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MODELLING THE ELECTRON WITH COSSERAT ELASTICITY

JAMES BURNETT AND DMITRI VASSILIEV

Abstract. We suggest an alternative mathematical model for the electron in dimension $1+2$. We think of our $(1+2)$-dimensional spacetime as an elastic continuum whose material points can experience no displacements, only rotations. This framework is a special case of the Cosserat theory of elasticity. Rotations of material points are described mathematically by attaching to each geometric point an orthonormal basis which gives a field of orthonormal bases called the coframe. As the dynamical variables (unknowns) of our theory we choose a coframe and a density. We then add an extra (third) spatial dimension, extend our coframe and density into dimension $1+3$, choose a conformally invariant Lagrangian proportional to axial torsion squared, roll up the extra dimension into a circle so as to incorporate mass and return to our original $(1+2)$-dimensional spacetime by separating out the extra coordinate. The main result of our paper is the theorem stating that our model is equivalent to the Dirac equation in dimension $1+2$. In the process of analysing our model we also establish an abstract result, identifying a class of nonlinear second order partial differential equations which reduce to pairs of linear first order equations.

§1. *Introduction.* In this paper we consider an electron living in $(1+2)$-dimensional Minkowski spacetime $\mathbb{M}^{1+2}$ with coordinates x^α, $\alpha=0,1,2$, and metric $g_{\alpha\beta}=\operatorname{diag}(-1,+1,+1)$. The reduction of total dimension from 4 $(=1+3)$ to 3 $(=1+2)$ makes dealing with spin easier. At a technical level this reduction of dimension manifests itself in the fact that we do not need a four-component complex bispinor for describing the electron, just a two-component complex spinor.

The Dirac equation in $\mathbb{M}^{1+2}$ is

$$[\sigma^\alpha{}_{\dot a b}(\mathrm{i}\partial+A)_\alpha \pm m\sigma^3{}_{\dot a b}]\eta^b=0. \tag{1}$$

Here m is the electron mass, σ^α are Pauli matrices (see (A2), (A3)), $\partial_\alpha=\partial/\partial x^\alpha$ and A_α is a given real electromagnetic covector potential. The tensor summation index α runs through the values 0, 1, 2, the spinor summation index b runs through the values 1, 2 and the free spinor index $\dot a$ runs through the values $\dot1$, $\dot2$. The spinor field $\eta:\mathbb{M}^{1+2}\to\mathbb{C}^2$ is the dynamical variable (unknown quantity). The two choices of sign give two versions of the Dirac equation corresponding to spin up and down.

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Equations (1) are, of course, a special case of the Dirac equation in dimension $1+3$. The latter is a system of four complex equations for four complex unknowns and if one assumes that $A_3 = 0$ and looks for solutions which do not depend on x^3 then this system splits into a pair of systems (1). The reduction from dimension $1+3$ to dimension $1+2$ is explained in Appendix B.

Throughout this paper all fields are assumed to be infinitely smooth with no assumptions on their behaviour at infinity. We focus on understanding the geometric meaning of equation (1) rather than on fitting it into the framework of operator theory.

We suggest a new geometric interpretation of equation (1). The basic idea is to view our $(1+2)$-dimensional spacetime as an elastic continuum whose material points can experience no displacements, only rotations, with rotations of different material points being independent. The idea of rotating material points may seem exotic; however, it has long been accepted in continuum mechanics within the Cosserat theory of elasticity [**8**]. This idea also lies at the heart of the theory of *teleparallelism* (also known as absolute parallelism and as fernparallelismus), a subject promoted by Einstein and Cartan [**6, 14, 15**]. With regards to the latter it is interesting that Cartan acknowledged [**5**] that he drew inspiration from the 'beautiful' work of the Cosserat brothers.

An elastic continuum with no displacements, only rotations, is, of course, a limit case of Cosserat elasticity. The other limit case is classical elasticity with displacements only and no (micro)rotations.

Rotations of material points of the $(1+2)$-dimensional elastic continuum are described mathematically by attaching to each geometric point of Minkowski spacetime $\mathbb{M}^{1+2}$ an orthonormal basis, which gives a field of orthonormal bases called the *frame* or *coframe*, depending on whether one prefers dealing with vectors or covectors. Our model will be built on the basis of exterior calculus so for us it will be more natural to use the coframe.

Our model is described in §2. Subsequent sections contain mathematical analysis culminating in Theorem 1 (see §5) which establishes that our model is equivalent to the Dirac equation (1).

The mathematical model presented in §2 is quite simple. However, seeing that this model generates the Dirac equation (1) is not easy. The main difficulties are as follows.

- The dynamical variables in our model and the Dirac model are different. We will overcome this difficulty by performing a nonlinear change of dynamical variables given by the explicit formulae (20)–(22).
- We incorporate mass and electromagnetic field into our model by means of a Kaluza–Klein extension, i.e. by adding an extra spatial dimension and then separating out the extra coordinate x^3. Now, our field equation (Euler–Lagrange equation) will turn out to be nonlinear so the fact that it admits separation of variables is nontrivial. We will establish separation of variables by performing explicit calculations. We suspect that the underlying group-theoretic reason for our nonlinear field equation admitting separation of variables is the U(1)-invariance of our model,

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i.e. invariance under multiplication of the spinor field by a complex constant of modulus 1. Hence, it is feasible that one could perform the separation of variables arguments without writing down the explicit form of the field equation.

- Our field equation will be second order so it is unclear how it can be reduced to a first-order equation (1). This issue is addressed in Appendix C. Namely, in this appendix we prove an abstract lemma showing that a certain class of nonlinear second-order partial differential equations reduces to pairs of linear first order equations. To our knowledge, this abstract lemma is a new result.

Our paper is a development of the publication [7] where a similar model was suggested for a massless fermion (neutrino).

§2. *Our model.* The coframe ϑ is a triple of orthonormal covector fields ϑ^j, $j = 0, 1, 2$, in $\mathbb{M}^{1+2}$. Each covector field ϑ^j can be written more explicitly as $\vartheta^j{}_\alpha$ where the tensor index $\alpha = 0, 1, 2$ enumerates the components. Of course, orthonormality is understood in the Lorentzian sense: the inner product $\vartheta^j \cdot \vartheta^k = g^{\alpha\beta}\vartheta^j{}_\alpha\vartheta^k{}_\beta$ is -1 if $j = k = 0$, $+1$ if $j = k = 1$ or $j = k = 2$, and zero otherwise.

The orthonormality condition for the coframe can be represented as a single tensor identity

$$g = o_{jk}\vartheta^j \otimes \vartheta^k \tag{2}$$

where

$$o_{jk} = o^{jk} := \mathrm{diag}(-1, +1, +1). \tag{3}$$

For the sake of clarity we repeat formula (2) giving tensor indices explicitly and performing summation over frame indices explicitly: $g_{\alpha\beta} = -\vartheta^0{}_\alpha\vartheta^0{}_\beta + \vartheta^1{}_\alpha\vartheta^1{}_\beta + \vartheta^2{}_\alpha\vartheta^2{}_\beta$ where α and β run through the values 0, 1, 2. We view the identity (2) as a kinematic constraint: the covector fields ϑ^j are chosen so that they satisfy (2), which leaves us with three real degrees of freedom at every point of $\mathbb{M}^{1+2}$. If one views $\vartheta^j{}_\alpha$ as a 3×3 real matrix-function, then condition (2) means that this matrix-function is pseudo-orthogonal, i.e. orthogonal with respect to the Lorentzian inner product.

We choose to work with coframes satisfying conditions

$$\det \vartheta^j{}_\alpha = +1 > 0, \qquad \vartheta^0{}_0 > 0 \tag{4}$$

which single out coframes that can be obtained from the trivial (aligned with coordinate lines) coframe $\vartheta^j{}_\alpha = \delta^j{}_\alpha$ by proper Lorentz transformations.

As dynamical variables in our model we choose the coframe ϑ and a positive density ρ. Our coframe and density are functions of coordinates x^α, $\alpha = 0, 1, 2$, in $\mathbb{M}^{1+2}$. At a physical level, making the density ρ a dynamical variable means that we view our continuum more like a fluid rather than a solid: we allow the material to redistribute itself so that it finds its equilibrium distribution. Note that the total number of real dynamical degrees of freedom contained in the coframe ϑ and positive density ρ is four, exactly as in a two-component complex-valued spinor field η.

