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On randomly weighted one-sided ergodic Hilbert transforms

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Abstract. In this paper we obtain (almost) optimal results concerning randomly weighted one-sided ergodic Hilbert transforms. Given an iid sequence of centered random variables (X_n) in $L\log L$, we show that there exists a universal set Ω' of probability 1 such that for any $\omega \in \Omega'$ the realization $(X_n(\omega))$ is good for the almost everywhere convergence of the weighted one-sided ergodic Hilbert transform associated with any dynamical system and any $g \in L\log L$. The method applies to powers along subsequences with 'small' growth and when considering Dunford–Schwartz operators instead of pointwise transformation. If the (X_n) are symmetric, but only in $L\log\log L$, we obtain a slightly weaker result.

1. *Introduction*

During recent decades many authors have tried to obtain necessary or sufficient conditions on a sequence (a_n) to ensure that this sequence is universally good for the weighted ergodic theorems on any dynamical system, or more specifically for the convergence of the weighted one-sided ergodic transform. The case of Dunford–Schwartz operators was also considered (an operator T on a probability space which is a contraction of L^1 and L^∞). More precisely, one wants to know if for any probability space (X, Σ, μ), any Dunford–Schwartz operator T on it and any function g in a certain class of measurable functions on X, we have that

$$\frac{1}{n}\sum_{k=1}^{n} a_k T^k g \tag{1}$$

converges μ-almost surely, or the series

$$\sum_{n\geq 1} \frac{a_n T^n g}{n} \tag{2}$$

converges μ-almost surely.

One may deduce the first convergence from the second one, using Kronecker's lemma. The case where T is given by an (ergodic) measure-preserving transformation τ is of particular interest.

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In [**9**], del Junco and Rosenblatt considered the general case where $(X, \Sigma, \mu) = ([0,1], \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra, λ is the Lebesgue measure and T is associated to any measure-preserving transformation τ (which may be chosen such that the system be ergodic). They proved that a necessary condition for the convergence of (2) for any bounded centered g on $X = [0,1]$ is that $\sup_{n\geq 1} \left|\sum_{k=1}^{n} a_k/k\right| < +\infty$. This is an easy consequence of [**9**, Theorem 3.2].

One may ask about sufficient conditions. The Brunel–Keane ergodic theorem along 'uniform sequences' led to the proof by Ryll–Nardzewski of almost sure convergence of (1) for (a_n) bounded Besicovich and $g \in L^1$ (for measure-preserving τ). Earlier Tempelman had proved the convergence in (1) for (a_n) (unbounded) p-Besicovich and $g \in L^q$, $q = p/(p-1) > 1$. See [**11**] for extensions to more general operators (including Dunford–Schwartz ones) and additional references.

Other sufficient conditions were obtained when the (a_n) are given 'randomly'. The deep return times theorem of Bourgain [**4**] (see [**5**] or [**16**]) asserts that for any sequence (X_n) of stationary bounded random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ there exists a set Ω', $\mathbb{P}(\Omega') = 1$, such that for every $\omega \in \Omega'$, the sequence $(a_n) := (X_n(\omega))$ is good for the almost sure convergence of (1) for any $g \in L^1$ on any dynamical system (i.e. T is given by a measure-preserving transformation). More generally, the result remains true with the duality $(X_1) \in L^p$, $g \in L^q$ with $1/p + 1/q = 1$.

Assani [**1**] investigated a possible break in the duality in Bourgain's theorem when the (X_n) are assumed iid. In [**2**] he looked at (2).

In this note we will extend results in this direction, proving convergence results for (2) for Dunford–Schwartz operators.

Sufficient conditions for almost sure convergence of series similar to (2), for Dunford–Schwartz operators) were given by Rosenblatt [**15**].

2. *The results*

Our aim is to prove the following.

THEOREM 1. *Let (X_n) be a sequence of iid random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, centered and in $L\log L$. Let (u_n) be a strictly increasing sequence of integers satisfying $u_n = O(e^{n^\gamma})$ for some $\gamma \in (0,1)$. Then, there exists a set Ω' with $\mathbb{P}(\Omega') = 1$, such that for any probability space (X, Σ, μ), any Dunford–Schwartz operator T on X, and any g with $\int_X |g| \log^+ |g|\, d\mu < +\infty$, the series*

$$\sum_{n\geq 1} \frac{X_n(\omega) T^{u_n} g}{n} \tag{3}$$

converges μ-almost surely for every $\omega \in \Omega'$.

Furthermore, for every $\omega \in \Omega'$, $\sup_{n\geq 1} \left|\sum_{k=1}^{n} [X_k(\omega) T^{u_k} g]/k\right| \in L^1(X)$.

If the (X_n) are assumed to be symmetric, we obtain the following result.

