Probability in the Engineering and Informational Sciences, **25**, 2011, 449–467.
doi:10.1017/S026996481100012X

AGING PROPERTIES OF SEQUENTIAL ORDER STATISTICS

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Sequential order statistics describe the ordered failure times in a k-out-of-n system, where the failures of components might affect the performance of remaining working components. In this article aging properties of sequential order statistics are examined and conditions are given such that the distribution of a sequential order statistic is ILR, IFR, IFRA, or NBU. Moreover, conditions for aging properties of spacings of sequential order statistics are obtained.

1. INTRODUCTION

Sequential order statistics (SOS) has been introduced in Kamps [16] (see also Cramer and Kamps [11,12] and Cramer [10]). They described the increasingly ordered failure times of components in a sequential k-out-of-n system. Sequential k-out-of-n systems extend the usual k-out-of-n systems by allowing failures of components to affect the performance of the remaining components. In order to reflect effects caused by failures (for instance, an increased load) in the model, the distribution functions of the residual lifetimes of the remaining components might change after every failure.

In the following, we assume throughout the article that $F_1, F_2, \ldots, F_n$ are absolutely continuous distribution functions with support $[0, \infty)$. Their probability density functions are denoted by $f_1, f_2, \ldots, f_n$, respectively. The corresponding reliability

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functions are given by $\overline{F}_i = 1 - F_i$ for $i = 1, 2, \ldots, n$. Their hazard rates are denoted by $h_i = f_i/(1 - F_i)$ for $i = 1, 2, \ldots, n$. Now, SOS can be introduced as follows: Let us assume that at time $t = 0$, all of the n components in a system start to work. The lifetimes of the components are described by independent random variables $X_1^{(1)}, X_2^{(1)}, \ldots, X_n^{(1)}$ that are identically distributed according to the distribution function F_1. The first failure time of a component is then given by

$$X_{1:n}^* = \min\{X_1^{(1)}, X_2^{(1)}, \ldots, X_n^{(1)}\}.$$

If this first failure occurs at time $t_1 = X_{1:n}^*$, then we assume that the residual lifetimes of the $n - 1$ remaining components are equal in law to the residual lifetimes of independent and identically distributed (i.i.d.) components with age t_1 and distribution function F_2 (instead of F_1). Here, the distribution function F_2 is used to model the influence of the first component failure on the behavior of the other components. The lifetimes of such components will be represented by random variables $X_1^{(2)}, X_2^{(2)}, \ldots, X_{n-1}^{(2)}$. Given that $X_{1:n}^* = t_1$, the random variables $X_1^{(2)}, X_2^{(2)}, \ldots, X_{n-1}^{(2)}$ are i.i.d. according to the truncated distribution function $F_2(\cdot|t_1)$ with $\overline{F}_2(x|t_1) = \overline{F}_2(x)/\overline{F}_2(t_1)$ for $x \geq t_1$. In particular, given that $X_{1:n}^* = t_1$, these random variables satisfy $X_i^{(2)} \geq t_1$ for $i = 1, 2, \ldots, n - 1$. Then the second failure time in the system is given by

$$X_{2:n}^* = \min\{X_1^{(2)}, X_2^{(2)}, \ldots, X_{n-1}^{(2)}\}.$$

We proceed this way and, by induction, we assume that the ith component failure occurs at time $t_i = X_{i:n}^*$. Then we assume that the residual lifetimes of the $n - i$ remaining components are equal in law to the residual lifetimes of i.i.d. components with age t_i and distribution function F_{i+1}. The lifetimes of such components will be represented by $X_1^{(i+1)}, X_2^{(i+1)}, \ldots, X_{n-i}^{(i+1)}$. Given that $X_{i:n}^* = t_i$, the random variables $X_1^{(i+1)}, X_2^{(i+1)}, \ldots, X_{n-i}^{(i+1)}$ are i.i.d. according to the truncated distribution function $F_{i+1}(\cdot|t_i)$ with $\overline{F}_{i+1}(x|t_i) = \overline{F}_{i+1}(x)/\overline{F}_{i+1}(t_i)$ for $x \geq t_i$. Then the $(i + 1)$st component failure time is

$$X_{i+1:n}^* = \min(X_1^{(i+1)}, X_2^{(i+1)}, \ldots, X_{n-i}^{(i+1)}).$$

Finally, if the $(n - 1)$st component failure occurs at time $t_{n-1} = X_{n-1:n}^*$, then the last failure time is given by $X_{n:n}^*$ with the reliability function $\overline{F}_n(x|t_{n-1}) = \overline{F}_n(x)/\overline{F}_n(t_{n-1})$ for $x \geq t_{n-1}$.

The ordered lifetimes $X_{1:n}^*, X_{2:n}^*, \ldots, X_{n:n}^*$ obtained using this procedure are called sequential order statistics based on $F_1, F_2, \ldots, F_n$. The distribution and reliability functions of $X_{i:n}^*$ will be represented by $F_{i:n}^*$ and $\overline{F}_{i:n}^*$, respectively. Note that if $F_1 = F_2 = \cdots = F_n$ (i.e., when the failure of a component does not affect the other components), then we obtain the usual order statistics based on i.i.d. random variables that are distributed according to F_1.

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It is seen from the previous construction that SOS satisfy (cf. Kamps [16, p. 29])

$$P(X^*_{r:n} > t | X^*_{r-1:n} = s) = \left(\frac{\overline{F}_r(t)}{\overline{F}_r(s)} \right)^{n-r+1}, \qquad s \le t, \tag{1.1}$$

for $r = 2, \ldots, n$; that is, the distribution of the rth SOS given the last failure takes place at time s is equal to the distribution of the minimum of $n - r + 1$ random variables that follow the distribution function F_r truncated on the left at s. By choosing $F_1, F_2, \ldots, F_n$ appropriately, the influences of failures on remaining components are described. A common choice of the underlying distributions $F_1, F_2, \ldots, F_n$ is given by

$$\overline{F}_i = \overline{F}^{\alpha_i}, \qquad i = 1, 2, \ldots, n, \tag{1.2}$$

where F denotes an absolutely continuous distribution function and $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$. Note that the underlying distributions satisfy the proportional hazard rate (PHR) model. The SOS under this model have been studied extensively in, for instance, Cramer and Kamps [11–13], Kamps and Cramer [17], Balakrishnan, Beutner, and Kamps [2] and Beutner [5]. Some ordering properties for SOS were obtained in Zhuang and Hu [31] and Navarro and Burkschat [21] without assumption (1.2).