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In order to incorporate into our model mass and electromagnetic field we perform a Kaluza–Klein extension: we extend our original $(1+2)$-dimensional Minkowski spacetime $\mathbb{M}^{1+2}$ to $(1+3)$-dimensional Minkowski spacetime $\mathbb{M}^{1+3}$ by adding the extra spatial coordinate x^3. The metric on $\mathbb{M}^{1+3}$ is $\mathbf{g}_{\boldsymbol{\alpha\beta}} = \operatorname{diag}(-1, +1, +1, +1)$. Here and further on we use **bold** type for extended quantities. Say, the use of bold type in the tensor indices of $\mathbf{g}_{\boldsymbol{\alpha\beta}}$ indicates that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ run through the values 0, 1, 2, 3.

We extend our coframe as

$$\boldsymbol{\vartheta}^{j}{}_{\boldsymbol{\alpha}} = \begin{pmatrix} \vartheta^{j}{}_{\alpha} \\ 0 \end{pmatrix}, \quad j = 0, 1, 2, \quad \boldsymbol{\vartheta}^{3}{}_{\boldsymbol{\alpha}} = \begin{pmatrix} 0_{\alpha} \\ 1 \end{pmatrix} \tag{5}$$

where the bold tensor index $\boldsymbol{\alpha}$ runs through the values 0, 1, 2, 3, whereas its nonbold counterpart α runs through the values 0, 1, 2. In particular, the 0_{α} in formula (5) stands for a column of three zeros.

Our original $(1+2)$-dimensional coframe ϑ, which was initially a function of (x^0, x^1, x^2) only, is now allowed to depend on x^3 in an arbitrary way, as long as the kinematic constraint (2) is maintained. Our only restriction on the choice of extended $(1+3)$-dimensional coframe $\boldsymbol{\vartheta}$ is the condition that the last element of the coframe is prescribed as the conormal to the original Minkowski spacetime $\mathbb{M}^{1+2}$; see the last formula (5).

We also extend our positive density ρ allowing arbitrary dependence on x^3. We retain the nonbold type for the extended ρ.

The coframe elements $\boldsymbol{\vartheta}^{\mathbf{j}}$ are different at different points $x \in \mathbb{M}^{1+3}$ and this causes deformations. As a measure of these "rotational deformations" we choose *axial torsion* which is the 3-form defined by the formula

$$\mathbf{T}^{\mathrm{ax}} := \tfrac{1}{3}\mathbf{o}_{\mathbf{jk}}\boldsymbol{\vartheta}^{\mathbf{j}} \wedge d\boldsymbol{\vartheta}^{\mathbf{k}} \tag{6}$$

where $\mathbf{o}_{\mathbf{jk}} = \mathbf{o}^{\mathbf{jk}} := \operatorname{diag}(-1, +1, +1, +1)$ (compare with formula (3)) and d denotes the exterior derivative on $\mathbb{M}^{1+3}$. Here "torsion" stands for "torsion of the teleparallel connection" with "teleparallel connection" defined by the condition that the covariant derivative of each coframe element $\boldsymbol{\vartheta}^{\mathbf{j}}$ is zero; see Appendix A of [4] for a concise exposition. Axial torsion (6) is the totally antisymmetric part of the torsion tensor

$$\mathbf{T} := \mathbf{o}_{\mathbf{jk}}\boldsymbol{\vartheta}^{\mathbf{j}} \otimes d\boldsymbol{\vartheta}^{\mathbf{k}}. \tag{7}$$

We choose the basic Lagrangian density of our mathematical model as

$$L(\boldsymbol{\vartheta}, \rho) := \|\mathbf{T}^{\mathrm{ax}}\|^2 \rho \tag{8}$$

where $\|\mathbf{T}^{\mathrm{ax}}\|^2 = \frac{1}{3!}\mathbf{T}^{\mathrm{ax}}_{\boldsymbol{\alpha\beta\gamma}}\mathbf{T}^{\mathrm{ax}}_{\boldsymbol{\kappa\lambda\mu}}\mathbf{g}^{\boldsymbol{\alpha\kappa}}\mathbf{g}^{\boldsymbol{\beta\lambda}}\mathbf{g}^{\boldsymbol{\gamma\mu}}$. The main motivation behind the choice of the Lagrangian density (8) is the observation that it is conformally invariant: it does not change if we rescale the coframe as $\boldsymbol{\vartheta}^{\mathbf{j}} \mapsto \mathrm{e}^{\mathbf{h}}\boldsymbol{\vartheta}^{\mathbf{j}}$, metric as $\mathbf{g}_{\boldsymbol{\alpha\beta}} \mapsto \mathrm{e}^{2\mathbf{h}}\mathbf{g}_{\boldsymbol{\alpha\beta}}$ and density as $\rho \mapsto \mathrm{e}^{2\mathbf{h}}\rho$ where $\mathbf{h} : \mathbb{M}^{1+3} \to \mathbb{R}$ is an arbitrary scalar function. At this point it is important to note that our Kaluza–Klein extension procedure does not actually allow for conformal rescalings because the last formula (5) is very specific. Thus, our logic is that we choose the

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Lagrangian density (8) which *would be* conformally invariant if not for the prescriptive nature of the Kaluza–Klein construction. This is in line with the view that mass breaks conformal invariance. The electron mass m will appear below in formulae (13) and (14).

The fact that axial torsion is conformally covariant was previously pointed out by Obukhov [**13**] and Nester [**12**].

Substituting (5) into (6) we get

$$\mathbf{T}^{\mathrm{ax}} = T^{\mathrm{ax}} - \boldsymbol{\vartheta}^3 \wedge D_3\vartheta \tag{9}$$

where

$$T^{\mathrm{ax}} := \tfrac{1}{3} o_{jk} \vartheta^j \wedge d\vartheta^k \tag{10}$$

is the axial torsion in original $(1+2)$-dimensional spacetime (with d now denoting the exterior derivative on $\mathbb{M}^{1+2}$) and $D_3\vartheta$ is the 2-form

$$D_3\vartheta := \tfrac{1}{3} o_{jk} \vartheta^j \wedge \partial_3 \vartheta^k. \tag{11}$$

The 2-form $D_3\vartheta$ characterizes the rotation of the coframe ϑ as we move along the coordinate x^3 and is, in effect, an analogue of angular velocity.

Substituting (9) into (8) we rewrite our basic Lagrangian density as

$$L(\vartheta, \rho) := (\|T^{\mathrm{ax}}\|^2 + \|D_3\vartheta\|^2)\rho. \tag{12}$$

We now incorporate the electron mass m into our model by imposing the periodicity conditions

$$\vartheta(x^0, x^1, x^2, x^3 + \pi/m) = \vartheta(x^0, x^1, x^2, x^3), \tag{13}$$

$$\rho(x^0, x^1, x^2, x^3 + \pi/m) = \rho(x^0, x^1, x^2, x^3). \tag{14}$$

Conditions (13) and (14) mean that we make the coordinate x^3 cyclic with period π/m. In other words, we effectively roll up our third spatial dimension into a circle of radius $1/2m$.

Finally, we incorporate the prescribed electromagnetic covector potential A into our model by formally mixing up the partial derivatives appearing in the definition of axial torsion (10) as

$$\partial_\alpha \mapsto \partial_\alpha + m^{-1} A_\alpha \partial_3, \quad \alpha = 0, 1, 2. \tag{15}$$

As a result, our Lagrangian density (12) turns into

$$L(\vartheta, \rho) := (\|T_A^{\mathrm{ax}}\|^2 + \|D_3\vartheta\|^2)\rho, \tag{16}$$

where

$$T_A^{\mathrm{ax}} := T^{\mathrm{ax}} - m^{-1} A \wedge D_3\vartheta. \tag{17}$$

Let us summarize the above construction. The Lagrangian density that we shall be studying is given by formula (16) where the 3-form T_A^{ax} and 2-form $D_3\vartheta$ are defined by formulae (10), (11) and (17). The corresponding action

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(variational functional) is

$$S(\vartheta,\,\rho) := \int_{\mathbb{M}^{1+3}} L(\vartheta,\,\rho)\,dx^0\,dx^1\,dx^2\,dx^3; \tag{18}$$

of course, the integral in (18) need not converge as we will be using it only for the purpose of deriving field equations (Euler–Lagrange equations). Our dynamical variables are the coframe ϑ and density ρ which live in the original $(1+2)$-dimensional spacetime but depend on the extra spatial coordinate x^3. We seek solutions which are periodic in x^3; see formulae (13) and (14).

