STABILITY ANALYSIS ON THE ALGORITHM OF CONSTITUTIVE RELATION IN VISCOPLASTIC MATERIALS

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ABSTRACT

The numerical stability of the explicit precise algorithm, which was developed for the viscoplastic materials, was analyzed. It was found that this algorithm is not absolutely stable. A necessary but not sufficient condition for the numerical stability was deduced. It showed that the time step in numerical calculation should be less than a certain value to guarantee the stability of explicit precise algorithm. Through a series of numerical examples, the stability analysis on the explicit precise algorithm was proved to be reliable. At last, an iterative algorithm was presented for viscoplastic materials. Both of the theoretical and numerical results showed that the iterative algorithm is unconditionally stable and its convergence rate is rapid. In practice, the explicit precise algorithm and iterative algorithm can be combined to obtain reliable results with the minimum computing costs.

Keywords: Viscoplastic constitutive relationship, Numerical stability, Explicit precise algorithm, Iterative algorithm.

1. INTRODUCTION

As one of the most important topics in solid mechanics, the constitutive relationships for plastic and viscoplastic materials have received intense attention in the last century. In 1950s, the Drucker postulate was proposed [1,2], which laid the basis of plasticity theory. Two main conclusions can be deduced from the Drucker postulate: (a) the plastic flow rule and (b) the convexity of yield surface. According to the plastic flow rule, the plastic flow should develop along the external normal direction of the yield surface. The shortcoming of Drucker postulate is that it's just applicable to stable materials, the strength of which increases with the plastic strain. In 1967, Palmer *et al*. [3] pointed out that the conclusions (a) and (b) could also be deduced for unstable materials. Inspired by Palmer's idea, a modified Drucker postulate was proposed by Li *et al*. [4]. The modified Drucker postulate, which was called as generalized Drucker postulate, is applicable to both stable and unstable materials.

The algorithm of plastic constitutive relation is always a central and hot topic in the fields of computational mechanics. In most of impact dynamic softwares [5,6], so far, the radial return method is used for the calculation of constitutive relationship. It consists of two steps [7-9]. In the first step, a trial stress is calculated based on the elasticity theory. In the second step, the stress is updated in accordance with a flow rule and a consistency condition. This method was originally proposed for the perfectly plastic solid. For the viscoplastic materials, the radial return method is an approximation algorithm. As an alternative to the radial return method, a multi-stage return method [10-12] and a sub-stepping stress integration algorithm using the forward Euler methods were developed [13,14]. The sub-stepping algorithm was found to be a reliable approach to increasing the accuracy of the stress integration method, but with more computing costs. A new exact algorithm for the elastoplastic model with linear isotropic hardening is presented by Szabó and Kossa [15]. Explicit expression for the algorithmically consistent tangent tensor is also derived. An implicit stress-update algorithm is proposed by Yuki

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Yamakawa *et al*. [16], which synthesized the phenomenological capability of the subloading surface model and the numerical advantage of the radial return algorithm. This algorithm is robustness and accurate even for large incremental steps. Mohammad Rezania *et al*. [17] presented a modified Newton-Raphson stress update algorithm for the elasto-plastic model. It was found that the convergence performance of the algorithm is sensitive to the expression of the yield function. It is shown that the performance of the numerical algorithm can be improved by changing the form of the yield function. Dorival M. Pedroso *et al*. [18] proposed an explicit algorithm for elastoplastic models, which is based on the Runge-Kutta embedded method of second-order accuracy. This algorithm is slower for non-convex yield surface than that for convex yield surface.

Different from the above methods, to avoid the errors in radial return method, a new algorithm called explicit precise algorithm was proposed by Li *et al*. [19]. Based on the generalized Drucker postulate and universal thermal viscoplastic constitutive equation, Li *et al*. [19] pointed out that the plastic flow factor is determined by the stress states of material. In practical computation, the plastic flow factor at present time can be calculated from the state variables at last time with first-order accuracy. Then the plastic strain increments and elastic strain increments at present time step can be obtained using the plastic flow rule, and the stress increments can be calculated using the Hooke's law.

In the present paper, the stability of the algorithm for constitutive equation is analyzed. It was found that the explicit precise algorithm, which was proposed for the viscoplastic materials by Li *et al*. [19], is conditionally stable. The time step in the computation should be less than a certain value to meet the deduced stability condition. A series of numerical examples were carried out to verify the stability analysis. Lastly, based on the plastic flow rule and Von Mises yield criterion, an iterative algorithm for the viscoplastic constitutive relationship was proposed. Through numerical examples, the iterative algorithm is proved to be unconditionally stable and have fast convergence.

2. EXPLICIT PRECISE ALGORITHM AND ITS STABILITY

The explicit precise algorithm was deduced on the basis of universal thermal viscoplastic constitutive relationship by Li *et al*. [19]. Its derivation process was described elsewhere [19] in detail. In the present paper, for simplicity, the formulas and calculation processes of explicit precise algorithm are deduced only for the materials with viscoplastic constitutive relationship and Von Mises yield criterion. Then the stability of the explicit precise algorithm is analyzed.

