THE SOLUTION OF BOUNDARY VALUE PROBLEMS FOR A GENERAL HODOGRAPH EQUATION

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ABSTRACT. In this paper the solution is found of certain boundary value problems appropriate to the flow of an inviscid, compressible fluid past a wedge-shaped profile. The most general hodograph equation in the form

$$\frac{\partial^2\Psi}{\partial\sigma^2}+k(\sigma)\frac{\partial^2\Psi}{\partial\theta^2}=0$$

is used. Expressions are found for the position coordinates and the drag. The resulting solutions are then discussed for functions $k(\sigma)$ appropriate to Chaplygin's equation, Tricomi's equation and the equation of Tomotika and Tamada. The case of Helmholtz flow past a wedge with sonic velocity upstream is treated in detail, and the resulting expressions for the drag enable an estimate to be made of the error which results in using approximations to Chaplygin's equation.

1. *Introduction.* The application of the hodograph transformation is now a well-established method of determining exact solutions in the two-dimensional, steady motion of a non-viscous gas. The equation for the stream function Ψ takes the form

$$\frac{\partial^2\Psi}{\partial\sigma^2}+k(\sigma)\frac{\partial^2\Psi}{\partial\theta^2}=0, \tag{1·1}$$

where σ is a function of the velocity q, and θ is the angle which the velocity vector makes with the positive direction of the x-axis. $k(\sigma)$ is a certain function of σ which is positive for subsonic and negative for supersonic values of the velocity. Corresponding to the equation of state of a polytropic gas, the form of $k(\sigma)$ is such that solutions of (1·1) may be obtained in terms of hypergeometric functions. The relative complexity of these solutions has encouraged many authors to use approximations for $k(\sigma)$, notably that for which $k(\sigma)=b\sigma$, where b is a constant, in which case (1·1) reduces effectively to Tricomi's equation. This equation is valid for small perturbations on a uniform sonic stream. Solutions can now be obtained in terms of Bessel functions, and some important examples have been discovered which represent flow past thin bodies. It should be noted that for certain problems concerning jets with free boundaries the exact solutions with a general value of $k(\sigma)$ may be obtained by comparatively elementary methods, provided some knowledge is assumed about the solution of the ordinary differential equation in σ which results from separating the variables in (1·1). However, in any attempt to find the flow of an unbounded stream of gas past a body two main difficulties are encountered. The first is the determination of the singularity in the hodograph plane which corresponds to the point at infinity in the physical plane. The second is the difficulty in finding a satisfactory representation for a 'second' solution of the equation valid in regions of the physical plane away from the stagnation point. This will be discussed more fully later. The first problem can be resolved by a determination of the Green's function of (1·1) for a suitable

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domain. An alternative approach, which in the cases we consider is fundamentally equivalent, is to establish an inversion formula associated with (1·1) which is similar to the Hankel inversion formula for Bessel functions. Such a formula has been obtained by Germain (3) and this is used in the present paper to obtain the solution for flow past a wedge-shaped body. The form of solution presents a suitable method of overcoming the second difficulty mentioned above.

The generality of Germain's results means that the scope of the investigation can be extended beyond that of finding an exact solution for the flow of a polytropic gas past a body. It will serve for greater clarity if in future (1·1) and associated equations are referred to as Chaplygin's equation whenever a polytropic gas is considered. Initially, therefore, $k(\sigma)$ will not be specified exactly. The equation (1·1) for each suitably restricted function $k(\sigma)$ is equivalent to an equation for the stream function Ψ of a gas with a certain equation of state. The solution of a given boundary value problem will therefore be obtained for a general $k(\sigma)$. A certain knowledge of the properties of $k(\sigma)$ is, of course, required, and we shall assume as for the jet problems that we know the solutions of the ordinary differential equation in σ obtained by separating the variables in (1·1). These solutions are known functions for Chaplygin's equation and for the approximations to it which are considered in this paper.

In the work which follows the velocity of the gas will always be subsonic or sonic. For such cases the problem is more easily formulated but apart from this the absence of a supersonic region means that no shock waves appear. This is essential to our purpose here, for in obtaining the analytic expression for a flow with shock waves some approximation is unavoidable and it is our object to compare exact solutions for the same problem with different functions $k(\sigma)$. Any discrepancy between the solution for a certain value of $k(\sigma)$ and that of the same problem for Chaplygin's equation can therefore be identified with certainty as due to the particular form of $k(\sigma)$ chosen and not to any approximating procedure used in obtaining the solution.

2. *Explanation of notation.* The notations used by different workers in the field show a wide variation, and it is therefore desirable to establish a definite notation at the outset. ρ and V are respectively the density and velocity of the gas, normalized in such a way that they are both unity when the Mach number M is unity. r and q are the actual values of the density and velocity so that $r = r^*\rho$, $q = q^*V$, where the asterisk denotes that the value is taken at $M = 1$. The mass flow m is defined by $m = \rho V$. Then σ and $k(\sigma)$ are defined by

$$\sigma = \int_V^1 \frac{\rho\, dV}{V}, \tag{2·1}$$

$$k(\sigma) = \frac{1}{\rho^3}\frac{dm}{dV}. \tag{2·2}$$

It is observed that $\sigma = 0$ at sonic velocity and is positive for subsonic velocities. In addition σ is infinite at the stagnation point. The velocity potential Φ and the stream function Ψ are then related by

$$\frac{\partial\Phi}{\partial\theta} = -\frac{\partial\Psi}{\partial\sigma}, \quad \frac{\partial\Phi}{\partial\sigma} = k(\sigma)\frac{\partial\Psi}{\partial\theta}, \tag{2·3}$$

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which lead to equation (1·1). When Φ and Ψ are known in terms of σ and θ the position coordinates x and y can be found by integrating

$$dx + i\,dy = e^{i\theta}\left(\frac{1}{V}\,d\Phi + \frac{i}{m}\,d\Psi\right). \tag{2·4}$$

It is now possible to start with a suitably arbitrary function $k(\sigma)$ which characterizes a fictitious gas. (2·1) may be rewritten as

$$\frac{d}{d\sigma}\left(\frac{1}{V}\right) = \frac{1}{m}, \tag{2·5}$$

and if ρ is eliminated from equations (2·2) and (2·5) we get

$$\frac{d^2}{d\sigma^2}\left(\frac{1}{V}\right) - \frac{k(\sigma)}{V} = 0. \tag{2·6}$$

The connexion between equations (2·6) and (1·1) is significant and will be discussed in more detail later. An interesting consequence is that $\Psi = V^{-1}\sin\theta$ is a solution of (1·1). For Chaplygin's equation this is usually referred to as Ringleb's solution.

