Math. Proc. Camb Phil. Soc. (1981), **90**, 33
Printed in Great Britain

On Moore geometries. II

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(*Received* 9 *September* 1980)

Abstract. In a previous paper, M. Georgiacodis and the author proved that there are no non-trivial Moore geometries of diameter greater than 12. Here, it is proved that there are none of diameter greater than 4.

1. *Introduction*

In a previous paper (2), M. Georgiacodis and the author proved that there are no non-trivial Moore geometries of diameter greater than 12. In this paper we shall prove that there are none of diameter greater than 4. In view of Fuglister's result ((3), theorem 3) that there are none of diameter 3, it seems likely that diameter 4 is also impossible, but we have not succeeded in proving this. This paper is not self-contained; we shall quote several results from (2). Also we shall use the same notation, and we refer the reader to (2) for the longer definitions.

The work of (2) was based on Fuglister's result (3) that if a Moore geometry exists, then the roots of a certain polynomial, called $F_d(x)$, are all rational or quadratic. In this paper, we need a stronger result, also due to Fuglister, which is the following:

THEOREM 1. *Suppose that there is a Moore geometry with N points and parameters (d, a, b). Then for every root $x = \alpha$ of the polynomial $F_d(x)$, the number*

$$c(\alpha) = \frac{(a+1)\,N\{\alpha^2 - 2(a+b)\,\alpha + (a-b)^2\}}{\{\alpha - (a+1)\,(b+1)\}\{(2d+1)\,\alpha + a - b\}} \tag{1·1}$$

is rational.

This is Theorem 2 of (3). Using this result, we shall prove the main theorem of this paper, which is

THEOREM 2. *Suppose that a, b are not both zero and $d > 4$. Then $F_d(x)$ has at least one root $x = \alpha$, say, for which $c(\alpha)$ is irrational. Hence there are no non-trivial Moore geometries of diameter greater than* 4.

We shall now give a sketch of the proof. Throughout this paper, we assume that $c(\alpha)$ is rational, and we eventually deduce a contradiction. We obtain this by considering the polynomials $F_d(x)$ and $H_d(z)$; in particular, we shall consider their irreducible factors over the field $\mathbf{Q}_2$ of 2-adic numbers. Most of the results of (2) were obtained on the assumption that a, b are coprime. This was justified by applying the transformation θ, say, sending a and b into a/h and b/h, where h is the highest common factor. If $x = \alpha$ is any root of $F_d(x)$, then θ sends α to α/h, so leaves the degree of α unchanged. Here, we cannot immediately apply the same device, because θ could possibly change the value of $c(\alpha)$ from irrational to rational or vice versa. We must therefore replace Theorem 2 by a result (Theorem 4 below) which is invariant under θ.

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In the rest of this paper, we distinguish two cases, defined as in (2), § 2, and we also define the numbers r and s as follows:

$$\left.\begin{aligned}&\text{Case 1, } a \text{ is even and } b \text{ is odd, then } d = 2^r.s,\\&\text{Case 2, } b \text{ is even and } a \text{ is odd, then } d = 2^r.s-1.\end{aligned}\right\} \tag{1·2}$$

In § 2 we shall show that other values of a, b may be neglected. From (2), theorems 3–5, it follows that the only values of d (> 4) not yet eliminated are the following:

$$\begin{aligned}d &= 5, 6, 12 \quad \text{in Case 1,}\\ d &= 5, 11 \quad\quad \text{in Case 2.}\end{aligned}$$

Going in order of increasing difficulty, we shall eliminate $d = 5$ (Case 1) in § 3, $d = 11$ and 12 in § 4, and $d = 6$ and 5 (Case 2) in § 6. We first consider (for each d) the polynomial $H_d(z)$. By (2), lemma 3·2, $H_d(z)$ has a stated number of quadratic factors (the so-called *special* factors) congruent to stated residues modulo 2. Applying the methods of (2), § 5, we deduce that a 'large' power of 2 divides ab. We now consider the polynomial $F_d(x)$. (It is convenient to change the variable to y, where $y = x-a-b$.) Part of this is coprime to 2, the rest is divisible by the 'large' power of 2 mentioned above, so it is small in the 2-adic topology. By Hensel's Lemma, it follows that the factors of $F_d(x)$ can be specified to within a small error. (In §§ 3 and 4, this can be done without an explicit appeal to the Lemma, but in § 6, we need a very elaborate version of Hensel's Lemma. This is given in § 5.) From the factors of $F_d(x)$, we can then select one which is irreducible, and whose coefficients violate the necessary condition (Theorem 3 below) for $c(\alpha)$ to be rational.