THEOREM 2. *Let (X_n) be a sequence of iid symmetric random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, in $L\log\log L$. Let (u_n) be a strictly increasing sequence of integers satisfying $u_n = O(e^{\log^\delta n})$ for some $\delta \geq 1$. Then, there exists a set Ω'*

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with $\mathbb{P}(\Omega') = 1$*, such that for any probability space* (X, Σ, μ)*, any Dunford–Schwartz operator* T *on* X*, any* $p > 1$ *and any* $g \in L^p(X)$*, the series*

$$\sum_{n\geq 1} \frac{X_n(\omega) T^{u_n} g}{n} \tag{4}$$

converges μ*-almost surely for every* $\omega \in \Omega'$*.*

Furthermore, for every $\omega \in \Omega'$*,* $\sup_{n\geq 1} \left|\sum_{k=1}^{n} [X_k(\omega) T^{u_k} g]/k\right| \in L^1(X)$*.*

Remarks. (1) The convergence of Theorem 1 for the special case $g \equiv 1$ and $T = \mathrm{Id}$ is given in Theorem 6 of [**12**].

(2) Applying Theorems 1 or 2, with $u_n = n$, to arbitrary rotations of the unit circle and $g(z) = z$ we obtain almost sure pointwise convergence of the random Fourier series $\sum_{n\geq 1} [X_n(\omega)\lambda^n]/n$. This was first proved in [**6**], where even the uniform convergence of the Fourier series is obtained. When the (X_n) are symmetric, the uniform convergence may be deduced from our theorem and Billard's theorem (see e.g. [**10**]).

(3) Let (Z_n) be positive iid random variables. Then, it is not difficult to show (see the discussion after Theorem 6 in [**12**]) that the convergence of the series (3) with $(X_n = Z_n - \mathbb{E}[Z_n])$, $g \equiv 1$ and $T = \mathrm{Id}$ (or more generally the condition $\sup_{n\geq 1} \left|\sum_{k=1}^{n} X_k/k\right| < +\infty$) implies that Z_1 (and so X_1) is in $L \log L$. So, the integrability condition on (X_n) in Theorem 1 is optimal even if we restrict ourselves to bounded g.

(4) As already observed by Assani [**2**], when $u_n = n$ the integrability condition on (X_n) in Theorem 2 is necessary even if we restrict ourselves to bounded g. It is also necessary if we restrict ourselves to T associated to ergodic transformation and centered g. By Remark (2), (for λ a root of unity we take the cyclic rotation on a finite group) we obtain almost sure convergence of the Fourier series in Remark (2) at any $\lambda \neq 1$. Now, if the conclusion of Theorem 2 is valid, by the result of del Junco and Rosenblatt the sequence (X_n/n) is bounded almost surely and X_1 must be in L^1 by the Borel–Cantelli lemma. Hence, by symmetry (and independence) $\sum_{n\geq 1} X_n/n$ converges almost surely (see [**12**]). So, the Fourier series converges at any λ. But a theorem of Talagrand [**19**] asserts that for a sequence of symmetric iid (X_n) the condition $X_1 \in L \log\log L$ is equivalent to the almost sure pointwise convergence of $\sum_{n\geq 1} [X_n(\omega)\lambda^n]/n$.

(5) If we restrict ourselves to T given by an *ergodic* transformation and centered g, the integrability condition on (X_n) in Theorem 1 is optimal. By the result of [**9**] mentioned in the introduction, a necessary condition for the convergence of (3) is that $\sup_{n\geq 1} \left|\sum_{k=1}^{n} X_k/k\right| < +\infty$ $\mathbb{P}$-almost surely, and by Remark (3) (if $X_1 = Z_1 - \mathbb{E}[Z_1]$), this implies that X_1 is in $L \log L$.

Theorem 1 extends results of Assani [**2**] and Boukhari and Weber [**3**], who worked in L^p spaces $p > 1$. See also Schneider and Weber [**18**], Assani [**1**] or Durand and Schneider [**7**] for versions of randomly weighted ergodic theorems, still in some L^p spaces.

More precisely, they proved the following.

THEOREM 3. (Assani, [**2**]) *Let* (X_n) *be iid centered random variables in* $L^p(\Omega, \mathcal{F}, \mathbb{P})$ $(p > 1)$*. Then there exists a set of full measure* Ω' $(\mathbb{P}(\Omega') = 1)$*, such that, for any dynamical system* (X, Σ, τ, μ) *and all* $g \in L^q(X)$ $(q > 1$*, not necessarily conjugate of* $p)$*,*

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the series

$$\sum_{n\geq 1}\frac{X_n(\omega)g\circ\tau^n}{n}$$

converges μ-almost surely for every $\omega\in\Omega'$.

Assani noticed there that his proof works when replacing (T^n) with (T^{u_n}) for some sequence (u_n) of polynomial growth.

THEOREM 4. (Durand and Schneider [**7**]) *Let (X_n) be iid centered random variables in $L^p(\Omega,\mathcal{F},\mathbb{P})$ $(p>1)$. Let (u_n) be a strictly increasing sequence of integers satisfying $u_n=O(2^{n^\gamma})$ for some $\gamma\in(0,1)$. Then there exists a set of full measure Ω' $(\mathbb{P}(\Omega')=1)$, such that, for any dynamical system (X,Σ,τ,μ) and all $g\in L^2(X)$,*

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}X_k(\omega)g\circ\tau^{u_k}=0,\quad \mu\text{-almost surely}$$

for every $\omega\in\Omega'$.