In this article, we focus on aging properties of SOS. Recall the following notions (see, e.g., Barlow and Proschan [3] and Lai and Xie [18]): Let X be a nonnegative random variable with the support $[0, \infty)$ and the distribution function F, reliability function $\overline{F} = 1 - F$, and hazard function $R_F = -\ln \overline{F}$. If necessary, it is assumed that F is absolutely continuous and has a density f with corresponding failure or hazard rate $h_F = f/\overline{F}$. Then X (or F) is called

- ILR (increasing likelihood ratio) if f is log-concave or, equivalently, $f(t)/f(t+x)$ is increasing in $t \ge 0$ for every $x > 0$,
- IFR (increasing failure rate) if R_F is convex on $[0, \infty)$ or, equivalently, h_F is increasing on $[0, \infty)$,
- IFRA (increasing failure rate average) if R_F is starshaped on $[0, \infty)$ or, equivalently, $\int_0^x h_F(t)\, dt/x$ is increasing in $x \ge 0$,
- NBU (new better than used) if R_F is superadditive on $[0, \infty)$ or, equivalently, $\overline{F}(x+t) \le \overline{F}(x)\overline{F}(t)$ for all $x, t \ge 0$.

Alternatively, log-concave functions are also called PF_2. It is well known that

$$\text{ILR} \implies \text{IFR} \implies \text{IFRA} \implies \text{NBU}. \tag{1.3}$$

Moreover, for our purposes here it is sufficient to recall only two of the corresponding dual notions of negative aging, namely the DLR and DFR properties: X (or F) is called DLR (decreasing likelihood ratio) if f is log-convex or, equivalently, $f(t)/f(t+x)$ is decreasing in $t \ge 0$ for every $x > 0$. X (or F) is called DFR (decreasing failure rate) if R_F is concave on $[0, \infty)$ or, equivalently, h_F is decreasing on $[0, \infty)$. Note that DLR implies DFR.

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There exist some results on aging properties of SOS based on the PHR model (1.2). Thus, if F is IFR (IFRA, NBU), then the rth SOS $X^*_{r:n}$ based on (1.2) also has the IFR (IFRA, NBU) property (see Kamps [16] and Cramer and Kamps [12,13]). Moreover, if F is DFR, then the spacings $X^*_{r:n} - X^*_{r-1:n}$ $(2 \leq r \leq n)$ of SOS based on (1.2) are known to be DFR.

In this article, we examine aging properties of SOS based on the distribution functions $F_1, F_2, \ldots, F_n$ without imposing (1.2). Our results extend and supplement the properties for the model (1.2) given earlier. The rest of the article is organized as follows: In Section 2 we show that IFR and IFRA classes are not closed under the formation of SOS and conditions to get the IFR, IFRA, and NBU properties of SOS are obtained. The derivations are based on notions of relative aging. The result for the IFRA property is extended to a multivariate IFRA notion. Furthermore, aging properties of spacings of SOS are given. In Section 3 conditions for the ILR property of SOS are established. Moreover, the ILR and DLR properties of spacings of SOS are considered.

Throughout the article we use the terms increasing and decreasing in a wide sense, that is, a function g is increasing (decreasing) if $g(x) \leq (\geq) g(y)$ for all $x < y$.

2. CONDITIONS FOR THE IFR, IFRA, AND NBU PROPERTIES

At first, we consider an example which illustrates that the IFR property of the underlying distributions does not imply the IFR property of SOS.

Example 2.1: Let $n = 2$. The reliability functions of the SOS $X^*_{1:2}$ and $X^*_{2:2}$ associated with the distribution functions F_1 and F_2 are given (see Cramer and Kamps [13, Lemma 2.4]) by

$$\overline{F}^*_{1:2}(t) = \overline{F}^2_1(t), \qquad \overline{F}^*_{2:2}(t) = \overline{F}^*_{1:2}(t) + \overline{F}_2(t) \int_0^t \frac{f^*_{1:2}(x)}{\overline{F}_2(x)}\, dx,$$

respectively, with the corresponding density functions

$$f^*_{1:2}(t) = 2f_1(t)\overline{F}_1(t), \qquad f^*_{2:2}(t) = f_2(t) \int_0^t \frac{f^*_{1:2}(x)}{\overline{F}_2(x)}\, dx.$$

Consequently, the hazard rates of $X^*_{1:2}$ and $X^*_{2:2}$ are given by

$$h^*_{1:2}(t) = 2\frac{f_1(t)}{\overline{F}_1(t)}, \qquad h^*_{2:2}(t) = \frac{f_2(t)I(t)}{\overline{F}^*_{1:2}(t) + \overline{F}_2(t)I(t)} \quad \text{with} \quad I(t) = \int_0^t \frac{f^*_{1:2}(x)}{\overline{F}_2(x)}\, dx.$$

Clearly, if F_1 is IFR, so is the distribution of $X^*_{1:2}$. However, if F_1 and F_2 are IFR, then the distribution of $X^*_{2:2}$ does not need to be IFR. This is illustrated by choosing

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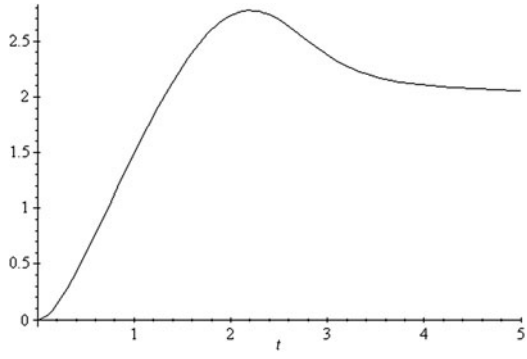


FIGURE 1. Hazard rate function of the sequential order statistic $X^*_{2:2}$ obtained in Example 2.1.

$\overline{F}_1(t) = e^{-t}$ (exponential) and $\overline{F}_2(t) = e^{-t^2}$ (Weibull) for $t \geq 0$. Then we obtain

$$h^*_{2:2}(t) = \frac{2te^{-t^2}I(t)}{e^{-2t} + e^{-t^2}I(t)} \qquad \text{with} \quad I(t) = \int_0^t 2e^{x^2-2x}\,dx.$$

From the plot of $h^*_{2:2}(t)$ (see Fig. 1) it is seen that the hazard rate is not an increasing function for every $t > 0$. For instance, $h^*_{2:2}(2) \approx 2.7311$ and $h^*_{2:2}(4) \approx 2.1042$. Hence, $X^*_{2:2}$ is not IFR. As F_1 and F_2 are ILR, this example also shows that the ILR class is not closed under the formation of SOS.