Our field equations are obtained by varying the action (18) with respect to the coframe ϑ and density ρ. Varying with respect to the density ρ is easy: this gives the field equation $\|T_A^{\mathrm{ax}}\|^2 + \|D_3\vartheta\|^2 = 0$ which is equivalent to $L(\vartheta,\,\rho) = 0$. Varying with respect to the coframe ϑ is more difficult because we have to maintain the kinematic constraint (2). A technique for varying the coframe with kinematic constraint (2) was described in [**4**, Appendix B] but we do not use it in the current paper.

§3. *Switching to the language of spinors.* As pointed out in the previous section, varying the coframe subject to the kinematic constraint (2) is not an easy task. This technical difficulty can be overcome by switching to a different dynamical variable. Namely, it is known that in dimension $1+2$ a coframe ϑ and a positive density ρ are equivalent to a two-component complex-valued spinor field $\xi = \xi^a = \binom{\xi^1}{\xi^2}$ satisfying the inequality

$$\bar{\xi}^{\dot{a}}\sigma_{3\dot{a}b}\xi^b > 0. \tag{19}$$

The explicit formulae establishing this equivalence are

$$\rho = \bar{\xi}^{\dot{a}}\sigma_{3\dot{a}b}\xi^b, \tag{20}$$

$$\vartheta^0{}_\alpha = \rho^{-1}\bar{\xi}^{\dot{a}}\sigma_{\alpha\dot{a}b}\xi^b, \tag{21}$$

$$(\vartheta^1 + \mathrm{i}\vartheta^2)_\alpha = \rho^{-1}\epsilon^{\dot{c}\dot{b}}\sigma_{3\dot{b}a}\xi^a\sigma_{\alpha\dot{c}d}\xi^d. \tag{22}$$

Here σ are Pauli matrices and ϵ is the "metric spinor" (see (A1)–(A3)), the free tensor index α runs through the values 0, 1, 2, and the spinor summation indices run through the values 1, 2 or $\dot{1}$, $\dot{2}$. The advantage of switching to a spinor field ξ is that there are no kinematic constraints on its components, so the derivation of field equations becomes straightforward.

Formulae (20)–(22) are a variant of those from [**7**]: in [**7**] these formulae were written for dimension three, i.e. for three-dimensional Euclidean space, whereas in the current paper we write them for dimension $1+2$, i.e. for $(1+2)$-dimensional Minkowski spacetime. Both the formulae from [**7**] and formulae (20)–(22) are a special case of those from [**10**].

Remark 1. The right-hand sides of formulae (20)–(22) are invariant under the change of sign of ξ. Hence, the correspondence between coframe and positive density on the one hand and spinor field satisfying condition (19) on the other is one to two. A spinor field is, effectively, a square root of a coframe and a density.

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The fact that the spinor field has indeterminate sign does not cause problems as long as we work on a simply connected open set, such as the whole Minkowski space $\mathbb{M}^{1+2}$; here and further on the notions of openness and connectedness of subsets of $\mathbb{M}^{1+2}$ are understood in the Euclidean sense, i.e. in terms of a positive three-dimensional metric. Note that a similar issue (extraction of a single-valued "square root" of a tensor) arises in the mathematical theory of liquid crystals [**1**].

We now need to express the differential forms (10), (11) and (17) via the spinor field ξ. This is done by direct substitution of formulae (20)–(22) giving

$$*T^{\mathrm{ax}} = -\frac{2\mathrm{i}(\bar{\xi}^{\dot{a}}\sigma^{\alpha}{}_{\dot{a}b}\partial_{\alpha}\xi^{b} - \xi^{b}\sigma^{\alpha}{}_{\dot{a}b}\partial_{\alpha}\bar{\xi}^{\dot{a}})}{3\bar{\xi}^{\dot{c}}\sigma_{3\dot{c}d}\xi^{d}}, \tag{23}$$

$$(*D_3\vartheta)_{\alpha} = \frac{2\mathrm{i}(\bar{\xi}^{\dot{a}}\sigma_{\alpha\dot{a}b}\partial_3\xi^{b} - \xi^{b}\sigma_{\alpha\dot{a}b}\partial_3\bar{\xi}^{\dot{a}})}{3\bar{\xi}^{\dot{c}}\sigma_{3\dot{c}d}\xi^{d}}, \tag{24}$$

$$*T^{\mathrm{ax}}_{A} = -\frac{2\mathrm{i}(\bar{\xi}^{\dot{a}}\sigma^{\alpha}{}_{\dot{a}b}(\partial_{\alpha} + m^{-1}A_{\alpha}\partial_3)\xi^{b} - \xi^{b}\sigma^{\alpha}{}_{\dot{a}b}(\partial_{\alpha} + m^{-1}A_{\alpha}\partial_3)\bar{\xi}^{\dot{a}})}{3\bar{\xi}^{\dot{c}}\sigma_{3\dot{c}d}\xi^{d}}. \tag{25}$$

The tensor summation index α in formulae (23) and (25) and the free tensor index α in formula (24) run through the values 0, 1, 2. Formulae (23) and (24) are, of course, a variant of those from [**7**]: we simply turned three-dimensional Euclidean space into $(1+2)$-dimensional Minkowski spacetime and replaced the extra coordinate x^0 with the extra coordinate x^3.

Substituting formulae (25) and (24) into (16) we arrive at the following self-contained explicit spinor representation of our Lagrangian density

$$\begin{aligned} L(\xi) = -\frac{4}{9\bar{\xi}^{\dot{c}}\sigma_{3\dot{c}d}\xi^{d}}([\mathrm{i}(\bar{\xi}^{\dot{a}}\sigma^{\alpha}{}_{\dot{a}b}(\partial_{\alpha} + m^{-1}A_{\alpha}\partial_3)\xi^{b} \\ - \xi^{b}\sigma^{\alpha}{}_{\dot{a}b}(\partial_{\alpha} + m^{-1}A_{\alpha}\partial_3)\bar{\xi}^{\dot{a}})]^2 \\ + \|\mathrm{i}(\bar{\xi}^{\dot{a}}\sigma_{\alpha\dot{a}b}\partial_3\xi^{b} - \xi^{b}\sigma_{\alpha\dot{a}b}\partial_3\bar{\xi}^{\dot{a}})\|^2). \end{aligned} \tag{26}$$

Here and further on we write our Lagrangian density and our action as $L(\xi)$ and $S(\xi)$ rather than $L(\vartheta, \rho)$ and $S(\vartheta, \rho)$, thus indicating that we have switched to spinors. The spinor field ξ satisfying condition (19) is the new dynamical variable.

The field equation for our Lagrangian density (26) is

$$\begin{aligned} \frac{4\mathrm{i}}{3}((*T^{\mathrm{ax}}_{A})\sigma^{\alpha}{}_{\dot{a}b}(\partial_{\alpha} + m^{-1}A_{\alpha}\partial_3)\xi^{b} \\ + \sigma^{\alpha}{}_{\dot{a}b}(\partial_{\alpha} + m^{-1}A_{\alpha}\partial_3)((*T^{\mathrm{ax}}_{A})\xi^{b}) - (*D_3\vartheta)_{\alpha}\sigma^{\alpha}{}_{\dot{a}b}\partial_3\xi^{b} \\ - \sigma^{\alpha}{}_{\dot{a}b}\partial_3((*D_3\vartheta)_{\alpha}\xi^{b})) - \rho^{-1}L\sigma_{3\dot{a}b}\xi^{b} = 0 \end{aligned} \tag{27}$$

where the quantities $*T^{\mathrm{ax}}_{A}$, $*D_3\vartheta$, ρ and L are expressed via the spinor field ξ in accordance with formulae (25), (24), (20) and (26).

We seek solutions of the field equation (27) which satisfy the periodicity condition

$$\xi(x^0, x^1, x^2, x^3 + \pi/m) = \xi(x^0, x^1, x^2, x^3), \tag{28}$$

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or the antiperiodicity condition

$$\xi(x^0, x^1, x^2, x^3 + \pi/m) = -\xi(x^0, x^1, x^2, x^3). \tag{29}$$

The above periodicity/antiperiodicity conditions are our original periodicity conditions (13) and (14) rewritten in terms of the spinor field. The splitting into periodicity/antiperiodicity occurs because the spinor field corresponding to a coframe and a density is determined uniquely modulo sign; see Remark 1.

