{i_1 j_1} & \cdots & b_{i_1 j_k} & b_{i_1 j} \\ \vdots & \cdots & \vdots & \vdots \\ b_{i_k j_1} & \cdots & b_{i_k j_k} & b_{i_k j} \\ b_{i j_1} & \cdots & b_{i j_k} & 0 \end{vmatrix}. \end{aligned} \tag{57}$$

Let $A_{sr} = (-1)^{s+r} M_{sr}, s, r = 1, 2, \ldots, k+1$, represent the cofactors of the elements of the determinant in (57). Using the Laplace decomposition of Det corresponding to column j, the following can be developed:

$$\text{Det} = \sum_{s=1}^{k} b_{i_s j} A_{s,k+1}. \tag{58}$$

Decomposing Det with the help of rows 1,2, ..., k,

$$\text{Det} = \sum_{r=1}^{k+1} b_{i_s j_r} A_{sr}, \qquad s = 1, 2, \ldots, k; \tag{59}$$

then, (43) and (44) imply that

$$k\,\text{Det} = \sum_{s=1}^{k} \sum_{r=1}^{k+1} b_{i_s j_r} A_{sr} = \sum_{s=1}^{k} \sum_{r=1}^{k+1} b_{i_s j_r} A_{sr} + \sum_{s=1}^{k} b_{i_s j} A_{s,k+1} = \sum_{s,r=1}^{k} b_{i_s j_r} A_{sr} + D$$

and

$$\text{Det} = \frac{1}{k-1} \sum_{s,r=1}^{k} (-1)^{s+r} b_{i_s j_r} M_{sr}, \tag{60}$$

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where the minors M_{sr} correspond to the cofactors A_{sr}. The minors M_{sr} have the same structure as the determinant Det, and thus it is easy to see that

$$f_{ij}[(j_1, i_1), \ldots, (j_{s-1}, i_{s-1}), (j_{s+1}, i_{s+1}), \ldots, (j_k, i_k)] = (-1)^{k-1} M_{sr}. \quad \textbf{(61)}$$

Thus, the relationships (51) and (60) provide, for $k > 1$, that

$$\begin{aligned} f_{ij}[(j_1, i_1), \ldots, (j_k, i_k)] = (-1)^k D &= \frac{(-1)^k}{k-1} \sum_{s,r=1}^{k} (-1)^{s+r} b_{i_s j_r} M_{sr} \\ &= \frac{1}{1-k} \sum_{s,r=1}^{k} (-1)^{s+r+1} b_{i_s j_r} f_{ij} \\ &\quad \times [(j_1, i_1), \ldots, (j_{s-1}, i_{s-1}), (j_{s+1}, i_{s+1}), \ldots, (j_k, i_k)]. \end{aligned} \quad \textbf{(62)}$$

or, in matrix form, the following familiar presentation can be provided:

$$\begin{aligned} &F[(j_1, i_1), \ldots, (j_k, i_k)] \\ &= \frac{1}{1-k} \sum_{s,r=1}^{k} (-1)^{s+r+1} b_{i_s j_r} \\ &\quad \times F_{ij}[(j_1, i_1), \ldots, (j_{s-1}, i_{s-1}), (j_{s+1}, i_{s+1}), \ldots, (j_k, i_k)], \qquad k \geq 2. \end{aligned} \quad \textbf{(63)}$$

7. FINE STRUCTURE OF TEMPORAL MULTIPLIERS

The fine structure of temporal multipliers M_L and M_R can be obtained easily from the properties of the fine structure of the temporal increment. Consider, for example, the left temporal multiplier, M_L (consideration of the right multiplier is similar). Formulas (2), (5), (7), and (8) imply that

$$\begin{aligned} M_L(B, E) &= B(E)(I - A) = [B + D(B, E)](I - A) = I + D(B, E)(I - A) \\ &= I + \frac{1}{Q(E)} \left[\sum_{k=1}^{n} \sum_{\substack{i_r < i_{r+1} \\ j_r \neq j_s}}' F(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k) e_{j_1 i_1} e_{j_2 i_2} \cdots e_{j_k i_k} \right] (I - A) \\ &= I + \left[\sum_{k=1}^{n} \sum_{\substack{i_r < i_{r+1} \\ j_r \neq j_s}}' G_L(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k) e_{j_1 i_1} e_{j_2 i_2} \cdots e_{j_k i_k} \right], \end{aligned} \quad \textbf{(64)}$$