Hence, Theorem 3 is contained in Theorem 1 and Theorem 8 is a corollary of it by Kronecker's lemma.

The method of proof of Theorems 1 and 2 (as in the proofs of the previous mentioned works) relies on the spectral theorem and on uniform estimates of random polynomials.

In Theorem 1 we will use results which are basically due to Salem and Zygmund [**17**]. In Theorem 2, we will need much stronger estimates due to Weber [**20**], which may also be derived from Fan and Schneider [**8**].

3. *The proofs*

Theorem 1 is based on the following observations.

LEMMA 5. *Let (X_n) be a sequence of iid centered random variables in $L\log L$, defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Let (u_n) be a strictly increasing sequence of integers satisfying $u_n=O(e^{n^\gamma})$ for some $\gamma\in(0,1)$. Let $\alpha\in(0,1)$. Define*

$$Y_n:=X_n\mathbf{1}_{\{|X_n|\leq n^\alpha\}}-\mathbb{E}[X_n\mathbf{1}_{\{|X_n|\leq n^\alpha\}}],\quad \text{for all } n\geq 1.$$

We have:

(i) $\sum_{n\geq 1}(|X_n|/n)\mathbf{1}_{\{|X_n|\geq n^\alpha\}}<+\infty$, $\mathbb{P}$*-almost surely;*

(ii) *for every $\beta>2/(1-\alpha)$, we have $\sup_{N\geq 1}(1/N^{\beta-1})\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}|Y_n|<+\infty$, $\mathbb{P}$-almost surely;*

(iii) *if $\alpha<1-\gamma$, then, for every $\beta>(1+\alpha)/(1-\alpha-\gamma)$ we have*

$$\sum_{N\geq 1}N^{\alpha/2}\sup_{t\in\mathbf{R}}\left|\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}\frac{Y_n}{n}e^{iu_nt}\right|<+\infty,\quad \mathbb{P}\text{-almost surely.}$$

Proof. (i) We have

$$\sum_{n\geq 1}\mathbb{E}\left[\frac{|X_n|}{n}\mathbf{1}_{\{|X_n|\geq n^\alpha\}}\right]=\mathbb{E}\left[|X_1|\sum_{1\leq n\leq|X_1|^{1/\alpha}}\frac{1}{n}\right]\leq C\mathbb{E}[|X_1|\log^+|X_1|]<+\infty.$$

Hence, the first point is true by Beppo Levi.

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(ii) We have $(1/N^{\beta-1})\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}|Y_n| \le (1/N^{\beta-1})\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}(|Y_n| - \mathbb{E}[|Y_n|]) + C\mathbb{E}[|X_1|]$.

It suffices to prove that the series $\sum_{N\ge1}(1/N^{\beta-1})\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}(|Y_n| - \mathbb{E}[|Y_n|])$ converges $\mathbb{P}$-almost surely. But, by independence of (Y_n) and by Komolgorov's theorem it suffices to show that

$$S := \sum_{N\ge1}\mathbb{E}\left[\left(\frac{1}{N^{\beta-1}}\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}(|Y_n| - \mathbb{E}[|Y_n|])\right)^2\right] < +\infty.$$

By orthogonality, we have

$$\begin{aligned} S &\le \sum_{N\ge1}\frac{1}{N^{2(\beta-1)}}\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}\mathbb{E}[|Y_n|^2] \le \sum_{N\ge1}\frac{1}{N^{2(\beta-1)}}\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} n^\alpha\mathbb{E}[|X_1|] \\ &\le C\sum_{N\ge1}\frac{(N+1)^{(\alpha+1)\beta} - N^{(\alpha+1)\beta}}{N^{2(\beta-1)}} \le C\sum_{N\ge1}\frac{1}{N^{2(\beta-1)-((\alpha+1)\beta-1)}} \\ &\le C\sum_{N\ge1}\frac{1}{N^{\beta(1-\alpha)-1}} < +\infty, \end{aligned}$$

which proves the second point.

(iii) We need the following estimation, whose proof may be found in [**1**] and which is a consequence of a result of Salem and Zygmund [**17**]: there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\sup_{t\in\mathbf{R}}\left|\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}\frac{Y_n}{n}e^{iu_nt}\right|\right] \le C\left(\log u_{[(N+1)^\beta]}\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}\frac{\mathbb{E}[|Y_n|^2]}{n^2}\right)^{1/2}.$$