With a given $k(\sigma)$ the procedure is to solve (2·6) subject to the conditions

$$\frac{1}{V} = \frac{d}{d\sigma}\left(\frac{1}{V}\right) = 1 \quad \text{when} \quad \sigma = 0.$$

Finally the relation between the (non-normalized) pressure p and the density is found by integrating Bernoulli's equation in the differential form

$$dp + r^*q^{*2}\rho V\,dV = 0.$$

For a polytropic gas it may be shown that $k(\sigma) \sim (\gamma+1)\,\sigma$ near $\sigma = 0$, where γ is the ratio of specific heats, and that it tends to the value $\{2/(\gamma+1)\}^{2/(\gamma-1)}$ as $\sigma \to \infty$. The choice of a suitable approximation for $k(\sigma)$ is governed by the range of V which might be expected to occur in the given problem. When this range includes the value $V = 1$ it is necessary that $k(\sigma) \sim (\gamma+1)\,\sigma$ near $\sigma = 0$.

The solutions of (1·1) obtained by separating the variables lead to two ordinary differential equations in σ, namely

$$\frac{d^2u}{d\sigma^2} + \beta^2 k(\sigma)\,u = 0 \tag{2·7}$$

and

$$\frac{d^2u}{d\sigma^2} - \nu^2 k(\sigma)\,u = 0. \tag{2·8}$$

The corresponding functions of θ are respectively exponential and harmonic. We denote by $S(\sigma, \beta)$ the solution of (2·7) which is such that

$$S(0, \beta) = 0, \quad \frac{\partial S(0, \beta)}{\partial \sigma} = 1,$$

and by $R(\sigma, \nu)$ the solution of (2·8) with similar properties. It follows from the general theory of the dependence of solutions of a differential equation on parameters occurring in the equation that $S(\sigma, \beta)$ is an integral function of the complex variable β^2

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and that consequently it is an even analytic function of β in the complex β-plane. Similarly, $R(\sigma, \nu)$ is an even analytic function of ν. Then clearly

$$S(\sigma, i\nu) = R(\sigma, \nu). \tag{2·9}$$

A second solution of each of the equations (2·7), (2·8) is also required. $L(\sigma, \nu)$ is defined as the solution of (2·8) which is -1 when $\sigma = 0$ and which tends to zero as $\sigma \to \infty$ for real positive values of ν. $K(\sigma, \beta)$ is defined as the solution of (2·7) which is -1 when $\sigma = 0$ and which tends to zero as $\sigma \to \infty$ for values of β with argument $-\frac{1}{2}\pi$. It follows therefore that

$$K(\sigma, \beta) = L(\sigma, i\beta). \tag{2·10}$$

Finally, the solutions $R(\sigma, \nu)$, $L(\sigma, \nu)$ are linearly independent. For it is easy to show from the differential equation (2·8) that their Wronskian $W(R, L)$ is a constant and by considering the value at $\sigma = 0$ we have

$$W(R, L) = R(\sigma, \nu) L'(\sigma, \nu) - R'(\sigma, \nu) L(\sigma, \nu) = 1, \tag{2·11}$$

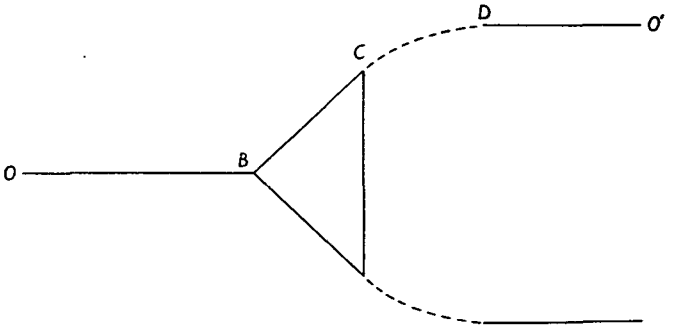


Fig. 1

where the prime denotes differentiation with respect to σ. Similarly

$$W(S, K) = 1.$$

For the forms of $k(\sigma)$ considered, these functions will all be readily identified. If we take as an example the simple case $k(\sigma) = 1$ corresponding to Laplace's equation and incompressible flow we have

$$S(\sigma, \beta) = \frac{\sin \beta\sigma}{\beta}, \quad R(\sigma, \nu) = \frac{\sinh \nu\sigma}{\nu},$$

$$K(\sigma, \beta) = -e^{-i\beta\sigma}, \quad L(\sigma, \nu) = -e^{-\nu\sigma}.$$

3. *The solution for the Roshko model cavity problem.* We shall now find a solution of equation (1·1) which represents the flow past a wedge-shaped body, placed symmetrically in an unbounded uniform stream in which the Mach number is M_1 (< 1) at infinity upstream. The suffix 1 will be used to denote conditions in the undisturbed stream. The model selected is based on the cavity solution first discussed by Roshko (11) for incompressible flow and is shown in Fig. 1. The dividing streamline has a stagnation point at the nose B and leaves the wedge shoulder C at sonic velocity which it retains along the curved portion CD. It then becomes straight and the velocity decreases

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along DO' to the value corresponding to $M = M_1$. Although not ideal, this is a reasonably satisfactory description of the flow past a wedge, incorporating as it does the essential features of zero velocity at the nose and sonic velocity at the shoulder. The most unsatisfactory part is that immediately downstream from the shoulder. From the point of view of determining the physical quantities such as the drag, for example, any inaccuracy in the description of the flow in this region will have very little influence on the regions upstream of it. It must be remembered that unless some comparable hypothesis is made concerning the downstream flow no exact solution can be obtained. As has been pointed out, the introduction of a shock wave causes the whole structure of strictly isentropic flow to collapse as far as exact solutions are concerned. Moreover, it is not possible to cut off the downstream flow by a limiting

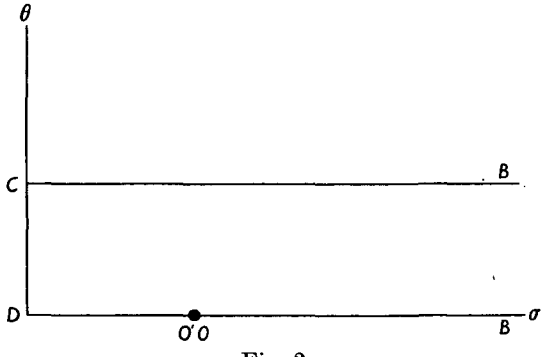


Fig. 2

characteristic from the shoulder unless $M_1 = 1$. For when $M_1 < 1$ the supersonic region is bounded and the effects of any disturbance occurring in it such as a shock wave are propagated over the whole flow field. It is true that the effects will be very small upstream but the object of the present work is to establish an *exact* solution of the differential equation which satisfies given boundary conditions. Another possibility would be to consider the Helmholtz flow past a wedge. In this case the streamline downstream from the wedge shoulder is one of constant velocity for which $M = M_1$. This is a less interesting problem from the mathematical point of view and is also less acceptable physically since it is felt desirable to demand that sonic velocity be attained at the shoulder.

