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Hedgehogs of Hausdorff dimension one

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Abstract. We present a construction of hedgehogs for holomorphic maps with an indifferent fixed point. We construct, for a family of commuting nonlinearizable maps, a common hedgehog of Hausdorff dimension one, the minimum possible.

1. *Introduction*

We consider the dynamics of a holomorphic diffeomorphism $f(z) = e^{2\pi i\alpha}z + O(z^2)$, $\alpha \in \mathbf{R} - \mathbf{Q}$, defined in a neighbourhood of the indifferent irrational fixed point 0. The map f is said to be *linearizable* if there is a holomorphic change of variables $z = h(w) = w + O(w^2)$ such that

$$h^{-1} \circ f \circ h = R_\alpha$$

in a neighbourhood of the origin, where $R_\alpha(w) = e^{2\pi i\alpha}w$ is the rigid rotation.

The problem of *linearization*, or determining when such an f is linearizable, has a long and interesting history, including the work of Cremer [**Cr1, Cr2**] in the 1920s, Siegel [**Si**] in 1942, A. D. Brjuno in the 1960s, and Yoccoz [**Y**], who resolved the problem of the optimal arithmetic condition on α for linearizability in 1987. In this article, however, we are primarily concerned with the structure of invariant sets for the dynamics near the fixed point, rather than the issue of linearizability.

When f is linearizable, f behaves like the rotation by angle $2\pi\alpha$ around 0 on the maximal domain of linearization, called the *Siegel disk* of f, which turns out to be a domain of the form $h(\{|w| < R\})$ where R is less than or equal to the radius of convergence of h around 0. In this case, the compact sets $h(\{|w| \leq r\})$ for $r < R$ are clearly completely invariant under f. However, even in the more general case where f is not necessarily linearizable, R. Perez-Marco found completely-invariant continua for f that persist.

THEOREM. (Perez-Marco [**PM1**]). *Let* $f(z) = e^{2\pi i\alpha}z + O(z^2)$ *and* $\alpha \in \mathbf{R} - \mathbf{Q}$ *be such that* f *and* f^{-1} *are defined and univalent on a neighbourhood of the closure of a Jordan domain* $\Omega \subset \mathbf{C}$ *containing* 0. *Then there exists a full compact connected set* K *contained in* $\overline{\Omega}$ *such that* $0 \in K$, $K \cap \partial\Omega \neq \varnothing$ *and* $f(K) = f^{-1}(K) = K$.

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The invariant compacts K thus obtained are called *Siegel compacta*. A classical topological theorem of Birkhoff [**Bi**] does in fact guarantee, for planar homeomorphisms near Lyapunov unstable points, the existence of compacts, but which are either positive or negative invariant, not necessarily totally invariant. In the holomorphic setting one can thus improve on this to obtain totally-invariant compacts. Moreover, though this will not be used in what follows, if the boundary of the Jordan domain Ω is C^1-smooth, then Perez-Marco has shown that such a compact K is in fact unique (see [**PM5**]). If K is not contained in the closure of a linearization domain it is called a *hedgehog*. A hedgehog is called linearizable or nonlinearizable depending on whether it contains a linearization domain or not. We will only deal with nonlinearizable hedgehogs in this article. Perez-Marco has shown that these have no interior (see [**PM5**]). The structure of such hedgehogs is topologically complex; for example, Perez-Marco shows in [**PM2**] that they are not locally connected at any point different from the fixed point. Objects such as *combs* (homeomorphs of the product of a Cantor set and an interval) had been expected to be found within hedgehogs. Indeed, this is stated by Perez-Marco in [**PM4**], and the author uses Perez-Marco's techniques of 'tube-log Riemann surfaces' to construct hedgehogs containing smooth combs in [**B**]. These techniques, first introduced by Perez-Marco in [**PM6**] and later used again in [**PM3**] and [**PM4**], are a powerful means of constructing a wide range of nonlinearizable dynamics with good control over the geometry of the corresponding hedgehogs.

In the present article, we use these techniques to construct a hedgehog having Hausdorff dimension one (the minimum possible for connected planar sets).

MAIN THEOREM. *There exists a Cantor set $C \subset \mathbf{R}$ and a family of commuting holomorphic maps $(f_t(z) = e^{2\pi i t} z + O(z^2))_{t \in C}$, all defined and univalent on a neighbourhood of the closed unit disk $\overline{\mathbf{D}}$, having a common hedgehog $K \subset \overline{\mathbf{D}}$, such that K has Hausdorff dimension one. The maps f_t are nonlinearizable for $t \in C \cap \mathbf{R} - \mathbf{Q}$.*

2. *The construction*

2.1. *General idea of the construction.* It is strongly advised to read this section in conjunction with [**PM3**] and [**PM4**]. As these techniques are explained in detail in [**PM3**] and especially in [**PM4**], and the basic elements of the construction are the same as in [**PM4**], to avoid repetition the description below is shorter and slightly informal.

We start with the linear flow of translations $(F_t(z) = z + t)_{t \in \mathbf{R}}$ on the upper half-plane $\mathbf{H}$. (Note: for convenience, instead of working with dynamics and dynamical objects such as Siegel disks and hedgehogs defined on the unit disk $\mathbf{D}$, we will actually work instead with their lifts to the upper half-plane via the universal covering $E : \mathbf{H} \to \mathbf{D} - 0$, $E(z) = e^{2\pi i z}$.)

2.1.1. *The Riemann surface $\mathcal{S}_0$.* The idea is to 'fold' the dynamics by introducing nonlinearities in the dynamics, such as fixed points; these result in a shrinking of the linearization domain. This is accomplished by means of the following Riemann surface with distinguished charts $\mathcal{S}_0$ (see [**PM4**, §1.2.1]).

The surface $\mathcal{S}_0$ (a 'tube-log Riemann surface') is formed from a copy of $\mathbf{C}$ with 'slits' $[n, n - i\infty]$ at each of the integers $n \in \mathbf{Z}$, above and below each of which is pasted,

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isometrically along the slits, a family (indexed by $\mathbf{Z}$) of slit cylinders $\mathbf{C}/\mathbf{Z}$ with slits $[0, 0 - i\infty]$ (see Figure 1). This construction gives not only a Riemann surface, but also a canonical set of charts for the surface, which allow us to write formulae for functions defined on the surface in terms of these canonical coordinates. We note here that the surface $\mathcal{S}_0$ has 'ramification points of infinite order' at the points corresponding to $\mathbf{Z}$ in the 0-plane and to 0 in the cylinders. These can be seen as points added in the completion of the flat metric on $\mathcal{S}_0$ (induced from the flat metrics on its building blocks). However, it is important to note that these points do not belong to the surface $\mathcal{S}_0$ itself, nor can the complex structure on $\mathcal{S}_0$ be extended to these points.