Remark 2.2: In Belzunce, Mercader, and Ruiz [4] a sufficient condition is given such that the epoch times of a nonhomogeneous pure birth (NHPB) process possess the multivariate IFR (MIFR) property defined by Shaked and Shantikumar [28]. Precisely, it is shown that the vector of the first n epoch times of a NHPB process with intensities $(r_k)_{k\in\mathbb{N}}$ is MIFR if r_k is increasing for all $k \in \mathbb{N}$. If the parametrization of Belzunce et al. [4] is applied, then the joint distribution of sequential order statistics $X^*_{1:n}, X^*_{2:n}, \ldots, X^*_{n:n}$ based on absolutely continuous $F_1, F_2, \ldots, F_n$ coincides with that of the first n epoch times of a NHPB process with the intensities

$$r_k = (n-k+1)h_k, \qquad k = 1, 2, \ldots, n.$$

In particular, the two-dimensional random vector of the sequential order statistics in the preceding example is MIFR, as both underlying distributions are IFR, that is, h_1 and h_2 are increasing. Consequently, Example 2.1 also illustrates that a MIFR random vector does not need to have IFR marginal distributions.

In the following, it turns out that for deriving sufficient conditions for the IFR, IFRA, and NBU properties of SOS, some notions of relative aging considered by

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Kalashnikov and Rachev [15], Sengupta and Deshpande [27], and Rowell and Siegrist [26] are useful.

We assume that X and Y are random variables with support $[0, \infty)$. The corresponding hazard functions are given by $R_F = -\ln \overline{F}$ and $R_G = -\ln \overline{G}$. It is said that (cf. Sengupta and Deshpande [27])

- X is aging faster than Y (denoted by $X \leq_{af} Y$ or $F \leq_{af} G$) if $R_G(X)$ is IFR,
- X is aging faster than Y in average (denoted by $X \leq_{afa} Y$ or $F \leq_{afa} G$) if $R_G(X)$ is IFRA,
- X is aging faster than Y in quantile (denoted by $X \leq_{afq} Y$ or $F \leq_{afq} G$) if $R_G(X)$ is NBU.

The conditions of the orders have the following meaning (cf. Rowell and Siegrist [26]): If time is transformed so that Y becomes ageless (i.e., exponential), then X is IFR (IFRA, NBU). Moreover, if E denotes a standard (with mean 1) exponential random variable, then

$$X \leq_{af} (\leq_{afa}, \leq_{afq}) E \quad \Longleftrightarrow \quad X \text{ IFR (IFRA, NBU).} \tag{2.1}$$

In Sengupta and Deshpande [27] it is shown that the three orders are transitive. Moreover, from (1.3), we have

$$X \leq_{af} Y \implies X \leq_{afa} Y \implies X \leq_{afq} Y.$$

The next characterizations of the preceding orders and the associated aging notions also follow from results in Sengupta and Deshpande [27]. Additionally, they justify the names of these orders.

PROPOSITION 2.3: *Let X and Y be random variables with continuous distributions F and G with support $[0, \infty)$, respectively.*

(a) $X \leq_{af} Y$ *if and only if* $R_F \circ R_G^{-1}$ *is convex on* $[0, \infty)$. *If F and G are absolutely continuous, then* $X \leq_{af} Y$ *if and only if* h_F/h_G *is increasing on* $[0, \infty)$.

(b) $X \leq_{afa} Y$ *if and only if* $R_F \circ R_G^{-1}$ *is star-shaped on* $[0, \infty)$ *or, equivalently,* R_F/R_G *is increasing on* $[0, \infty)$.

(c) $X \leq_{afq} Y$ *if and only if* $R_F \circ R_G^{-1}$ *is superadditive on* $[0, \infty)$ *or, equivalently,*

$$\overline{F}^{-1}\left(\frac{\overline{F}(x+t)}{\overline{F}(t)}\right) \geq \overline{G}^{-1}\left(\frac{\overline{G}(x+t)}{\overline{G}(t)}\right) \qquad \text{for every } x, t > 0.$$

The following result states that the previous orders are closed under strictly increasing transformations with a fixed point in the origin (cf. Sengupta and Deshpande [27, Thm. 2.1]).

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PROPOSITION 2.4: *Let X and Y be nonnegative random variables with support $[0,\infty)$ and let $\phi : [0,\infty) \to [0,\infty)$ be strictly increasing with $\phi(0) = 0$. If $X \leq_{af} (\leq_{afa}, \leq_{afq}) Y$, then $\phi(X) \leq_{af} (\leq_{afa}, \leq_{afq}) \phi(Y)$.*

The next two lemmas (cf., e.g., Righter, Shaked, and Shanthikumar [25, Cor. 5.1]) will be helpful for deriving aging properties of SOS.

LEMMA 2.5: *Let Z be a nonnegative random variable with support $[0,\infty)$ and let F and G be continuous distribution functions with support $[0,\infty)$ such that $F \leq_{af} (\leq_{afa}, \leq_{afq}) G$. If Z is IFR (IFRA, NBU), then $R_G \circ R_F^{-1}(Z)$ is IFR (IFRA, NBU).*

LEMMA 2.6: *Let Z be a nonnegative random variable with support $[0,\infty)$ and let F be a continuous distribution function with support $[0,\infty)$. If F and Z are IFR (IFRA, NBU), then $R_F^{-1}(Z)$ is IFR (IFRA, NBU).*

In order to obtain conditions for the IFR, IFRA, and NBU properties of SOS, we use a distribution theoretical result from Cramer and Kamps [13]. According to that article, we can assume that

$$X^*_{1:n} = F_1^{-1}(1 - V_1), \tag{2.2}$$

$$X^*_{r:n} = F_r^{-1}(1 - V_r \overline{F}_r(X^*_{r-1:n})), \qquad r = 2, 3, \ldots, n, \tag{2.3}$$

where $V_1, V_2, \ldots, V_n$ denote independent random variables that satisfy $V_k \sim \text{Beta}(n - k + 1, 1)$; that is, V_k follows a Beta distribution with the parameters $n - k + 1$ and 1. Rewriting these relations in terms of the hazard functions $R_k = -\ln \overline{F}_k, 1 \leq k \leq n$, leads to the representations

$$X^*_{1:n} = R_1^{-1}(Z_1), \tag{2.4}$$

$$X^*_{r:n} = R_r^{-1}(Z_r + R_r(X^*_{r-1:n})), \qquad r = 2, 3, \ldots, n, \tag{2.5}$$

with independent random variables $Z_1, Z_2, \ldots, Z_n$ that satisfy $Z_k \sim \text{Exp}(n - k + 1)$; that is, Z_k follows an exponential distribution with mean $E(Z_k) = 1/(n - k + 1)$. Utilizing this result, we can state our first result on aging properties of SOS. It extends some results for the relevation transform given in Shanthikumar and Baxter [29] to the case of SOS.