2.1 Explicit Precise Algorithm

For the viscoplastic materials with Von Mises yield criterion, the constitutive relationship has a general form

$$f \equiv \bar{\sigma} - F(\dot{\bar{\varepsilon}}^p) = 0 \tag{1}$$

The effective stress and effective plastic strain are defined by

$$\bar{\sigma} = \sqrt{\frac{3}{2} s_{ij}\, s_{ij}} \tag{2}$$

$$\dot{\bar{\varepsilon}}^p = \sqrt{\frac{2}{3} \dot{\varepsilon}_{ij}^p\, \dot{\varepsilon}_{ij}^p} \tag{3}$$

where σ_{ij} is the stress tensor, $s_{ij} = \sigma_{ij} - 1/3\,\sigma_{ii}$ is deviatoric stress tensor, $\dot{\varepsilon}_{ij}^p$ is the plastic strain rate tensor. In general, the plastic volume strain is equal to zero, so the plastic deviatoric strain tensor is equal to the plastic strain tensor.

The plastic flow rule is expressed as

$$\dot{\varepsilon}_{ij}^p = \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}} \tag{4}$$

where $\dot{\lambda}$ is the plastic flow factor.

Using the chain rule and the plastic flow rule (4), we have

$$\dot{\varepsilon}_{ij}^p = \frac{3}{2} \dot{\lambda} \frac{s_{ij}}{\bar{\sigma}} \tag{5}$$

According to the definition of effective plastic strain (3) and Eq. (5), we have

$$\dot{\bar{\varepsilon}}^p = \dot{\lambda} \tag{6}$$

The effective plastic strain can be obtained by solving the constitutive relationship (1),

$$\dot{\bar{\varepsilon}}^p = F^{-1}(\bar{\sigma}) = g(\bar{\sigma}) \tag{7}$$

Combing (6) and (7), we have

$$\dot{\lambda} = g(\bar{\sigma}) \tag{8}$$

The formula (8) shows that the plastic flow factor is determined by the state of material. It relies on the present stress states of material.

Substituting $\dot{\lambda}$ from Eq. (8) into Eq. (5) gives

$$\dot{\varepsilon}_{ij}^p = \frac{3}{2} \frac{s_{ij}}{\bar{\sigma}} g(\bar{\sigma}) \tag{9}$$

According to the Hooke's law, the stress tensors at time $n + 1$ and time n meet the relationship

$$s_{ij}^{n+1} = s_{ij}^n + 2Gdt(\dot{e}_{ij}^e)^{n+1/2} = s_{ij}^n + 2Gdt\,(\dot{e}_{ij} - \dot{\varepsilon}_{ij}^p)^{n+1/2} \tag{10}$$

where G is the shear module, dt is the time step, $\dot{e}_{ij}^{n+1/2}$ and $(\dot{e}_{ij}^e)^{n+1/2}$ are respectively the total strain rate tensor and elastic strain rate tensor in the time interval from n to $n + 1$. Applying the forward difference scheme on Eq. (9), we have

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$$(\dot{\varepsilon}_{ij}^{p})^{n+1/2} = \frac{3}{2}\frac{s_{ij}^{n}}{\bar{\sigma}^{n}} g(\bar{\sigma}^{n}) \tag{11}$$

Substituting it into Eq. (10) gives

$$s_{ij}^{n+1} = s_{ij}^{n} + 2Gdt\left(\dot{e}_{ij}^{n+1/2} - \frac{3s_{ij}^{n}}{2\bar{\sigma}^{n}} g(\bar{\sigma}^{n})\right) \tag{12}$$

Now we have obtained the formula to update the deviatoric stress tensor for viscoplastic constitutive relationship. It is an explicit expression. By using the formula (12), the deviatoric stress tensor at time $n + 1$ (s_{ij}^{n+1}) can be calculated with the deviatoric stress tensor at time n (s_{ij}^{n}) and the total strain rate tensor $\dot{e}_{ij}^{n+1/2}$. Thus the most important procedure in the calculation for viscoplastic constitutive equation can be accomplished.

2.2 Stability Analysis

It is difficult to analyze the stability of Eq. (12), which is in tensor form. To accomplish the work in an analytical framework, the stability of explicit precise algorithm was analyzed under a special condition.