2.1.2. *Uniformization of the Riemann surface* $\mathcal{S}_0$. The upper ends of the cylinders, on the other hand, do correspond to points for the complex structure. Adding in these points gives a simply-connected Riemann surface, which it is then not too hard to see is biholomorphic to $\mathbf{C}$. So $\mathcal{S}_0$ should be biholomorphic to the complex plane minus a doubly-infinite discrete set of points; in fact, there is an explicit formula for the uniformization (see [**PM4**, §1.2.3]), which in terms of the distinguished charts on $\mathcal{S}_0$ is given by

$$K(z) = \frac{1}{2\pi i} \log(-\log(1 - e^{2\pi i z})).$$

The map K satisfies the properties

$$K(z+1) = K(z) + 1 \quad \text{on all of the 0-sheet of } \mathcal{S}_0$$
$$K(z) - z \to 0 \quad \text{as } \operatorname{Im} z \to +\infty \text{ in the 0-sheet of } \mathcal{S}_0,$$

for the appropriate choices of the branches of the logarithms.

The images under K of horizontal lines just above the real line $\mathbf{R}$ in the 0-plane are thus 1-periodic (with respect to parametrization by the x-coordinate) curves in $\mathbf{C}$; since the ramification points at $\mathbf{Z}$ in the 0-plane correspond to ends at infinity under the uniformization, these curves oscillate with amplitudes that tend to infinity as we take lines tending to the real line. The upper ends of the cylinders correspond under K to a discrete set of points $P \subset \mathbf{C}$; among these, those that correspond to the upper ends of the 1-cylinders lie below, but close to, the image of the real line. Figure 2 (reproduced here courtesy of Arnaud Cheritat) illustrates the image of the foliation of horizontal lines on $\mathcal{S}_0$ under K.

2.1.3. *Lifting and folding of dynamics.* Consider the linear flow of translations $(F_t(z) = z + t)_{t\in\mathbf{R}}$ on the upper half-plane $\mathbf{H}$. The upper half-plane of the 0-plane of $\mathcal{S}_0$ projects univalently onto the upper half-plane $\mathbf{H}$, so we can lift each F_t to a map $\tilde{F}_t(z)$ from the upper half-plane of the 0-plane of $\mathcal{S}_0$ to itself, given in terms of the distinguished chart by the same formula $\tilde{F}_t(z) = z + t$. We would like to extend the lift analytically to a univalent map $\tilde{F}_t$ of the surface to itself; however, it is not possible to extend to the points $\mathbf{Z} - t$ of the 0-plane or the points $0 - t$ of the cylinders, as these would have to be mapped to the ramification points, and moreover there is a non-trivial monodromy when continuing analytically around these points.

To overcome this problem, we can remove from the surface a small ϵ-neighbourhood (for the flat metric on $\mathcal{S}_0$) $\mathcal{I}(\epsilon) \subset \mathcal{S}_0$ of the ramification points. On $\mathcal{S}_0 - \mathcal{I}(\epsilon)$, for $|t| < \epsilon$,

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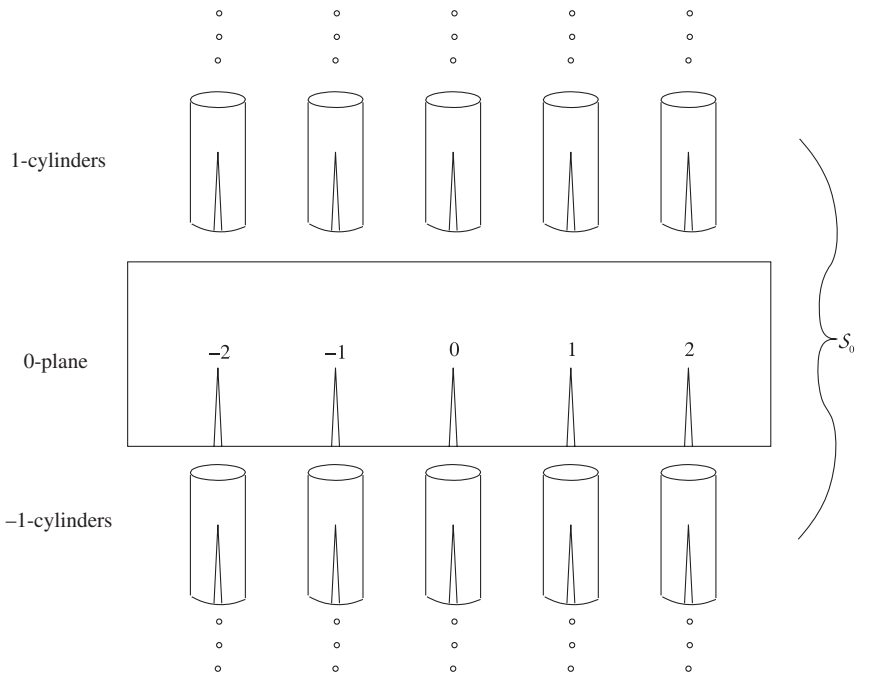


FIGURE 1. The tube-log Riemann surface $\mathcal{S}_0$.

the aforementioned problem does not arise for the maps $\tilde{F}_t$, each of which then extends to a univalent map $\tilde{F}_t : \mathcal{S}_0 - \mathcal{I}(\epsilon) \to \mathcal{S}_0$, which is expressed in terms of the distinguished charts on $\mathcal{S}_0$ by the same formula $\tilde{F}_t(z) = z + t$ (for this first step it suffices, in fact, to remove only a set of horizontal slits $\mathbf{Z} + (\epsilon, \epsilon)$ near the ramification points, but later in order to iterate the 'folding' procedure we will have to remove neighbourhoods of the ramification points).

Let $T : \mathcal{S}_0 \to \mathcal{S}_0$ be the automorphism given in the charts on $\mathcal{S}_0$ by $T(z) = z + 1$. We note that the lifts of the integer translations $z \mapsto z + n$ extend to all of $\mathcal{S}_0$ as the automorphisms $T^n : \mathcal{S}_0 \to \mathcal{S}_0$. These automorphisms commute with the maps $\tilde{F}_t$, so we can also define for times $t' = n + t \in \mathbf{Z} + (-\epsilon, \epsilon)$ the lifts $\tilde{F}_{t'} : \mathcal{S}_0 - \mathcal{I}(\epsilon) \to \mathcal{S}_0$, by $\tilde{F}_{n+t} := T^n \circ \tilde{F}_t$. Hence we obtain a semi-flow $(\tilde{F}_t)$ defined for times $t \in \mathbf{Z} + (-\epsilon, \epsilon)$ on a large part $\mathcal{S}_0 - \mathcal{I}(\epsilon)$ of the surface.

The 'upper ends' of the cylinders on the surface are points for the complex structure, which become fixed points (modulo the automorphism $T : \mathcal{S}_0 \to \mathcal{S}_0$) for the dynamics on the surface. Hence lifting the dynamics to the surface has the effect not only of extending its domain of definition, but also of introducing nonlinearities, such as fixed points.