2. *Basic results*

We first recall some notation and results from (2).

Definitions. Define $G_d(y)$ to be $F_d(x)$ expressed as a polynomial in y, where $y = x-a-b$. Then (2), (2·2) gives

$$G_d(y) = \frac{(ab)^{d/2}}{t^d(t^2-1)}\{t^{2d+2}-1+t\sqrt{(b/a)}\,(t^{2d}-1)\}, \tag{2·1}$$

where

$$y = \sqrt{(ab)}\,(t+t^{-1}). \tag{2·2}$$

If α is a root of $F_d(x)$, we define $\beta = \alpha-a-b$ as the corresponding root of $G_d(y)$. Define

$$e = \operatorname{ord}\{d(d+1)\}, \quad p = |\operatorname{ord}(a)-\operatorname{ord}(b)|, \quad q = \min\{\operatorname{ord}(a), \operatorname{ord}(b)\}$$

where ord denotes the 2-adic order. Note that if a and b are coprime, then $p = \operatorname{ord}(ab)$, as in (2).

THEOREM 3. *Let $F_d(y)$ have an irrational root $x = \alpha$ such that $c(\alpha)$ is rational. Let $\beta = \alpha-a-b$, and let*

$$P(y) = y^2+fy+g$$

be the minimal polynomial of α. Suppose also that

$$p > e \quad \textit{and} \quad \operatorname{ord}(4ab) > \operatorname{ord}(g). \tag{2·3}$$

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Then

$$\text{ord}\,(g/f) = 1 + \text{ord}\,(a(d+1)+bd). \tag{2·4}$$

If also a and b are coprime, then

$$\text{ord}\,(g/f) = r+1 \tag{2·5}$$

where r is defined as in (1·2).

Proof. Define

$$A = 2d(a-1)(b-1)+ab-2a+1,$$
$$B = 2a(ab+1)(d+1)+2db(ab+1)+4ab(2d+1).$$

On expressing $c(\alpha)$ in terms of β and rearranging, (1·1) gives us

$$\frac{N(a+1)}{c(\alpha)} - (2d+1) = \frac{A\beta+B}{4ab-\beta^2} = \frac{A\beta+B}{f\beta+g+4ab} \tag{2·6}$$

since $P(\beta) = 0$. Since the right-hand side of (2·6) is rational and β is not, this fraction must be independent of β. Hence $f/A = (g+4ab)/B$. On taking orders, we have

$$\text{ord}\,(g+4ab) + \text{ord}\,(A) = \text{ord}\,(f) + \text{ord}\,(B). \tag{2·7}$$

Now by (2), theorem 3, a and b cannot both be odd, so A is odd. Also by (2·3), $\text{ord}\,(g+4ab) = \text{ord}\,(g)$.

Now suppose $\text{ord}\,(a) > \text{ord}\,(b)$. Then the three terms of B have orders respectively $\geqslant 1+p+q$, $\leqslant 1+e+q$, $2+p+2q$. By (2·3), the middle term has order less than the two other terms, so

$$\text{ord}\,(B) = \text{ord}\,(2bd) = \text{ord}\,(2a(d+1)+2bd).$$

Then (2·7) implies (2·4), and if a, b are coprime then Case 1 applies, so $\text{ord}\,(2bd) = r+1$, which implies (2·5). If $\text{ord}\,(a) < \text{ord}\,(b)$, the proof is similar. ∎

We shall actually prove Theorem 2 in the following form:

THEOREM 4. *Suppose that $d > 4$, a, b are not both zero and suppose that $F_d(x)$ splits into factors of degree $\leqslant 2$. Then $p > e$ and $G_d(y)$ has at least one irreducible factor $P(y)$ for which* $\text{ord}\,(g) < \text{ord}\,(4ab)$ *and equation* (2·4) *is false. (Therefore c is irrational, so there is no non-trivial Moore geometry with parameters (d, a, b)).*

LEMMA 2·1. *In the proof of Theorem* 4, *we may assume without loss of generality that a and b are coprime.*

Proof. If not, let h be the highest common factor. If β is any root of $G_d(y)$, then the transformation θ defined above maps β into β/h. If β is a quadratic root and $P(y) = y^2+fy+g$ is its minimal polynomial, then θ will map f and g into f/h and g/h^2 respectively. Therefore θ decreases both sides of equation (2·4) by $\text{ord}\,(h)$, so the truth of (2·4) is unaffected. Also θ does not affect the truth of (2·3). So if Theorem 4 is true for a/h and b/h which are coprime, it will also be true for arbitrary a, b. ∎

3. *Diameter* 5, *Case* 1

We first recall some further definitions from (2). For integer m, $L(m)$ is the residue of $H_d(z)$ on division by z^2+mz+1. Also we recall that **o** denotes an odd number of

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undefined value. If $d = 5$ and Case 1 holds, then $r = 0$, $s = 5$, b is odd and $a = 2^p\mathbf{o}$. By (2), lemma 5·2, one of $L(1)$ or $L(-1)$ (we do not know which) is divisible by $2^{2r+2} = 16$. Now,

$$L(1) = (-16b - 24a)z - 16b - 45a \equiv -8az - 5a \pmod{16},$$
$$L(-1) = (8a + 4b)z - 4b - 3a.$$

Therefore $L(-1)$ can never be divisible by 16, and if 16 divides $L(1)$ then

$$p = \text{ord}\,(a) \geqslant 4. \tag{3·1}$$

By (2), lemma 3·2, if $H_d(z)$ splits into quadratic factors, it has two special factors of type (d). Let $R(z)$ be one of these. By (2), lemma 2·1,

$$R(z) = z^2 + (2g - f^2)z/ab + g^2/a^2b^2 \equiv z^2 + z + 1 \pmod{2}.$$

Hence

$$\text{ord}\,(f^2) = \text{ord}\,(ab) = \text{ord}\,(g).$$

So by Theorem 3, we have

$$p = \text{ord}\,(ab) = \text{ord}\,(g^2/f^2) = 2.\text{ord}\,(g/f) = 2(r+1) = 2.$$

This contradiction with (3·1) proves Theorem 4 for $d = 5$, Case 1. ∎

4. *Diameter* 12 *or* 11

Here $s = 3$ and $r = 2$, so by (2), lemma 3·2, $H_d(z)$ has four special factors of type (c). We now prove

LEMMA 4·1. (1) $H_d(z)$ *has at least two special factors satisfying*

$$R(z) \equiv z^2 + 4z + 1 \pmod{8}.$$

(2) $$p \geqslant 8.$$

Proof. First take $d = 12$. Suppose that three (or four) of the type (c) factors were congruent to $z^2 + 2z + 1 \pmod 4$. Then for the correct choice of $m = 2$ or 6, $H_d(z)$ will have

$$\text{two factors} \equiv z^2 + mz + 1 \pmod{8},$$
$$\text{one factor} \equiv z^2 + mz + 1 \pmod{4},$$

and

$$\text{one factor} \equiv z^2 + mz + 1 \pmod{2}.$$

So by (2), lemma 5·1, with $i = j = 1$ and $k = 2$, 2^9 divides $L(6)$ or $L(2)$ (we do not know which). By computer calculations, we have

$$L(6) = (-779\,536\,000b - 6\,001\,283\,060a)z - 133\,747\,200b + 1\,029\,657\,339a$$
$$= (2^7\cdot\mathbf{o} + 2^{2+p}.\mathbf{o})z + (2^9.\mathbf{o} + 2^p.\mathbf{o}).$$
$$L(2) = (-107\,136b - 304\,764a)z - 86\,400b - 250\,475a$$
$$= (2^7.\mathbf{o} + 2^{2+p}.\mathbf{o})z + (2^7.\mathbf{o} + 2^p.\mathbf{o}).$$

As in (2), § 5, neither of these is divisible by 2^9. Hence H_d has at most two factors congruent to $z^2 + 2z + 1 \pmod 4$. The other two must be congruent to $z^2 + 1 \pmod 4$, because their constant terms are odd squares.