The tensor s_{ij}^{n} and $\dot{e}_{ij}^{n+1/2}$ are supposed to be collinear, namely

$$\dot{e}_{ij}^{n+1/2} = \dot{e} l_{ij} \tag{13}$$

$$s_{ij}^{n} = s^{n} l_{ij} \tag{14}$$

After contraction of tensor, the Eq. (14) can be simplified through setting $l_{ij}\, l_{ij} = \frac{2}{3}$.

$$\bar{\sigma}^{n} = s^{n} \tag{15}$$

Substitution of Eqs. (13), (14) and (15) into Eq. (12) gives

$$s_{ij}^{n+1} = s^{n} l_{ij} + 2Gdt\left(\dot{e} l_{ij} - \frac{3s^{n} l_{ij}}{2s^{n}} g(s^{n})\right) \tag{16}$$

This formula indicates that the deviatoric stress tensor at time $n + 1$ is also collinear with the tensor $\dot{e}_{ij}^{n+1/2}$,

$$s_{ij}^{n+1} = s^{n+1} l_{ij} \tag{17}$$

where

$$s^{n+1} = 2Gdt\left(\dot{e} - \frac{3}{2} g(s^{n})\right) + s^{n} \tag{18}$$

Until now, the formula (12), which is in the tensor form, is simplified to be an algebraic Eq. (18) under the special conditions (13) and (14).

To analyze the stability of explicit precise algorithm, small perturbations are introduced into the stresses s^{n} and s^{n+1}. In Eq. (18), s^{n} is replaced by $s^{n} + \Delta^{n}$ and s^{n+1} is replaced by $s^{n+1} + \Delta^{n+1}$.

$$s^{n+1} + \Delta^{n+1} = 2Gdt\left(\dot{e} - \frac{3}{2} g(s^{n} + \Delta^{n})\right) + s^{n} + \Delta^{n} \tag{19}$$

The function $g(s^{n} + \Delta^{n})$ can be approximated by using the Taylor series expansion,

$$g(s^{n} + \Delta^{n}) = g(s^{n}) + \Delta^{n} g'(s^{n}) + O\left((\Delta^{n})^{2}\right) \tag{20}$$

Substituting Eq. (20) into Eq. (19) and neglecting the second order terms, we have

$$\Delta^{n+1} = \Delta^{n}\left(1 - 3Gdtg'(s^{n})\right) \tag{21}$$

To guarantee the stability of calculation, the perturbations of stress can't be magnified. Therefore,

$$0 < dt < \frac{2}{3Gg'(s^{n})} \tag{22}$$

Now we have deduced the stability condition of explicit precise algorithm. It indicates that the algorithm of constitutive equation is not unconditionally stable. The time step should be positive and less than a certain value to guarantee the stability of the algorithm of constitutive equation.

It should be noted that the stability condition (22) is deduced under special conditions (13) and (14), so it is just a necessary but not sufficient condition for the stability of explicit precise algorithm.

While the viscoplastic constitutive relationship is taken as

$$\bar{\sigma} = Y^{*}\left(1 + \beta \ln\left(1 + \frac{\dot{\bar{\varepsilon}}^{p}}{\dot{\bar{\varepsilon}}_{0}}\right)\right) \tag{23}$$

The stability condition (22) is simply

$$0 < dt < \frac{2\beta Y^{*}}{3G\dot{\bar{\varepsilon}}_{0}\left(\exp\left(\frac{1}{\beta}\left(\frac{\bar{\sigma}}{Y^{*}} - 1\right)\right) - 1\right)} = \frac{2\beta Y^{*}}{3G\dot{\bar{\varepsilon}}^{p}} \tag{24}$$

where $\dot{\bar{\varepsilon}}_{0}$ is the reference strain rate, Y^{*} and b are material parameters.

2.3 Numerical Examples

In order to verify the above stability analysis on the explicit precise algorithm, a series of numerical examples were carried out.

In the numerical examples, a single quadrilateral element, which is 20mm on each side, was used. The material 4340 steel, which is widely used in industry, was selected. The viscoplastic constitutive relationship (23) was used to describe the mechanical behavior of 4340 steel [20]. All of the parameters needed in the numerical calculations are listed in Table 1.

The deformation processes of the element are showed in Fig. 1. The movements of nodal points 1, 2,

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3 and 4 are restricted in X direction and the movements of nodal points 1 and 2 are restricted in Y direction. At the beginning, both of the nodal points 3 and 4 move 1mm along Y direction with a uniform velocity of –50m/s. Then they move back with a uniform velocity of 50m/s. The whole process takes exactly 40μs. In the numerical examples, the element experiences complicated deformation processes: ① elastic loading, ② plastic loading, ③ elastic unloading, ④ reverse elastic loading, ⑤ reverse plastic loading. In the plastic deformation processes ② and ⑤, the plastic strain rate is about 1700/s. By using the formula (24), the stability condition is obtained: $0 < dt < 0.56 \times 10^{-7}$s. In the following, three time steps were selected to verify the stability analysis in Section 2.2.

(1) The time step is set as $dt = 0.3 \times 10^{-7}$s.

The calculated effective stress and effective plastic strain histories are plotted in Fig. 2. The numerical results exactly describe all the deformation processes of the quadrilateral element.

① Elastic loading stage: The effective stress increases with a constant speed. No plastic strain appears.

② Plastic loading stage: The effective stress is equal to yield stress while the material suffers plastic deformation. Because the effective plastic strain rate keeps in constant, the effective stress remains unchanged in the present plastic loading stage. The effective plastic strain increases uniformly.

