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# The escaping set of the exponential

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*Abstract*. We show that the set $I(f)$ of points that converge to infinity under iteration of the exponential map $f(z) = \exp(z)$ is a connected subset of the complex plane.

## 1. *Introduction*

If $f : \mathbb{C} \to \mathbb{C}$ is an entire transcendental function, then its *escaping set* $I(f)$ is the set of points that tend to infinity under iteration:

$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty\}.$$

For the dynamically simplest entire functions, such as exponential maps of the form $f(z) = \exp(z) + a$ with $a < -1$, the escaping set is the disjoint union of uncountably many curves to infinity, each of which is a connected component of $I(f)$. (In particular, $I(f)$ is disconnected while $I(f) \cup \{\infty\}$ is connected and path-connected.) Eremenko [**E**] conjectured that every connected component of $I(f)$ is unbounded for every transcendental entire function $f$. Despite recent progress (compare, for example, [**R1, RS1, R$^3$S**]), this question is still very much open.

In view of this, the escaping set is usually regarded very much as a set that is likely to be disconnected. However, Rippon and Stallard [**RS1**] proved that the escaping set of an entire function with a multiply-connected wandering domain is in fact connected. They have since extended this result to much larger classes of entire functions [**RS2**].

These examples are quite different from the exponential maps mentioned above in that they do not belong to the *Eremenko–Lyubich class*

$$\mathcal{B} := \{f : \mathbb{C} \to \mathbb{C} \text{ transcendental, entire} : \operatorname{sing}(f^{-1}) \text{ is bounded}\},$$

where $\operatorname{sing}(f^{-1})$ denotes the set of critical and asymptotic values of $f$. We note that, by [**EL**, Theorem 1], the escaping set $I(f)$ is a subset of the Julia set $J(f)$ when $f \in \mathcal{B}$. Bergweiler (personal communication) asked whether the escaping set of a function in $\mathcal{B}$ can be connected, and more precisely whether this might be the case for some function of the form $f(z) = a\cos(z) + b\sin(z)$ whose critical points are strictly preperiodic. While Mihaljević-Brandt [**M-B**] has given a negative answer to the latter, Rippon and Stallard observed that, for the function $f(z) = (\cosh(z))^2$, the escaping set is connected.

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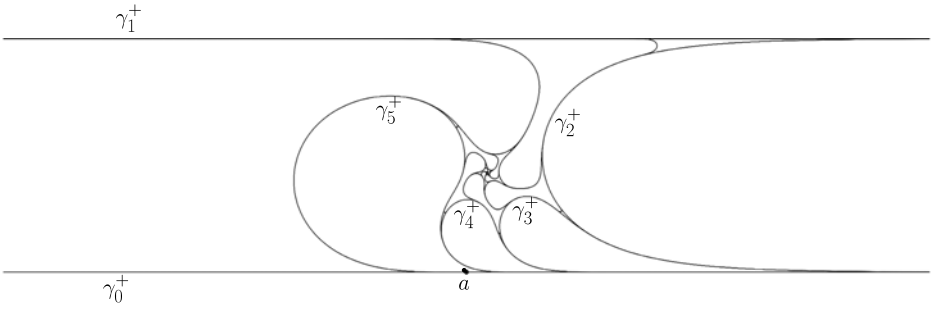



FIGURE 1. The set $X^+$ (for the case $a = 0$).

Indeed, the union of the real axis with all its iterated preimages is path-connected (and clearly dense in $I(f)$).

In contrast to this example, for the exponential map $f(z) = \exp(z)$ every path-connected component of the escaping set is known to be a single curve to $\infty$ that is relatively closed and nowhere dense in $I(f)$ (see Proposition 3.2). It may seem plausible that these path-connected components are also the connected components of $I(f)$, but we show that the situation is rather different.

THEOREM 1.1. (Escaping set of the exponential) *Let $a \in (-1, \infty)$ and consider the function $f(z) = \exp(z) + a$. Then $I(f)$ is a connected subset of the plane.*

The proof is elementary; the main idea is to consider a countable sequence of preimage components of the negative real axis that was studied by Devaney [**D**] in his construction of an indecomposable continuum (see Figure 1). Each of these components is an arc tending to infinity in both directions, but we shall show that their union is connected. Theorem 1.1 then follows relatively easily.

*Basic notation.* As usual, we denote the complex plane by $\mathbb{C}$, and the Riemann sphere by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Given $A \subset \mathbb{C}$, we denote by $\overline{A}$ the closure of $A$ in $\mathbb{C}$.

## 2. *The Devaney continuum*

For the rest of this paper, fix $a \in (-1, \infty)$ and set $f(z) = \exp(z) + a$. Then $f^n(x) \to \infty$ for all $x \in \mathbb{R}$.