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Now suppose that one of these factors were congruent to $z^2+1 \pmod 8$. By (2), lemma 5·1, with $i=2, j=k=1$, 2^7 divides $L(0)$. But

$$\begin{aligned} L(0) &= (-1600b-2600a)z-2079a \\ &= (2^6.\mathbf{o}+2^{3+p}.\mathbf{o})z+2^p.\mathbf{o}. \end{aligned}$$

which is not divisible by 2^7. This proves Part 1.

Now by (2), lemma 5·1, with $i=k=2$ and $j=0$, we see that 2^8 divides $L(-4)$ which equals a. Hence $p \geqslant 8$, as stated.

This proves the lemma for $d=12$. For $d=11$ the proof is similar, with

$$\begin{aligned} L(6) &= (-10\,257\,740b-779\,536\,000a)z \\ &\quad -17\,373\,059b-133\,747\,200a \\ &= (2^{2+p}.\mathbf{o}+2^7.\mathbf{o})z+(2^p.\mathbf{o}+2^9.\mathbf{o}), \\ L(2) &= (-37\,380b-107\,136a)z-29\,451b-86\,400a \\ &= (2^{2+p}.\mathbf{o}+2^7.\mathbf{o})z+(2^p.\mathbf{o}+2^7.\mathbf{o}), \\ L(0) &= (-600b-1600a)z+481b \\ &= (2^{3+p}.\mathbf{o}+2^6.\mathbf{o})z+2^p.\mathbf{o}, \end{aligned}$$

and

$$L(-4)=b.$$

This completes the proof. ∎

We proceed to prove Theorem 4 for $d=11$ or 12. Let $R(z)$ be one of the factors given by Lemma 4·1 (1). Then by (2), lemma 2·1,

$$R(z)=z^2+(2g-f^2)z/ab+g^2/a^2b^2 \equiv z^2+4z+1 \pmod 8.$$

Hence $\operatorname{ord}(g)=\operatorname{ord}(ab)=p$ and $\operatorname{ord}(f^2)=\operatorname{ord}(2g)=p+1$. By Theorem 3,

$$p-1=\operatorname{ord}(g^2/f^2)=2r+2=6.$$

This contradicts Lemma 4·1 (2), so Theorem 4 is proved for $d=11$ or 12.

5. *Hensel's Lemma*

In order to prove Theorem 4 for diameter $d=6$ and 5 (Case 2), we must first give an elaborate version of Hensel's Lemma. This is essentially due to Artin (1), and we shall use his notation.

Definition. Let $\mathbf{K}$ be a complete non-Archimedean valued field, and let $|\ |$ be the valuation. We extend $|\ |$ to the polynomial ring $\mathbf{K}[x]$ as follows: let c be a real positive constant, and let

$$u(x)=a_0+a_1x+a_2x^2+\ldots+a_nx^n,$$

then define $|x|=c$ and

$$|u(x)| = \max_{0 \leqslant i \leqslant n} |a_ix^i| = \max_{0 \leqslant i \leqslant n} |a_i|\,c^i. \tag{5·1}$$

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THEOREM 5. *Let $f(x)$ be a polynomial in $\mathbf{K}[x]$. Suppose that there exist polynomials $\phi(x)$, $\psi(x)$, $h(x)$, $A(x)$, $B(x)$, $C(x)$, and a number d such that*

$$f(x) = \phi(x)\psi(x) + h(x), \tag{5·2}$$

$$\phi(x) \neq 0 \text{ and its value is equal to that of its highest term,} \tag{5·3}$$

$$A(x)\phi(x) + B(x)\psi(x) = d + C(x), \tag{5·4}$$

$$|\phi(x)| \leqslant 1, \quad |B(x)| \leqslant 1, \quad |C(x)| \leqslant |d| \leqslant 1, \tag{5·5}$$