Then, using the previous computations, we obtain

$$\begin{aligned} &\sum_{N\ge1}N^{\alpha/2}\mathbb{E}\left[\sup_{t\in\mathbf{R}}\left|\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}\frac{Y_n}{n}e^{iu_nt}\right|\right] \\ &\quad\le C\sum_{N\ge1}N^{\alpha/2}\left(\frac{(N+1)^{\gamma\beta}}{N^{2\beta}}\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}\mathbb{E}[|Y_n|^2]\right)^{1/2} \\ &\quad\le C\sum_{N\ge1}N^{\alpha/2+\gamma\beta/2-1}\left(\frac{1}{N^{2(\beta-1)}}\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}\mathbb{E}[|Y_n|^2]\right)^{1/2} \\ &\quad\le C\sum_{N\ge1}\frac{1}{N^{\frac{1}{2}(1-\alpha+\beta(1-\alpha-\gamma))}} < +\infty. \end{aligned}$$

Hence, the point (iii) of the lemma is true, by Beppo Levi. □

Proof of Theorem 1. Since $\gamma \in (0,1)$, one can choose α such that $0 < \alpha < 1-\gamma$. Then, one can choose $\beta > \max(2/(1-\alpha), (1+2\alpha)/(1-\alpha-\gamma))$. Then, let Ω' be the set on which the assertions (i), (ii) and (iii) of the previous lemma are satisfied. Let $\omega \in \Omega'$.

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Let (X, Σ, T, μ) be a dynamical system and g be a Σ-measurable function on X in $L \log L$ (i.e. $\int_X |g| \log^+ |g| \, d\mu < +\infty$).

By point (i) of Lemma 5 (and the proof of it), we have (since T contracts L^1)

$$\sum_{n\geq 1} \int_X \frac{|X_n|(\omega)}{n} \mathbf{1}_{\{|X_n|(\omega)\geq n^\alpha\}} |T^{u_n} g| \, d\mu < +\infty, \tag{5}$$

$$\sum_{n\geq 1} \int_X \mathbb{E}\left[\frac{|X_n|}{n} \mathbf{1}_{\{|X_n|\geq n^\alpha\}}\right] |T^{u_n} g| \, d\mu < +\infty. \tag{6}$$

Hence, by Beppo Levi, we only need to prove that the series $\sum_{n\geq 1} (Y_n(\omega) g \circ T^{u_n}/n)$ converges μ-almost surely.

We want also to truncate the function g, in order to use the spectral theorem. We have, using point (ii) of Lemma 5,

$$\begin{aligned}
&\sum_{N\geq 1} \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \int_X |T^{u_n}(g\mathbf{1}_{\{|g|\geq N^\alpha\}})|(x)\mu(dx) \frac{|Y_n(\omega)|}{n} \\
&\quad \leq \sum_{N\geq 1} \int_X \frac{1}{N} |g\mathbf{1}_{\{|g|\geq N^\alpha\}}|(x)\mu(dx) \frac{1}{N^{\beta-1}} \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} |Y_n(\omega)| \\
&\quad \leq \sup_{N\geq 1} \frac{1}{N^{\beta-1}} \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} |Y_n|(\omega) \sum_{N\geq 1} \int_X \frac{1}{N} |g\mathbf{1}_{\{|g|\geq N^\alpha\}}|(x)\mu(dx) \\
&\quad \leq C \sum_{N\geq 1} \int_X |g| \sum_{1\leq N\leq |g|^{1/\alpha}} \frac{1}{n} \, d\mu \\
&\quad \leq C \int_X |g| \log^+ |g| \, d\mu.
\end{aligned} \tag{7}$$

So $\sum_{N\geq 1} \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} (|Y_n||T^{u_n}(g\mathbf{1}_{\{|g|\geq N^\alpha\}})|/n) < +\infty$ μ-almost surely, by point (ii) of Lemma 5. Hence, it suffices to prove that the series

$$\sum_{N\geq 1} \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{Y_n(\omega) T^{u_n}(g\mathbf{1}_{\{|g|\leq N^\alpha\}})}{n}$$

converges μ-almost surely and that

$$\sup_{[N^\beta]+1\leq n\leq [(N+1)^\beta]} \left| \sum_{k=[N^\beta]+1}^{n} \frac{Y_n(\omega) T^{u_n}(g\mathbf{1}_{\{|g|\leq N^\alpha\}})}{n} \right| \to 0.$$

Since T is Dunford–Schwartz it also contracts $L^2(X)$. For every $h \in L^2(X)$, define a sequence by $c_l := \langle T^l h, h\rangle$ for $l \geq 0$ and $c_l := \overline{\langle T^l h, h\rangle}$ for $I < 0$. Then (see, e.g., [**14**]), $(c_l)_{l\in\mathbf{Z}}$ is positive semi-definite, hence represents the Fourier coefficients of a measure μ_h, and, by the unitary dilation theorem (see [**14**]) we have the following (see also Lemma 2.1 of [**3**]).