§4. *Separating out the extra spatial coordinate.* Our field equation (27) is highly nonlinear and one does not expect it to admit separation of variables. Nevertheless, we seek solutions of the form

$$\xi(x^0, x^1, x^2, x^3) = \eta(x^0, x^1, x^2)\mathrm{e}^{\mp imx^3}. \tag{30}$$

Note that such solutions automatically satisfy the antiperiodicity condition (29): according to formulae (20)–(22), the coframe corresponding to a spinor field of the form (30) experiences one full turn (counterclockwise or clockwise) in the $(\vartheta^1, \vartheta^2)$-plane as x^3 runs from 0 to π/m.

Substituting formula (30) into (25), (24), (20) and (26) we get

$$*T^{\mathrm{ax}}_{A\pm} = -\frac{2(\bar\eta^{\dot a}\sigma^{\alpha}{}_{\dot a b}(\mathrm{i}\partial \pm A)_\alpha \eta^b - \eta^b \sigma^{\alpha}{}_{\dot a b}(\mathrm{i}\partial \mp A)_\alpha \bar\eta^{\dot a})}{3\bar\eta^{\dot c}\sigma_{3\dot c d}\eta^d}, \tag{31}$$

$$(*D_3\vartheta)_\alpha = \pm\frac{4m\bar\eta^{\dot a}\sigma_{\alpha\dot a b}\eta^b}{3\bar\eta^{\dot c}\sigma_{3\dot c d}\eta^d}, \tag{32}$$

$$\rho = \bar\eta^{\dot a}\sigma_{3\dot a b}\eta^b, \tag{33}$$

$$L_\pm(\eta) = -\frac{16}{9\bar\eta^{\dot c}\sigma_{3\dot c d}\eta^d}([\tfrac{1}{2}(\bar\eta^{\dot a}\sigma^{\alpha}{}_{\dot a b}(\mathrm{i}\partial \pm A)_\alpha \eta^b - \eta^b\sigma^{\alpha}{}_{\dot a b}(\mathrm{i}\partial \mp A)_\alpha \bar\eta^{\dot a})]^2 - (m\bar\eta^{\dot a}\sigma_{3\dot a b}\eta^b)^2) \tag{34}$$

where the signs agree with those in (30) (upper sign corresponds to upper sign and lower sign corresponds to lower sign).

Note that the quantities (31)–(34) do not depend on x^3, which simplifies the next step: substituting (30) into our field equation (27) and dividing through by the common factor $\mathrm{e}^{\mp imx^3}$ we get

$$\frac{4}{3}((*T^{\mathrm{ax}}_{A\pm})\sigma^{\alpha}{}_{\dot a b}(\mathrm{i}\partial \pm A)_\alpha \eta^b + \sigma^{\alpha}{}_{\dot a b}(\mathrm{i}\partial \pm A)_\alpha((*T^{\mathrm{ax}}_{A\pm})\eta^b)) + \frac{32m^2}{9}\sigma^3{}_{\dot a b}\eta^b - \rho^{-1}L_\pm\sigma_{3\dot a b}\eta^b = 0. \tag{35}$$

Observe that formulae (31)–(35) do not contain x^3. Thus, we have shown that our field equation (27) admits separation of variables, i.e. one can seek solutions of the form (30).

Consider now the action

$$S_\pm(\eta) := \int_{\mathbb{M}^{1+2}} L_\pm(\eta)\, dx^0\, dx^1\, dx^2 \tag{36}$$

where $L_\pm(\eta)$ is the Lagrangian density (34). It is easy to see that equation (35) is the field equation (Euler–Lagrange equation) for the action (36).

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In the remainder of the paper we do not use the explicit form of the field equation (35), dealing only with the Lagrangian density (34) and action (36). We needed the explicit form of field equations, (27) and (35), only to justify separation of variables.

We give for reference a more compact representation of our Lagrangian density (34) in terms of axial torsion $T^{\mathrm{ax}}_{A\pm}$ (see formula (31)) and density ρ (see formula (33)):

$$L_{\pm}(\eta) = -\left((*T^{\mathrm{ax}}_{A\pm})^2 - \frac{16}{9}m^2\right)\rho. \tag{37}$$

Of course, formula (37) is our original formula (16) with x^3 separated out. The choice of dynamical variables in the Lagrangian density (37) is up to the user: one can either use the x^3-independent spinor field η or, equivalently, the corresponding x^3-independent coframe and x^3-independent density (the latter are related to η by formulae (20)–(22) with ξ replaced by η).

§5. *Main result.* Let D_{rs} be the linear differential operator mapping undotted spinor fields into dotted spinor fields in accordance with formula

$$\eta \mapsto D_{rs}\eta := \sigma^{\alpha}{}_{\dot{a}b}(\mathrm{i}\partial_{\alpha} + rA_{\alpha})\eta^b + sm\sigma^3{}_{\dot{a}b}\eta^b \tag{38}$$

where the tensor summation index α runs through the values 0, 1, 2 and the letters r and s take, independently, symbolic values $\pm$ (as in D_{rs}) or numerical values ± 1 (as in the right-hand side of formula (38)), depending on the context.

The main result of our paper is the following theorem.

THEOREM 1. *Let Ω be an open subset of $\mathbb{M}^{1+2}$ and let $\eta : \Omega \to \mathbb{C}^2$ be a spinor field satisfying the condition*

$$\bar{\eta}^{\dot{a}}\sigma_{3\dot{a}b}\eta^b > 0 \tag{39}$$

(compare with (19)). Then η is a solution of the field equation for the Lagrangian density L_+ if and only if it is a solution of the Dirac equation $D_{++}\eta = 0$ or the Dirac equation $D_{+-}\eta = 0$, and a solution of the field equation for the Lagrangian density L_- if and only if it is a solution of the Dirac equation $D_{-+}\eta = 0$ or the Dirac equation $D_{--}\eta = 0$.

Proof. Put

$$\begin{aligned} L_{rs}(\eta) := {} & \tfrac{1}{2}[\bar{\eta}^{\dot{a}}\sigma^{\alpha}{}_{\dot{a}b}(\mathrm{i}\partial_{\alpha} + rA_{\alpha})\eta^b - \eta^b\sigma^{\alpha}{}_{\dot{a}b}(\mathrm{i}\partial_{\alpha} - rA_{\alpha})\bar{\eta}^{\dot{a}}] \\ & + sm\bar{\eta}^{\dot{a}}\sigma^3{}_{\dot{a}b}\eta^b. \end{aligned} \tag{40}$$

This is the Lagrangian density for the Dirac equation $D_{rs}\eta = 0$. Formula (40) can be rewritten in more compact form as

$$L_{rs}(\eta) = (-\tfrac{3}{4} * T^{\mathrm{ax}}_{Ar} + sm)\rho \tag{41}$$

where $*T^{\mathrm{ax}}_{Ar}$, $r = \pm$, is the Hodge dual of axial torsion defined by formula (31) and ρ is the density defined by formula (33). Comparing formulae (37) and (41)

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we get

$$L_r(\eta) = -\frac{32m}{9}\,\frac{L_{r+}(\eta)\,L_{r-}(\eta)}{L_{r+}(\eta) - L_{r-}(\eta)}. \tag{42}$$

Note that the denominator in the above formula is nonzero because condition (39) can be equivalently rewritten as $L_{r+}(\eta) > L_{r-}(\eta)$.

The result now follows from formula (42) and Lemma C.1 (see Appendix C). □

§6. *The sign in the inequality (19).* In §3, when switching to the language of spinors, we chose to work with spinor fields ξ satisfying the inequality (19). It is natural to ask the question what happens if we choose to work with spinor fields $\tilde{\xi}$ satisfying the inequality

$$\bar{\tilde{\xi}}^{\dot{a}}\sigma_{3\dot{a}b}\tilde{\xi}^{b} < 0. \tag{43}$$

One can check that in this case all our arguments can be repeated with minor changes. Namely, in dimension $1+2$ a coframe ϑ and a positive density ρ are equivalent to a two-component complex-valued spinor field $\tilde{\xi}$ satisfying the inequality (43), with this equivalence described by a slightly modified version of formulae (20)–(22). In the end we get an analogue of Theorem 1 for such spinors.

In fact, there is no need to repeat our arguments because there is a bijection between spinor fields ξ satisfying the inequality (19) and spinor fields $\tilde{\xi}$ satisfying the inequality (43):

$$\xi \mapsto \tilde{\xi}^{c} = \epsilon^{cb}\sigma_{3\dot{a}b}\bar{\xi}^{\dot{a}}, \qquad \tilde{\xi} \mapsto \xi^{c} = \epsilon^{cb}\sigma_{3\dot{a}b}\bar{\tilde{\xi}}^{\dot{a}}. \tag{44}$$

We do not view the transformation (44) as physically significant because the primary dynamical variables in our model are coframe and positive density, not the spinor field. We view the spinor field merely as a convenient change of dynamical variables. If two different spinor fields correspond to the same coframe and positive density we interpret them as the same particle. In group-theoretical language this means that our model is built on the basis of the pseudo-orthogonal group SO(1, 2) rather than the spin group Spin(1, 2).