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where

$$Q(E) = \frac{\det B}{\det B(E)} = \det(I - BE) = \det(I - EB) \tag{65}$$

and

$$G_L(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k) = F(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k)(I - A) \tag{66}$$

are the multiplier fields of influence. Equation (61) immediately implies that

$$G_L(i_1, j_1) = F(i_1, j_1)(I - A) = \begin{pmatrix} b_{1j_1} \\ b_{2j_1} \\ \vdots \\ b_{nj_1} \end{pmatrix} (b_{i_1 1} \quad b_{i_1 2} \quad \cdots \quad b_{i_1 n})(I - A)$$

$$= \begin{pmatrix} b_{1j_1} \\ b_{2j_1} \\ \vdots \\ b_{nj_1} \end{pmatrix} (0, \ldots, 1, 0, \ldots, 0) = \begin{pmatrix} 0 & \cdots & 0 & b_{1j_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{2j_1} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & b_{nj_1} & 0 & \cdots & 0 \end{pmatrix}. \tag{67}$$

This matrix includes only one nonzero i_1th column,

$$\begin{pmatrix} b_{1j_1} \\ b_{2j_2} \\ \vdots \\ b_{nj_1} \end{pmatrix}.$$

Further, the components $g_{ij}(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k)$ of the multiplier fields of influence $G_L(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k)$ can be calculated using the following expression:

$$g_{ij}(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k) = \sum_{s=1}^{k} f_{is}(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k) c_{sk}, \tag{68}$$

where $f_{is}(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k)$ are the components of the field of influence $F(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k)$ and c_{sj} are the components of the matrix $I - A$. Therefore,

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$$
\begin{aligned}
&g_{ij}(i_1, i_2, \ldots, i_k;\, j_1, j_2, \ldots, j_k) \\
&\quad = \sum_{s=1}^{k} (-1)^k \begin{vmatrix} b_{i_1 j_1} & b_{i_1 j_2} & \cdots & b_{i_1 j_k} & b_{i_1 j} \\ b_{i_2 j_1} & b_{i_2 j_2} & \cdots & b_{i_2 j_k} & b_{i_2 j} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ b_{i_k j_1} & b_{i_k j_2} & \cdots & b_{i_k j_k} & b_{i_k j} \\ b_{i j_1} & b_{i j_2} & \cdots & b_{i j_k} & 0 \end{vmatrix} c_{sj} \\
&\quad = (-1)^k \begin{vmatrix} b_{i_1 j_1} & b_{i_1 j_2} & \cdots & b_{i_1 j_k} & \sum b_{i_1 j} c_{sj} \\ b_{i_2 j_1} & b_{i_2 j_2} & \cdots & b_{i_2 j_k} & \sum b_{i_2 j} c_{sj} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ b_{i_k j_1} & b_{i_k j_2} & \cdots & b_{i_k j_k} & \sum b_{i_k j} c_{sj} \\ b_{i j_1} & b_{i j_2} & \cdots & b_{i j_k} & 0 \end{vmatrix} \\
&\quad = (-1)^k \begin{vmatrix} b_{i_1 j_1} & b_{i_1 j_2} & \cdots & b_{i_1 j_k} & d_{i_1 j} \\ b_{i_2 j_1} & b_{i_2 j_2} & \cdots & b_{i_2 j_k} & d_{i_2 j} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ b_{i_k j_1} & b_{i_k j_2} & \cdots & b_{i_k j_k} & d_{i_k j} \\ b_{i j_1} & b_{i j_2 k} & \cdots & b_{i j_k} & 0 \end{vmatrix} \\
&\quad = (-1)^{k+1} \sum_{s=1}^{k} d_{i_s j} B_{\mathrm{or}}(i_1, i_2, \ldots, i_{s-1}, i, i_{s+1}, \ldots, i_k\, j_1, j_2, \ldots, j_k), \qquad \textbf{(69)}
\end{aligned}
$$