THEOREM 2.7: *Let $1 \leq r \leq n$.*

(a) If $F_1 \leq_{af} F_2 \leq_{af} \ldots \leq_{af} F_r$ and F_r is IFR, then $X^*_{r:n}$ is IFR.
(b) If $F_1 \leq_{afa} F_2 \leq_{afa} \ldots \leq_{afa} F_r$ and F_r is IFRA, then $X^*_{r:n}$ is IFRA.
(c) If $F_1 \leq_{afq} F_2 \leq_{afq} \ldots \leq_{afq} F_r$ and F_r is NBU, then $X^*_{r:n}$ is NBU.

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PROOF: We only prove (a); the other parts follow analogously. Let $r \geq 2$ (for $r = 1$, the assertion is directly seen from (2.4) and Lemma 2.6). At first, we will show by induction that the random variables

$$R_{i+1}(X^*_{i:n}), \qquad i = 1, 2, \ldots, r-1,$$

are IFR. Let $i = 1$. Note that (2.5) yields $R_2(X^*_{1:n}) = R_2 \circ R_1^{-1}(Z_1)$. Because Z_1 follows an exponential distribution, it is IFR. Thus, the assertion follows from Lemma 2.5. Now, we assume that $R_i(X^*_{i-1:n})$ is IFR and we prove the same for $R_{i+1}(X^*_{i:n})$. Let $Z_i \sim \mathrm{Exp}(n-i+1)$ be independent of $R_i(X^*_{i-1:n})$. Because $R_i(X^*_{i-1:n})$ is IFR by assumption and Z_i is also IFR, it is well known that $Z_i' = Z_i + R_i(X^*_{i-1:n})$ is IFR. Then Lemma 2.5 yields that $R_{i+1} \circ R_i^{-1}(Z_i')$ is also IFR. Thus, we obtain from (2.5) that $R_{i+1}(X^*_{i:n})$ is IFR.

Now, it follows that $R_r(X^*_{r-1:n})$ is IFR under the given assumptions. Let $Z_r \sim \mathrm{Exp}(n-r+1)$ be independent of $R_r(X^*_{r-1:n})$. Then $Z_r' = Z_r + R_r(X^*_{r-1:n})$ is also IFR. By utilizing the assumption on F_r, Lemma 2.6 yields that $R_r^{-1}(Z_r')$ is IFR. Therefore, (2.5) leads to the claimed result.

Note that (b) and (c) are shown along the lines of the preceding proof by utilizing that IFRA and NBU are also closed under convolutions (see Lai and Xie [18, p. 38]). ■

Remark 2.8:

(a) Observe that due to Proposition 2.3 and the transitivity of the involved orders, all underlying distributions $F_1, F_2, \ldots, F_r$ in Theorem 2.7 are IFR (IFRA, NBU).

(b) In Sengupta and Deshpande [27] and Rowell and Siegrist [26], conditions on the parameters of several distribution classes (for instance, Weibull, Pareto, and Gamma distributions) are given such that their members satisfy the $\leq_{af}$-order.

Under the particular assumption (1.2) on the underlying distributions, we obtain the following results stated in Cramer and Kamps [12] and Cramer [9].

COROLLARY 2.9: *Let $1 \leq r \leq n$ and let assumption (1.2)—that is, $F_k = 1 - (1-F)^{\alpha_k}$ for $k = 1, 2, \ldots, r$—be valid.*

(a) If F is IFR, then $X^*_{r:n}$ is IFR.

(b) If F is IFRA, then $X^*_{r:n}$ is IFRA.

(c) If F is NBU, then $X^*_{r:n}$ is NBU.

PROOF: Under the assumption on the F_ks, we obtain

$$R_k \circ R_{k+1}^{-1}(t) = \frac{\alpha_k}{\alpha_{k+1}}\, t, \qquad t \geq 0,$$

for $k = 1, 2, \ldots, r-1$. Hence, $F_1 \leq_{af} F_2 \leq_{af} \ldots \leq_{af} F_r$. Moreover, because $R_r = \alpha_r R_F$, the result follows from Theorem 2.7. ■

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Remark 2.10: Takahasi [30] showed for the usual order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ from i.i.d. random variables that $X_{r+1:n}$ is IFR if $X_{r:n}$ is IFR. It follows from this result that the IFR property of the sample distribution F is transfered to the distribution of $X_{r:n}$ for $1 \leq r \leq n$ (note that $X_{1:n}$ is IFR if and only if F is IFR). This property was given by Esary and Proschan [14]. Kamps [16] extended Takahasi's result to m-generalized order statistics. In fact, these results are obtained by proving that (m-generalized) order statistics satisfy $X_{r+1:n} \leq_{af} X_{r:n}$, which yields the desired implication. Therefore, it is of interest to check if the result of Theorem 2.7(a) can be strengthened to $X^*_{r+1:n} \leq_{af} X^*_{r:n}$ under the given assumptions. However, the following example illustrates that even under assumption (1.2), SOS do not need to satisfy this order relation.

Example 2.11: Let $r = 3, n = 4$ and choose in model (1.2) the parameters

$$\alpha_1 = \alpha_2 = 1, \qquad \alpha_3 = \frac{11}{8}, \qquad \alpha_4 = \frac{7}{4}.$$

Moreover, let F be distributed according to the standard exponential distribution. Using results from Kamps and Cramer [17] on the distribution theory of SOS, it can be shown that the corresponding hazard rates of $X^*_{3:4}$ and $X^*_{4:4}$ are given by

$$h^*_{3:4}(t) = \frac{(528/5)e^{-(11/4)t} - 132e^{-3t} + (132/5)e^{-4t}}{(192/5)e^{-(11/4)t} - 44e^{-3t} + (33/5)e^{-4t}},$$

$$h^*_{4:4}(t) = \frac{(308/15)e^{-(7/4)t} + (924/5)e^{-3t} - (308/15)e^{-4t} - (924/5)e^{-(11/4)t}}{(176/15)e^{-(7/4)t} + (308/5)e^{-3t} - (77/15)e^{-4t} - (336/5)e^{-(11/4)t}}$$

It is easily seen that

$$\lim_{x\to\infty} \frac{h^*_{3:4}(t)}{h^*_{4:4}(t)} = \frac{11}{7} \approx 1.5714.$$

However, it can be shown numerically that $h^*_{3:4}(5)/h^*_{4:4}(5) \approx 1.5253$. Hence, $h^*_{3:4}/h^*_{4:4}$ is not decreasing on $(0, \infty)$ and therefore $X^*_{4:4} \not\leq_{af} X^*_{3:4}$.

It will turn out that the result for the IFRA property can be extended to a multivariate setting. We recall the following multivariate IFRA notion introduced by Block and Savits [7].