$$|h(x)| < |d|^2 |\phi(x)|, \tag{5·6}$$

Then there exist polynomials $\Phi(x)$, $\Psi(x)$ in $\mathbf{K}[x]$ satisfying:

$$f(x) = \Phi(x)\Psi(x), \tag{5·7}$$

$$\Phi(x) \text{ has the same degree and leading coefficient as } \phi(x), \tag{5·8}$$

$$|\Phi(x) - \phi(x)| \leqslant |h(x)/d|. \tag{5·9}$$

Proof. Most of this is proved by Artin ((1), chapter 2, theorem 5), except that instead of (5·8) and (5·8), he only states the weaker results:

$$\Phi(x) \text{ and } \phi(x) \text{ have the same degree,} \tag{5·8}'$$

$$|\Phi(x) - \phi(x)| < |d\phi(x)|. \tag{5·9}'$$

We have to show, by inspection of Artin's proof, that we can substitute the stronger conclusions (5·8) and (5·9). In the proof, $\Phi(x)$ is constructed as the sum of a series

$$\Phi(x) = \phi(x) + \sum_{i=1}^{\infty} \beta_i(x),$$

where the degree of β_1 is less than that of ϕ, and $|\beta_1(x)| \leqslant |h(x)/d|$. By induction, all the β_i satisfy the same conditions. Hence so does their sum, proving (5·8) and (5·9).

6. *Diameter* 6 *and* 5 (*case* 2)

In this section we shall prove Theorem 4 for diameter 6. The proof for diameter 5 (case 2) is similar. For diameter 6, we have $r = 1$, $s = 3$, b is odd, and $a = 2^p . o$.

LEMMA 6·1 $$p \geqslant 5.$$

Proof. By (2), lemma 3·2, $H_d(z)$ has two special factors of type (*c*) having the form

$$R(z) = z^2 + (2g - f^2)z/ab + g^2/a^2b^2 \equiv z^2 + 1 \pmod 2.$$

If either of these special factors is congruent to $z^2 + 1 \pmod 4$, then by (2), lemma 5·1 with $i = j = 1$,

$$8 \text{ divides } L(-4) = (4a - 4b)z - a,$$

which is impossible. Since their constant terms are odd squares, both special factors of type (*c*) are congruent to $z^2 \pm 2z + 1 \pmod 8$. Therefore by (2), lemma 5·1, with $j = k = 1$, 32 divides one of $L(2)$ or $L(-2)$.

Now

$$\left.\begin{aligned} L(2) &= (-160b - 494a)z - 96b - 325a \equiv -2az - 5a \pmod{32} \\ \text{and} \qquad L(-2) &= -2az + 3a. \end{aligned}\right\} \tag{6·1}$$

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So if 32 divides either, then 32 divides a, as stated. For $d = 5$ (Case 2), the proof is similar; the corresponding formulae are:

$$L(-4) = (4b-4a)z-b,$$
$$L(2) = (-50b-160a)z-25b-96a,$$
$$L(-2) = 2bz-b.$$

We shall now apply Theorem 5 to $G_6(y)$. Put $\mathbf{K} = \mathbf{Q}_2$, and define the valuation $|\ |$ on $\mathbf{K}[y]$ by (5·1) with

$$|y| = \sqrt{|ab|} = 2^{-p/2}.$$

By equations (2·1) and (2·2),

$$G_6(y) = y^6 + by^5 - 5aby^4 - 4ab^2y^3 + 6a^2b^2y^2 + 3a^2b^3y - a^3b^3.$$

Hence

$$yG_6(y)/a^3b^4 = (y^2/ab)(y^2/ab-1)(y^2/ab-3) - h(y) \tag{6·2}$$

(say), where

$$h(y) = (y/b)(y^6/a^3b^3 - 5y^4/a^2b^2 + 6y^2/ab - 1).$$

Therefore $|h(y)| = |y| = 2^{-p/2} \leqslant 2^{-5/2}$. Now define

$$\phi(y) = \phi_1(y) = y^2/ab - 1 \quad \text{and} \quad \psi(y) = y^2/ab(y^2/ab-3).$$