The boundary value problem in the hodograph plane (Fig. 2) can now be set up without difficulty. The upper half of the physical plane in Fig. 1 is mapped into the rectangle $BCDB$. BC is the line $\theta = \alpha$, where 2α is the angle at the nose. $O'O$ is the point $(\sigma_1, 0)$ which corresponds to the point at infinity in the physical plane. Ψ has a singularity at this point since all the streamlines must begin and end at it. Apart from this $\Psi = 0$ on the boundaries of the rectangle.

Because of the condition $\Psi = 0$ when $\sigma = 0$ we look for solutions of the type $S(\sigma, \beta)\, e^{\pm\beta\theta}$. A solution containing an arbitrary function $f(\beta)$ and satisfying conditions on BC, CD is

$$\Psi = \int_0^\infty \beta f(\beta) \frac{\sinh \beta(\alpha - \theta)}{\sinh \beta\alpha} S(\sigma, \beta)\, d\beta.$$

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$f(\beta)$ is determined from the boundary conditions on DB. In fact we require

$$\int_0^\infty \beta f(\beta)\, S(\sigma, \beta)\, d\beta = A\delta(\sigma - \sigma_1), \tag{3·1}$$

where $\delta(\sigma - \sigma_1)$ is the delta function with origin at $\sigma = \sigma_1$ and A is a constant. This constant is associated with the strength of the singularity which in turn determines the length of the straight part of the wedge section.

We now state the fundamental inversion theorem as given by Germain (3). The solution of the integral equation

$$\int_0^\infty \beta f(\beta)\, S(\sigma, \beta)\, d\beta = g(\sigma)$$

is

$$f(\beta) = \frac{2w(\beta)}{\pi} \int_0^\infty k(\sigma)\, g(\sigma)\, S(\sigma, \beta)\, d\sigma.$$

In this formula

$$w(\beta) = \Im \frac{\partial K(0, \beta)}{\partial \sigma}.$$

It may also be given by the relation

$$w(\beta)\, S(\sigma_1, \beta) = \Im K(\sigma_1, \beta)$$

for any value of σ_1, which is equivalent to

$$2iw(\beta)\, S(\sigma_1, \beta) = L(\sigma_1, i\beta) - L(\sigma_1, -i\beta) \tag{3·2}$$

when use is made of (2·10).

It is now possible to solve the boundary value problem completely. By means of the inversion formula (3·1) can be solved to give

$$f(\beta) = \frac{2Aw(\beta)}{\pi} k(\sigma_1)\, S(\sigma_1, \beta)$$

and hence

$$\Psi = \frac{2Ak(\sigma_1)}{\pi} \int_0^\infty \frac{\beta \sinh \beta(\alpha - \theta)}{\sinh \beta\alpha} w(\beta)\, S(\sigma, \beta)\, S(\sigma_1, \beta)\, d\beta. \tag{3·3}$$

Before proceeding further we point out that this solution can be generalized to give the stream function for a similar model in which the wedge is placed symmetrically in a channel with parallel walls. It is not difficult to see that in the hodograph of this new flow the points O', O are no longer coincident on the σ-axis. If σ_2 ($< \sigma_1$) is the value of σ corresponding to the velocity far downstream in the channel, then O' is the point $(\sigma_2, 0)$ and the boundary value problem is $\Psi =$ constant on $O'O$ and zero elsewhere on the boundary of the rectangle. The solution is then obtained by integrating the right-hand side of (3·3) with respect to σ_1 and, when use is made of the fact that $S(\sigma, \beta)$ satisfies (2·7), we obtain

$$\Psi = \frac{2A}{\pi} \int_0^\infty \frac{\beta \sinh \beta(\alpha - \theta)}{\sinh \beta\alpha} w(\beta)\, S(\sigma, \beta) \{S'(\sigma_1, \beta) - S'(\sigma_2, \beta)\}\, d\beta. \tag{3·4}$$

An alternative procedure would have been to obtain this expression directly from the inversion formula and then to deduce (3·3) by letting $\sigma_2 \to \sigma_1$, the constant A being adjusted to prevent the expression from vanishing.

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An important special case is obtained by letting $\sigma_1 \to 0$ in (3·3). This corresponds to the Helmholtz flow past a wedge with sonic velocity at infinity in which the streamline breaking away from the shoulder has sonic velocity on it at all points downstream. This flow has been discussed with reference to Tricomi's equation in (5) and (6) and for Chaplygin's equation in (10). It is obtained by taking the derivative of the integrand in (3·3) with respect to σ_1 and letting $\sigma_1 \to 0$. If the multiplying constant is suitably adjusted we obtain

$$\Psi = \frac{2A}{\pi} \int_0^\infty \frac{\beta \sinh \beta(\alpha - \theta)}{\sinh \beta\alpha} w(\beta) S(\sigma, \beta) \, d\beta. \tag{3·5}$$

Finally, we can obtain the solution of the free stream flow past a wedge in a channel by replacing $S'(\sigma_1, \beta)$ by 1 in (3·4).

While the integral expressions (3·3)–(3·5) are concise forms of solution, they are not very convenient from the point of view of further investigation. This is due to convergence difficulties associated with the solution near the stagnation point which arise because $S(\sigma, \beta)$ is infinite when $\sigma \to \infty$. We shall therefore return to consider equation (3·3) specifically in more detail and obtain an alternative representation. In the work which follows it is necessary to have some knowledge of the asymptotic behaviour of the particular solutions of equations (2·7), (2·8) which have been defined. In practice, it is more convenient to verify these *a posteriori* for particular cases of $k(\sigma)$ rather than to attempt to develop rigorously a general theory. However, the following general considerations will be helpful.