§7. *Plane wave solutions.* In this section we construct a special class of explicit solutions of the field equations for our Lagrangian density (16). This construction is presented, initially, in the language of spinors and under the additional assumption that the electromagnetic covector potential A is zero.

We seek solutions of the form

$$\xi(x^0, x^1, x^2, x^3) = \mathrm{e}^{-\mathrm{i}(p\cdot x + rmx^3)}\zeta \tag{45}$$

where $p = (p_0, p_1, p_2)$ is a real constant covector, r takes the values ± 1 and $\zeta \neq 0$ is a constant spinor. We shall call solutions of the type (45) *plane wave*. In seeking plane wave solutions what we are doing is separating out all the variables, namely, the original variables $x = (x^0, x^1, x^2)$ (coordinates on $\mathbb{M}^{1+2}$) and the extra variable x^3 (Kaluza–Klein coordinate).

As usual, our spinor field ξ is assumed to satisfy the inequality (19). As explained in §6, this assumption does not lead to the loss of solutions.

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Our field equation (27) is highly nonlinear so it is not *a priori* clear that one can seek solutions in the form of plane waves. However, plane wave solutions are a special case of solutions of the type (30), and these have already been analysed in preceding sections. Namely, Theorem 1 gives us an algorithm for the calculation of all plane wave solutions (45) by reducing the problem to Dirac equations

$$D_{rs}\eta = 0 \tag{46}$$

for the x^3-independent spinor field

$$\eta(x^0, x^1, x^2) = \mathrm{e}^{-\mathrm{i}p\cdot x}\zeta. \tag{47}$$

Here r is the same as in formula (45), i.e. a number taking the values ± 1, and s is another number, also taking, independently, the values ± 1. By D_{rs} we denote the differential operators (38).

Clearly, a Dirac equation (46) has a nontrivial plane wave solution η if and only if the momentum p satisfies the condition $\|p\|^2 + m^2 = 0$, so p is timelike. Our model is invariant under proper Lorentz transformations of coordinates (x^0, x^1, x^2) so without loss of generality we can assume that

$$p_1 = p_2 = 0. \tag{48}$$

Combining formulae (38), (A2), (A3), (47) and (48) we see that the Dirac equation (46) takes the form

$$\begin{pmatrix} -p_0 + sm & 0 \\ 0 & -p_0 - sm \end{pmatrix} \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} = 0. \tag{49}$$

Equation (47) has a nontrivial solution satisfying the inequality (19) only if

$$p_0 = sm \tag{50}$$

with the corresponding ζ given, up to scaling by a nonzero complex factor, by the formula

$$\zeta^d = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{51}$$

Combining formulae (45), (48), (50) and (51) we conclude that our model admits, up to a proper Lorentz transformation of the coordinate system in $\mathbb{M}^{1+2}$ and complex scaling, four plane wave solutions and that these plane wave solutions are given by the explicit formula

$$\xi^d = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{-\mathrm{i}m(sx^0 + rx^3)}. \tag{52}$$

Here the numbers r and s can, independently, take values ± 1.

Let us now rewrite the plane wave solutions (52) in terms of our original dynamical variables, coframe ϑ and density ρ. Substituting formulae (A2), (A3) and (52) into formulae (20)–(22) we get $\rho = 1$, ${\vartheta^0}_\alpha = {\delta^0}_\alpha$ and

$${\vartheta^1}_\alpha = \begin{pmatrix} 0 \\ \cos 2m(sx^0 + rx^3) \\ \sin 2m(sx^0 + rx^3) \end{pmatrix}, \qquad {\vartheta^2}_\alpha = \begin{pmatrix} 0 \\ -\sin 2m(sx^0 + rx^3) \\ \cos 2m(sx^0 + rx^3) \end{pmatrix}. \tag{53}$$

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Table 1: Classification of solutions (53).

	$s=+1$	$s=-1$
$r=+1$	Electron with spin up	Positron with spin down
$r=-1$	Positron with spin up	Electron with spin down

In order to distinguish the two spins we fix x^3 and examine how the covectors ϑ^1 and ϑ^2 evolve as a function of time x^0. We say that spin is up if the rotation is counterclockwise and spin is down if the rotation is clockwise. Examination of formula (53) shows that we have spin up if $s=+1$ and spin down if $s=-1$.

We will now establish which of the solutions (53) describe the electron and which describe the positron. Let us introduce a weak constant positive electric field, $0<A_0<m$ and $A_1=A_2=0$. Then we can repeat the calculation leading up to formula (53), only now we get

$$\begin{aligned}\vartheta^1{}_\alpha &= \begin{pmatrix} 0 \\ \cos 2[(sm-rA_0)x^0+rmx^3] \\ \sin 2[(sm-rA_0)x^0+rmx^3] \end{pmatrix}, \\ \vartheta^2{}_\alpha &= \begin{pmatrix} 0 \\ -\sin 2[(sm-rA_0)x^0+rmx^3] \\ \cos 2[(sm-rA_0)x^0+rmx^3] \end{pmatrix}.\end{aligned} \tag{54}$$

We define quantum mechanical energy as

$$\varepsilon := |sm-rA_0| \tag{55}$$

which is half the angular frequency (as a function of time x^0) of the solution (54). Note that our energy (55) is by definition positive. The fact that we do not have to deal with negative energies is a characteristic feature of our model.

We say that we are dealing with an electron if $\varepsilon<m$ and with a positron if $\varepsilon>m$. Examination of formula (55) shows that we are looking at an electron if the signs of r and s are the same and at a positron if the signs of r and s are opposite. This means that the electron is described by a wave travelling in the negative x^3-direction whereas the positron is described by a wave travelling in the positive x^3-direction.

Our classification of plane wave solutions is summarized in Table 1.

§8. *Discussion.*

8.1. *Distinguishing the electron from the positron.* The mathematical model presented in this paper allows us to clearly distinguish the electron from the positron without resorting to notions such as "negative energy" or the "Dirac sea". This is achieved by using the coframe and positive density as our primary dynamical variables rather than the more traditional spinor field. As explained in the end of §6, our model is built on the basis of the pseudo-orthogonal group SO(1, 2) rather than the spin group Spin(1, 2), and this leads to a clearer

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definition of quantum mechanical energy as half the angular frequency with which the coframe rotates as a function of time.

8.2. *Problem of vanishing density.* The only technical assumption in our analysis is that the density ρ does not vanish. Rephrased in terms of the spinor field, this assumption reads as

$$\bar{\xi}^{\dot{a}}\sigma_{3\dot{a}b}\xi^{b} \neq 0; \tag{56}$$

compare with (19) and (43). We do not know how to drop the assumption (56).

8.3. *Curved spacetime.* One of the advantages of our mathematical model is that it does not use covariant differentiation, only exterior differentiation, so generalization to the case of a curved (with "curved" referring to the curvature of the Levi-Civita connection generated by the metric g, as is customary in general relativity) $(1+2)$-dimensional spacetime is absolutely straightforward. Covariant derivatives appear only when we switch from coframe and density to a spinor field. All our analysis, including Theorem 1, carries over to the case of curved spacetime. We chose our $(1+2)$-dimensional spacetime to be flat only to make the exposition clearer.

8.4. *Exclusion of gravity.* We assumed the $(1+2)$-dimensional metric g to be prescribed (fixed) and the coframe ϑ to be chosen so as to satisfy the kinematic constraint (2). As explained in §8.3, the fact that we chose the metric g to be Minkowski is irrelevant and all our analysis carries over to the case of an arbitrary Lorentzian metric in dimension $1+2$. The important thing is that the metric g is not treated as a dynamical variable. This means that we chose to exclude gravity from our model.

On the other hand, in teleparallelism it is traditional to view the metric as a dynamical variable. In other words, in teleparallelism it is customary to view the identity (2) not as a kinematic constraint but as a definition of the metric and, consequently, to vary the coframe ϑ without any constraints. This is not surprising as most, if not all, authors who contributed to teleparallelism came to the subject from general relativity.

It appears that the idea of working with a coframe subject to the kinematic constraint (2) is new.