Let $\mathbb{H}_+$ and $\mathbb{H}_-$ denote the upper and lower half-planes, respectively. We also define the strips

$$S_+ := \{z \in \mathbb{C} : \mathrm{Im}(z) \in (0, \pi)\} \quad \text{and} \quad S_- := \{z \in \mathbb{C} : \mathrm{Im}(z) \in (-\pi, 0)\}.$$

For $\sigma \in \{+, -\}$, let

$$L_\sigma : \mathbb{H}_\sigma \to S_\sigma$$

be the branch of $f^{-1}$ taking values in $S_\sigma$. $L_\sigma$ is a conformal isomorphism that extends to a homeomorphism between $\overline{\mathbb{H}_\sigma} \setminus \{a\}$ and $\overline{S_\sigma}$; we denote this extension also by $L_\sigma$.

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Define $\gamma_0^\sigma := (-\infty, a)$, and inductively

$$\gamma_{k+1}^\sigma := L_\sigma(\gamma_k^\sigma).$$

Then each $\gamma_k^\sigma$, $k \geq 1$, is an injective curve tending to infinity in both directions. (Also, $\gamma_k^-$ is the reflection of $\gamma_k^+$ in the real axis for all $k$.)

We define sets $\Gamma^\sigma$ and $X^\sigma$ by

$$\Gamma^\sigma := \bigcup_{k\geq 0} \gamma_k^\sigma, \quad X^\sigma := \overline{\Gamma^\sigma}.$$

Figure 1 shows a picture of the set $X^+$ (for the standard exponential map; i.e. the case $a = 0$). We require the following key fact [**D**, p. 631].

PROPOSITION 2.1. (Hausdorff limit of $\gamma_k^\pm$) *Let $\sigma \in \{+, -\}$. The set $X^\sigma \cup \{\infty\}$ is the Hausdorff limit, with respect to the spherical metric of $\hat{\mathbb{C}}$, of the sequence $(\gamma_k^\sigma \cup \{\infty\})$. In particular, $\bigcup_{k\geq k_0} \gamma_k^\sigma$ is dense in $X^\sigma$ for all $k_0 \geq 0$.*

*Proof.* Let $z_0 \in X^\sigma \cup \infty$, and let $U$ be a neighborhood of $z_0$ in $\hat{\mathbb{C}}$. We need to show that $\gamma_k^\sigma \cap U \neq \emptyset$ for all sufficiently large $k$.

By definition of $X^\sigma$, the set $U$ contains some $z_1 \in \Gamma^\sigma$. Let $D_n$ denote the (Euclidean) disk of radius $2\pi$ around $f^n(z_1)$. It is elementary to see—using the fact that $f$ is expanding in a suitable right half-plane—that there is $n_0$ with $f^j(a) \notin D_n$ for all $n \geq n_0$ and $j < n$. We may assume that $n_0$ is chosen sufficiently large to ensure that also $f^n(z_1) \in \mathbb{R}$ for $n \geq n_0$.

Hence, for $n \geq n_0$, there is a branch $\varphi_n : D_n \to \mathbb{C}$ of $f^{-n}$ with $\varphi_n(f^n(z_1)) = z_1$. Clearly $\max_{z\in D_n} |\varphi_n'(z)| \to 0$ as $n \to \infty$ (again due to the expansion of $f$ in a right half-plane). In particular, there is $k_0 \geq n_0 + 1$ such that $\varphi_{k-1}(D_{k-1}) \subset U$ for $k \geq k_0$. We then have $\varphi_{k-1}\left(f^{k-1}(z_1) + \pi i\right) \in \gamma_k^\sigma \cap U$, and the claim follows. □

PROPOSITION 2.2. ($\Gamma^\pm$ connected) *The sets $\Gamma^+$ and $\Gamma^-$ are connected.*

*Proof.* Let $\sigma \in \{+, -\}$. Suppose that $U \subset \mathbb{C}$ is an open set with $\Gamma^\sigma \cap U \neq \emptyset$ and $\Gamma^\sigma \cap \partial U = \emptyset$. We need to show that $\Gamma^\sigma \subset U$.