To see this dynamics on the surface as dynamics on the plane, we use the uniformization $K : \mathcal{S}_0 \to \mathbf{C} - P$.

Conjugating the lifts $\tilde{F}_t$ by K, we obtain dynamics $K \circ \tilde{F}_t \circ K^{-1}$ on a large part $K(\mathcal{S}_0 - \mathcal{I}(\epsilon)) \subset \mathbf{C}$ of the plane. The dynamics flows along the foliation shown in Figure 2, and has as fixed points (modulo $\mathbf{Z}$) the points P corresponding under K to the upper ends of the cylinders. This dynamics is what we mean by the 'folded' dynamics.

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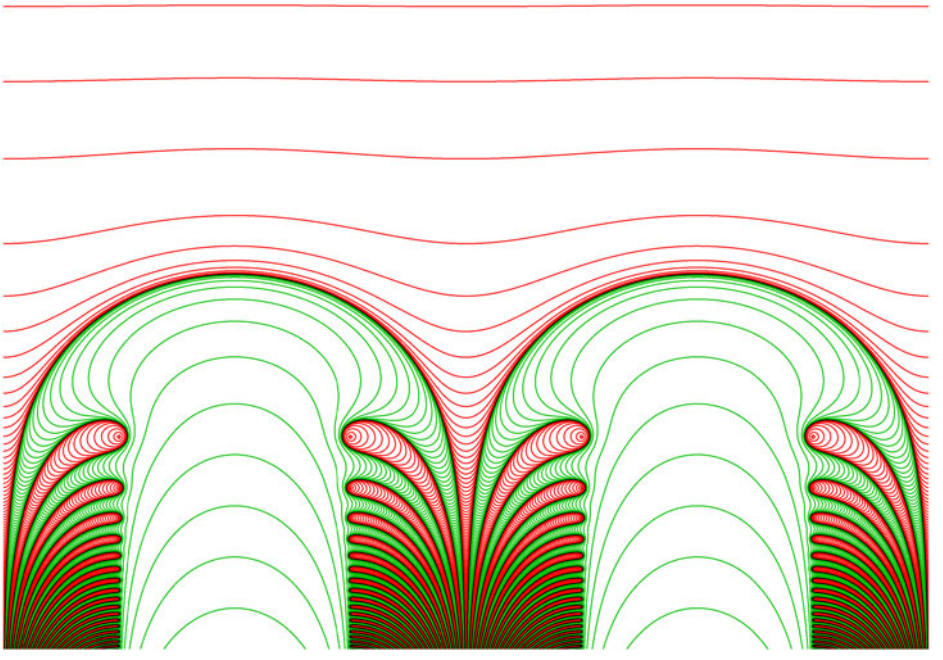


FIGURE 2. Image of the horizontal foliation on $\mathcal{S}_0$ after uniformization.

A key property of the construction is that small neighbourhoods of the infinite ramification points in $\mathcal{S}_0$ correspond under K_0 to domains in the plane with large negative imaginary part; any half-plane $\{\mathrm{Im}\, z \geq -y\} \subset \mathbf{C}$ where $y \geq 0$ is contained in $K(\mathcal{S}_0 - \mathcal{I}(\epsilon))$ for ϵ small enough (see [**PM4**, §1.2.4]), and so we can always ensure a large domain of definition for the dynamics.

An equally important property of the uniformization K_0 is that the lower ends of the cylinders and points with large negative imaginary part in the 0-plane in $\mathcal{S}_0$ correspond to points with large negative imaginary part in the plane. As explained later (in §2.1.5), these two properties make it possible to iterate the 'folding' construction indefinitely, all the while maintaining a common domain of definition, so that one can obtain dynamics in the limit.

2.1.4. *Normalization of the uniformization.* The images under K of horizontal lines in the upper half-plane of the 0-sheet of $\mathcal{S}_0$ are graphs over the x-axis that are 1-periodic with respect to parametrization by the x-coordinate. For the image of a horizontal line at a height $\delta > 0$ in the 0-sheet, peaks and troughs occur at the points $K(n + (1/2) + i\delta)$ and $K(n + i\delta)$ respectively (where n runs over all integers). The amplitude of oscillation (the difference in height between peaks and troughs) is a strictly decreasing function of δ that tends to $+\infty$ as δ tends to 0, and tends to 0 as δ tends to $+\infty$.

We define a family of normalized uniformizations as follows.

First we place the ramification points of $\mathcal{S}_0$ slightly below the real axis instead of on the real axis. More precisely, for $\delta > 0$ we denote by $\mathcal{S}_\delta$ the Riemann surface obtained by

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the same construction as $\mathcal{S}_0$, pasting a plane and cylinders isometrically along slits, but with ramification points placed at $\mathbf{Z} - i\delta$ in the 0-plane and $0 - i\delta$ in the cylinders. The uniformization of this surface is given by the map $z \mapsto K(z + i\delta)$ instead of $z \mapsto K(z)$. The image under this uniformization of the real line $\mathbf{R}$ in the 0-sheet of $\mathcal{S}_\delta$ is a 1-periodic graph with an amplitude of oscillation that is large when δ is small. For small δ, the troughs of this curve have large negative imaginary parts and lie well below the real axis; in applications, however, we would like the image of the real line to lie above the x-axis, with the troughs touching the x-axis, and so we define the uniformization $K_\delta : \mathcal{S}_\delta \to \mathbf{C}$ of $\mathcal{S}_\delta$ by

$$K_\delta(z) := K(z + i\delta) - i \times \text{‘height of the troughs’},$$

where by the ‘height of the troughs’ we mean

$$\text{‘height of the troughs’} = -\operatorname{Im} K(n + i\delta) = \frac{1}{2\pi} \log(-\log(1 - e^{-2\pi\delta})),$$

(where n here is any integer). The curve $K_\delta(\mathbf{R})$ then lies in the upper half-plane of $\mathbf{C}$ with troughs lying on the real axis at the points $K_\delta(n + 0i) = n + 0i$, $n \in \mathbf{Z}$; the image of the upper half-plane of the 0-sheet of $\mathcal{S}_\delta$ under K_δ, which we denote by $K_\delta(\mathbf{H})$, is the region in the upper half-plane of $\mathbf{C}$ bounded by $K_\delta(\mathbf{R})$.