where $d_{i_s j}$ are Kronecker's symbols. Thus, the multiplier field of influence $G_L(i_1, i_2, \ldots, i_k;\, j_1, j_2, \ldots, j_k)$ includes only k nonzero $i_1, i_2, \ldots, i_k$ columns. Moreover, the following recurrent formula holds:

$$
\begin{aligned}
G_L(i_1, i_2, \ldots, i_k;\, j_1, j_2, \ldots, j_k) &= \frac{1}{k-1} \sum_{s=1}^{k} \sum_{r=1}^{k} (-1)^{r+s+1} b_{i_s j_r} \\
&\times G_L(i_1, i_2, \ldots, i_{s-1}, i_{s+1}, \ldots, i_k;\, j_1, j_2, \ldots, j_{r-1}, j_{r+1}, \ldots, j_k). \qquad \textbf{(70)}
\end{aligned}
$$

8. APPLICATION TO STRUCTURAL-PATH ANALYSIS

The field-of-influence approach provides a way to obtain the matrix form of structural-path analysis elaborated for economic systems several decades ago [see Defourny and Thorbecke (1984)], Structural-path analysis operates on the basis of the identification of an *influence graph* the elements of which are the vertices that comprise the set of sectors in the social accounting system. Interpretation of the associated coefficients in this system, such as the direct coefficients, a_{ij}, is different from that usually associated with interindustry and social accounting matrices. The influence in a graph is the strength of the connection; the magnitude of the direct influence, $J^D(j, i)$, transmitted from vertex (i.e., sector) j to vertex i through the arc $(j \rightarrow i)$ is equal to a_{ij}. In a fashion analogous to the ripple effects

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associated with a multiplier, the flow-on effects of individual transactions can be identified by defining an appropriate path (a sequence of arcs). An elementary path $p(j, i)$ from vertex j to vertex i might appear as

$$(j, k_1), (k_1, k_2), \ldots, (k_r, i), \qquad k_r \neq k_s$$

from which the total influence $J^T p(j, i)$ of vertex j on vertex i along the elementary path $p(j, i)$ is given by

$$J^T p(j, i) = a_{k_1 j} a_{k_1 k_2} \cdots a_{ik} \frac{M[p(j, i)]}{\Delta}, \tag{71}$$

where $a_{kj} \in A$ (the matrix of direct inputs); $\Delta = \det(I - A)$; and $M[p(j, i)]$ denotes the minor of the matrix $(I - A)$ by removing rows and columns $j, k_1, k_2, \ldots, k_r$ and i; $M(p)/\Delta$ is the path multiplier. The *global influence* of the vertex j on the vertex i, $J^G(j, i)$, is the component b_{ij} of the associated inverse matrix $B = (I - A)^{-1}$. Thus, the Leontief inverse of an interindustry, or the corresponding matrix in the social accounting system, can be considered as a matrix of global influences.

Because, for most interindustry and social accounting sytems, the flows exhibit complex patterns, the global influence of vertex j on vertex i can be considered equal to the sum of the total influences of vertex j on vertex i along all elementary paths, $p(j, i)$ joining j to i:

$$b_{ij} = J^G(j, i) = \sum_{p(j,i)} J^T p(j, i). \tag{72}$$

Thus, unlike the usual inverse matrix, which only summarizes the paths from j to i in terms of a scalar, b_{ij}, structural-path analysis provides the analyst with the opportunity to view the myriad pattern of linkages or ripple effects—essentially, unraveling the inverse matrix and decomposing it into a set of elementary paths. The spatial dimension appears to be important when the elementary path connects the origin from one region with a destination in another region.