8.5. *Our choice of Lagrangian.* We chose a very particular Lagrangian density (8) containing only one irreducible piece of torsion (axial) whereas in teleparallelism it is traditional to choose a more general Lagrangian containing all three pieces (axial, vector and tensor) of the torsion tensor (7); see formula (26) in [**11**] or formula (4.18) in [**3**]. Note that when Einstein introduced teleparallelism [**15**] he failed to identify axial torsion as a separate irreducible piece: his Lagrangian contained only two terms, namely the square of the full torsion tensor and the square of its vector piece.

In choosing our particular Lagrangian density (8) we were guided by the principles of conformal invariance, simplicity and analogy with Maxwell's theory. The analogy with Maxwell's theory is that we characterize the field strength by a differential form, replacing the electromagnetic tensor (2-form)

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by axial torsion (3-form). It appears that the Lagrangian density (8) was never examined.

8.6. *Density as a dynamical variable.* We took the positive density of our continuum to be a dynamical variable whereas in teleparallelism the tradition is to prescribe it as $\rho = \sqrt{|\det g|}$. Taking ρ to be a dynamical variable is, of course, equivalent to introducing an extra real positive scalar field into our model. It appears that the idea of making the density a dynamical variable is also new.

8.7. *Electron in dimension* $1+3$. The major outstanding issue is whether we can reformulate the Dirac equation in dimension $1+3$ using our approach. This would mean starting from $(1+3)$-dimensional spacetime, performing a Kaluza–Klein extension to dimension $1+4$, choosing the conformally invariant Lagrangian density (8), and so on, as described in §2.

It seems that the equation we get starting from $(1+3)$-dimensional spacetime and performing the construction described in §2 is not the Dirac equation in dimension $1+3$. Our analysis is heavily dependent on dimension and, when starting from $(1+3)$-dimensional spacetime, we do not appear to get a factorization of the Lagrangian density of the type (42).

However, the equation we get in dimension $1+3$, although nonlinear, seems to be very similar to the Dirac equation. The natural way of testing how close our equation is to the Dirac equation would be to calculate the energy spectrum of the electron in a given static electromagnetic field, starting with the case of the Coulomb potential (hydrogen atom).

A. *Appendix. Notation.* Our notation follows [**3**, **4**, **7**, **16**]. The only difference with [**4**, **16**] is that in [**4**, **16**] the Lorentzian metric has opposite signature. In [**3**, **7**] the signature is the same as in the current paper, i.e. the $(1+3)$-dimensional metric has signature $-+++$.

We use Greek letters for tensor (holonomic) indices and Latin letters for frame (anholonomic) indices.

We identify differential forms with covariant antisymmetric tensors. Given a pair of real covariant antisymmetric tensors P and Q of rank r we define their dot product as $P \cdot Q := \frac{1}{r!} P_{\alpha_1 \dots \alpha_r} Q_{\beta_1 \dots \beta_r} g^{\alpha_1 \beta_1} \dots g^{\alpha_r \beta_r}$. We also define $\|P\|^2 := P \cdot P$.

We define the action of the Hodge star on a rank r antisymmetric tensor R as $(*R)_{\alpha_{r+1} \dots \alpha_3} := (r!)^{-1} R^{\alpha_1 \dots \alpha_r} \varepsilon_{\alpha_1 \dots \alpha_3}$ where ε is the totally antisymmetric quantity, $\varepsilon_{012} := +1$.

We use two-component complex-valued spinors (Weyl spinors) whose indices run through the values $1, 2$ or $\dot{1}, \dot{2}$. Complex conjugation makes the undotted indices dotted and vice versa.

We define the 'metric spinor'

$$\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{A1}$$

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and choose Pauli matrices

$$\sigma_{0\dot{a}b} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\sigma^{0}{}_{\dot{a}b}, \tag{A2}$$

$$\sigma_{1\dot{a}b} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^{1}{}_{\dot{a}b}, \qquad \sigma_{2\dot{a}b} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \sigma^{2}{}_{\dot{a}b},$$
$$\sigma_{3\dot{a}b} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^{3}{}_{\dot{a}b}. \tag{A3}$$

Here the first spinor index enumerates rows and the second enumerates columns.

B. *Appendix. The Dirac equation in dimension* $1+2$. The aim of this appendix is to explain how the Dirac equation in dimension $1+2$, given by formula (1), is derived from the more customary Dirac equation in dimension $1+3$. The notation in this appendix is self-contained and clashes with that in the main text.

The Dirac equation in $\mathbb{M}^{1+3}$ is

$$\sigma^{\alpha}{}_{\dot{a}b}(\mathrm{i}\partial + A)_{\alpha}\xi^{b} + m\eta_{\dot{a}} = 0, \tag{B1}$$
$$\sigma^{\alpha\dot{a}b}(\mathrm{i}\partial + A)_{\alpha}\eta_{\dot{a}} + m\xi^{b} = 0, \tag{B2}$$

where the tensor summation index α runs through the values 0, 1, 2, 3, the spinor indices $\dot{a}$ and b (which play the roles of either free indices or repeated summation indices) run through the values $\dot{1}$, $\dot{2}$ and 1, 2 respectively, and ξ^{b}, $\eta_{\dot{a}}$ is a bispinor field which is a function of coordinates x^{α}, $\alpha = 0, 1, 2, 3$. The "contravariant" (with regards to the spinor indices) Pauli matrices appearing in (B2) are expressed via our original "covariant" Pauli matrices (A2), (A3) and the "metric spinor" (A1) as

$$\sigma_{\beta}{}^{\dot{a}c} := \epsilon^{\dot{a}\dot{e}}\sigma_{\beta\dot{e}d}\epsilon^{cd}. \tag{B3}$$

Note the order of spinor indices in (B3). The explicit formulae for these "contravariant" Pauli matrices are

$$\sigma_{0}{}^{\dot{a}b} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\sigma^{0\dot{a}b}, \tag{B4}$$

$$\sigma_{1}{}^{\dot{a}b} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \sigma^{1\dot{a}b}, \qquad \sigma_{2}{}^{\dot{a}b} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \sigma^{2\dot{a}b},$$
$$\sigma_{3}{}^{\dot{a}b} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma^{3\dot{a}b}, \tag{B5}$$

where, as usual, the first spinor index enumerates rows and the second enumerates columns.

In writing the Dirac equation (B1), (B2) one can put an arbitrary sign, plus or minus, in front of A and, independently, an arbitrary sign, plus or minus, in front of m. This gives four different versions of the Dirac equation. It is easy to see that these four versions of the Dirac equation are related by simple transformations

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such as time reversal, spatial inversion and charge conjugation and the only things that change under such transformations are the physical interpretations of which solutions describe an electron and which describe a positron, as well as which solutions describe spin up and which describe spin down. We take the view that each of the four Dirac equations is a model for the same physical reality and stick with the choice of signs as in (B1), (B2).

For the sake of clarity, we rewrite the Dirac equation (B1), (B2) in explicit matrix form, using formulae (A2), (A3), (B4), (B5). We get

$$\begin{pmatrix} -m & 0 & P_0+P_3 & P_1-\mathrm{i}P_2 \\ 0 & -m & P_1+\mathrm{i}P_2 & P_0-P_3 \\ P_0-P_3 & -P_1+\mathrm{i}P_2 & -m & 0 \\ -P_1-\mathrm{i}P_2 & P_0+P_3 & 0 & -m \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \\ \eta_{\dot{1}} \\ \eta_{\dot{2}} \end{pmatrix} = 0, \tag{B6}$$

where we denoted $P_\alpha := (\mathrm{i}\partial + A)_\alpha$. Note that our formula (B6) agrees (possibly, up to spatial inversion) with the displayed formula below formula (21.3) in [**2**].

As explained in the beginning of [**2**, §21], the Dirac equation can be written in a variety of equivalent ways. One can, for example, perform the following linear transformation of the unknown bispinor field

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \\ \eta_{\dot{1}} \\ \eta_{\dot{2}} \end{pmatrix}; \tag{B7}$$

see formula (21.17) in [**2**]. Note that the 4×4 matrix O appearing in (B7) is a special orthogonal matrix which also happens to be symmetric (i.e. it is a square root of the identity matrix). If we now express the bispinor $(\xi^1\ \xi^2\ \eta_{\dot{1}}\ \eta_{\dot{2}})^T$ appearing in the Dirac equation (B6) via the bispinor $(\phi_1\ \phi_2\ \chi_1\ \chi_2)^T$ and then multiply the equation by O from the left we get

$$\begin{pmatrix} P_0-m & 0 & -P_3 & -P_1+\mathrm{i}P_2 \\ 0 & P_0-m & -P_1-\mathrm{i}P_2 & P_3 \\ P_3 & P_1-\mathrm{i}P_2 & -P_0-m & 0 \\ P_1+\mathrm{i}P_2 & -P_3 & 0 & -P_0-m \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \chi_1 \\ \chi_2 \end{pmatrix} = 0. \tag{B8}$$

Note that our formula (B8) agrees (possibly, up to spatial inversion) with formula (21.19) in [**2**].