③ Elastic unloading stage: Since the time 20μs, the nodal points 3 and 4 move back. The element suffers elastic unloading. The effective stress decreases to zero with a constant speed. In this elastic process, no plastic strain increments appears.

④ Reverse elastic loading: After the effective stress decreases to zero, the element suffers elastic loading for the second time. The effective stress increases uniformly. There is no plastic strain in this stage.

⑤ Reverse plastic loading: While the effective stress reaches the yield stress, the element goes into plastic state again. The effective stress keeps in constant because of the constant plastic strain rate.

These results elucidate that while the stability condition (24) is met, reliable and stable results can be obtained with the explicit precise algorithm.

(2) The time step is set as $dt = 1.0 \times 10^{-7}$s.

This time step doesn't meet the stability condition (24). The calculated effective stress and effective plastic strain histories are showed in Fig. 3. After the element goes into the plastic state, the effective stress becomes unstable. The numerical calculation gives unstable results. This proves that while the stability condition is unsatisfied, the calculation with explicit precise algorithm will not give correct and stable results.

Table 1 The constitutive parameters for 4340 steel [20].

Bulk module K (GPa)	Shear module G (GPa)	Y^* (GPa)	Strain-rate sensitivity coefficient β	Reference strain-rate $\dot{\bar{\varepsilon}}_0$ (s^{-1})
164.0	77.5	0.792	0.014	1.0

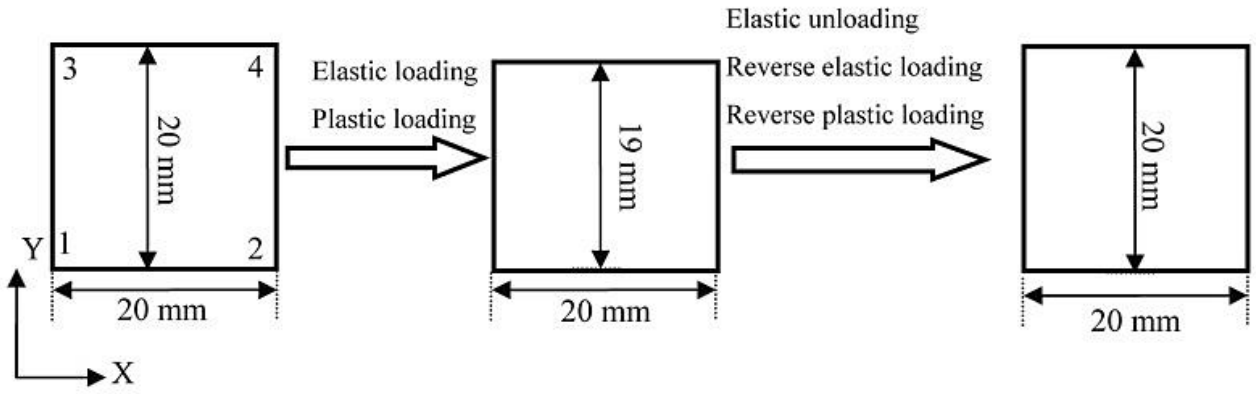


Fig. 1 Computational model with a single element.

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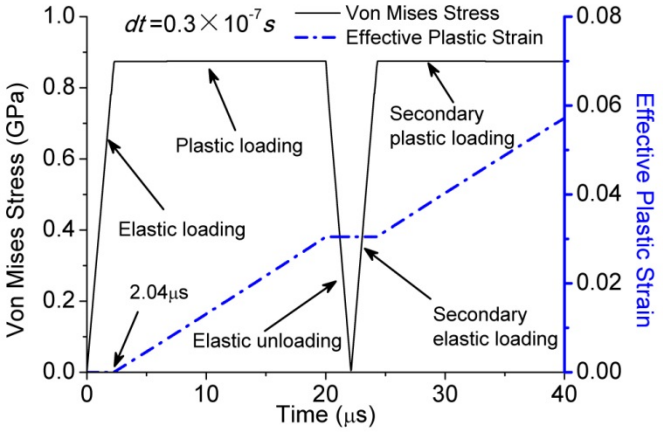


Fig. 2 Computational results with explicit precise algorithm (time step $dt = 0.3 \times 10^{-7}$s).

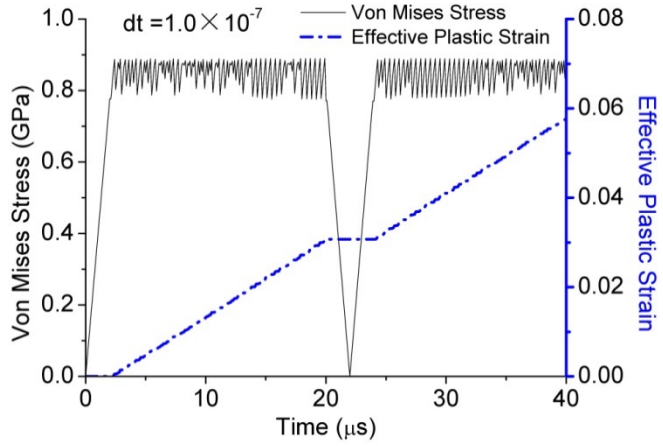


Fig. 3 Computational results with explicit precise algorithm (time step $dt = 1.0 \times 10^{-7}$s).