The matrix form of structural-path analysis can be presented as follows: If, in (64) and (65), the matrix B is replaced by the unit matrix I, and the matrix E by A, then

$$B = (I - A)^{-1}$$

$$= I + \frac{1}{\det(I - A)} \left[\sum_{k=1}^{n} \sum_{\substack{i_r < i_{r+1} \\ j_r \neq j_s}}{}' G_L^I(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k) a_{j_1 i_1} a_{j_2 i_2} \cdots a_{j_k i_k} \right], \tag{73}$$

where matrices $G_L^I(i_1, i_2, \ldots, i_k; j_1, j_2, \ldots, j_k)$ are the multiplier fields of influence of the unit matrix I and can be computed through the use of (26) and (27).

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It can be shown that the fields of influence, $G_L^I(i;\, j)$, of the first order for the unit matrix are equal to the matrix P_{ji} with the only nonzero-unit component in the jth row and the ith column. Therefore,

$$\sum_{i_1 j_1} G_L^I(i;\, j) a_{j_1 i_1} = A$$

and

$$\begin{aligned} B &= (I - A)^{-1} \\ &= I + \frac{1}{\det(I - A)} \\ &\quad \times \left[A + \sum_{k=2}^{n} {\sum_{\substack{i_r < i_{r+1} \\ j_r \neq j_s}}}' G_L^I(i_1, i_2, \ldots, i_k;\, j_1, j_2, \ldots, j_k) a_{j_1 i_1} a_{j_2 i_2} \cdots a_{j_k i_k} \right]. \end{aligned} \tag{74}$$

Further, (70) implies that the multiplier fields of influence $G_L^I(i_1, i_2, \ldots, i_k;\, j_1, j_2, \ldots, j_k)$ are linear combinations of the matrix of the form P_{ji} with the coefficients $1/(k-1)!,\ -[1/(k-1)!], 0$.

9. PROCESS OF COMPLICATION OF STRUCTURAL PATHS

Structural-path analysis represents, in a highly disaggregated form, the complexity of multisectoral economic interdependencies; by employing a block generalization formulation, it is possible to view these interdependencies in a hierarchical fashion. A typical application would be to the case of a multiregional system of economies. Consider, for example, a two-region system represented by the following block matrix, A, of direct inputs:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{75}$$

from which it is possible to derive the well-known Schur (1917) formula,

$$B = (I - A)^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{11} A_{12} B_1 \\ B_2 A_{21} B_1 & B_{22} \end{pmatrix}, \tag{76}$$

where $B_1 = (I - A_{11})^{-1}$; $B_2 = (I - A_{22})^{-1}$ are the internal multipliers and $B_{11} = (I - A_{11} - A_{12} B_2 A_{21})^{-1}$; $B_{22} = (I - A_{22} - A_{21} B_1 A_{12})^{-1}$ are the extended Leontief multipliers for the first and second region. The expressions

$$S_1 = A_{11} + A_{12} B_2 A_{21}; \quad S_2 = A_{22} + A_{21} B_1 A_{12} \tag{77}$$

were introduced by Schur (1917) and usually are referred to as Schur complements. They can be interpreted as agumented inputs for the first and second region [Yamada and Ihara (1969); see Figure 1].

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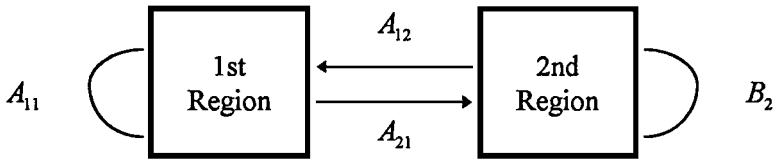


FIGURE 1. Augmentation of inputs into the first region of a two-region economy.