LEMMA 6. *For every $h \in L^2(X)$ and any polynomial $P(z) = \sum_{k=1}^n \alpha_k z^k$, we have*

$$\|P(T)h\|_2 \leq \left(\int_{-\pi}^{\pi} |P(e^{it})|^2 \mu_h(dt) \right)^{1/2} \leq \sup_{t\in[-\pi,\pi]} |P(e^{it})| \|h\|_2.$$

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Using the previous lemma with $h := g\mathbf{1}_{\{|g|\le N\}}$, we obtain

$$\begin{aligned}
&\int_X \left| \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{Y_n(\omega) T^{u_n}(g\mathbf{1}_{\{|g|\le N^\alpha\}})(x)}{n} \right| \mu(dx) \\
&\quad \le \left(\int_X \left| \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{Y_n(\omega) T^{u_n}(g\mathbf{1}_{\{|g|\le N^\alpha\}})}{n} \right|^2 \mu(dx) \right)^{1/2} \\
&\quad \le \sup_{t\in\mathbf{R}} \left| \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{Y_n(\omega)}{n} e^{iu_n t} \right| \, \|g\mathbf{1}_{\{|g|\le N^\alpha\}}\|_2 \\
&\quad \le \|g\|_1 N^\alpha \sup_{t\in\mathbf{R}} \left| \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{Y_n(\omega)}{n} e^{iu_n t} \right|.
\end{aligned}$$

Hence

$$\sum_{N\ge 1} \int_X \left| \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{Y_n(\omega) T^{u_n}(g\mathbf{1}_{\{|g|\le N^\alpha\}})(x)}{n} \right| \mu(dx) < +\infty. \tag{8}$$

By point (ii) of Lemma 5, we have, using the fact that T contracts $L^\infty(X)$,

$$\begin{aligned}
&\sup_{[N^\beta]+1\le n\le [(N+1)^\beta]} \left| \sum_{k=[N^\beta]+1}^{n} \frac{Y_n(\omega) T^{u_n}(g\mathbf{1}_{\{|g|\le N^\alpha\}})}{n} \right| \\
&\quad \le N^\alpha \frac{1}{N^\beta} \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} |Y_n|(\omega) \\
&\quad \le \frac{1}{N^{1-\alpha}} \sup_{N\ge 1} \frac{1}{N^{\beta-1}} \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} |Y_n|(\omega).
\end{aligned} \tag{9}$$

This proves the desired almost sure convergence. To obtain the integrability of the maximal function, notice that (9) is uniform on X, and combine this with (8), (5), (6) and (7). □

Remark. Notice that we used the parameter α for the truncation of (X_n) and those of g. We did not see any benefit in using different truncations. The main reason is the limitation imposed by the computation (7) (which is the only place where we use that g is in $L\log L$, everywhere else only $g \in L^1$ is used). Hence it looks legitimate to conjecture that the integrability condition assumed on g in Theorem 1 is optimal.

The proof of Theorem 2 is based on the following lemma. It will not be possible here to use the rather crude estimate used in (9). Hence we need to replace point (iii) of Lemma 5 by points (iii) and (iv) of the next lemma which will allow us to use results of Móricz [**13**].

LEMMA 7. *Let (X_n) be a sequence of iid symmetric random variables in $L\log\log L$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let (u_n) be a strictly increasing sequence of integers satisfying $u_n = O(e^{\log^\delta n})$ for some $\delta \ge 1$. For $\varepsilon > 0$, define*

$$Y_{n,\varepsilon} := X_n \mathbf{1}_{\{|X_n|\le n/\log^\varepsilon n\}}, \quad \textit{for all } n \ge 1.$$

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We have:

(i) *for every* $\varepsilon > 0$, $\sum_{n\geq 1}(|X_n|/n)\mathbf{1}_{\{|X_n|\geq n/\log^{\varepsilon} n\}} < +\infty$, $\mathbb{P}$-*almost surely;*

(ii) *for every* $\varepsilon > 1$, $\sup_{N\geq 1}(1/2^N)\sum_{k=2^N+1}^{2^{N+1}} |Y_{k,\varepsilon}| < +\infty$, $\mathbb{P}$-*almost surely;*

(iii) *for every* $\varepsilon > 0$, *for* $\mathbb{P}$-*almost every* $\omega \in \Omega$, *there exists* $C_\varepsilon(\omega) > 0$ *such that*

$$\sup_{t\in\mathbf{R}}\left|\sum_{k=l+1}^{n}\frac{Y_{k,\varepsilon}}{k}e^{iu_kt}\right| \leq C_\varepsilon(\omega)\left(\log u_n\sum_{k=l+1}^{n}\frac{|Y_{k,\varepsilon}|^2}{k^2}\right)^{1/2}, \quad \text{for all } 0\leq l<n;$$

(iv) *for every* $\alpha > 0$ *and* $\varepsilon > 2+\delta+\alpha$, *we have*

$$\sum_{N\geq 1}\left(N^\alpha \log u_{2^{N+1}}\sum_{k=2^N+1}^{2^{N+1}}\frac{|Y_{k,\varepsilon}|^2}{k^2}\right)^{1/2} < +\infty, \quad \mathbb{P}\text{-almost surely}.$$

Proof. (i) We give here an argument due to Cuzick and Lai [**6**]. Since $X_1 \in L^1$, then (by the Borel–Cantelli lemma), almost surely, the events $\{|X_n| \geq n\}$ take place a finite number of times. On the other hand, we have

$$\begin{aligned}\sum_{n\geq 1}\frac{\mathbb{E}\big[|X_n|\mathbf{1}_{\{n/\log^{\varepsilon} n\leq|X_n|\leq n\}}\big]}{n} &\leq C\mathbb{E}\left[|X_1|\sum_{|X_1|\leq n\leq|X_1|(\log^+|X_1|)^{\varepsilon}}\frac{1}{n}\right]\\ &\leq C\mathbb{E}[|X_1|\log\log(e+|X_1|)].\end{aligned}$$

Hence, (i) is true by Beppo Levi.