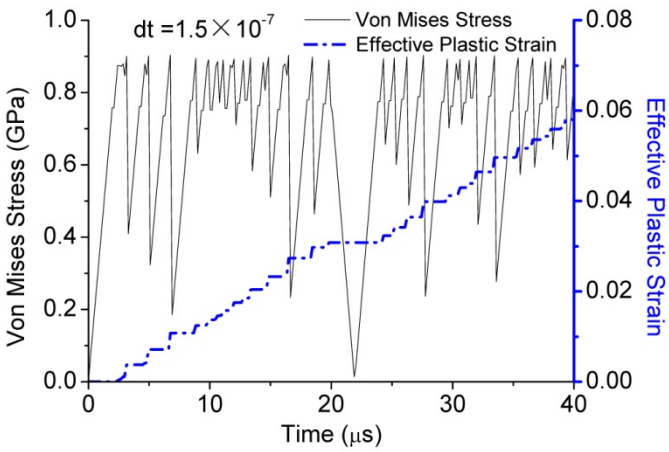


Fig. 4 Computational results with explicit precise algorithm (time step $dt = 1.5 \times 10^{-7}$s).

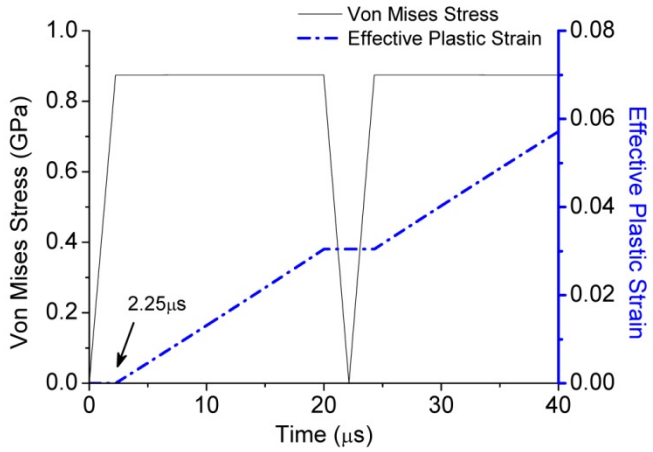


Fig. 5 Computational results with radial return algorithm (time step $dt = 1.0 \times 10^{-8}$s).

(3) The time step is set as $dt = 1.5 \times 10^{-7}$s.

The time step increases to be a much bigger value. The calculated results are showed in Fig. 4. After the element goes into the plastic state, the effective stress becomes extremely unstable. The effective plastic strain increases with a stepped characteristic. This means that the element alternates between elastic state and plastic state continually. These results are far away from the real deformation processes.

The numerical results above prove the correctness of the stability analysis on explicit precise algorithm. The deduced formula (22) accurately predicts the stability condition in the present numerical examples, but it is just a necessary condition for the stability of explicit precise algorithm. In practical application, a safety factor can be used to improve the reliability of the formula (22).

(4) Computation results with radial return algorithm.

An extra computation with radial return algorithm is carried out using the commercial software of ANSYS/LS-DYNA and the time step of 1.0×10^{-8}s is used. The results are shown in Fig. 5 to compare with the results of explicit precise algorithm (see Fig. 2).

On the whole, the computational results with radial return algorithm are nearly the same with those with explicit precise algorithm (time step $dt = 0.3 \times 10^{-7}$s). This implies that these two algorithms are both acceptable. But the radial return algorithm is just an approximation algorithm, in which the total effective strain rate is taken as the effective plastic strain rate and the elastic strain rate is ignored. So the initial yield strength in radial return algorithm (about 0.874GPa) is higher than that in explicit precise algorithm (0.792GPa), and the time of initial yield in radial return algorithm is later than that in explicit precise algorithm. The time of initial yield is 2.25μs and 2.04μs in radial return algorithm, explicit precise algorithm, respectively.

3. THE ITERATIVE ALGORITHM AND DISCUSSION

Generally speaking, the numerical stability is only analyzed for the difference schemes of continuity equation, momentum equation and energy equation. The stability of the algorithm for the constitutive equation

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has rarely been discussed. This is because that, in the traditional calculating framework, the stress increments are assumed to be accurately calculated from the strain increments. But in the explicit precise algorithm, the plastic strain rate at the current time is calculated with the stress states at the previous time. The constitutive equation is just approximately equal. To avoid magnifying the errors, the time step should be less than a certain value.