The relationship to the temporal Leontief inverse can be revealed by considering the extended Leontief inverses, B_{11}, B_{22}, in the form

$$\begin{aligned} B_{11} &= (I - A_{11} - A_{12}B_2A_{21})^{-1} = (I - A_{11} - E_{11})^{-1}, \\ B_{22} &= (I - A_{22} - A_{21}B_1A_{12})^{-1} = (I - A_{22} - E_{22})^{-1}, \end{aligned} \tag{78}$$

where $E_{11} = A_{12}B_2A_{21}$, $E_{22} = A_{21}B_1A_{12}$. Therefore,

$$\begin{aligned} B_{11} &= B_1 + D(B_1, E_{11}), \\ B_{22} &= B_2 + D(B_2, E_{22}), \end{aligned} \tag{79}$$

where $B_1 = (I - A_{11})^{-1}$, $B_2 = (I - A_{22})^{-1}$ are the internal (domestic) Leontief inverses and the increments $D(B_1, E_{11})$, $D(B_2, E_{22})$ represent the self-influence of regions caused by their own economic interaction. The decomposition (79) offers additional content to the description of the structural path of transfer of influence from region 2 to region 1. This influence is presented by the components

$$\begin{aligned} B_{12} &= B_{11}A_{12}B_2 = [B_1 + D(B_1, E_{11})]A_{12}B_2 \\ &= B_1A_{12}B_2 + D(B_1, E_{11})A_{12}B_2. \end{aligned}$$

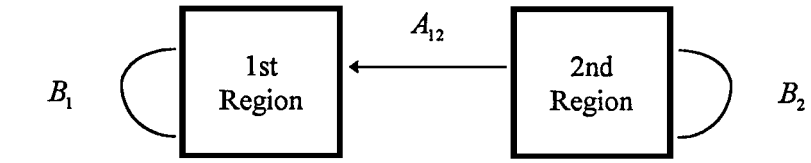
Therefore, the total transfer of influence includes the direct transfer of influences, $B_1A_{12}B_2$, and the augmented, indirect transfer of influence, $D(B_1, E_{11})A_{12}B_2$ (see Figure 2).

Further extensions to the case of three and n regions can be found in Sonis and Hewings (1996); in these more complicated cases, the increments have both a temporal and a spatial manifestation. [See Sonis and Hewings (1995) for a presentation of the block matrix fields of influence.]

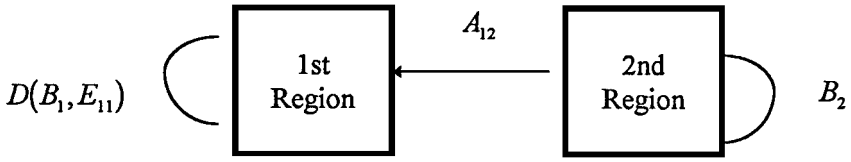
10. APPLICATIONS TO TEMPORAL CHANGES IN MIYAZAWA PARTITIONED MATRIX MULTIPLIERS

In this section, attention is directed to the case of temporal changes in a partitioned input-output system. The partitioning enables explicit recognition of hierarchical structures; Miyazawa (1976) exploited this formalism in the two-region case and introduced the notions of internal and external multipliers. The following presentation moves this consideration to issues of temporal changes. Consider a two-region

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A) Network of the direct transfer of influence



B) Augmented transfer of influence

FIGURE 2. Transfer of influence (direct and augmented) from the second region to the first.