If in equation (2·8) the substitutions

$$t = \int^\sigma \{k(\sigma)\}^{\frac{1}{2}} \, d\sigma, \quad \chi = \{k(\sigma)\}^{\frac{1}{4}} u$$

are made, we obtain

$$\frac{d^2\chi}{dt^2} - \nu^2 \chi = F(t) \chi.$$

Thus for a fixed value of $\sigma > 0$ it is to be expected that when $|\nu|$ is large, two linearly independent solutions will have asymptotic behaviour like $e^{\pm \nu t}$. From the definition of $L(\sigma, \nu)$ it is clear that it behaves like $e^{-\nu t}$ and consequently $R(\sigma, \nu)$ behaves like $e^{\nu t}$ when $\Re\nu > 0$. Moreover, t is a positive, monotonically increasing function of σ. It can therefore be expected that

$$\lim_{|\nu| \to \infty} R(\sigma, \nu) L(\sigma_1, \nu) = 0, \tag{3·6}$$

when $\sigma < \sigma_1$ and $\Re\nu > 0$.

Let us suppose, further, that $k(\sigma)$ tends to a positive constant k_0^2 as $\sigma \to \infty$. Then $L(\sigma, \nu) \sim e^{-\nu k_0 \sigma}$ for large values of σ if $\Re\nu > 0$. Now V^{-1} satisfies (2·6) and is infinite when $\sigma \to \infty$ so that $V^{-1} \sim e^{k_0 \sigma}$ as $\sigma \to \infty$. Combining these two results we may state that

$$\lim_{\sigma \to \infty} V^{-1} L(\sigma, \nu) = 0 \tag{3·7}$$

if $\Re\nu > 1$. The same result can be shown to hold when $k(\sigma) \sim \sigma$ as $\sigma \to \infty$ by an examination of the asymptotic forms of the solutions of Airy's equation.

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The first step in finding an alternative expression for (3·3) is to substitute for $w(\beta)$ from (3·2). The integral then becomes

$$\Psi = \frac{A}{2\pi i}\int_0^\infty \frac{\beta \sinh \beta(\alpha-\theta)}{\sinh \beta\alpha} S(\sigma,\beta)\{L(\sigma_1, i\beta) - L(\sigma_1, -i\beta)\}\,d\beta,$$

where the factor $2k(\sigma_1)$ has been absorbed for convenience in the scale factor A.

If we substitute $\beta = -i\nu$ in the first and $\beta = i\nu$ in the second integral and recall that $S(\sigma,\beta)$ is an even function of β we obtain

$$\Psi = -\frac{A}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\nu \sin \nu(\alpha-\theta)}{\sin \nu\alpha} R(\sigma,\nu)\, L(\sigma_1,\nu)\,d\nu. \tag{3·8}$$

Because the integrand in (3·3) is symmetric in σ and σ_1 this may also be written

$$\Psi = -\frac{A}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\nu \sin \nu(\alpha-\theta)}{\sin \nu\alpha} R(\sigma_1,\nu)\, L(\sigma,\nu)\,d\nu. \tag{3·9}$$

It will be useful later to consider this integral on a contour C defined by

$$\int_C = \int_{-i\infty+\xi_0}^{i\infty+\xi_0}$$

where $1 < \xi_0 < \pi/\alpha$. This shift of contour crosses no singularities of the integrand and it can be readily seen to be justified because of the strong exponential term in the denominator together with the oscillating behaviour of $R(\sigma,\nu)$ and $L(\sigma,\nu)$ on the imaginary axis which cause the contributions at $\pm i\infty$ to vanish.

If $\sigma < \sigma_1$, equation (3·6) shows that the integral in (3·8) taken round a suitably chosen infinite semicircle in the right-hand half-plane vanishes. Thus (3·8) may be written as a series generated by the residues at the poles of cosec $\nu\alpha$. This gives

$$\Psi = -\frac{A\pi}{\alpha^2}\sum_{s=1}^{\infty} s \sin\frac{s\pi\theta}{\alpha} R\left(\sigma, \frac{s\pi}{\alpha}\right) L\left(\sigma_1, \frac{s\pi}{\alpha}\right) \quad (\sigma < \sigma_1), \tag{3·10}$$

and similarly

$$\Psi = -\frac{A\pi}{\alpha^2}\sum_{s=1}^{\infty} s \sin\frac{s\pi\theta}{\alpha} L\left(\sigma, \frac{s\pi}{\alpha}\right) R\left(\sigma_1, \frac{s\pi}{\alpha}\right) \quad (\sigma > \sigma_1). \tag{3·11}$$

It is easy to confirm from these series forms that the solution satisfies the boundary conditions.

4. *Position coordinates and drag.* Corresponding to the solution

$$\Psi = L(\sigma,\nu)\sin\nu(\alpha-\theta),$$

the velocity potential is given by

$$\Phi = -\frac{1}{\nu} L'(\sigma,\nu)\cos\nu(\alpha-\theta);$$

and the position coordinates, found by integrating (2·4), are expressed in the form

$$z = x + iy = \frac{e^{i\theta}}{i(\nu^2-1)}\left[\frac{i\nu L(\sigma,\nu)}{m}\{\cos\nu(\alpha-\theta) + i\nu\sin\nu(\alpha-\theta)\} + \frac{L'(\sigma,\nu)}{V}\{\sin\nu(\alpha-\theta) - i\nu\cos\nu(\alpha-\theta)\}\right],$$

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apart from an arbitrary constant. Of particular interest is the value of z when $\theta = \alpha$. This is given by

$$z = \frac{\nu e^{i\alpha}}{\nu^2 - 1}\left\{\frac{L(\sigma,\nu)}{m} - \frac{L'(\sigma,\nu)}{V}\right\}. \tag{4·1}$$

Similar results are obtained by replacing $L(\sigma,\nu)$ by $R(\sigma,\nu)$ in the above.