In applications, it will also be useful to have the image of the real line oscillate at higher frequencies, more precisely with period $1/a$ instead of period 1 for integers $a \geq 1$. To do this, we consider the uniformizations $(1/a)K_\delta$ obtained by dividing K_δ by integers $a \geq 1$. The image of the real line, the curve $(1/a)K_\delta(\mathbf{R})$, is then a $1/a$-periodic curve. Finally, in order to control the amplitude of oscillation, we note that given integer $a \geq 1$ and real $h > 2$, there exists a unique $\delta = \delta(a, h)$ (as is shown in [**PM4**, §1.2.5]), with $0 < \delta < 1/2$, such that the amplitude of oscillation of $K_\delta(\mathbf{R})$ is $a \cdot h$, and hence that of $(1/a)K_\delta(\mathbf{R})$ is h. Moreover $\delta(a, h) \to 0$ for fixed h when $a \to \infty$. The constant δ satisfies (see [**PM4**, §1.2.5])

$$\frac{\log(1 + e^{-2\pi\delta})}{-\log(1 - e^{-2\pi\delta})} = e^{-2\pi a h}.$$

The peaks and troughs of the curve $(1/a)K_\delta(\mathbf{R})$ occur at the points $(1/a)K_\delta(n + (1/2) + 0i) = n + (1/2) + ih$ and $(1/a)K_\delta(n + 0i) = n + 0i$ respectively, and we have $\operatorname{Max}_{x \in \mathbf{R}} \operatorname{Im}(1/a)K_\delta(x) = \operatorname{Im}(1/a)K_\delta(n + (1/2)) = h$ and $\operatorname{Min}_{x \in \mathbf{R}} \operatorname{Im}(1/a)K_\delta(x) = \operatorname{Im}(1/a)K_\delta(n) = 0$, for any integer $n \in \mathbf{Z}$ (see Figure 3).

We thus have a family of normalized uniformizations $(1/a)K_\delta$ depending on two parameters $a \geq 1$ (an integer) and $h > 2$ (a real), where here δ is understood to be $\delta = \delta(a, h)$. The maps $(1/a)K_\delta$ have the following properties.

(1) $(1/a)K_\delta$ is univalent on a neighbourhood of $\overline{\mathbf{H}}$, mapping $\mathbf{H}$ into itself.
(2) $(1/a)K_\delta(z + 1) = (1/a)K_\delta(z) + (1/a)$.
(3) $(1/a)K_\delta(i\mathbf{R}_{\geq 0}) = i\mathbf{R}_{\geq 0}$ and $((1/a)K_\delta)'(z) > 0$ for $z \in i\mathbf{R}_{\geq 0}$.

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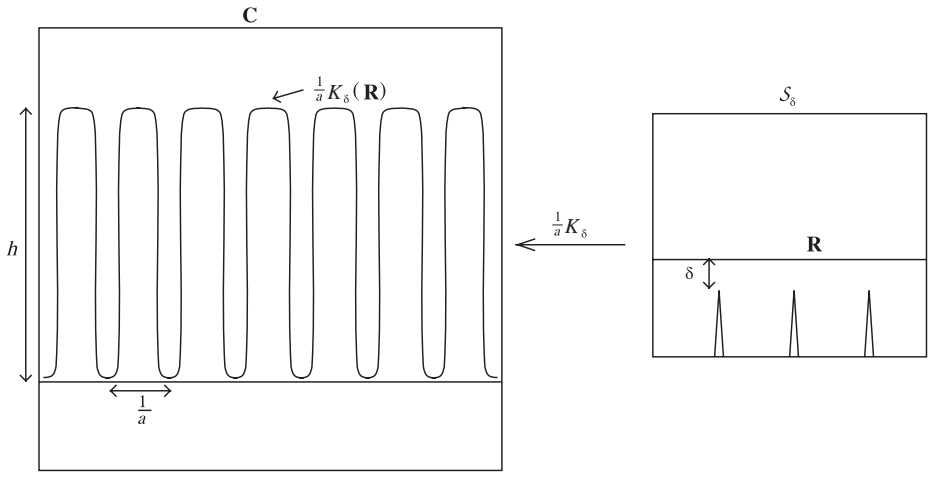


FIGURE 3. Image of the real line under the uniformization $1/aK_\delta$.

(4) There exists a constant C independent of a and $h > 2$ such that

$$\left|\frac{1}{a}K_\delta(z) - \left(\frac{1}{a}z + ih\right)\right| \leq \frac{C}{a}$$

for all z with $\operatorname{Im} z > 1$.

(5) Let $z_0 \in \mathbf{C}$ denote the limit of $(1/a)K_\delta(z)$ when $z \to \infty$ in $\mathcal{S}_\delta$ through the upper end of one of the 1-cylinders. There exists a constant C_1 independent of a and $h > 2$ such that

$$\operatorname{Im} z_0 \geq h - \frac{C_1}{a}$$

(note that the images under $(1/a)K_\delta$ of the upper ends of the 1-cylinders differ from each other by $1/a$-translations, so $\operatorname{Im} z_0$ is independent of the choice of 1-cylinder). We note that, for the dynamics obtained by conjugation by $(1/a)K_\delta$, these points are no longer necessarily fixed points (modulo $\mathbf{Z}$), but rather periodic points (modulo $\mathbf{Z}$) of period a.

We note that the properties (1)–(4) refer to $(1/a)K_\delta(z)$ for z in the 0-plane of $\mathcal{S}_\delta$, while (5) refers to the 1-cylinders. These properties appear in [**PM4**, §1.2.5], but can also be checked by elementary computations. We note that condition (5) allows us to control the heights in the upper half-plane of the nonlinearities of the dynamics obtained by conjugation by $(1/a)K_\delta$; indeed (5) says that the height of the periodic orbits is approximately h, so larger choices of h result in somewhat smaller linearization domains.

2.1.5. *Construction of the limit dynamics.* Now we iterate the procedure described in §2.1.3 above. We take a surface $\mathcal{S}_{\delta_1}$, lift the linear flow of translations to $\mathcal{S}_{\delta_1}$ and conjugate by a uniformization $K_0 = (1/a_0)K_{\delta_1}$ to obtain folded dynamics as described in §2.1.3. Having fixed $\mathcal{S}_{\delta_1}$ and K_0, we take a surface $\mathcal{S}_{\delta_2}$, lift the linear flow of translations to $\mathcal{S}_{\delta_2}$ and conjugate by a uniformization $K_1 = (1/a_1)K_{\delta_2}$; we then lift this

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folded dynamics to $\mathcal{S}_{\delta_1}$ and conjugate by the uniformization K_0, to obtain a twice-folded dynamics, given by conjugating by the map K_0K_1. We continue in this way, choosing at each stage a surface $\mathcal{S}_{\delta_n}$ and a uniformization K_{n-1}, to obtain at stage n a dynamics given by conjugation by $K_0 \dots K_{n-1}$. At each stage the dynamics obtained is linearizable, but the linearization domain decreases; the point is that, by choosing the appropriate uniformizations and controlling the geometry of the domains obtained at each stage, we should be able to obtain, in the limit, dynamics having an invariant domain with the desired geometry.