(or the case of a region versus the rest of the economy) system, represented by the block matrix A, of direct inputs shown in (75). Then, the following decomposition of the Leontief inverse, $B = (I - A)^{-1}$, holds [see Miyazawa (1966), Sonis and Hewings (1993)]:

$$B = (I - A)^{-1} = \left(\begin{array}{c|c} \Delta_{11} B_1 & \Delta_{11} B_1 A_{12} B_2 \\ \hline \Delta_{22} B_2 A_{21} B_1 & \Delta_{22} B_2 \end{array}\right)$$
$$= \left(\begin{array}{c|c} \Delta_{11} & 0 \\ \hline 0 & \Delta_{22} \end{array}\right) \left(\begin{array}{c|c} I & B_1 A_{12} \\ \hline B_2 A_{21} & I \end{array}\right) \left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right), \tag{80}$$

where $B_1 = (I - A_{11})^{-1}$, and $B_2 = (I - A_{22})^{-1}$ are the Miyazawa internal matrix multipliers for the first and second regions, and

$$\begin{aligned} \Delta_{11} &= (I - B_1 A_{12} B_2 A_{21})^{-1}, \\ \Delta_{22} &= (I - B_2 A_{21} B_1 A_{12})^{-1} \end{aligned} \tag{81}$$

are the Miyazawa external matrix multipliers for the first and second regions.

Let

$$E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$$

be the block matrix of the incremental changes in direct inputs. From (80), the following representation of the Leontief inverse, $B(E) = (I - A - E)^{-1}$, can be provided:

$$B(E) = \left[\begin{array}{c|c} \Delta_{11}(E) & 0 \\ \hline 0 & \Delta_{22}(E) \end{array}\right] \left[\begin{array}{c|c} I & B_1(E_{11})(A_{12} + E_{12}) \\ \hline B_2(E_{22})(A_{21} + E_{21}) & I \end{array}\right]$$
$$\times \left[\begin{array}{c|c} B_1(E_{11}) & 0 \\ \hline 0 & B_2(E_{22}) \end{array}\right], \tag{82}$$

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where

$$\begin{aligned} B_1(E_{11}) &= (I - A_{11} - E_{11})^{-1}, \\ B_2(E_{22}) &= (I - A_{22} - E_{22})^{-1}, \\ \Delta_{11}(E) &= [I - B_1(E_{11})(A_{12} + E_{12})B_2(E_{22})(A_{21} + E_{21})]^{-1}, \\ \Delta_{22}(E) &= [I - B_2(E_{22})(A_{21} + E_{21})B_1(E_{11})(A_{12} + E_{12})]^{-1}. \end{aligned} \tag{83}$$

It is easy to see that

$$\begin{aligned} B_1(E_{11}) &= B_1 M_R(B_1, E_{11}), \\ B_2(E_{22}) &= B_2 M_R(B_2, E_{22}). \end{aligned} \tag{84}$$

Let us introduce the following increments:

$$\begin{aligned} E^*_{11} &= B_1(E_{11})(A_{12} + E_{12})B_2(E_{22})(A_{21} + E_{21}) - B_1 A_{12} B_2 A_{21}, \\ E^*_{22} &= B_2(E_{22})(A_{21} + E_{21})B_1(E_{11})(A_{12} + E_{12}) - B_2 A_{21} B_1 A_{12}. \end{aligned} \tag{85}$$

Then, the external multipliers $\Delta_{11}(E)$ and $\Delta_{22}(E)$ have the form

$$\begin{aligned} \Delta_{11}(E) &= M_L(\Delta_{11}, E^*_{11})\Delta_{11}, \\ \Delta_{22}(E) &= M_L(\Delta_{22}, E^*_{22})\Delta_{22}. \end{aligned} \tag{86}$$

Let us introduce the increments

$$\begin{aligned} E^*_{12} &= B_1(E_{11})(A_{12} + E_{12}) - B_1 A_{12}, \\ E^*_{22} &= B_2(E_{22})(A_{21} + E_{21}) - B_2 A_{21}. \end{aligned} \tag{87}$$