Let $z_0 \in U \cap \Gamma^\sigma$. Then by Proposition 2.1, there is $k_0$ such that $\gamma_k^\sigma \cap U \neq \emptyset$ for all $k \geq k_0$. Since $\gamma_k^\sigma$ is connected, in fact $\gamma_k^\sigma \subset U$. Thus

$$\Gamma^\sigma \subset X^\sigma = \overline{\bigcup_{k\geq k_0} \gamma_k^\sigma} \subset \overline{U}.$$

By choice of $U$, we hence have $\Gamma^\sigma \subset U$, as desired. □

## 3. *Proof of the theorem*

By Proposition 2.2, the union of $\Gamma^+$ and $\Gamma^-$ is connected. Also note that $\gamma_1^+ \subset \Gamma^+$ is the $2\pi i$-translate of $\gamma_1^- \subset \Gamma^-$. Hence it follows that the set

$$Y := \bigcup_{\sigma\in\{+,-\},k\in\mathbb{Z}} (\Gamma^\sigma + 2\pi i k)$$

is a connected subset of $I(f)$. In particular, $Y$ contains all points whose imaginary parts are odd multiples of $\pi$; i.e. $f^{-1}((-\infty, a)) \subset Y$.

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PROPOSITION 3.1. (Preimages of $Y$) *Inductively define a sequence $Y_j \subset I(f)$ by $Y_0 := Y$ and $Y_{j+1} := f^{-1}(Y_j) \cup Y_j$. Then $Y_j$ is connected for all $j \geq 0$.*

*Proof.* The proof is by induction on $j$. Note that $Y_j$ contains $Y$ for all $j$.

Let $k \in \mathbb{Z}$, and let $L_k : \mathbb{C} \setminus [a, \infty) \to \mathbb{C}$ be the branch of $f^{-1}$ that takes values with imaginary parts between $2\pi k$ and $2\pi(k+1)$. Set $z_k := (2k+1)\pi i$. Then both $z_k$ and $f(z_k)$ are elements of $Y \subset Y_j$. Hence we see that $z_k \in Y_j \cap L_k(Y_j)$. As $L_k$ is a continuous function (and $Y_j$ is contained in its domain of definition), it follows from the induction hypothesis that $L_k(Y_j) \cup Y_j$ is connected. Hence

$$Y_{j+1} = \bigcup_{k \in \mathbb{Z}} L_k(Y_j) \cup Y_j$$

is connected, as claimed. □

*Proof of Theorem 1.1.* The set $\bigcup_{j \geq 0} f^{-n}(-1) \subset \bigcup_{j \geq 0} Y_j$ is dense in the Julia set, and hence in the escaping set. Since $\bigcup_{j \geq 0} Y_j$ is a connected subset of $I(f)$, the claim follows. □

We contrast our result with the following fact, mentioned in the introduction.

PROPOSITION 3.2. (Path-connected components of $I(f)$) *Let $P$ be a path-connected component of $I(f)$, where again $f(z) = \exp(z) + a$, $a \in (-1, \infty)$. Then $P$ is relatively closed and nowhere dense in $I(f)$.*

*Proof.* The path-connected components of $I(f)$ (in fact, of the escaping set of any exponential map) are completely described in [**FRS**, Corollary 4.3]. First of all, for any $n \geq 0$, each connected component of $f^{-n}(\mathbb{R})$ is a path-connected component of $I(f)$.

Every one of these is nowhere dense and closed in $\mathbb{C}$, and in particular relatively closed in $I(f)$.

Now suppose that $z_0 \in I(f)$ never maps to the positive real axis under iteration. Then there exists a unique sequence $\underline{s} = s_0 s_1 s_2 \ldots$ of integers that satisfy $\operatorname{Im} f^n(z_0) \in \big((2s_n - 1)\pi, (2s_n + 1)\pi\big)$ for all $n$. It follows from the assumption on $z_0$ that $\underline{s}$ must contain infinitely many non-zero entries (see [**D**, Theorem on p. 632]).

Let $K = K_{\underline{s}}$ be the set of all points $z \in J(f)$ with

$$\operatorname{Im} f^n(z) \in [(2s_n - 1)\pi, (2s_n + 1)\pi]$$

for all $n \geq 0$. Clearly $K$ is closed and nowhere dense. It is known [**DK, SZ**] that $K \cap I(f)$ is path-connected; in fact, $K \cap I(f)$ is the trace of an injective curve $g_{\underline{s}} : [0, \infty) \to \mathbb{C}$ or $g_{\underline{s}} : (0, \infty) \to \mathbb{C}$ with $g_{\underline{s}}(t) \to \infty$ as $t \to \infty$. (This curve is called a *Devaney hair* or a *dynamic ray*.) We remark that, for certain addresses $\underline{s}$, the limit set of $g_{\underline{s}}(t)$ as $t \to 0$ will not consist of a single point [**DJ**] (compare also [**R2**]). Any escaping points in this limit set must necessarily lie on $g_{\underline{s}}$ themselves.

By [**FRS**, Corollary 4.3], the curve $P := K \cap I(f) = g_{\underline{s}}$ is the path-connected component of $I(f)$ containing $z_0$. Since $K$ is closed and nowhere dense, the claim follows. □

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