Formulae (B6) and (B8) are two equivalent representations of the same equation, the Dirac equation in dimension $1+3$. According to [**2**, §21] these two representations of the Dirac equation bear the names *spinor representation* and *standard representation* respectively. The representation most commonly used in literature is the standard representation (B8). However, we choose to work with the spinor representation (B6). As pointed out in the beginning of [**2**, §21], the advantage of using the spinor representation is that invariance properties of the Dirac operator are most apparent in this representation.

Further on we work with the original form (B1), (B2) of the Dirac equation in its spinor representation, i.e. we use notation involving summation over tensor

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and spinor indices rather than matrix notation. We assume that $A_3 = 0$ and seek solutions of the Dirac equation (B1), (B2) which do not depend on x^3, i.e. we assume that $P_3 = (\mathrm{i}\partial + A)_3 = 0$.

It will be convenient for us to write the Dirac equation (B1), (B2) in operator-theoretic form as

$$D\psi = 0 \tag{B9}$$

where D is the linear differential operator mapping bispinor fields into bispinor fields in accordance with formula

$$\psi = \begin{pmatrix} \xi^b \\ \eta_{\dot{a}} \end{pmatrix} \mapsto \begin{pmatrix} \sigma^{\alpha\dot{a}b}(\mathrm{i}\partial + A)_\alpha \eta_{\dot{a}} + m\xi^b \\ \sigma^\alpha{}_{\dot{a}b}(\mathrm{i}\partial + A)_\alpha \xi^b + m\eta_{\dot{a}} \end{pmatrix} = D\psi$$

where the tensor summation index α runs through the values 0, 1, 2 (it no longer takes the value 3). Let Ψ be the complex vector space of all bispinor fields $\psi = \begin{pmatrix} \xi^b \\ \eta_{\dot{a}} \end{pmatrix}$ depending on coordinates x^α, $\alpha = 0, 1, 2$. This vector space can be written as a direct sum of two subspaces $\Psi = \Psi_+ \oplus \Psi_-$ where $\Psi_\pm$ is the complex vector space of all bispinor fields $\psi = \begin{pmatrix} \xi^b \\ \eta_{\dot{a}} \end{pmatrix}$ satisfying the condition

$$\eta_{\dot{a}} = \pm\sigma_{3\dot{a}b}\xi^b. \tag{B10}$$

Using the commutator identities

$$\sigma_{\alpha\dot{a}b}\sigma_3{}^{\dot{a}d} + \sigma_{3\dot{a}b}\sigma_\alpha{}^{\dot{a}d} = 0, \qquad \alpha = 0, 1, 2, \tag{B11}$$

it is easy to check that the operator D maps Ψ_+ to Ψ_+ and Ψ_- to Ψ_-. Thus, solving the equation (B9) is equivalent to solving two separate equations: $D\psi_+ = 0$ with $\psi_+ \in \Psi_+$ and $D\psi_- = 0$ with $\psi_- \in \Psi_-$. This means that the problem of finding all solutions of the Dirac equation (B1), (B2) reduces to finding solutions satisfying conditions (B10).

Substituting (B10) into (B1), (B2) we get

$$\sigma^\alpha{}_{\dot{a}b}(\mathrm{i}\partial + A)_\alpha \xi^b \pm m\sigma_{3\dot{a}b}\xi^b = 0, \tag{B12}$$

$$\pm\sigma^{\alpha\dot{a}b}(\mathrm{i}\partial + A)_\alpha \sigma_{3\dot{a}d}\xi^d + m\xi^b = 0. \tag{B13}$$

However, in view of the commutator identities (B11), the left-hand side of equation (B13) can be rewritten as

$$\pm\sigma^{\alpha\dot{a}b}(\mathrm{i}\partial + A)_\alpha \sigma_{3\dot{a}d}\xi^d + m\xi^b = \mp\sigma_3{}^{\dot{a}b}(\sigma^\alpha{}_{\dot{a}d}(\mathrm{i}\partial + A)_\alpha \xi^d \pm m\sigma_{3\dot{a}d}\xi^d)$$

which means that equations (B12) and (B13) are equivalent. Thus, the system (B12), (B13) is equivalent to the single equation (B12). It remains to note that equation (B12) differs from equation (1) only in the notation used for the spinor field.

For the sake of clarity, we rewrite equation (B12) in explicit matrix form, using formulae (A2), (A3). We get

$$\begin{pmatrix} -P_0 + m & P_1 - \mathrm{i}P_2 \\ P_1 + \mathrm{i}P_2 & -P_0 - m \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = 0 \tag{B14}$$

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and

$$\begin{pmatrix} -P_0 - m & P_1 - \mathrm{i}P_2 \\ P_1 + \mathrm{i}P_2 & -P_0 + m \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = 0 \tag{B15}$$

for the case of upper sign and lower sign respectively.

Similar arguments can be carried out starting with the standard representation of the Dirac equation (B8). In fact, in the standard representation the arguments are much simpler. Namely, direct inspection of the Dirac equation in its standard representation (B8) shows that when $P_3 = 0$ this system of four equations for four unknowns splits into the following two subsystems of two equations for two unknowns:

$$\begin{pmatrix} P_0 - m & -P_1 + \mathrm{i}P_2 \\ P_1 + \mathrm{i}P_2 & -P_0 - m \end{pmatrix} \begin{pmatrix} \phi_1 \\ \chi_2 \end{pmatrix} = 0 \tag{B16}$$

and

$$\begin{pmatrix} -P_0 - m & P_1 - \mathrm{i}P_2 \\ -P_1 - \mathrm{i}P_2 & P_0 - m \end{pmatrix} \begin{pmatrix} \chi_1 \\ \phi_2 \end{pmatrix} = 0. \tag{B17}$$

We have

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -P_0 + m & P_1 - \mathrm{i}P_2 \\ P_1 + \mathrm{i}P_2 & -P_0 - m \end{pmatrix} = \begin{pmatrix} P_0 - m & -P_1 + \mathrm{i}P_2 \\ P_1 + \mathrm{i}P_2 & -P_0 - m \end{pmatrix},$$

so equation (B14) is equivalent to equation (B16). Similarly, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -P_0 - m & P_1 - \mathrm{i}P_2 \\ P_1 + \mathrm{i}P_2 & -P_0 + m \end{pmatrix} = \begin{pmatrix} -P_0 - m & P_1 - \mathrm{i}P_2 \\ -P_1 - \mathrm{i}P_2 & P_0 - m \end{pmatrix},$$

so equation (B15) is equivalent to equation (B17).

If we now set $\phi_+ = \phi_1$, $\phi_- = \phi_2$, $\chi_- = \chi_1$ and $\chi_+ = \chi_2$, then the pair of equations (B16) and (B17) can be rewritten in compact form as

$$\begin{pmatrix} P_0 - m & -P_1 \pm \mathrm{i}P_2 \\ P_1 \pm \mathrm{i}P_2 & -P_0 - m \end{pmatrix} \begin{pmatrix} \phi_\pm \\ \chi_\pm \end{pmatrix} = 0. \tag{B18}$$

Note that our formula (B18) agrees, up to spatial inversion, with formula (5.1) from [**9**].

Formula (B18) is the representation of the Dirac equation in dimension $1 + 2$ most commonly used in literature. However, we choose to work with the equivalent representation (B12). We went to great length in carefully deriving (B12) from (B1), (B2) and comparing with standard notation in order to convince the reader that our Dirac equation in dimension $1 + 2$ is the usual one, only written in a different representation more suitable for the aims of our paper.

C. *Appendix. Nonlinear second order equations which reduce to pairs of linear first order equations.* Let Ω be an open subset of $\mathbb{R}^n$. We work with (infinitely) smooth vector functions $\Omega \to \mathbb{C}^m$ writing these as columns of m complex scalars. In this appendix "vector" does not carry a differential geometric meaning because we are not interested in coordinate transformations. We use Cartesian coordinates $x^1, \ldots, x^n$.