In the construction that follows, we define a sequence

$$K_n = \frac{1}{a_n} K_{\delta_{n+1}}, \quad \delta_{n+1} = \delta(a_n, h_n), \quad n \geq 0,$$

of uniformizations depending on two sequences of parameters $(a_n)_{n\geq 0}$ and $(h_n)_{n\geq 0}$. The whole construction is completely determined by the sequences of parameters a_n and h_n, which we will indicate how to choose in the course of the construction. For the moment we assume that they are given, and construct the desired limit dynamics as follows.

Let $n \geq 1$. We let $\mathcal{F}_{n,n} = (F_{n,n,t}(z) = z + t)_{t\in\mathbf{R}}$ be the linear flow of translations. As explained in §2.1.3 above, we can lift the elements of this flow to univalent maps $\tilde{F}_{n,n,t}$ defined on the part of the surface $\mathcal{S}_{\delta_n}$ obtained by deleting an ϵ-neighbourhood $\mathcal{I}(\epsilon)$ of the ramification points and for times $t \in \mathbf{Z} + (-\epsilon, \epsilon)$. We take $\epsilon = 3a_n^{-1}$, and conjugate by the uniformization K_{n-1} of $\mathcal{S}_{\delta_n}$ to obtain a semi-flow $\mathcal{F}_{n-1,n} = (F_{n-1,n,t})_{t\in A_{n-1,n}}$ defined by

$$F_{n-1,n,a_{n-1}^{-1}t} := K_{n-1} \circ \tilde{F}_{n,n,t} \circ K_{n-1}^{-1},$$

where

$$a_{n-1}^{-1}t \in A_{n-1,n} := a_{n-1}^{-1}(\mathbf{Z} + [-2a_n^{-1}, 2a_n^{-1}])$$

(we could in this step have taken a bigger set of times for $A_{n-1,n}$, such as $a_{n-1}^{-1}(\mathbf{Z} + (-3a_n^{-1}, 3a_n^{-1}))$, but in the steps that follow it will be necessary to restrict to smaller sets of times). We note that the time has been reparametrized in order to have the right behaviour at $+i\infty$: when $\operatorname{Im} z \to +\infty$,

$$F_{n-1,n,t}(z) = z + t + o(1).$$

These maps are defined on $K_{n-1}(\mathcal{S}_{\delta_n} - \mathcal{I}(3a_n^{-1})) \subset \mathbf{C}$, which contains a large half-plane $\mathbf{H}_{-y} = \{\operatorname{Im} z \geq -y\}$ where $y \to +\infty$ as $a_n \to +\infty$.

We would now like to lift the semi-flow $\mathcal{F}_{n-1,n}$ to the surface $\mathcal{S}_{\delta_{n-1}}$ and conjugate by the uniformization K_{n-2} to obtain a semi-flow $\mathcal{F}_{n-2,n}$. The difference with the previous case is that we are now starting with a nonlinear flow instead of the linear flow of translations. However, by giving up some space in the lower region of the half-plane $\mathbf{H}_{-y}$, we can work with maps close to linear ones; more precisely we have (see [**PM4**, Lemma 1.1.6 and Proposition 2.2.6]), for $t \in A_{n-1,n}$, if $\operatorname{Im} z \geq -y + (1/(2\pi)) \log a_{n-1} + C$ (where C is a universal constant), then

$$|F_{n-1,n,t}(z) - (z + t)| < \frac{1}{a_{n-1}}.$$

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Since $y \to +\infty$ as $a_n \to +\infty$, we observe that once a_{n-1} is fixed, by choosing a_n large enough we can ensure that this inequality holds in as large a half-plane $\mathbf{H}_{-\tilde{y}}$ as desired, where $-\tilde{y} = -y + (1/(2\pi)) \log a_{n-1} + C$.

Now consider the region $\mathcal{J}(-\tilde{y}) = \{z \in \mathcal{S}_{\delta_{n-1}} : \operatorname{Im} z \le -\tilde{y}\} \subset \mathcal{S}_{\delta_{n-1}}$ (note that there is a well-defined function $\operatorname{Im} z$ on $\mathcal{S}_{\delta_{n-1}}$). Since the estimate above holds on the complement $\mathcal{S}_{\delta_{n-1}} - \mathcal{J}(-\tilde{y})$ of this region, it should be clear that if we now remove as well an ϵ-neighbourhood $\mathcal{I}(\epsilon)$ of the ramification points, this time with $\epsilon = 3a_{n-1}^{-1}$, then we can define, for $t \in \mathbf{Z} + A_{n-1,n} \cap [-2a_{n-1}^{-1}, 2a_{n-1}^{-1}]$, the lifts of elements of $\mathcal{F}_{n-1,n}$ as univalent maps $\tilde{F}_{n-1,n,t} : \mathcal{S}_{\delta_{n-1}} - (\mathcal{J}(-\tilde{y}) \cup \mathcal{I}(3a_{n-1}^{-1})) \to \mathcal{S}_{\delta_{n-1}}$.

We can now proceed as before and define the semi-flow $\mathcal{F}_{n-2,n} = (F_{n-2,n,t})_{t \in A_{n-2,n}}$ by

$$F_{n-2,n,a_{n-2}^{-1}t} := K_{n-2} \circ \tilde{F}_{n-1,n,t} \circ K_{n-2}^{-1},$$

where

$$a_{n-2}^{-1}t \in A_{n-2,n} := a_{n-2}^{-1}(\mathbf{Z} + A_{n-1,n} \cap [-2a_{n-1}^{-1}, 2a_{n-1}^{-1}])$$

(the times have again been reparametrized appropriately so that $F_{n-2,n,t}(z) = z + t + o(1)$ when $\operatorname{Im} z \to +\infty$).

These maps are defined on $K_{n-2}(\mathcal{S}_{\delta_{n-1}} - (\mathcal{J}(-\tilde{y}) \cup \mathcal{I}(3a_{n-1}^{-1}))) \subset \mathbf{C}$, which, as remarked earlier in §2.1.3, contains a large half-plane $\mathbf{H}_{-y'} = \{\operatorname{Im} z \ge -y'\}$ where $y' \to +\infty$ as $a_{n-1} \to +\infty$ and $\tilde{y} \to \infty$ (see [**PM4**, Proposition 2.2.8]).