To eliminate the additional restriction on time step, the constitutive equation should hold true rigorously in the current time. Therefore, an iterative algorithm should be used to calculate the stress increments from the strain increments. The core of the calculation for viscoplastic constitutive equation is to obtain the plastic strain in the present time step accurately. The developing direction of the plastic strain is governed by the plastic flow rule. According to the constitutive equation, the value of effective plastic strain rate is directly related to the yield stress; on the other hand, the stress increments are related to the elastic strain increments according to the Hooke's law. So the effective stress and plastic strain rate should be calculated by solving the constitutive equation and Hooke's law simultaneously. Generally speaking, an iterative method is needed to solve the equations due to the complexity of the form of viscoplastic constitutive equation.

For viscoplastic constitutive equation with Von Mises yield criterion (1), the Eqs. (5), (6) can be deduced from the plastic flow rule. Substituting Eq. (5) into Eq. (6) and using the backward difference scheme give

$$(\dot{\varepsilon}_{ij}^{p})^{n+1/2}=\frac{3(\dot{\bar{\varepsilon}}^{p})^{n+1/2}}{2\bar{\sigma}^{n+1}}s_{ij}^{n+1} \tag{25}$$

According to the Hooke's law, we have Eq. (10). Substitution of Eq. (25) into Eq. (10) gives

$$s_{ij}^{n+1}=s_{ij}^{*}-3Gdt(\dot{\bar{\varepsilon}}^{p})^{n+1/2}\frac{s_{ij}^{n+1}}{\bar{\sigma}^{n+1}} \tag{26}$$

where the superscript, *, denotes a trial stress value. The trial stress s_{ij}^{*} is defined as $s_{ij}^{*}=s_{ij}^{n}+2Gdt\dot{e}_{ij}$.

Solving for the trial stress gives

$$s_{ij}^{n+1}\left(1+\frac{3Gdt(\dot{\bar{\varepsilon}}^{p})^{n+1/2}}{\bar{\sigma}^{n+1}}\right)=s_{ij}^{*} \tag{27}$$

This formula proves that the stress at time $n+1$ and the trial stress s_{ij}^{*} are collinear. It also can be deduced that the plastic strain increments and the trial stress are collinear from Eqs. (25) and (27).

After contraction of tensor, the Eq. (27) can be simplified as

$$\bar{\sigma}^{n+1}=\bar{\sigma}^{*}-3Gdt(\dot{\bar{\varepsilon}}^{p})^{n+1/2} \tag{28}$$

where $\bar{\sigma}^{n+1}=\sqrt{\frac{3}{2}s_{ij}^{n+1}s_{ij}^{n+1}}$, $\bar{\sigma}^{*}=\sqrt{\frac{3}{2}s_{ij}^{*}s_{ij}^{*}}$.

We should pay more attention to Eq. (28), which gives a relationship between the effective stress at time $n+1$ ($\bar{\sigma}^{n+1}$) and the effective plastic strain rate ($(\dot{\bar{\varepsilon}}^{p})^{n+1/2}$). It was deduced for viscoplastic constitutive relationship. But actually, it doesn't only hold true for viscoplastic materials. It can be deduced under the conditions (2), (3) and (4), namely, it's true for any material with Von Mises yield criterion.

The $\bar{\sigma}^{n+1}$ and $(\dot{\bar{\varepsilon}}^{p})^{n+1/2}$ can be calculated by solving Eq. (28) and constitutive relationship. In general, the form of the constitutive relationship is relatively complicated, so we can't get analytical solutions. In this situation, an iterative method can be used to obtain numerical solutions.

The difference scheme of Eq. (23) is

$$\bar{\sigma}^{n+1}=Y^{*}\left(1+\beta\ln\left(1+\frac{(\dot{\bar{\varepsilon}}^{p})^{n+1/2}}{\dot{\bar{\varepsilon}}_{0}}\right)\right) \tag{29}$$

Combing Eqs. (28) and (29) gives

$$Y^{*}\left(1+\beta\ln\left(1+\frac{(\dot{\bar{\varepsilon}}^{p})^{n+1/2}}{\dot{\bar{\varepsilon}}_{0}}\right)\right)+3Gdt(\dot{\bar{\varepsilon}}^{p})^{n+1/2}-\bar{\sigma}^{*}=0 \tag{30}$$

while $\bar{\sigma}^{*}>Y^{*}$, the material is in plastic state. We can't obtain an analytical solution of effective plastic strain rate from Eq. (30). An iterative algorithm is used to get a numerical solution. Letting

$$\begin{aligned}&H\left((\dot{\bar{\varepsilon}}^{p})^{n+1/2}\right)\\&=Y^{*}\left(1+\beta\ln\left(1+\frac{(\dot{\bar{\varepsilon}}^{p})^{n+1/2}}{\dot{\bar{\varepsilon}}_{0}}\right)\right)+3Gdt(\dot{\bar{\varepsilon}}^{p})^{n+1/2}-\bar{\sigma}^{*}\end{aligned} \tag{31}$$