Definition 2.12: Let $(X_1, X_2, \ldots, X_n)$ be a nonnegative random vector. Then $(X_1, X_2, \ldots, X_n)$ is said to have a multivariate IFRA (MIFRA) distribution if and only if

$$E(\phi(X_1, X_2, \ldots, X_n)) \leq E^{1/\alpha}(\phi^\alpha(X_1/\alpha, X_2/\alpha, \ldots, X_n/\alpha))$$

for all continuous nonnegative increasing functions ϕ and all $\alpha \in (0, 1)$.

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Remark 2.13:

(a) The above MIFRA definition is a natural extension of the univariate IFRA property, because a univariate random variable X can be shown to be IFRA if and only if the inequality

$$E(\phi(X)) \leq E^{1/\alpha}(\phi^{\alpha}(X/\alpha))$$

is satisfied for every nonnegative increasing function ϕ and all $\alpha \in (0,1)$. In fact, the continuity assumption in the preceding definition can be removed (see Block and Savits [7]).

(b) It is shown in Block and Savits [7] that MIFRA is closed under marginalisation. In particular, a MIFRA random vector has univariate IFRA marginal distributions.

In order to prove the extension, we make use of the following result given in Block and Savits [7]. Note that a function $\psi : [0,\infty)^n \to \mathbb{R}$ is called subhomogeneous if

$$\alpha\psi(z_1, z_2, \ldots, z_n) \leq \psi(\alpha z_1, \alpha z_2, \ldots, \alpha z_n)$$

for every $z_1, z_2, \ldots, z_n \geq 0$ and $\alpha \in (0,1)$.

LEMMA 2.14: *Let $(Z_1, Z_2, \ldots, Z_n)$ be MIFRA and let the functions $\psi_i : [0,\infty)^n \to [0,\infty)$ be continuous, increasing, and subhomogeneous for $i = 1, 2, \ldots, r$. If $X_i = \psi_i(Z_1, Z_2, \ldots, Z_n)$ for $i = 1, 2, \ldots, r$, then $(X_1, X_2, \ldots, X_r)$ is MIFRA.*

By utilizing a representation as in Lemma 2.14, we obtain the following multivariate IFRA property of SOS.

THEOREM 2.15: *If $F_1 \leq_{afa} F_2 \leq_{afa} \ldots \leq_{afa} F_r$ and F_r is IFRA, then $(X^*_{1:n}, X^*_{2:n}, \ldots, X^*_{r:n})$ is MIFRA.*

PROOF: Let $\psi_1(z_1, z_2, \ldots, z_n) = R_1^{-1}(z_1)$ and define recursively

$$\psi_i(z_1, z_2, \ldots, z_n) = R_i^{-1}(z_i + R_i(\psi_{i-1}(z_1, z_2, \ldots, z_n))) \tag{2.6}$$

for $i = 2, 3, \ldots, r$ (note that ψ_i depends in fact only on $z_1, z_2, \ldots, z_i$). Moreover, let $Z_1, Z_2, \ldots, Z_n$ denote independent random variables with $Z_k \sim \mathrm{Exp}(n-k+1)$. Then (2.4) and (2.5) yield $X^*_{i:n} = \psi_i(Z_1, Z_2, \ldots, Z_n)$ for $i = 1, 2, \ldots, r$. Since random vectors consisting of independent IFRA random variables are MIFRA (see Block and Savits [7, Thm. 4.1(i)]), it follows that $(Z_1, Z_2, \ldots, Z_n)$ is MIFRA. Moreover, under the given general conditions, it is easily seen that the ψ_is are continuous and increasing. We will argue that the ψ_is are also subhomogeneous. Note that a star-shaped function ϕ satisfies $\phi(\alpha t) \leq \alpha\phi(t)$ for $0 < \alpha < 1$ (see Barlow and Proschan [3]). Thus, the assumption $F_{i-1} \leq_{afa} F_i$ implies that $R_i \circ R_{i-1}^{-1}$ is subhomogeneous ($2 \leq i \leq r$). In particular, $R_2 \circ$

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ψ_1 is subhomogeneous and we will now show by induction that the same holds for $R_3 \circ \psi_2, R_4 \circ \psi_3, \ldots, R_r \circ \psi_{r-1}$. Assume that $R_i \circ \psi_{i-1}$ is subhomogeneous. Then

$$\begin{aligned} &R_{i+1} \circ \psi_i(\alpha z_1, \alpha z_2, \ldots, \alpha z_n) \\ &\quad = R_{i+1} \circ R_i^{-1}(\alpha z_i + R_i(\psi_{i-1}(\alpha z_1, \alpha z_2, \ldots, \alpha z_n)) \\ &\quad \geq R_{i+1} \circ R_i^{-1}(\alpha z_i + \alpha R_i(\psi_{i-1}(z_1, z_2, \ldots, z_n)) \\ &\quad \geq \alpha R_{i+1} \circ R_i^{-1}(z_i + R_i(\psi_{i-1}(z_1, z_2, \ldots, z_n)) \\ &\quad = \alpha R_{i+1} \circ \psi_i(z_1, z_2, \ldots, z_n) \end{aligned}$$

for $\alpha \in (0, 1)$. The first inequality follows from the induction assumption and the fact that $R_{i+1} \circ R_i^{-1}$ is increasing. The second inequality holds because $R_{i+1} \circ R_i^{-1}$ is subhomogeneous. Thus, we have proven that the functions $R_2 \circ \psi_1, R_3 \circ \psi_2, \ldots, R_r \circ \psi_{r-1}$ are subhomogeneous. Furthermore, we observe that $R_1, R_2, \ldots, R_r$ are star-shaped by assumption (see Remark 2.8(a)) and, consequently, $R_1^{-1}, R_2^{-1}, \ldots, R_r^{-1}$ are subhomogeneous. Hence, ψ_1 is subhomogeneous. Moreover, utilizing definition (2.6), it is now immediately seen that $\psi_2, \psi_3, \ldots, \psi_r$ are also subhomogeneous. Thus, the result follows from Lemma 2.14. ■

Block and Savits [7] also give the next criterion for the MIFRA property of a random vector.

THEOREM 2.16: *Let $(X_1, X_2, \ldots, X_n)$ be a nonnegative random vector such that*

(a) X_1 is IFRA

(b) for $k = 2, 3, \ldots, n$, X_k is stochastically increasing and continuous in $X_1, X_2, \ldots, X_{k-1}$; that is, the conditional probability

$$\overline{G}_k(x_k|x_1, x_2, \ldots, x_{k-1}) = P(X_k > x_k | X_1 = x_1, X_2 = x_2, \ldots, X_{k-1} = x_{k-1})$$

is increasing and continuous for $x_1, x_2, \ldots, x_{k-1} \geq 0$, and, moreover, the inequality

$$\begin{aligned} &\overline{G}_k(x_k|x_1, x_2, \ldots, x_{k-1}) \leq \overline{G}_k^{1/\alpha}(\alpha x_k|\alpha x_1, \alpha x_2, \ldots, \alpha x_{k-1}), \\ &\qquad x_1, x_2, \ldots, x_{k-1} \geq 0, \end{aligned} \tag{2.7}$$

is satisfied for $\alpha \in (0, 1)$.