In order to find the length of the straight section of the profile in Fig. 1 we use both forms of solution (3·8) and (3·9). That is to say, if $l(\sigma)$ is the length at a given point measured from the stagnation point and $l(0) = l^*$ is the total length then we write

$$l^* = [l(\sigma)]_\infty^0 = [l(\sigma)]_{\sigma_1}^0 + [l(\sigma)]_\infty^{\sigma_1},$$

and use (4·1) (replacing $L(\sigma,\nu)$ by $R(\sigma,\nu)$) with (3·8) for the first term and with (3·9) for the second. If the contour $\int_{-i\infty}^{i\infty}$ is replaced by C on which $\Re\nu > 1$, the lower limit of the second integral will disappear because of the relation (3·7) and an associated relation involving $L'(\sigma,\nu)$. Thus

$$\begin{aligned} l^* &= -\frac{A}{2\pi i}\int_C \frac{\nu^2}{(\nu^2-1)\sin\nu\alpha}\left\{-L(\sigma_1,\nu) + \frac{1}{V_1}(R'L - RL')_{\sigma=\sigma_1}\right\}d\nu \\ &= \frac{A}{2\pi i}\int_C \frac{\nu^2}{(\nu^2-1)\sin\nu\alpha}\left\{\frac{1}{V_1} + L(\sigma_1,\nu)\right\}d\nu \end{aligned}$$

because of (2·11). If this is written as two integrals, the contour in the first can be moved back to the imaginary axis after taking into account the contribution from the pole at $\nu = 1$. On the imaginary axis the integral then vanishes since the integrand is an odd function. Thus

$$l^* = \frac{A}{2V_1\sin\alpha} + \frac{A}{2\pi i}\int_C \frac{\nu^2 L(\sigma_1,\nu)\,d\nu}{(\nu^2-1)\sin\nu\alpha}.$$

If the total drag on the upper side of the wedge is defined by

$$D = \sin\alpha\int_0^{l^*} p\,dl,$$

then

$$\begin{aligned} \frac{D}{\sin\alpha} &= p^*l^* - \int l\,dp \\ &= p^*l^* + r^*q^{*2}\int_0^1 l\rho V\,dV. \end{aligned}$$

l is found from (3·9) and (4·1) and hence

$$\int_0^1 l\rho V\,dV = \int_0^1\left[-\frac{A}{2\pi i}\int_C \frac{\nu^2}{(\nu^2-1)\sin\nu\alpha}R(\sigma_1,\nu)\left\{\frac{L(\sigma,\nu)}{m} - \frac{L'(\sigma,\nu)}{V}\right\}d\nu\right]\rho V\,dV.$$

If the orders of integration are reversed this becomes

$$\begin{aligned} \int_0^1 l\rho V\,dV &= -\frac{A}{2\pi i}\int_C \frac{\nu^2 R(\sigma_1,\nu)}{(\nu^2-1)\sin\nu\alpha}\left[\int_0^\infty \{L(\sigma,\nu) - \rho L'(\sigma,\nu)\}\frac{V}{\rho}d\sigma\right]d\nu \\ &= -\frac{A}{2\pi i}\int_C \frac{\nu^2 R(\sigma_1,\nu)\,[-VL(\sigma,\nu)]_0^\infty}{(\nu^2-1)\sin\nu\alpha}\,d\nu \\ &= \frac{A}{2\pi i}\int_C \frac{\nu^2 R(\sigma_1,\nu)\,d\nu}{(\nu^2-1)\sin\nu\alpha}. \end{aligned}$$

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The contour can now be moved back onto the imaginary axis provided the contribution from the pole at $\nu = 1$ is allowed for. Then, as before, the integral along the imaginary axis vanishes because the integrand is an odd function of ν. Thus finally

$$\frac{D}{\sin\alpha} = p^*l^* + \frac{Ar^*q^{*2}R(\sigma_1, 1)}{2\sin\alpha}.$$

The drag coefficient C_D will be defined by

$$C_D = \frac{\sin\alpha}{l^*}\int_0^{l^*}\frac{(p-p^*)\,dl}{\frac{1}{2}r^*q^{*2}}.$$

It is more customary to define the drag coefficient with the excess pressure measured relative to conditions in the undisturbed stream. However, it is convenient here to refer the pressure to its value at $M = 1$. This is also quite acceptable from the physical point of view as the pressure behind the wedge would be interpreted as p^* rather than p_1 on the basis of a 'wake' theory.

Accordingly

$$C_D = \frac{2}{r^*q^{*2}l^*}(D - p^*l^*\sin\alpha)$$

$$= \frac{AR(\sigma_1, 1)}{l^*},$$

or

$$C_D = \frac{R(\sigma_1, 1)\sin\alpha}{\frac{1}{2V_1} + \frac{\sin\alpha}{2\pi i}\int_C \frac{\nu^2 L(\sigma_1, \nu)\,d\nu}{(\nu^2-1)\sin\nu\alpha}}. \tag{4·2}$$

Special cases of all the above results occur when $\sigma_1 = 0$, which corresponds to the Helmholtz flow past a wedge with sonic velocity in the undisturbed stream. Thus (3·5) becomes

$$\Psi = -\frac{A}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\nu\sin\nu(\alpha-\theta)}{\sin\nu\alpha}L(\sigma, \nu)\,d\nu,$$

and this gives the series expansion

$$\Psi = -\frac{A\pi}{\alpha^2}\sum_{s=1}^{\infty} s\sin\frac{s\pi\theta}{\alpha}L\left(\sigma, \frac{s\pi}{\alpha}\right).$$

This could have been obtained directly by more elementary methods. The value of C_D^* (the value of C_D when $M_1 = 1$) can now be found by a process similar to the above or it may be obtained as the limit of (4·2), evaluated by differentiating numerator and denominator with respect to σ_1 and then putting $\sigma_1 = 0$. The final form is

$$\frac{1}{C_D^*} = \frac{1}{2\sin\alpha} + \frac{1}{2\pi i}\int_C \frac{\nu^2 L'(0, \nu)\,d\nu}{(\nu^2-1)\sin\nu\alpha}. \tag{4·3}$$

Some of the results of this section were obtained in (10) for the special case of Chaplygin's equation.

5. *Some special cases.* It is now shown how the general theory developed in the last section is applied to special examples. The simplest case is $k(\sigma) = 1$ which corresponds to incompressible flow. Any fixed velocity U may be taken as the 'sonic'

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velocity corresponding to $\sigma = 0$ and then $\sigma = -\log(q/U)$. The particular solutions of the differential equations (2·7) and (2·8) have already been identified at the end of § 2. However, it is not necessary to discuss this case further as the physical problems considered in the last section can be solved alternatively by the method of the complex potential and conformal transformation.