Then

$$\psi(y) - (y^2/ab-2)\phi(y) = 2.$$

So the conditions (5·2)–(5·6) of Theorem 5 all hold, with $|h| = 2^{-p/2}$ and $|d| = \frac{1}{2}$. Therefore $yG_6(y)$ has a factor, say $\Phi_1(y)$, of degree 2, with leading term y^2/ab, satisfying

$$|\Phi_1(y) - \phi_1(y)| \leqslant |h(y)/d| = 2^{1-p/2}. \tag{6·3}$$

Now define

$$P_1(y) = ab\Phi_1(y) = y^2 + f_1y + g_1, \quad \text{(say)}$$

Then from (6·3) it follows that $\operatorname{ord}(g_1/ab+1) \geqslant p/2-1 \geqslant \frac{3}{2}$. Since this order is an integer, it is at least 2, so

$$g_1/ab \equiv -1 \pmod 4. \tag{6·4}$$

In particular the constant term of P_1 is non-zero, so $P_1(y)$ divides $G_6(y)$. Also

$$\operatorname{ord}(g_1) = \operatorname{ord}(ab) = p. \tag{6·5}$$

Similarly from (6·3) we have $\operatorname{ord}(f_1y/ab) \geqslant p/2-1$, so

$$\operatorname{ord}(f_1) \geqslant p/2 - 1 + \operatorname{ord}(ab) - \operatorname{ord}(y) = p/2 - 1 + p - p/2 = p-1. \tag{6·6}$$

We now repeat the argument, with

$$\phi(y) = \phi_3(y) = y^2/ab - 3$$

in place of ϕ_1. Then $G_6(y)$ has a factor $P_3(y) = y^2 + f_3y + g_3$, satisfying (6·5) and (6·6), but

$$g_3/ab \equiv -3 \pmod 4 \tag{6·4}$$

in place of (6·4).

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LEMMA 6·2. *At least one of the polynomials $P_1(y)$, $P_3(y)$ is irreducible in $\mathbf{Q}_2[y]$.*

Proof. Suppose not. Then they have four roots, say $\beta_1 - \beta_4$, in $\mathbf{Q}_2$. Put $\gamma_i = \beta_i^2/ab$. Let $ab = 2^p.c$, so c is odd. By (6·5) and (6·6), ord $(\gamma_i) = 0$, so

$$\gamma_i \equiv 1/c \equiv c \pmod 8.$$

Hence the two type (c) special factors of $H_d(z)$ are

$$(z-\gamma_1)(z-\gamma_2) \quad \text{and} \quad (z-\gamma_3)(z-\gamma_4),$$

and both of these are congruent to $z^2 - 2cz + 1 \pmod 8$. So by (2), lemma 5·1, with $m = -2c, k = 2, i = j = 0$, 2^6 divides one or other of $L(2), L(-2)$. By (6·1), 2^6 divides a, i.e. $p \geqslant 6$.

Now let D_1, D_3 be the discriminants of P_1 and P_3. Then

$$D_1/4ab = f_1^2/4ab - g_1/ab \equiv 1 \pmod 4$$

by (6·6) and (6·4). Similarly $D_3/4ab \equiv 3 \pmod 4$. Therefore D_1 and D_3 cannot both be 2-adic squares, because $D_1/D_3 \equiv 3 \pmod 4$. ∎

Now let $P(y) = y^2 + fy + g$ be whichever of P_1 or P_3 is irreducible. By (6·5) and (6·6), ord $(g/f) \leqslant p - (p-1) = 1$, whereas Theorem 3 requires that ord (g/f) should be equal to $r + 1 = 2$. This contradiction proves Theorem 4 for $d = 6$. For $d = 5$ (Case 2), the proof is similar; the equation corresponding to (6·2) is

$$y.G_5(y)/a^3b^3 = (y^2/ab)(y^2/ab - 3)(y^2/ab - 1) + h(y),$$

where

$$h(y) = (y/a)(y^4/a^2b^2 - 3y^2/ab + 1).$$

This completes the proof of Theorem 4, and hence of Theorem 2.

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