Then $(X_1, X_2, \ldots, X_n)$ is MIFRA.

The preceding result is used this to show that although the IFR property of underlying distributions is not transfered to the rth SOS (see Example 2.1), it can still be said that the resulting rth SOS is IFRA.

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THEOREM 2.17: *Let F_1 be IFRA and $F_2, \ldots, F_r$ be IFR. Then $(X^*_{1:n}, X^*_{2:n}, \ldots, X^*_{r:n})$ is MIFRA. In particular, $X^*_{r:n}$ is IFRA.*

PROOF: It is easily seen that $X^*_{1:n}$ is IFRA if F_1 is IFRA. It remains to show condition (b) of Theorem 2.16. Since SOS forms a Markov chain (cf. Kamps [16, p. 29]), we obtain from (1.1) that

$$P(X^*_{k:n} > x_k | X^*_{1:n} = x_1, X^*_{2:n} = x_2, \ldots, X^*_{k-1:n} = x_{k-1}) = \begin{cases} \left(\dfrac{\overline{F}_k(x_k)}{\overline{F}_k(x_{k-1})} \right)^{n-k+1}, & x_k \geq x_{k-1} \\ 1, & x_k < x_{k-1}, \end{cases}$$

for $k = 2, 3, \ldots, r$. Recall that F_k is absolutely continuous on $[0, \infty)$ by assumption. Hence, the previous conditional probability is increasing and continuous in $x_{k-1} \geq 0$. Furthermore, note that for $x_k \geq x_{k-1}$ (the other case is trivial) inequality (2.7) reduces to

$$\frac{\overline{F}^{\alpha}_k(x_k)}{\overline{F}_k(\alpha x_k)} \leq \frac{\overline{F}^{\alpha}_k(x_{k-1})}{\overline{F}_k(\alpha x_{k-1})}, \qquad \alpha \in (0,1).$$

By rewriting this inequality in terms of the hazard function $R_k = -\ln \overline{F}_k$, we arrive at

$$\alpha R_k(x_k) - R_k(\alpha x_k) \geq \alpha R_k(x_{k-1}) - R_k(\alpha x_{k-1}), \qquad \alpha \in (0,1).$$

Thus, to prove the assertion, it is sufficient to show that $g_\alpha(x) = \alpha R_k(x) - R_k(\alpha x)$ is an increasing function in $x > 0$ for every $\alpha \in (0,1)$. Observe that

$$\frac{d}{dx} g_\alpha(x) = \alpha(h_k(x) - h_k(\alpha x)) \geq 0, \qquad x > 0,$$

because F_k is IFR and, consequently, the hazard rate h_k is increasing. Hence, the result is proven. ■

The following example illustrates that the assumptions on $F_1, F_2, \ldots, F_r$ of Theorem 2.17 cannot be weakened to IFRA; that is, it shows that IFRA property is not preserved under the formation of SOS.

Example 2.18: Let $\overline{F}_1 = e^{-t}, t \geq 0$, and $\overline{F}_2 = e^{-R_2(t)}, t \geq 0$, with hazard function

$$R_2(t) = (1 - e^{-t^{20}})t, \qquad t \geq 0.$$

Since $R_2(t)/t$ is an increasing function for $t \geq 0$, F_2 has the IFRA property. The survival function of $X^*_{2:2}$ based on F_1 and F_2 is given by

$$\begin{aligned} \overline{F}^*_{2:2}(t) &= \overline{F}^*_{1:2}(t) + \overline{F}_2(t) \int_0^t \frac{f^*_{1:2}(x)}{\overline{F}_2(x)}\, dx \\ &= e^{-2t} + 2e^{-R_2(t)} \int_0^t e^{-2x + R_2(x)}\, dx, \qquad t \geq 0. \end{aligned}$$

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Let $R^*_{2:2}(t) = -\ln \overline{F}^*_{2:2}$ denote the hazard function of $X^*_{2:2}$. It can be shown numerically that $g(t) = R^*_{2:2}(t)/t$ is not increasing for $t \in (1.1, 1.2)$ since $g(1.1) \approx 0.7797$ and $g(1.2) \approx 0.7784$. Hence, $X^*_{2:2}$ is not IFRA.

The distribution in the preceding example appears to be NBU. So the example does not seem to be applicable to show that NBU is not preserved under the formation of SOS. In the following theorem, we give another sufficient condition for the NBU property of SOS. It is based on the notion of the proportional hazard rate l_F, which is defined by (see, e.g., Righter et al. [25, Sect. 2.1.3])

$$l_F(x) = xh_F(x), \qquad x \geq 0.$$

In the next theorem, let $l_1, l_2, \ldots, l_n$ denote the proportional hazard rates of $F_1, F_2, \ldots, F_n$, respectively; that is, $l_i(x) = xh_i(x)$ for $i = 1, 2, \ldots, n$.

THEOREM 2.19: *Let F_1 be NBU and $l_2, \ldots, l_r$ be superadditive. Then $X^*_{r:n}$ is NBU.*

PROOF: It is easily seen that $X^*_{1:n}$ is NBU if F_1 is NBU. Now, the proof proceeds via induction. Assume that $X^*_{r-1:n}$ is NBU for some $r - 1 \in \mathbb{N}$. It follows from (2.3) that

$$P(X^*_{r:n} > x) = \int_0^\infty \overline{G}_r(x|y)\, dP^{X^*_{r-1:n}}(y)$$

with

$$\overline{G}_r(x|y) = \begin{cases} \left(\dfrac{\overline{F}_r(x)}{\overline{F}_r(y)}\right)^{n-r+1}, & x \geq y \\ 1, & x < y. \end{cases}$$