Following this same procedure of lifting and conjugating the dynamics, we can define successively the semi-flows $\mathcal{F}_{n-3,n}, \mathcal{F}_{n-4,n}, \ldots, \mathcal{F}_{0,n}$, with $\mathcal{F}_{j,n} = (F_{j,n,t})_{t \in A_{j,n}}$ defined for times $t \in A_{j,n}$, where $A_{j,n}$ is defined inductively by

$$A_{j,n} := a_j^{-1}(\mathbf{Z} + A_{j+1,n} \cap [-2a_{j+1}^{-1}, 2a_{j+1}^{-1}]),$$

for $j = n-1, \ldots, 0$ and $A_{n,n} = \mathbf{R}$. We note here that the sets $A_{0,n}$ form a decreasing sequence, ie $A_{0,n+1} \subset A_{0,n}$, $n \ge 1$; indeed,

$$\begin{aligned} A_{n-1,n+1} &= a_{n-1}^{-1}(\mathbf{Z} + A_{n,n+1} \cap [-2a_n^{-1}, 2a_n^{-1}]) \\ &\subset a_{n-1}^{-1}(\mathbf{Z} + [-2a_n^{-1}, 2a_n^{-1}]) = A_{n-1,n}, \end{aligned}$$

from which it follows similarly that

$$\begin{gathered} A_{n-2,n+1} \subset A_{n-2,n} \\ A_{n-3,n+1} \subset A_{n-3,n} \\ \cdots \\ A_{0,n+1} \subset A_{0,n}. \end{gathered}$$

Thus we obtain a sequence of semi-flows $\mathcal{F}_{0,n} = (F_{0,n,t})_{t \in A_{0,n}}$, where each one is given by 'folding' the linear flow n times. Moreover, the a_n can be chosen inductively to grow fast enough so that all the semi-flows $\mathcal{F}_{0,n}$ are defined in the upper half-plane $\mathbf{H}$, or indeed in any fixed half-plane $\mathbf{H}_{-M} = \{\operatorname{Im} z > -M\}$, as large as desired; indeed, we can state as a proposition the following (which follows directly from [**PM4**, §§1.3.1 and 2.2.3]).

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PROPOSITION 2.1.1. *Given $M > 0$, there exists a sequence of conditions $(\mathcal{C}_n)_{n\geq 1}$ on the integers a_n of the form*

$$(\mathcal{C}_n) \quad a_n \geq C_n(a_0, \ldots, a_{n-1}, h_0, \ldots, h_{n-1}, M),$$

such that when they are fulfilled then all the semi-flows $(\mathcal{F}_{0,n})_{n\geq 1}$ are defined in the half-plane $\mathbf{H}_{-M} = \{\operatorname{Im} z > -M\}$.

We observe thus a feature of the construction that is crucial when passing to the limit, namely that it allows us to successively paste nonlinearities to the dynamics without decreasing its domain of definition too much, allowing us to retain a half-plane $\mathbf{H}_{-M}$.

Assume that the a_n have been chosen so that all the semi-flows $\mathcal{F}_{0,n}$ are defined on $\mathbf{H}_{-M}$. From the sequence of semi-flows $\mathcal{F}_{0,n}$ we can extract a convergent subsequence (it always exists—see [**PM4**, §1.1.2]) converging normally for times $t \in A = \bigcap_{n\geq 0} A_{0,n}$, to get a limit semi-flow $\mathcal{F} = (F_t)_{t\in A}$ defined on the half-plane $\mathbf{H}_{-M}$ for times $t \in A$. We note that elements of the limit semi-flow do indeed commute, since we can pass to the limit in the equation

$$F_{0,n_k,t} \circ F_{0,n_k,s}(z) = F_{0,n_k,s} \circ F_{0,n_k,t}(z)$$

(using uniform convergence in compact neighbourhoods of the points z, $F_t(z)$ and $F_s(z)$).

The Cantor set of times C in the main theorem is given by

$$C := \bigcap_{n\geq 0} A_{0,n}/\mathbf{Z}.$$

If all the a_n are large enough, for example $a_n \geq 5$ for all n, then the set C is indeed a Cantor set.

The semi-flow $(f_t)_{t\in C}$ of the main theorem is defined on a common neighbourhood $E(\mathbf{H}_{-M})$ of the closed unit disk $\overline{\mathbf{D}}$ via the equation

$$f_t \circ E = E \circ F_t, \quad t \in C$$

(where E is the universal covering $E : \mathbf{C} \to \mathbf{C} - \{0\}$, $E(z) = e^{2\pi i z}$).

As stated in §2.1.4, each uniformization K_n maps the upper half-plane of the 0-plane of $\mathcal{S}_{\delta_{n+1}}$ into the upper half-plane $\mathbf{H}$. So we can consider its restriction to the upper half-plane of the 0-plane as a map $K_n : \mathbf{H} \to \mathbf{H}$ of $\mathbf{H}$ into itself. If we consider the composition of these maps $K_0 \circ K_1 \circ \cdots \circ K_n$, then it is clear that the set $K_0K_1 \ldots K_n(\overline{\mathbf{H}})$ is totally invariant under the semi-flow $\mathcal{F}_{0,n+1}$, and hence the set

$$\mathcal{H} = \bigcap_{n=0}^{\infty} K_0K_1 \ldots K_n(\overline{\mathbf{H}}),$$

is totally invariant under the limit semi-flow $\mathcal{F}$. Moreover, $\mathcal{H}$, being the decreasing limit of the domains $K_0K_1 \ldots K_n(\overline{\mathbf{H}})$, is full, and $\mathcal{H} \cap \partial\mathbf{H} \neq \varnothing$ (since $0 \in \mathcal{H}$). Thus $\mathcal{H}$ is in fact a common hedgehog for irrational elements of the limit semi-flow $\mathcal{F}$. The hedgehog K of the semi-flow $(f_t)_{t\in C}$ in the main theorem is given by

$$K = E(\mathcal{H}).$$

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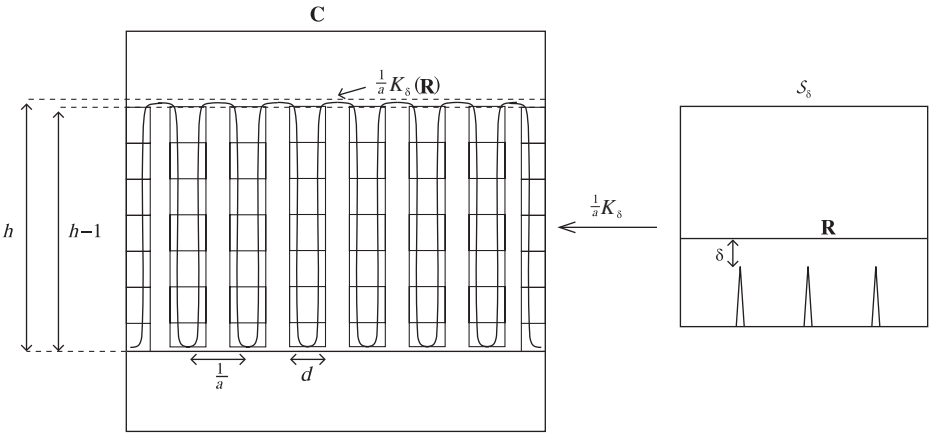


FIGURE 4. A covering of part of the hedgehog by squares.

It is important to note in this construction that the only requirement imposed on the a_n so that the limit dynamics is defined on the upper half-plane is that they grow fast enough, as expressed by Proposition 2.1.1 above. Subject to this restriction, the construction still offers enough flexibility to construct hedgehogs with quite different geometries by varying the choices of the parameters a_n and h_n.