Then, using (84), we have

$$\begin{aligned} B(E) &= \left[\begin{array}{c|c} \Delta_{11}(E) & 0 \\ \hline 0 & \Delta_{22}(E) \end{array}\right] \left(\begin{array}{c|c} I & B_1 A_{12} + E^*_{12} \\ \hline B_2 A_{21} + E^*_{21} & I \end{array}\right) \left[\begin{array}{c|c} B_1(E_{11}) & 0 \\ \hline 0 & B_2(E_{22}) \end{array}\right] \\ &= \left[\begin{array}{c|c} M_L\left(\Delta_{11}, E^*_{11}\right) & 0 \\ \hline 0 & M_L\left(\Delta_{22}, E^*_{22}\right) \end{array}\right] \begin{pmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{pmatrix} \\ &\quad \times \left(\begin{array}{c|c} I & B_1 A_{12} \\ \hline B_2 A_{21} & I \end{array}\right) \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \left[\begin{array}{c|c} M_R(B_1, E_{11}) & 0 \\ \hline 0 & M_R(B_2, E_{22}) \end{array}\right] \\ &\quad + \left[\begin{array}{c|c} \Delta_{11}(E) & 0 \\ \hline 0 & \Delta_{22}(E) \end{array}\right] \begin{pmatrix} 0 & E^*_{12} \\ E^*_{21} & 0 \end{pmatrix} \left[\begin{array}{c|c} B_{11}(E_{11}) & 0 \\ \hline 0 & B_{22}(E_{22}) \end{array}\right] \\ &= \left[\begin{array}{c|c} M_L(\Delta_{11}, E^*_{11}) & 0 \\ \hline 0 & M_L\left(\Delta_{22}, E^*_{22}\right) \end{array}\right] B \left[\begin{array}{c|c} M_R(B_1, E_{11}) & 0 \\ \hline 0 & M_R(B_2, E_{22}) \end{array}\right] \\ &\quad + \begin{bmatrix} 0 & \Delta_{11}(E) E^*_{12} B_2(E_{22}) \\ \Delta_{22}(E) E^*_{21} B_1(E_{11}) & 0 \end{bmatrix}. \end{aligned} \tag{88}$$

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Here, the diagonal matrix

$$M_L^{\text{Intra}} = \left[\begin{array}{c|c} M_L\left(\Delta_{11}, E_{11}^*\right) & 0 \\ \hline 0 & M_L\left(\Delta_{22}, E_{22}^*\right) \end{array}\right] \tag{89}$$

represents the left external intraregional temporal multiplier, and the diagonal matrix

$$M_R^{\text{Intra}} = \left[\begin{array}{c|c} M_R(B_1, E_{11}) & 0 \\ \hline 0 & M_R(B_2, E_{22}) \end{array}\right] \tag{90}$$

represents the right diagonal intraregional temporal multiplier, whereas the matrix

$$\Delta^{\text{Inter}} = \begin{bmatrix} 0 & \Delta_{11}(E)E_{12}^*B_2(E_{22}) \\ \Delta_{22}(E)E_{21}^*B_1(E_{11}) & 0 \end{bmatrix} \tag{91}$$

represents the interregional temporal increment of change in the Leontief inverse. Hence, in abbreviated form, we have

$$B(E) = M_L^{\text{Intra}} B M_R^{\text{Intra}} + \Delta^{\text{Inter}}. \tag{92}$$

11. CONCLUSIONS

Although the temporal Leontief inverse and associated properties do not provide the rich theoretical foundations that the Leontief dynamic system and some of its derivatives offer [see Xia and Yang (1991) for examples], the methodology provides the capability for implementation and for exploration of the role of structural changes in a time series of input-output tables. Furthermore, as noted earlier, the issue of changes in the current account matrices within the Leontief dynamic system has not been explored fully; hence, the methodology described in this paper also may provide for applications within a fully specified dynamic system. As the development of time-based and dynamic computable general equilibrium models proceeds, the opportunities to extract input-output tables will increase. The temporal Leontief inverse methodology will provide a set of techniques to explore the nature of these time series and assist in the extraction of important insights about the nature of technological change or changes in trading patterns (in the case of regional and interregional systems). As noted by Takayama (1994), input-output analysis and the study of the stability of the competitive market were developed in parallel. It is expected, therefore, that the new inverse-matrix perturbation theory also will find a suitable parallel development in other economic analyses.

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