(ii) We have

$$\sup_{N\geq 1}\frac{1}{2^N}\sum_{k=2^N+1}^{2^{N+1}}|Y_{k,\varepsilon}| \leq \sup_{N\geq 1}\frac{1}{2^N}\sum_{k=2^N+1}^{2^{N+1}}(|Y_{k,\varepsilon}|-\mathbb{E}[|Y_{k,\varepsilon}|])+\mathbb{E}[|X_1|].$$

As in the proof of Lemma 5, by Kolmogorov's theorem it suffices to prove that

$$S := \sum_{N\geq 1}\mathbb{E}\left[\left(\frac{1}{2^N}\sum_{n=2^N+1}^{2^{N+1}}(|Y_{n,\varepsilon}|-\mathbb{E}[|Y_{n,\varepsilon}|])\right)^2\right] < +\infty.$$

By orthogonality, we have

$$S \leq \sum_{N\geq 1}\frac{1}{2^{2N}}\sum_{n=2^N+1}^{2^{N+1}}\mathbb{E}[|Y_{n,\varepsilon}|^2] \leq C\sum_{N\geq 1}\frac{1}{2^{2N}}\sum_{n=2^N+1}^{2^{N+1}}\mathbb{E}[|X_1|]\frac{2^{N+1}}{N^{\varepsilon}} \leq C\sum_{N\geq 1}\frac{1}{N^{\varepsilon}} < +\infty.$$

Hence, point (ii) is true.

(iii) This follows from Theorem 7 of Weber [**20**], since the $(Y_{n,\varepsilon})$ are independent symmetric variables. It may be derived by applying Theorem 1 of Fan and Schneider [**8**] to the processes $(c_n\xi_n)$ (where (ξ_n) is the Rademacher system and (c_n) any sequence of complex numbers) and noticing that the processes $(Y_{n,\varepsilon})$ and $(\varepsilon_n Y_{n,\varepsilon})$ are stochastically equivalent.

(iv) We have

$$\begin{aligned}\sum_{k=2^N+1}^{2^{N+1}}\frac{\mathbb{E}[|Y_{k,\varepsilon}|^2]}{k^2} &\leq \mathbb{E}[|X_1|]\sum_{k=2^N+1}^{2^{N+1}}\frac{1}{k\log^{\varepsilon} k}\\ &\leq \frac{C}{N^{\varepsilon}}.\end{aligned}$$

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By the Cauchy–Schwarz inequality in $L^2(\Omega)$, we deduce

$$\sum_{N\geq 1}\mathbb{E}\bigg(N^{\alpha}\log u_{2^{N+1}}\sum_{k=2^N+1}^{2^{N+1}}\frac{|Y_{k,\varepsilon}|^2}{k^2}\bigg)^{1/2}\leq C\sum_{N\geq 1}\bigg(N^{\alpha+\delta}\sum_{k=2^N+1}^{2^{N+1}}\frac{\mathbb{E}[|Y_{k,\varepsilon}|^2]}{k^2}\bigg)^{1/2}$$
$$\leq C\sum_{N\geq 1}\frac{1}{N^{(\varepsilon-\alpha-\delta)/2}}<+\infty.$$

Hence (iv) is true, by Beppo Levi. □

Proof of Theorem 2. A little care here is to be taken to obtain the universality for all L^p spaces ($p>1$). Let (p_n) be a sequence decreasing to 1 ($p_n>1$). Obviously, for any probability space X, $\bigcup_{p>1}L^p(X)=\bigcup_{n\geq 1}L^{p_n}(X)$. Hence, it suffices to find Ω' such that the theorem is true for all the spaces $L^{p_n}(X)$.

Let $\delta>0$ be the real number in Theorem 2. For every $n\geq 1$, define $\alpha_n := 2+1/(p_n-1)$, $\varepsilon_n := 3+\alpha_n+\delta$. For every $n\geq 1$, let Ω_n be the set of measure 1 on which, the points (i), (ii), (iii) and (iv) of Lemma 7 are satisfied for (α_n,ε_n). Finally, define $\Omega' := \bigcap_{n\geq 1}\Omega_n$.

Let $\omega\in\Omega'$.

Let $g\in\bigcup_{p>1}L^p(X)$. Hence there exists n_0 such that $g\in L^{p_{n_0}}(X)$. For the sake of clarity we will write $p:=p_{n_0}$, $\alpha:=\alpha_{n_0}$ and $\varepsilon:=\varepsilon_{n_0}$.