According to Block, Li, and Savits [6, Thm. 3.2], then $X^*_{r:n}$ is NBU if $\overline{G}_r(x|y)$ is increasing in y for each $x \geq 0$ and satisfies

$$\overline{G}_r(x|y) \leq \overline{G}_r(\alpha x|\alpha y)\overline{G}_r(\beta x|\beta y) \tag{2.8}$$

with $\beta = 1 - \alpha$ for all $0 < \alpha < 1$ and all $x, y \geq 0$. Observe that the monotonicity condition is obviously satisfied. In the present situation, (2.8) is equivalent to

$$\left(\frac{\overline{F}_r(x)}{\overline{F}_r(y)}\right)^{n-r+1} \leq \left(\frac{\overline{F}_r(\alpha x)}{\overline{F}_r(\alpha y)}\right)^{n-r+1} \left(\frac{\overline{F}_r(\beta x)}{\overline{F}_r(\beta y)}\right)^{n-r+1}$$

for $x \geq y$ (the case $x < y$ is trivial). This inequality holds iff the function

$$d_\alpha(z) = \frac{\overline{F}_r(z)}{\overline{F}_r(\alpha z)\overline{F}_r(\beta z)}, \qquad z \geq 0, \tag{2.9}$$

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is decreasing. By examining the derivative of d_α, it turns out that this property is equivalent to the inequality

$$\alpha h_r(\alpha z) + \beta h_r(\beta z) \leq h_r(z), \qquad z > 0,$$

for the hazard rate h_r of F_r. By rewriting the preceding inequality in terms of the proportional hazard rate l_r, we arrive at

$$l_r(\alpha z) + l_r(\beta z) \leq l_r(z), \qquad z > 0.$$

Because this condition is equivalent to the superadditivity of l_r, the result follows. ■

Remark 2.20: Superadditivity of the proportional hazard rate l_F is a weaker assumption than IFR but stronger than NBU; that is,

$$F \text{ IFR} \implies l_F \text{ is superadditive} \implies F \text{ NBU}.$$

The first implication follows immediately, since the hazard rate h_F is increasing. The second implication is seen from (2.9), because d_α is decreasing with $d_\alpha(0) = 1$.

We close this section with an aging property for spacings of SOS. It generalizes a result given in Kamps [16] for generalized order statistics. Note that no further assumptions are imposed on $F_1, F_2, \ldots, F_{r-1}$ in the following theorem.

THEOREM 2.21: *Let $1 \leq r \leq n$ and let $X^*_{0:n} = 0$. If F_r is DFR, then $X^*_{r:n} - X^*_{r-1:n}$ is DFR.*

PROOF: For $r = 1$, the result is easily seen. For $2 \leq r \leq n$, observe that (2.5) yields

$$P(X^*_{r:n} - X^*_{r-1:n} \leq t) = \int_0^\infty F_{r|r-1}(t \mid s)\, dF^*_{r-1:n}(s),$$

where $F^*_{r-1:n}$ denotes the distribution function of $X^*_{r-1:n}$ and

$$F_{r|r-1}(t \mid s) = P(X^*_{r:n} - s \leq t \mid X^*_{r-1:n} = s) = 1 - \left(\frac{\overline{F}_r(t+s)}{\overline{F}_r(s)}\right)^{n-r+1} \tag{2.10}$$

denotes the distribution of $(X^*_{r:n} - s \mid X^*_{r-1:n} = s)$. If F_r is DFR, then $F_{r|r-1}(t \mid s)$ is DFR for all $s \geq 0$. Hence, as the DFR property is closed under mixtures (see Lai and Xie [18, p. 38]), the result follows. ■

Remark 2.22: A result similar to Theorem 2.21 can be obtained for the IMRL (increasing mean residual life) property. A nonnegative random variable X with support $[0, \infty)$ and finite mean satisfies the IMRL property if the mean residual life function

$$m_X(t) = E(X - t | X > t)$$

is increasing for $t \geq 0$. If F_r is IMRL for some $r \geq 2$, then $F_{r|r-1}(t \mid s)$ (see the previous proof) is also IMRL for all $s \geq 0$. Because the IMRL property is closed

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under mixtures, by arguing along the lines of the proof of Theorem 2.21 we obtain the following analogous result: If F_r is IMRL, then $X^*_{r:n} - X^*_{r-1:n}$ is IMRL.

3. CONDITIONS FOR THE ILR PROPERTY

In this section we study the ILR property of SOS. It is well known that the ILR class is closed under the formation of order statistics from i.i.d. random variables. However, it is not closed under the formation of order statistics from independent random variables. Some conditions to have the ILR property for order statistics from independent ILR random variables were given in Navarro [20]. Example 2.1 proves that the ILR class is not closed under the formation of SOS. Related results for series and parallel systems based on random variables with joint log-concave distributions have been obtained recently in Navarro and Shaked [22]. Let us recall the definition of (multivariate) log-concavity and log-convexity: Let $g : \mathbb{R}^n \to [0, \infty)$ be a function such that $B := \{x \in \mathbb{R}^n : g(x) > 0\}$ is convex. Then g is called log-concave (in B) if $\ln g$ is concave in B; that is,

$$\ln g(\lambda x + (1-\lambda)y) \geq \lambda \ln g(x) + (1-\lambda) \ln g(y), \qquad x, y \in B,\ \lambda \in [0,1]. \quad \textbf{(3.1)}$$

The function g is log-convex in B if (3.1) holds with reversed inequality sign.

THEOREM 3.1: *Let f_i be log-concave for $i = 1, 2, \ldots, r$ and $h_{j+1} - h_j$ be decreasing for $j = 1, 2, \ldots, r-1$. Then the joint density of $X^*_{1:n}, X^*_{2:n}, \ldots, X^*_{r:n}$ is log-concave. In particular, $X^*_{r:n}$ is ILR.*

PROOF: The joint density of $X^*_{1:n}, X^*_{2:n}, \ldots, X^*_{r:n}$ is given by (see Kamps [16, p. 29])

$$f^{X^*_{1:n}, X^*_{2:n}, \ldots, X^*_{r:n}}(x_1, x_2, \ldots, x_r)$$
$$= \frac{n!}{(n-r)!} \left(\prod_{i=1}^{r-1} \left(\frac{\overline{F}_i(x_i)}{\overline{F}_{i+1}(x_i)} \right)^{n-i} f_i(x_i) \right) \overline{F}_r(x_r)^{n-r} f_r(x_r)$$

for $x_1 \leq x_2 \leq \ldots \leq x_r$. Note that products of log-concave functions are again log-concave. Therefore, it is sufficient to show that the different factors in the preceding density are log-concave. Since f_i is log-concave (and so is $\overline{F}_i$) by assumption for $i = 1, 2, \ldots, r$ and

$$\ln \left(\frac{\overline{F}_i(x)}{\overline{F}_{i+1}(x)} \right) = R_{i+1}(x) - R_i(x), \qquad x > 0,$$

is concave iff the corresponding derivative

$$\frac{d}{dx}(R_{i+1}(x) - R_i(x)) = h_{i+1}(x) - h_i(x), \qquad x > 0,$$

is decreasing, we obtain that the joint density of $X^*_{1:n}, X^*_{2:n}, \ldots, X^*_{r:n}$ is log-concave. In particular, the marginal density of $X^*_{r:n}$ is also log-concave (see Prékopa [24]). ■

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Remark 3.2: If the hazard rates satisfy $h_{j+1} \geq h_j$ for $j = 1, 2, \ldots, r-1$ (which appears to be natural in the model of sequential order statistics), then the assumptions of Theorem 3.1 are stronger than those of Theorem 2.7 for the IFR property: Since f_i is log-concave, the hazard rate h_i is increasing ($1 \leq i \leq r$). Therefore, under the assumptions of Theorem 3.1,

$$(h_{j+1}(x) - h_j(x)) \cdot \frac{1}{h_j(x)} = \frac{h_{j+1}(x)}{h_j(x)} - 1$$

is decreasing for $j = 1, 2, \ldots, r-1$. Hence, h_j/h_{j+1} is increasing for $j = 1, 2, \ldots, r-1$, and the conditions of Theorem 2.7 are satisfied.