In the following sections we show how to choose the uniformizations K_n appropriately (by proper choices of the parameters a_n and h_n), and to control the geometries of the domains $K_0K_1 \dots K_n(\overline{\mathbf{H}})$ so that the hedgehog $\mathcal{H}$ has Hausdorff dimension one.

2.2. *A hedgehog of Hausdorff dimension one.* We fix some notation first. For a countable family $\mathcal{A} = \{U_i\}_{i\in I}$ of planar sets U_i, for $s > 0$ the *s-length* is defined to be

$$\mathcal{L}^s(\mathcal{A}) = \sum_{i\in I} \operatorname{diam}(U_i)^s$$

(recall that this is the quantity used to define Hausdorff s-measure).

We recall the definition of the uniformization $(1/a)K_\delta$,

$$\frac{1}{a}K_\delta(z) = \frac{1}{a}\left(\frac{1}{2\pi i}\log(-\log(1-e^{2\pi i(z+i\delta)})) - iC_\delta\right),$$

where $C_\delta = (1/(2\pi))\log(-\log(1-e^{-2\pi\delta}))$. We consider the part of the domain $(1/a)K_\delta(\overline{\mathbf{H}})$ below the height $h-1$.

For the parametrization $t \mapsto (1/a)K_\delta(t) = x(t) + iy(t)$ of the curve $(1/a)K_\delta(\mathbf{R})$, it is straightforward to check from the formulae that:

(1) $x'(t) > 0$ for all t, so the curve is a graph over $\mathbf{R}$, that is, it has a parametrization of the form $y = f(x)$;

(2) since $(1/a)K_\delta(t+1) = (1/a)K_\delta(t) + (1/a)$, the graph is $(1/a)$-periodic, that is,

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$f(x+(1/a)) = f(x)$; moreover, in any period $n \le t \le n+1$, $n \in \mathbf{Z}$, we have

$$x(n) + iy(n) = \frac{n}{a} + i0, \quad x\left(n+\frac{1}{2}\right) + iy\left(n+\frac{1}{2}\right) = \frac{n+1/2}{a} + ih,$$

$$y'(t) > 0 \quad \text{for } n < t < n+\frac{1}{2}, \quad y'(t) < 0 \quad \text{for } n+\frac{1}{2} < t < n+1, \quad \text{and}$$

$$y\left(\left(n+\frac{1}{2}\right)-u\right) = y\left(\left(n+\frac{1}{2}\right)+u\right) \quad \text{for all } u \in \mathbf{R}.$$

So in every period $n \le t \le n+1$, there are exactly two points $t = n+t_0$ and $t = n+1-t_0$, $0 < t_0 < 1$, where the curve crosses the line $\{\operatorname{Im} z = h-1\}$. Moreover $0 \le y(t) \le h-1$ for $n - t_0 \le t \le n + t_0$. Thus taking $d = 2x(t_0)$, we can, given an integer $M > 0$, cover the region

$$\begin{aligned}(1/a)K_\delta(\overline{\mathbf{H}}) \cap \{\operatorname{Im} z \le h-1,\ 0 \le \operatorname{Re} z \le M\} \\ = \{z = x+iy \mid f(x) \le y \le h-1,\ 0 \le x \le M\}\end{aligned}$$

by squares of side-length d, as shown in Figure 4 above.

We denote this covering by $\mathcal{A}(a, h, M)$.

The key to the whole construction is the following lemma, which says that the region being covered can be made arbitrarily 'thin' by taking a large enough, keeping h and M fixed.

LEMMA. *Fix $s > 1$. With the same notation as above, we have*

$$d = O\left(\frac{1}{a}e^{-2\pi a}\right)$$

and

$$\mathcal{L}^s(\mathcal{A}(a, h, M)) \to 0 \quad \text{as } a \to +\infty,$$

keeping h and M fixed.

Proof. We first prove the estimate on $d = 2x(t_0)$.

t_0 was defined as the smallest $t > 0$ such that $y(t) = \operatorname{Im}(1/a)K_\delta(t) = h-1$. The formula for $(1/a)K_\delta$ gives

$$\frac{1}{2\pi}\log\left|\frac{\log(1-e^{-2\pi\delta})}{\log(1-e^{2\pi i(t_0+i\delta)})}\right| = a(h-1).$$

But $\delta = \delta(a, h)$ is chosen so that

$$\frac{1}{2\pi}\log\left|\frac{\log(1-e^{-2\pi\delta})}{\log(1+e^{-2\pi\delta})}\right| = ah;$$

substituting this into the previous equation, rearranging terms and exponentiating gives

$$|\log(1-e^{2\pi i(t_0+i\delta)})| = e^{2\pi a}\log(1+e^{-2\pi\delta}).$$

For fixed $h > 0$, as $a \to +\infty$, $\delta \to 0$ (see [**PM4**]), so

$$|\log(1-e^{2\pi i(t_0+i\delta)})| \sim e^{2\pi a}\log(2)$$

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as $a \to +\infty$. Thus

$$\begin{aligned} d &= 2x(t_0) = 2\,\mathrm{Re}\,\frac{1}{a}K_\delta(t_0) \\ &= 2\frac{1}{a}\frac{1}{2\pi}\arg(-\log(1-e^{2\pi i(t_0+i\delta)})) \\ &= 2\frac{1}{a}\frac{1}{2\pi}\arcsin\left(\frac{\mathrm{Im}\,w}{|w|}\right) \quad (\text{where } w = -\log(1-e^{2\pi i(t_0+i\delta)})) \\ &= 2\frac{1}{a}\frac{1}{2\pi}O\left(\frac{\mathrm{Im}\,w}{|w|}\right) \quad (\text{note that } \mathrm{Im}\,w/|w| \to 0 \text{ in this case}) \\ &= O\left(\frac{1}{a}e^{-2\pi a}\right) \quad (\text{since } \mathrm{Im}\,w \text{ stays bounded, and } |w| \sim e^{2\pi a}\log(2)). \end{aligned}$$

The covering $\mathcal{A}(a, h, M)$ has a number N of squares of the order of $M \cdot (h/d) \cdot a$, so

$$\begin{aligned} \mathcal{L}^s(\mathcal{A}(a, h, M)) &= N \cdot (\sqrt{2}\,d)^s \\ &\sim M \cdot \frac{h}{d} \cdot a \cdot (\sqrt{2}\,d)^s \\ &= M \cdot h \cdot O\left(\left(\frac{1}{a}e^{-2\pi a}\right)^{s-1} \cdot a\right) \\ &\to 0 \quad \text{as } a \to +\infty. \end{aligned}$$

□

Remark. To conclude that $\mathcal{L}^s(\mathcal{A}(a, h, M)) \to 0$ as $a \to +\infty$, one only needs, in fact, $d = o(1/a)$, and not the stronger exponential decay proved above.