The most interesting case is that of Chaplygin's equation which is usually written in the form

$$4\tau^2(1-\tau)\frac{\partial^2\Psi}{\partial\tau^2} + 4\tau\{1+(\lambda-1)\tau\}\frac{\partial\Psi}{\partial\tau} + \{1-(2\lambda+1)\tau\}\frac{\partial^2\Psi}{\partial\theta^2} = 0, \tag{5·1}$$

where $\lambda = (\gamma-1)^{-1}$ and τ is defined by

$$\tau = \frac{q^2}{q_m^2} = \frac{\gamma-1}{\gamma+1}V^2,$$

q_m being the maximum velocity attainable by the gas due to adiabatic expansion. The relationship between τ and σ is given by

$$\frac{d\sigma}{d\tau} = -\left(\frac{\gamma+1}{2}\right)^{\lambda}\frac{(1-\tau)^{\lambda}}{2\tau}, \tag{5·2}$$

and we note in particular

$$\left(\frac{d\sigma}{d\tau}\right)_{\tau=\tau^*} = -\frac{\gamma+1}{2(\gamma-1)} = -\frac{1}{2\tau^*},$$

where τ^* is the value of τ at sonic velocity. The solution of (5·1) obtained by separating the variables is

$$\Psi = \psi_\nu(\tau)\sin\nu(\theta-\theta_0),$$

where $\psi_\nu(\tau)$ satisfies the ordinary differential equation

$$4\tau^2(1-\tau)\frac{d^2T}{d\tau^2} + 4\tau\{1+(\lambda-1)\tau\}\frac{dT}{d\tau} - \nu^2\{1-(2\lambda+1)\tau\}T = 0. \tag{5·3}$$

It can be shown that $\psi_\nu(\tau)$ is the function $\tau^{\nu/2}\,{}_2F_1(a_\nu, b_\nu;\ \nu+1;\ \tau)$ where a_ν and b_ν are the roots of the quadratic equation

$$2x^2 - 2(\nu-\lambda)x - \lambda\nu(\nu+1) = 0,$$

and the usual notation for hypergeometric functions is used. In general $\psi_{-\nu}(\tau)$ is also a solution of (5·3) and the functions $R(\sigma,\nu)$ and $L(\sigma,\nu)$ are therefore linear combinations of $\psi_\nu(\tau)$ and $\psi_{-\nu}(\tau)$. When, as here, functions of σ are said to be equal to certain functions of τ it is understood that σ and τ are related by integrating the equation (5·2), the constant of integration being fixed by the fact that $\sigma = 0$ when $\tau = \tau^*$.

An account of the properties of $\psi_\nu(\tau)$ as a function of the complex variable ν has been given by Lighthill (7). The principal results required in this paper are that $\psi_\nu(\tau) \sim \tau^{\nu/2}$ for small values of τ and that the asymptotic behaviour of $\psi_\nu(\tau)$ for large values of $|\nu|$ is $e^{\nu s}$, where $\tau < \tau^*$ and $\arg\nu$ is bounded away from π. s is a function of τ which increases from $-\infty$ to a negative constant s^* (σ in Lighthill's notation) as τ increases from 0 to τ^*. Knowledge of this asymptotic behaviour is required to verify that certain integrals in the complex plane vanish but we shall omit the details here.

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The Wronskian of $\psi_\nu(\tau)$ and $\psi_{-\nu}(\tau)$ can be found from the differential equation (5·3). After a little algebra we obtain

$$W\{\psi_\nu(\tau), \psi_{-\nu}(\tau)\} = -\frac{\nu(1-\tau)^\lambda}{\tau}.$$

From these results it may be seen that

$$L(\sigma, \nu) = -\frac{\psi_\nu(\tau)}{\psi_\nu(\tau^*)} \tag{5·4}$$

and

$$R(\sigma, \nu) = R_\nu(\tau) = -\frac{\psi_\nu(\tau)\,\psi_{-\nu}(\tau^*) - \psi_{-\nu}(\tau)\,\psi_\nu(\tau^*)}{2\nu(1-\tau^*)^\lambda}.$$

The solution of the Roshko problem for a polytropic gas can be written down from (3·10) and (3·11) as

$$\Psi = \frac{A\pi}{\alpha^2} \sum_{s=1}^{\infty} s \sin\frac{s\pi\theta}{\alpha} \frac{\psi_{s\pi/\alpha}(\tau)\, R_{s\pi/\alpha}(\tau_1)}{\psi_{s\pi/\alpha}(\tau^*)} \quad (\tau < \tau_1), \tag{5·5}$$

$$\Psi = \frac{A\pi}{\alpha^2} \sum_{s=1}^{\infty} s \sin\frac{s\pi\theta}{\alpha} \frac{R_{s\pi/\alpha}(\tau)\, \psi_{s\pi/\alpha}(\tau_1)}{\psi_{s\pi/\alpha}(\tau^*)} \quad (\tau^* \geqslant \tau > \tau_1). \tag{5·6}$$

It is not obvious that (5·6) is the correct analytic continuation of (5·5) in the region $\tau > \tau_1$. This is a consequence of the general theory developed in a previous section. The problem of continuing analytically a series solution such as (5·5) is one of considerable difficulty and leads in general to involved expressions. A general theory has been given by Lighthill (8) (see also (4)) and applied to a particular case by Mackie and Pack (9). It seems that the difficulty is caused in part by requiring the solution in terms of $\psi_\nu(\tau)$ and $\psi_{-\nu}(\tau)$. In obtaining solutions of (5·3) it is natural to look for expansions about $\tau = 0$. However, the function $\psi_{-\nu}(\tau)$ (for $\Re\nu > 0$) is unsatisfactory as a second solution to use with $\psi_\nu(\tau)$ because it has poles at $\nu = 2, 3, \ldots$, and these lead to undesirable complications in operations in the complex ν-plane. It is suggested there that $R_\nu(\tau)$ is a more suitable solution to use with $\psi_\nu(\tau)$. In the first place it is an integral function of ν. Secondly, it may be easily computed from the differential equation as it is the function which is zero and has derivative $-1/2\tau^*$ at $\tau = \tau^*$. It is true that in this example the expression (5·6) is simple because of the particularly favourable boundary conditions of the problem. Nevertheless, it seems probable that for such problems in which there is a free stream singularity at $\tau = \tau_1$ it would be helpful to look for a second solution of the equation (5·3) defined about $\tau = \tau^*$ as origin and that this should be used with $\psi_\nu(\tau)$ instead of $\psi_{-\nu}(\tau)$.