By (i) and the fact that $g\in L^1$, we have

$$\sum_{N\geq 1}\sum_{n=2^N+1}^{2^{N+1}}\int_X\frac{|T^{u_n}g||X_n(\omega)\mathbf{1}_{\{|X_n(\omega)|\geq n/\log^{\varepsilon}n\}}}{n}\,d\mu<+\infty. \tag{10}$$

Hence, by Beppo Levi, it suffices to prove that the series $\sum_{n\geq 1}((Y_{n,\varepsilon}(\omega)g\circ T^n)/n)$ converges μ-almost surely.

Again, we truncate g. We have, using point (ii) of Lemma 7,

$$\sum_{N\geq 1}\sum_{n=2^N+1}^{2^{N+1}}\int_X|T^{u_n}(g\mathbf{1}_{\{|g|\geq N^{1/(p-1)}\}})|(x)\frac{|Y_{n,\varepsilon}|(\omega)}{n}\mu(dx)$$
$$\leq\sum_{N\geq 1}\int_X(|g|\mathbf{1}_{\{|g|\geq N^{1/(p-1)}\}})(x)\mu(dx)\frac{1}{2^N}\sum_{n=2^N+1}^{2^{N+1}}|Y_{n,\varepsilon}|(\omega)$$
$$\leq\sup_{N\geq 1}\frac{1}{2^N}\sum_{n=2^N+1}^{2^{N+1}}|Y_{n,\varepsilon}|(\omega)\int_X\sum_{1\leq N\leq|g|^{p-1}}|g|\,d\mu\leq C\int_X|g|^p\,d\mu<+\infty. \tag{11}$$

Hence, the series $\sum_{N\geq 1}\sum_{n=2^N+1}^{2^{N+1}}|T^{u_n}(g\mathbf{1}_{\{|g|\geq N^{1/(p-1)}\}})|((|Y_{n,\varepsilon}|(\omega))/n)$ converges μ-almost surely and it suffices to prove that the series

$$\sum_{n\geq 1}\sum_{n=2^N+1}^{2^{N+1}}\frac{Y_{n,\varepsilon}(\omega)T^{u_n}(g\mathbf{1}_{\{|g|\leq N^{1/(p-1)}\}})}{n} \tag{12}$$

converges μ-almost surely and that

$$\sup_{2^N+1\leq n\leq 2^{N+1}}\bigg|\sum_{k=2^N}^{n}\frac{Y_{n,\varepsilon}(\omega)T^{u_n}(g\mathbf{1}_{\{|g|\leq N^{1/(p-1)}\}})}{n}\bigg|\to 0. \tag{13}$$

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Let $N \geq 1$ and $2^N \leq l < n \leq 2^{N+1}$. By Lemma 6, we obtain

$$\int_X \left| \sum_{k=l+1}^{n} \frac{Y_{k,\varepsilon}(\omega) T^{u_k}(g\mathbf{1}_{\{|g|\leq N^{1/(p-1)}\}})}{k} \right|^2 \mu(dx)$$
$$\leq \sup_{t\in\mathbf{R}} \left| \sum_{k=l+1}^{n} \frac{Y_{k,\varepsilon}(\omega)}{k} e^{iu_k t} \right|^2 \|g\mathbf{1}_{\{|g|\leq N^{1/(p-1)}\}}\|_2^2$$
$$\leq \|g\|_1 N^{1/(p-1)} \sup_{t\in\mathbf{R}} \left| \sum_{k=l+1}^{n} \frac{Y_{k,\varepsilon}(\omega)}{k} e^{iu_k t} \right|^2.$$

By point (iii) of Lemma 7, we deduce that, for all integers n, l such that $2^N + 1 \leq l < n \leq 2^{N+1}$, we have, using that T contracts $L^\infty(X)$,

$$\int_X \left| \sum_{k=l+1}^{n} \frac{Y_{k,\varepsilon}(\omega) T^{u_k}(g\mathbf{1}_{\{|g|\leq N^{1/(p-1)}\}})}{k} \right|^2 \mu(dx)$$
$$\leq C_\varepsilon(\omega) N^{1/(p-1)} \log u_{2^{N+1}} \sum_{k=l+1}^{n} \frac{|Y_{k,\varepsilon}|^2(\omega)}{k^2}. \tag{14}$$

At this stage we need Theorem 3 of [**13**]. We state a simplified version which is sufficient for our purpose and we use notation consistent with ours (hence slightly different from the original).