The idea of the proof is now simple. We choose the maps K_n inductively. Suppose $K_0, \ldots, K_{n-1}$ have been chosen; by fixing h_n to be quite large, and then choosing a_n very large, the lemma above ensures that a large part of the domain $K_n(\overline{\mathbf{H}})$ can be made very 'thin'. Since $K_0 \ldots K_{n-1}$ is a fixed holomorphic diffeomorphism not depending on the choice of a_n, we can ensure, by choosing a_n large enough, that it stays thin even after transporting by $K_0 \ldots K_{n-1}$. Thus, at each stage n, a large part of the domain $K_0 \ldots K_n(\overline{\mathbf{H}})$ can be made very thin, so making the domains thinner and thinner at each stage, we can obtain in the limit a hedgehog with Hausdorff dimension one.

Proof of main theorem. Fix three decreasing sequences $s_n \downarrow 1$, $\epsilon_n \downarrow 0$ and $d_n \downarrow 0$. We choose the parameters a_n and h_n, $n \geq 0$, inductively as follows.

For $n = 0$: let $h_0 = 10$. Applying the above lemma to $K_0 = (1/a_0)K_{\delta(a_0,h_0)}$, we choose $a_0 \geq 1$ large enough so that

$$\begin{gathered} \mathrm{diam}(U) \leq d_0 \quad \text{for all } U \in \mathcal{A}(a_0, h_0, 1) \quad \text{and} \\ \mathcal{L}^{s_0}(\mathcal{A}(a_0, h_0, 1)) \leq \epsilon_0. \end{gathered}$$

For $n \geq 1$: assume $a_0, \ldots, a_{n-1}$ and $h_0, \ldots, h_{n-1}$ have been chosen, so the functions $K_0, \ldots, K_{n-1}$ are determined and fixed.

By property (4) of the maps $(1/a)K_\delta$, we have

$$K_0 \circ \cdots \circ K_{n-1}(z) = z + i(h_0 + \cdots + h_{n-1}) + O(1/a_0 \ldots a_{n-1}) \quad \text{as } \mathrm{Im}\,z \to +\infty.$$

We choose $h_n > 0$ large enough so that $\mathrm{Im}\,K_0 \ldots K_{n-1}(x + i(h_n - 1)) \geq h_0 + n$ for all x.

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By the above choice of h_n, $K_0 \dots K_{n-1}(\overline{\mathbf{H}}) \cap \{\operatorname{Im} z \le h_0 + n\}$ is contained in $K_0 \dots K_{n-1}(\{0 \le \operatorname{Im} z \le h_n - 1\})$. Thus the covering $\mathcal{A}(a_n, h_n, a_0 \dots a_{n-1})$ of the region $K_n(\overline{\mathbf{H}}) \cap \{\operatorname{Im} z \le h_n - 1,\ 0 \le \operatorname{Re} z \le a_0 \dots a_{n-1}\}$ is transported by $K_0 \dots K_{n-1}$ to a covering

$$\mathcal{A}_n = K_0 \dots K_{n-1}(\mathcal{A}(a_n, h_n, a_0 \dots a_{n-1}))$$

of the region $K_0 \dots K_n(\overline{\mathbf{H}}) \cap \{\operatorname{Im} z \le h_0 + n,\ 0 \le \operatorname{Re} z \le 1\}$. By the lemma, for the covering $\mathcal{A}(a_n, h_n, a_0 \dots a_{n-1})$, for fixed s both its *s-length* and the maximum diameter of the sets it contains go to 0 as $a_n \to +\infty$. Since $K_0 \dots K_{n-1}$ is a fixed holomorphic diffeomorphism independent of the choice of a_n, the same is true for the covering $\mathcal{A}_n$. We thus choose a_n large enough so that

$$\operatorname{diam}(U) \le d_n \quad \text{for all } U \in \mathcal{A}_n \quad \text{and}$$
$$\mathcal{L}^{s_n}(\mathcal{A}_n) \le \epsilon_n.$$

Finally, we also choose a_n large enough, depending on $a_0, \dots, a_{n-1}, h_0, \dots, h_n$ and as given by Proposition 2.1.1, so that we obtain limit dynamics defined on a half-plane strictly containing $\overline{\mathbf{H}}$.

We can now prove that with the choice of the parameters a_n, h_n, $n \ge 0$ as above, the resulting hedgehog

$$\mathcal{H} = \bigcap_{n=0}^{\infty} K_0 \dots K_n(\overline{\mathbf{H}})$$

has Hausdorff dimension one.

It is enough to show that $H^s(\mathcal{H}) = 0$ for all $s > 1$, H^s denoting s-dimensional Hausdorff measure. So fix $s > 1$.

For $M > 0$, writing $\mathcal{H}_M = \mathcal{H} \cap \{\operatorname{Im} z \le M,\ 0 \le \operatorname{Re} z \le 1\}$, we have

$$\begin{aligned}
H^s(\mathcal{H}_M) &= \lim_{\delta \to +0} H^s_\delta(\mathcal{H}_M) \\
&= \lim_{n \to \infty} H^s_{d_n}(\mathcal{H}_M) \quad (\text{since } d_n \to 0) \\
&\le \limsup_{n \to \infty} H^{s_n}_{d_n}(\mathcal{H}_M) \quad (\text{since } s_n \le s \text{ eventually, } H^s_\delta \text{ decreasing in } s) \\
&\le \limsup_{n \to \infty} \mathcal{L}^{s_n}(\mathcal{A}_n) \quad (\text{since } \mathcal{A}_n \text{ covers } \mathcal{H}_M \text{ once } h_0 + n \ge M) \\
&\le \limsup_{n \to \infty} \epsilon_n \\
&= 0.
\end{aligned}$$

So

$$\begin{aligned}
H^s(\mathcal{H} \cap \{0 \le \operatorname{Re} z \le 1\}) &= \lim_{M \to \infty} H^s(\mathcal{H}_M) \\
&= 0.
\end{aligned}$$

Since $\mathcal{H} = \mathcal{H} + 1$, it follows that $H^s(\mathcal{H}) = 0$, and hence $H^s(K) = 0$. □

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3. *Remarks*

(1) Note that, since the hedgehog has Hausdorff dimension one, it has no interior and cannot contain a Siegel disk, so the limit dynamics $(f_t)_{t\in C}$ is nonlinearizable.

(2) The rotation numbers $t \in C$ of the limit dynamics are of the form

$$t = \sum_{n=0}^{\infty} \frac{\varepsilon_n}{a_0 \dots a_n},$$

where each ε_n is 0, 1 or -1. Since our construction above requires the a_n to be growing very fast, this means that we obtain very Liouvillean rotation numbers.

(3) It is possible, by modifying the construction appropriately, to obtain linearizable dynamics having a common Siegel disk, but such that the portion of the hedgehog outside the closure of the Siegel disk has Hausdorff dimension one. Thus linearizability is not an obstruction to the hedgehog being 'thin'.

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