The drag coefficient for the Roshko problem is given by substituting appropriately in (4·2). Some simplification is possible as the function $\psi_1(\tau)$ can be written in the closed form

$$\psi_1(\tau) = (\gamma - 1)\{1 - (1-\tau)^{\lambda+1}\}/\gamma\tau^{\frac{1}{2}},$$

while

$$\psi_{-1}(\tau) = \tau^{-\frac{1}{2}} + \tfrac{1}{2}\lambda\psi_1(\tau).$$

From these results it follows that

$$R(\sigma, 1) = R_1(\tau) = \frac{\gamma - 1}{2\gamma(\tau\tau^*)^{\frac{1}{2}}(1-\tau^*)^\lambda}\{(1-\tau)^{\lambda+1} - (1-\tau^*)^{\lambda+1}\}.$$

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From these results, too, we may verify that V^{-1} satisfies the same differential equation as $\psi_1(\tau)$ and $\psi_{-1}(\tau)$. In fact

$$\frac{1}{V} = \left(\frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{2}} \tau^{-\frac{1}{2}} = \left(\frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{2}} \{\psi_{-1}(\tau) - \tfrac{1}{2}\lambda\psi_1(\tau)\}.$$

As the next example we consider the Tricomi approximation which is obtained by putting $k(\sigma) = (\gamma+1)\sigma$. Equation (1·1) is now reduced to Tricomi's equation

$$\frac{\partial^2\Psi}{\partial\eta^2} + \eta\frac{\partial^2\Psi}{\partial\theta^2} = 0 \tag{5·7}$$

by introducing a new variable η defined by

$$\eta = (\gamma+1)^{\frac{1}{3}}\sigma.$$

The solution of (5·7) obtained by separating the variables is

$$\Psi = \eta^{\frac{1}{2}}\{AK_{\frac{1}{3}}(\tfrac{2}{3}\nu\eta^{\frac{3}{2}}) + BI_{\frac{1}{3}}(\tfrac{2}{3}\nu\eta^{\frac{3}{2}})\}\sin\nu(\theta-\theta_0).$$

It is now straightforward to verify from the definitions that

$$R(\sigma,\nu) = \frac{3^{\frac{1}{3}}\Gamma(\frac{4}{3})}{\nu^{\frac{1}{3}}(\gamma+1)^{\frac{1}{3}}}\eta^{\frac{1}{2}}I_{\frac{1}{3}}(\tfrac{2}{3}\nu\eta^{\frac{3}{2}}),$$

$$L(\sigma,\nu) = -\frac{3^{\frac{1}{3}}\nu^{\frac{1}{3}}\Gamma(\frac{2}{3})}{\pi}\eta^{\frac{1}{2}}K_{\frac{1}{3}}(\tfrac{2}{3}\nu\eta^{\frac{3}{2}}). \tag{5·8}$$

V^{-1} may be found as a linear combination of these functions when $\nu = 1$ but the expression does not simplify appreciably and there is no particular advantage in obtaining V^{-1} explicitly. The series solutions for the Roshko problem can now be obtained by substituting in (3·10) and (3·11). This problem has been discussed by Helliwell and Mackie (5).

Finally we consider the approximation introduced by Tomotika and Tamada (12). This consists of putting $k(\sigma) = a(1-e^{-2\kappa\sigma})$, where

$$a = \left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)}, \quad \kappa = \left(\frac{\gamma+1}{2}\right)^{(\gamma+1)/(\gamma-1)}.$$

The advantage of this choice of $k(\sigma)$ is that $k(\sigma) \sim (\gamma+1)\sigma$ near $\sigma = 0$ and so is a good approximation near the sonic velocity while in addition $k(\sigma) \to \{2/(\gamma+1)\}^{2/(\gamma-1)}$ as $\sigma \to \infty$ so that the approximation is good near the stagnation point also. By contrast the Tricomi approximation is good only near the sonic velocity.

If we put

$$t = e^{-\kappa\sigma}, \tag{5·9}$$

equation (1·1) reduces in this case to

$$\frac{\partial^2\Psi}{\partial t^2} + t\frac{\partial\Psi}{\partial t} + a\kappa^{-2}(1-t^2)\frac{\partial^2\Psi}{\partial\theta^2} = 0,$$

and this has solutions

$$\Psi = \{AJ_n(nt) + BY_n(nt)\}\sin\nu(\theta-\theta_0),$$

where

$$n^2 = a\kappa^{-2}\nu^2. \tag{5·10}$$

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Then in the same manner as for Tricomi's equation it can be verified that

$$R(\sigma, \nu) = \frac{\pi}{2K}\{J_n(nt)\,Y_n(n) - Y_n(nt)\,J_n(n)\}$$

and

$$L(\sigma, \nu) = -\frac{J_n(nt)}{J_n(n)}, \tag{5·11}$$

where t, n are related to σ, ν respectively by the equations (5·9), (5·10).

6. *Application to the Helmholtz flow with* $M_1 = 1$. As an illustration of the manner in which the general theory can be applied to particular problems we shall examine in some detail the expression for the drag coefficient for the flow of gas past a wedge with sonic velocity upstream and with a free streamline, also at sonic velocity, breaking away from the wedge shoulder. More specifically, asymptotic expansions will be found for the drag coefficient as given by (4·3) for small values of α and these are obtained for the three cases corresponding to the special values of $k(\sigma)$ discussed in the last section. If that which corresponds to Chaplygin's equation is regarded as the exact solution, then an estimate can be made of the range for which the other two approximations may be considered good.

It is simplest to consider first the Tricomi approximation since the expression $L'(0, \nu)$ which occurs in (4·3) is found from (5·8) to be given by

$$L'(0, \nu) = \frac{\Gamma(\frac{2}{3})\,(\gamma+1)^{\frac{1}{3}}\,\nu^{\frac{2}{3}}}{3^{\frac{2}{3}}\Gamma(\frac{4}{3})}.$$

If α is small, the first pole of $\operatorname{cosec} \nu\alpha$ in the integrand in (4·3) is at a large distance from the origin. Consequently the contour C can be deformed without passing over any singularities to a new contour on which $|\nu|$ is everywhere large. The expression $\nu^2/(\nu^2-1)$ can now be expanded in powers of ν^{-1}, the new contour replaced once more by C and the resulting expressions evaluated. We make repeated use of the result

$$\frac{1}{2i}\int_C \frac{\nu^{-k}\,d\nu}{\sin \nu\alpha} = \left(\frac{\alpha}{\pi}\right)^{k-1}(1 - 2^{1-k})\,\zeta(k), \tag{6·1}$$

where $\zeta(k)$ is the Riemann zeta function.