The effective strain rate can be obtained through calculating the zero of the function H. The first and second derivatives of function H are respectively,

$$H'=\frac{1}{\dot{\bar{\varepsilon}}_{0}+(\dot{\bar{\varepsilon}}^{p})^{n+1/2}}Y^{*}\beta+3Gdt>0 \tag{32}$$

$$H''=\frac{-1}{\left(\dot{\bar{\varepsilon}}_{0}+(\dot{\bar{\varepsilon}}^{p})^{n+1/2}\right)^{2}}Y^{*}\beta<0 \tag{33}$$

while the material is in plastic state, we have $H(0)=Y^{*}-\bar{\sigma}^{*}<0$ according to Eq. (31). We have $H'>0$ and $H''<0$ according to Eqs. (32) and (33). So H is a monotonically increasing concave function with a negative y intercept. The function H is schematically plotted in Fig. 6. The Newton iteration method is very suitable to calculate the zero of such a function. The initial value is set as $\left((\dot{\bar{\varepsilon}}^{p})^{n+1/2}\right)_{0}=0$. The principle of the Newton iteration method is schematically showed in Fig. 5. It reveals that the convergence rate of the Newton iteration method for calculating the zero of function H is very quick. The detailed formula for iterative algorithm is

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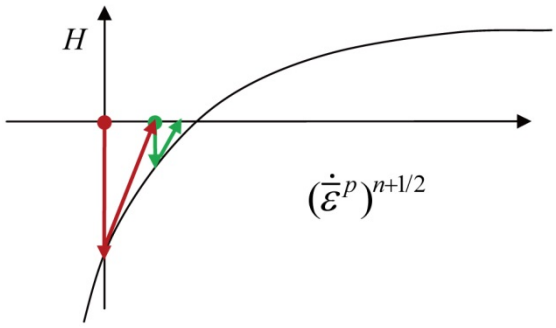


Fig. 6 Schematic diagram for Newton iteration method.

$$\left((\dot{\bar{\varepsilon}}^p)^{n+1/2}\right)_{i+1}=\left((\dot{\bar{\varepsilon}}^p)^{n+1/2}\right)_i-\frac{H\left(\left((\dot{\bar{\varepsilon}}^p)^{n+1/2}\right)_i\right)}{H'\left(\left((\dot{\bar{\varepsilon}}^p)^{n+1/2}\right)_i\right)} \tag{34}$$

where $i = 0, 1, 2, \ldots$, is the number of iterations.

Now we have obtained all the formulas of the iterative algorithm. The calculation process is summarized as follows:

Known: s_{ij}^n, $\dot{e}_{ij}^{n+1/2}$, dt; Solve for: s_{ij}^{n+1}

① Calculate the trial stress tensor $s_{ij}^* = s_{ij}^n + 2Gdt\dot{e}_{ij}$.

② Calculate the trial effective stress $\bar{\sigma}^* = \sqrt{\frac{3}{2}s_{ij}^* s_{ij}^*}$.

③ While $\bar{\sigma}^* \le Y^*$, the material is in elastic state, so $s_{ij}^{n+1} = s_{ij}^*$. Calculation ends.

④ While $\bar{\sigma}^* > Y^*$, the material is in plastic state. Substituting $\bar{\sigma}^{n+1}$ from Eq. (28) into constitutive relationship leads to an equation about effective strain rate $(\dot{\bar{\varepsilon}}^p)^{n+1/2}$.

⑤ Solve the equation about effective strain rate with the Newton iteration method.

⑥ Substituting the effective stain rate $(\dot{\bar{\varepsilon}}^p)^{n+1/2}$ into Hooke's law gives the value of s_{ij}^{n+1}. Calculation ends.

Now, we use the iterative algorithm introduced above to simulate the examples described in Section 2.3. The time steps are set as $dt = 1.0 \times 10^{-6}$s and $dt = 1.0 \times 10^{-8}$s, respectively. The former is one order higher than the critical time step of explicit precise algorithm and the latter is one order lower than it. The number of iterations is set as 8. The simulation results are showed in Figs. 7 and 8. The effective stress and effective plastic strain histories are nearly the same for these two time steps. They both correctly describe the deformation process of the element. In the calculations, the number of iterations is fixed as 8, which is very small. This illustrates that the convergence rate of the iterative algorithm is very fast.

The iterative algorithm is absolutely stable for different time steps. Its convergence rate is very fast, so it will not cost too much additional computing time.