Remark 3.3:

(a) By applying a result obtained by Pellerey, Shaked, and Zinn [23] in the setting of relevation counting processes, an alternative set of conditions for the ILR property of SOS can be given (see Pellerey et al. [23, Thm. 3.4]): Let $k_i = (n-i+1)f_i(\overline{F}_i)^{n-i}$ for $i = 1, 2, \ldots, n$. If h_i is log-concave for $i = 1, 2, \ldots, r$ and k_j/k_{j+1} is log-concave for $j = 1, 2, \ldots, r-1$, then the rth SOS is ILR. The same condition also yields the ILR property of single spacings (see Pellerey et al. [23, Thm. 4.2]).

(b) Under assumption (1.2), an alternative set of conditions for the log-concavity of the joint density function of SOS is found from results in Chen, Xie, and Hu [8] for generalized order statistics. Note that in model (1.2), the parameters must satisfy $\alpha_{j+1} \leq \alpha_j$ for $j = 1, 2, \ldots, r-1$ in order to satisfy the assumptions of Theorem 3.1.

(c) From results obtained by Belzunce et al. [4] in the NHBP setting, conditions for the MPF_2 property (see Shaked and Shantikumar [28]) of SOS can be derived. The MPF_2 property represents another extension of univariate log-concavity (PF_2) to the multivariate setting.

The following example illustrates that there are situations where, for instance, the IFR property of SOS can be obtained by applying Theorem 3.1, although the assumptions of Theorem 2.7(a) are not satisfied.

Example 3.4: Consider the reliability functions

$$\overline{F}_1(t) = \exp(-t^2/2 - 2t), \qquad \overline{F}_2(t) = \exp(-t^2/2 - t)$$

for $t \geq 0$. The corresponding hazard rates are given by $h_1(t) = t + 2$ and $h_2(t) = t + 1$. Hence, F_1 and F_2 are IFR. Since

$$\frac{h_1(t)}{h_2(t)} = 1 + \frac{1}{t+1}, \qquad t \geq 0,$$

is a decreasing function, Theorem 2.7 cannot be applied to conclude that $X^*_{2:2}$ based on F_1 and F_2 is IFR. However, it is easily seen that the densities of F_1 and F_2 are

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log-concave. Moreover, since $h_2 - h_1 = -1$, Theorem 3.1 yields that $X^*_{2:2}$ is ILR and therefore also IFR.

In fact, it can be shown by arguing along the lines of the proof of Theorem 2.1 in Chen et al. [8] that the result of Theorem 3.1 can be generalized, for instance, to the vector of spacings in multiply Type-II censored samples of SOS. Here, we just note the following result that should be contrasted with Theorem 2.21.

THEOREM 3.5: *Let $1 \leq r \leq n$ and let $X^*_{0:n} = 0$. Let f_i be log-concave (log-convex) for $i = 1, 2, \ldots, r$ and $h_{j+1} - h_j$ be decreasing (increasing) for $j = 1, 2, \ldots, r-1$. Then the joint density of $X^*_{1:n}, X^*_{2:n} - X^*_{1:n}, \ldots, X^*_{r:n} - X^*_{r-1:n}$ is log-concave (log-convex). In particular, $X^*_{r:n} - X^*_{r-1:n}$ is ILR (DLR).*

Remark 3.6: The proof of the DLR case in Theorem 3.5 is carried out analogously to the ILR case by exploiting that the considered random vector has also log-convex marginal densities (see An [1, Remark 8]). However, there does not exist such a DLR analogue in the situation of Theorem 3.1. For instance, it is easily seen that the density of the second order statistic from a standard exponential distribution in a sample of size $n = 2$ has the ILR property but not the DLR property. Note that the result given in An [1] does not necessarily apply to situations in which the interval of integration for obtaining the marginal density depends on the remaining arguments of the density function. This can be seen from the proof given in that work.

In fact, in order to get the DLR property of the univariate distribution of a single spacing, the assumptions of Theorem 3.5 can be strongly weakened.

THEOREM 3.7: *Let $1 \leq r \leq n$ and let $X^*_{0:n} = 0$. If F_r is DLR, then $X^*_{r:n} - X^*_{r-1:n}$ is DLR.*

PROOF: For $r = 1$, the result is easily seen. For $r = 2, 3, \ldots, n$, we can argue as in the proof of Theorem 2.21. Note that if F_r is DLR, then the distribution function $F_{r|r-1}(t \mid s)$ given in (2.10) is DLR for all $s \geq 0$. Moreover, the DLR property is closed under mixtures (see Marshall and Olkin [19, p. 182]). ■

Finally, we would like to note that a result similar to that included in Theorem 2.7 cannot be obtained for the ILR class. Note that to this purpose we need to define the following aging ordering: $X \leq_{alr} Y$ if and only if $R_G(X)$ is ILR. Clearly, as ILR implies IFR, then $X \leq_{alr} Y$ implies $X \leq_{af} Y$. However, unfortunately, the *alr*-order is not a proper ordering because it does not satisfy the transitive property. For instance, it is easy to see that if X, Y, and Z have the probability density functions $f_X(t) = ce^{-t^2+3t}$, $f_Y(t) = e^{-t}$, and $f_Z(t) = (1+t)^{-2}$, respectively, for $t \geq 0$, then $X \leq_{alr} Y$ and $Y \leq_{alr} Z$ hold, but $X \leq_{alr} Z$ does not hold.

Acknowledgments

The authors thank an anonymous referee for the careful reading of the manuscript. JN is partially supported by Ministerio de Ciencia y Tecnología under grant MTM2009-08311 and Fundación Séneca (Región de Murcia) under grant No. 08627/PI/08.

https://doi.org/10.1017/S026996481100012X Published online by Cambridge University Press

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