After some algebra we obtain

$$\frac{1}{C_D^*} = \frac{(2^{\frac{5}{3}}-1)\,\zeta(\frac{5}{3})\,\Gamma(\frac{5}{6})}{3^{\frac{2}{3}}\Gamma(\frac{1}{3})\,\pi^{\frac{1}{2}}}\,\frac{(\gamma+1)^{\frac{1}{3}}}{\alpha^{\frac{5}{3}}} + \frac{1}{2\alpha} + O(\alpha^{\frac{1}{3}}).$$

The leading term agrees with that obtained by Helliwell and Mackie (5) as the limiting case of the drag in the Roshko problem as $M_1 \to 1$. Further evaluation gives

$$C_D^* = \frac{1{\cdot}89\alpha^{\frac{5}{3}}}{(\gamma+1)^{\frac{1}{3}}}\{1 - 0{\cdot}70\alpha^{\frac{2}{3}} + 0{\cdot}49\alpha^{\frac{4}{3}} + O(\alpha^2)\}. \tag{6·2}$$

A point of theoretical interest arises here. If the Tricomi approximation is derived by considering perturbations on a uniform sonic stream and neglecting higher-order terms, then it is customary to use a first-order approximation for the pressure also and

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this procedure would lead to the first term in (6·2) only. However if, as here, the equation for Ψ is thought of as the exact equation for a fictitious gas, then the additional terms in (6·2) are obtained. These further terms are of the correct form as far as the powers of α are concerned, but it will be seen that the numerical coefficients are too much in error to make them significant.

The same method is used to calculate C_D^* for the equations of Chaplygin and of Tomotika and Tamada, $L'(0, \nu)$ being found from (5·4) and (5·11) respectively. The only essential difference is that the expression for $L'(0, \nu)$ has now itself to be expanded in (fractional) powers of ν^{-1}. For Chaplygin's equation the asymptotic forms of $\psi_\nu(\tau)$, $\psi'_\nu(\tau)$ as given by Cherry (1) are used together with the asymptotic expansions of $J_\nu(\nu)$ and $J'_\nu(\nu)$, these latter being also required when the equation of Tomotika and Tamada is used. It is not necessary to present the work in detail, the procedure being tedious but straightforward. All the resulting integrals can be evaluated in terms of zeta functions by means of (6·1) and the first few terms in the expansions for C_D^* are as follows:

Chaplygin
$$C_D^* = \frac{1{\cdot}89\alpha^{\frac{5}{3}}}{(\gamma+1)^{\frac{1}{3}}}\{1 - 0{\cdot}155\alpha^{\frac{2}{3}} - 0{\cdot}25\alpha^{\frac{4}{3}} + O(\alpha^2)\}. \tag{6·3}$$

Tomotika and Tamada
$$C_D^* = \frac{1{\cdot}89\alpha^{\frac{5}{3}}}{(\gamma+1)^{\frac{1}{3}}}\{1 - 0{\cdot}283\alpha^{\frac{2}{3}} + 0{\cdot}065\alpha^{\frac{4}{3}} + O(\alpha^2)\}.$$

The leading term is the same in all three cases. (This is true not only of the approximate numerical value given above but also of the algebraic form before it is computed.) Equation (6·3) represents the exact solution. There is no justification for retaining terms other than the first in the Tricomi approximation (6·2), but it is interesting to note that the error which arises when using the first term by itself is about the same as that in the form of C_D^* given by using the equation of Tomotika and Tamada. It must be remembered that this applies only to small angle wedges, which is equivalent to a small perturbation on a uniform sonic stream, and that the approximation of Tomotika and Tamada can be used for a variety of other problems for which the Tricomi approximation is quite inapplicable. It may, however, be concluded that the error in using Tricomi's equation which is introduced by the very bad approximation near the stagnation point is not as serious as might be expected. For total wedge angles less than 20° the leading term gives an estimate which is accurate to within about 7 %.

7. *Concluding remarks.* The theory described in this paper presents a fairly complete account of the flow of a gas past a wedge provided the velocity is nowhere allowed to become supersonic. The restriction $\Psi = 0$ on the portion DC of the sonic line in Fig. 2 is not serious. For if instead $\Psi = h(\theta)$ there it is only necessary to add a solution

$$\Psi = -\sum_{n=1}^{\infty} a_n L\left(\sigma, \frac{n\pi}{\alpha}\right) \sin\frac{n\pi\theta}{\alpha},$$

where the a_n are the coefficients of the Fourier sine series of $h(\theta)$ in the range $0 < \theta < \alpha$. By combining these results an expression for Ψ can be found in the interior of the rectangle $BCDB$ when arbitrary values are specified on the boundary.

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Certain generalizations and extensions are suggested. Germain (3) has developed a theory analogous to that used in § 2 for which the conditions $\Psi = 0$ on DC is replaced by $\partial\Psi/\partial\sigma = 0$. By means of (2·3) and (2·4) it is easily verified that the Jacobian of the transformation from x, y to σ, θ is given by

$$J = \frac{\partial(x, y)}{\partial(\sigma, \theta)} = \frac{k(\sigma)\,(\partial\Psi/\partial\theta)^2 + (\partial\Psi/\partial\sigma)^2}{mV}.$$

It follows that if $\partial\Psi/\partial\sigma = 0$ on $\sigma = 0$, $J = 0$ on the sonic line (since $k(0) = 0$) which is therefore a limit line. It would not then be possible to continue the solution across the sonic line, although a good representation of the flow upstream from it might be obtained. In two of the earliest papers on transonic flow past wedges this model was adopted by Cole (2) and Weinstein (13), who used the Tricomi approximation.

Finally it should be pointed out that problems similar to those discussed can be solved in a rectangular region in Fig. 2 bounded on the left by $\sigma = \sigma_0\ (> 0)$ instead of $\sigma = 0$. The resulting algebra is more complicated and it does not seem desirable to develop this theory in any detail as more direct alternative methods are available for strictly subsonic flows.

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