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Given a pair of vector functions $u, v : \Omega \to \mathbb{C}^m$ we define their inner product in the standard Euclidean manner as $(u, v) := \int_\Omega v^* u \, dx^1 \cdots dx^n$ where the star $*$ denotes Hermitian conjugation. This integral need not converge as we will be using it only for the purpose of defining the formal adjoint of a differential operator; see next paragraph.

Let $A_\pm$ be a pair of formally self-adjoint (symmetric) first order linear partial differential operators (differential expressions) with smooth coefficients acting on smooth vector functions $\Omega \to \mathbb{C}^m$. We do not introduce any boundary conditions.

Put

$$L_\pm(u) := \mathrm{Re}(u^* A_\pm u). \tag{C1}$$

It is easy to see that $L_\pm(u)$ is the Lagrangian density for the partial differential equation $A_\pm u = 0$. Namely, if one writes down the action (variational functional) $S_\pm(u) := \int_\Omega L_\pm(u) \, dx^1 \cdots dx^n$ then the corresponding field equation (Euler–Lagrange equation) is $A_\pm u = 0$.

Let us now define a new Lagrangian density

$$L(u) := \frac{L_+(u) \, L_-(u)}{L_+(u) - L_-(u)} \tag{C2}$$

and corresponding action $S(u) := \int_\Omega L(u) \, dx^1 \ldots dx^n$. The field equation for the Lagrangian density (C2) is, of course, second order and nonlinear.

Note that the notation in this appendix is self-contained and the Lagrangian densities (C1), (C2) should not be confused with the Lagrangian densities (34), (40) introduced in the main text (the latter have an extra subscript).

The main result of this appendix is the following lemma.

LEMMA C.1. *Let $u : \Omega \to \mathbb{C}^m$ be a vector function satisfying the condition*

$$L_+(u) \neq L_-(u). \tag{C3}$$

Then u is a solution of the field equation for the Lagrangian density L if and only if it is a solution of the equation $A_+ u = 0$ or the equation $A_- u = 0$.

Proof. The explicit formula for the operator $A_\pm$ is

$$A_\pm = \mathrm{i} B^\alpha_\pm \partial_\alpha + \frac{\mathrm{i}}{2}(\partial_\alpha B^\alpha_\pm) + C_\pm \tag{C4}$$

where $B^\alpha_\pm$ and $C_\pm$ are some smooth Hermitian $m \times m$ matrix functions and the index α runs through the values $1, \ldots, n$. Substituting (C4) into (C1) we get

$$L_\pm(u) = \frac{i}{2}[u^* B^\alpha_\pm \partial_\alpha u - (\partial_\alpha u^*) B^\alpha_\pm u] + u^* C_\pm u. \tag{C5}$$

Now take an arbitrary smooth function $h : \Omega \to \mathbb{R}$. Examination of formula (C5) shows that

$$L_\pm(\mathrm{e}^h u) = \mathrm{e}^{2h} L_\pm(u). \tag{C6}$$

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We call the property (C6) *scaling covariance*. Scaling covariance is a remarkable feature of the Lagrangian density of a formally self-adjoint first order linear partial differential operator.

Formulae (C2) and (C6) imply that the Lagrangian density L also possesses the property of scalar covariance, i.e. $L(\mathrm{e}^h u) = \mathrm{e}^{2h} L(u)$ for any smooth $h : \Omega \to \mathbb{R}$. Thus, all three of our Lagrangian densities, L, L_+ and L_-, have this property.

Observe now that if the vector function u is a solution of the field equation for some Lagrangian density $\mathcal{L}$ possessing the property of scaling covariance then $\mathcal{L}(u) = 0$. Indeed, let us perform a scaling variation of our vector function

$$u \mapsto u + \delta u = u + hu = \mathrm{e}^h u + O(h^2) \tag{C7}$$

where $h : \Omega \to \mathbb{R}$ is an arbitrary "small" smooth function with compact support, $h \in C_0^\infty(\Omega; \mathbb{R})$. Then $0 = \delta \int \mathcal{L}(u) = 2 \int h\mathcal{L}(u)$ which holds for arbitrary h only if $\mathcal{L}(u) = 0$.

In the remainder of the proof, the variation $\delta u : \Omega \to \mathbb{C}^m$ of the vector function $u : \Omega \to \mathbb{C}^m$ is arbitrary and not necessarily of the scaling type (C7). The only assumption is that $\delta u \in C_0^\infty(\Omega; \mathbb{C}^m)$.

Suppose that u is a solution of the field equation for the Lagrangian density L_+. (The case when u is a solution of the field equation for the Lagrangian density L_- is handled similarly.) Then $L_+(u) = 0$ and, in view of formula (C3), $L_-(u) \neq 0$. Varying u we get

$$\begin{aligned}\delta \int L(u) &= \int \frac{L_-(u)}{L_+(u) - L_-(u)} \delta L_+(u) + \int L_+(u) \delta \frac{L_-(u)}{L_+(u) - L_-(u)} \\ &= -\int \delta L_+(u) = -\delta \int L_+(u)\end{aligned}$$

so

$$\delta \int L(u) = -\delta \int L_+(u). \tag{C8}$$

We assumed that u is a solution of the field equation for the Lagrangian density L_+ so $\delta \int L_+(u) = 0$ and formula (C8) implies that $\delta \int L(u) = 0$. As the latter is true for an arbitrary variation of u this means that u is a solution of the field equation for the Lagrangian density L.

Suppose that u is a solution of the field equation for the Lagrangian density L. Then $L(u) = 0$ and formula (C2) implies that either $L_+(u) = 0$ or $L_-(u) = 0$; note that in view of (C3) we cannot have simultaneously $L_+(u) = 0$ and $L_-(u) = 0$. Assume for definiteness that $L_+(u) = 0$. (The case when $L_-(u) = 0$ is handled similarly.) Varying u and repeating the argument from the previous paragraph we arrive at (C8). We assumed that u is a solution of the field equation for the Lagrangian density L so $\delta \int L(u) = 0$ and formula (C8) implies that $\delta \int L_+(u) = 0$. As the latter is true for an arbitrary variation of u this means that u is a solution of the field equation for the Lagrangian density L_+. □

Remark C.1. It may seem that the variational proof presented above is "insufficiently rigorous". An alternative "completely rigorous" way of

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proving Lemma C.1 is to write down the field equation for the Lagrangian density (C2), (C5) explicitly and analyse this second order nonlinear partial differential equation. The result, of course, remains the same, only the calculations become much longer.

Remark C.2. Examination of the proof of Lemma C.1 shows that the fact that the differential operators $A_\pm$ are linear and first order is not important. What is important is that their Lagrangian densities possess the scaling covariance property (C6). As the Lagrangian density (C2) possesses this property as well, our construction admits an obvious extension which gives a hierarchy of nonlinear partial differential equations which reduce to several separate equations.

Example C.1. Let us give an elementary example illustrating the use of Lemma C.1. Consider the pair of linear first order ordinary differential equations

$$\mathrm{i}u' \pm u = 0 \tag{C9}$$

where $u : \mathbb{R} \to \mathbb{C}$ is a scalar function. Let us write down the corresponding Lagrangian densities $L_\pm(u) = (\mathrm{i}/2)(\bar{u}u' - u\bar{u}') \pm |u|^2$ in accordance with formula (C1) and form a new Lagrangian density

$$-2L(u) = \left(\frac{\bar{u}u' - u\bar{u}'}{2|u|}\right)^2 + |u|^2$$

in accordance with formula (C2). The latter gives the field equation (Euler–Lagrange equation)

$$\left(\frac{\bar{u}u' - u\bar{u}'}{2|u|^2}\, u\right)' + \frac{(\bar{u}u')^2 - (u\bar{u}')^2}{4|u|^4}\, u + u = 0. \tag{C10}$$

Lemma C.1 tells us that a smooth nonvanishing function u is a solution of equation (C10) if and only if it is a solution of one of the two equations (C9). Of course, this fact can be checked directly by switching to the polar representation $u = r\mathrm{e}^{-\mathrm{i}\varphi}$ where $r : \mathbb{R} \to (0, +\infty)$ and $\varphi : \mathbb{R} \to \mathbb{R}$.

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James Burnett,
Department of Mathematics and Institute of Origins,
University College London,
Gower Street, London WC1E 6BT,
U.K.
E-mail: J.Burnett@ucl.ac.uk

Dmitri Vassiliev,
Department of Mathematics and Institute of Origins,
University College London,
Gower Street, London WC1E 6BT,
U.K.
E-mail: D.Vassiliev@ucl.ac.uk

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