THEOREM 8. (Móricz [**13**]) *Let $r < s$ be integers. Let $(h_m)_{r+1\leq m\leq s}$ be any measurable functions on a probability space (X, Σ, μ). Assume there exist positive numbers $(b_m)_{r+1\leq m\leq s}$ and a constant $D > 0$ such that*

$$\int_X \left| \sum_{k=l+1}^{n} h_k \right|^2 d\mu \leq D \sum_{k=l+1}^{n} b_k, \quad \text{for all } r \leq l < n \leq s.$$

Then we have

$$\int_X \sup_{r<n\leq s} \left| \sum_{k=r+1}^{n} h_k \right|^2 d\mu \leq D(\log_2(2s))^2 \sum_{k=r+1}^{s} b_k.$$

We apply Theorem 8 to the family of inequalities given by (14). We obtain

$$\int_X \sup_{2^N+1\leq n\leq 2^{N+1}} \left| \sum_{k=2^N+1}^{n} \frac{Y_{k,\varepsilon}(\omega) T^{u_k}(g\mathbf{1}_{\{|g|\leq N^{1/(p-1)}\}})}{k} \right|^2 \mu(dx)$$
$$\leq C N^{1/(p-1)} N^2 \log u_{2^{N+1}} \sum_{k=2^N+1}^{2^{N+1}} \frac{|Y_{k,\varepsilon}|^2(\omega)}{k^2}.$$

By point (iv) of Lemma 5 (and our choice of α), we deduce that

$$\sum_{N\geq 1} \left\| \sup_{2^N+1\leq n\leq 2^{N+1}} \left| \sum_{k=2^N+1}^{n} \frac{Y_{k,\varepsilon}(\omega) T^{u_k}(g\mathbf{1}_{\{|g|\leq N^{1/(p-1)}\}})}{k} \right| \right\|_{L^2(X)} < +\infty.$$

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In particular,

$$\sum_{N\geq 1}\left\|\sup_{2^N+1\leq n\leq 2^{N+1}}\left|\sum_{k=2^N+1}^{n}\frac{Y_{k,\varepsilon}(\omega)T^{u_k}(g\mathbf{1}_{\{|g|\leq N^{1/(p-1)}\}})}{k}\right|\right\|_{L^1(X)}<+\infty. \tag{15}$$

which proves that the series in (12) converges μ-almost surely and that (13) is satisfied.

Now, combining (15), (10) and (11) yields the integrability of the maximal function. □

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REFERENCES

[1] I. Assani. A weighted pointwise ergodic theorem. *Ann. Inst. H. Poincaré Prob. Statist.* **34** (1998), 139–150.
[2] I. Assani. Duality and the one-sided ergodic Hilbert transform. *Contemp. Math. (Chapel Hill Ergodic Workshop)* **357** (2004), 81–90.
[3] F. Boukhari and M. Weber. Almost sure convergence of weighted series of contractions. *Illinois J. Math.* **46** (2002), 1–21.
[4] J. Bourgain. *Return Times of Dynamical System*, 1988, unpublished manuscript.
[5] J. Bourgain, Y. Katznelson, H. Furstenberg and D. Ornstein. Appendix to J. Bourgain, Pointwise ergodic theorems for arithmetic sets. *Publ. Math. Inst. Hautes Études Sci.* **69** (1989), 4–45.
[6] J. Cuzick and T. L. Lai. On random Fourier series. *Trans. Amer. Math. Soc.* **261** (1980), 53–80.
[7] S. Durand and D. Schneider. Random ergodic theorems and regularizing random weights. *Ergod. Th. & Dynam. Sys.* **23** (2003), 1059–1092.
[8] A. Fan and D. Schneider. Sur une inégalité de Littlewood–Salem. *Ann. Inst. H. Poincaré Prob. Statist.* **39**(2) (2003), 193–216.
[9] A. del Junco and J. Rosenblatt. Counterexamples in ergodic theory and number theory. *Math. Ann.* **245** (1979), 185–197.
[10] J. P. Kahane. *Some Random Series of Functions*, 2nd edn. Cambridge University Press, Cambridge, 1985 (1st edn.: D. C. Heath, Lexington, MA, 1968).
[11] M. Lin, J. Olsen and A. Tempelman. On modulated ergodic theorems for Dunford–Schwartz operators. *Illinois J. Math.* **43** (1999), 542–567.
[12] J. Marcinkiewicz and A. Zygmund. Sur les fonctions indépendantes. *Fund. Math.* **29** (1937), 60–90.
[13] F. Móricz. Moment inequalities and the strong laws of large numbers. *Z. Wahr. Verw. Gebiete* **35** (1976), 299–314.
[14] F. Riesz and B. Sz. Nagy. *Functional Analysis*. Translated from the 2nd French edn by Leo F. Boron. Dover Publications, New York, 1990.
[15] J. Rosenblatt. Almost everywhere convergence of series. *Math. Ann.* **280** (1989), 565–577.
[16] D. Rudolph. A joining proof of Bourgain's return time theorem. *Ergod. Th. & Dynam. Sys.* **14**(1) (1994), 197–203.
[17] R. Salem and A. Zygmund. Some properties of trigonometric series whose terms have random signs. *Acta Math.* **91** (1954), 245–301.
[18] D. Schneider and M. Weber. Weighted averages of contractions along subsequences. *Convergence in Ergodic Theory and Probability (Ohio State University)*. W. de Gruyter, Berlin, New York, 1996.
[19] M. Talagrand. A borderline random Fourier series. *Ann. Prob.* **23** (1995), 776–785.
[20] M. Weber. Estimating random polynomials by means of metric entropy methods. *Math. Inequal. Appl.* **3** (2000), 443–457.

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