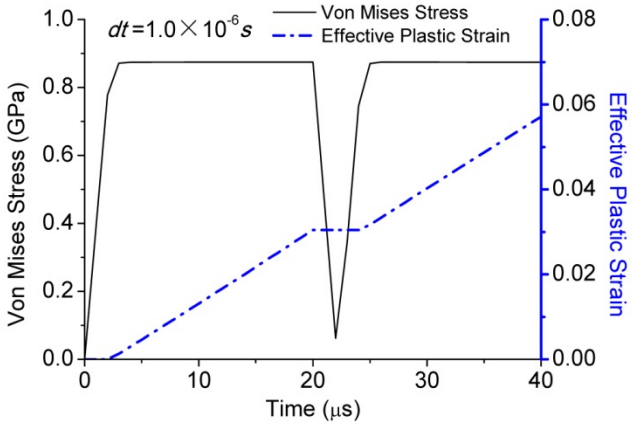


Fig. 7 Computational results with iterative algorithm (time step $dt = 1.0 \times 10^{-6}$s).

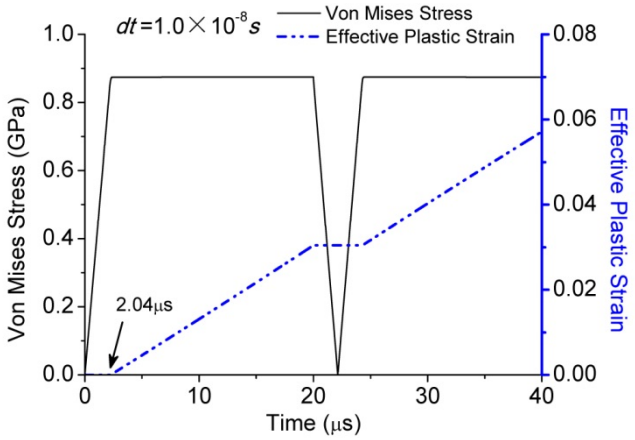


Fig. 8 Computational results with iterative algorithm (time step $dt = 1.0 \times 10^{-8}$s).

The explicit precise algorithm is conditionally stable. The time step should be less than a certain value to guarantee its stability. But it's worth noting that the explicit precise algorithm gives the same results with the iterative algorithm while the stability condition is satisfied. The initial yield strength in iterative algorithm (0.792GPa) equals to that in explicit precise algorithm (0.792Gpa), which is less than that in radial return algorithm (about 0.874GPa). The time of initial yield in radial return algorithm is later than that in explicit precise algorithm and iterative algorithm. The time of initial yield is 2.25μs, 2.04μs and 2.04μs in radial return algorithm, explicit precise algorithm and iterative algorithm, respectively. This also proves the correctness of explicit precise algorithm and the approximate feature of the radial return algorithm.

In the numerical calculations, as we known, the time step is associated with the mesh sizes. In some large-scale simulations with fine structures, the mesh sizes should be very small to obtain more reliable and precise results, as a result, the corresponding time step should be also very small. The stability condition of explicit precise algorithm may be automatically satisfied. In

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this situation, the explicit precise algorithm displays its superiority. It's more efficient than the iterative algorithm because it doesn't need the iterative process. So we can use the explicit precise algorithm in this situation.

In practice, the two algorithms, explicit precise algorithm proposed by Li *et al*. [19] and iterative algorithm introduced above, can be combined to use. While the stability condition of explicit precise algorithm is automatically satisfied with the time step calculated from the mesh sizes, the explicit precise algorithm is adopted, otherwise the iterative algorithm is adopted. Thus we can obtain reliable and stable results with the minimum computing costs.

4. CONCLUSIONS

Based on the universal thermal viscoplastic constitutive relationship, Li *et al*. [19] developed an explicit precise algorithm for constitutive calculation. We found that this algorithm is conditionally stable.

In the present work, the formulas and computing processes of explicit precise algorithm were derived on the basis of Von Mises yield criterion at first. Then the stability of explicit precise algorithm was analyzed. Under a special condition, a necessary condition for the stability of explicit precise algorithm was obtained. Some numerical examples were carried out to verify the correctness of the stability analysis. The results showed that while the time steps are too large to violate the stability condition, the numerical results are not stable and some fluctuates appear in the effective stress-time curve. While the stability condition is satisfied, the numerical results exactly describe the complicated deformation processes of the element: ① elastic loading, ② plastic loading, ③ elastic unloading, ④ reverse elastic loading, ⑤ reverse plastic loading.

It should be noted that the stability condition is just a necessary condition. It certainly proved that the explicit precise algorithm is conditionally stable. But even if the stability condition is satisfied, the numerical results will not be guaranteed to be stable. In practice, a safety factor can be introduced to improve the reliable of the stability condition.

Because the explicit precise algorithm is conditionally stable, the time step should be less a certain value to guarantee its stability. This will greatly increase the computing costs. To improve the efficient of calculation, an iterative algorithm, which is unconditionally stable, was proposed. All the formulas and computing processes of the iterative algorithm were presented. Numerical examples proved that this iterative algorithm is absolutely stable and has fast convergence.

In practice, the explicit precise algorithm and iterative algorithm can be combined to use. While the stability condition of explicit precise algorithm is automatically satisfied, the explicit precise algorithm is adopted, otherwise the iterative algorithm is adopted. Thus reliable results will be obtained with the minimum computing costs.

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