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LIOUVILLE NUMBER GROUPS: STRUCTURE, CARDINALITY AND STRONG GENERATORS

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Abstract

We introduce and study Liouville number groups, namely additive subgroups of the real numbers in which every nonzero element is a Liouville number. Using continued-fraction methods and linear independence over the field of algebraic numbers, we establish the existence of large families of such groups with rich algebraic and topological structure. We prove that there exist $(2^{\mathfrak{c}})$ pairwise distinct Liouville number groups generated by strong Liouville numbers and that, among these, there are $(2^{\mathfrak{c}})$ pairwise nonhomeomorphic groups. We further show that there exist continuum many countable Liouville number groups, each generated by countably many strong Liouville numbers and homeomorphic, as topological spaces, to the rational numbers. In addition, we prove that no subgroup of the real numbers is homeomorphic to the space of Liouville numbers, thereby highlighting a strong topological distinction between the space of Liouville numbers and Liouville number groups.

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1. Preliminaries

We begin with a little notation: $\mathbb{N}$ for the set of positive integers; $\mathbb{Z}$ for the group of all integers; $\mathbb{P}$ for the set of all real irrational numbers; $\mathbb{R}$ for the additive group or field of all real numbers; $\mathbb{R}^{>0}$ for the set of positive real numbers; $\mathbb{A}$ for the field of all algebraic numbers; $\mathbb{C}$ for the additive group or field of all complex numbers; $\mathbb{T}$ for the set of all real transcendental numbers.

In 1844, Joseph Liouville (1809–1882) proved the existence of transcendental numbers [5]. He introduced the set $\mathcal{L}$ of real numbers, now known as Liouville numbers, and showed that they are all transcendental. A real number x is said to be a *Liouville number* if for every positive integer n, there exists a pair of integers (p, q) with $q > 1$ such that $0 < |x - p/q| < 1/q^n$.

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In 1962, Paul Erdős (1913–1996) proved that every real number is the sum of two Liouville numbers (and also, if nonzero, is the product of two Liouville numbers) [3]. This implies, for example, that the smallest group containing the set $\mathcal{L}$ of Liouville numbers is the group $\mathbb{R}$ of all real numbers. A set S of numbers is said to have the *Erdős property* if every real number is a sum of two members of S.

Erdős wondered whether any proper subset of $\mathcal{L}$ has the Erdős property. In particular, he speculated that the subset of $\mathcal{L}$ called the strong Liouville numbers might have this property. György Petruska in 1992 proved that the set of strong Liouville numbers does not have this property [9]. However, it was proved in 2023 that $\mathcal{L}$ does have $2^{\mathfrak{c}}$ subsets with the Erdős property [1].

In 2021, the author of this paper in [8] (published in 2024) introduced the notion of a transcendental group and proved that there exist $2^{\mathfrak{c}}$ such topological groups.

DEFINITION 1.1. A topological group G is said to be a *transcendental group* if it is a subgroup of the topological group $\mathbb{C}$ of all complex numbers and such that every element of G except 0 is a transcendental number.

We now introduce the notion of a Liouville number group.

DEFINITION 1.2. A topological group G is said to be a *Liouville number group* if it is a subgroup of the additive topological group $\mathbb{R}$ of all real numbers and such that every element of G except 0 is a Liouville number.

It is apparent that every Liouville number group is a transcendental group.

THEOREM 1.3. *There exist* $\mathfrak{c}$ *Liouville number groups each isomorphic as a topological group to the discrete group* $\mathbb{Z}$ *of integers.*

PROOF. By [2, remark after Proposition 2], every nonzero integer multiple of a Liouville number is a Liouville number. (See also Remark 2.8.) Thus, the group $\mathrm{gp}\{g\}$ generated by each Liouville number is a discrete cyclic subgroup of $\mathbb{R}$ and is a Liouville number group. Further, $\mathrm{gp}\{g\}$ is isomorphic as a topological group to $\mathbb{Z}$. Since each cyclic subgroup of $\mathbb{R}$ is countable, it contains only countably many Liouville numbers. As there are $\mathfrak{c}$ Liouville numbers, the family $\{\mathrm{gp}\{g\} : g \in \mathcal{L}\}$ contains $\mathfrak{c}$ distinct groups. □

2. Liouville number groups: structure, cardinality and strong generators

We first recall a continued-fraction characterisation of Liouville numbers [4, Ch. X]. Let $x \in \mathbb{R} \setminus \mathbb{Q}$ and let p_k/q_k be the kth convergent of the continued fraction expansion of x. Then, x is a *Liouville number* if and only if for each fixed $n \in \mathbb{N}$, there exist infinitely many k such that $q_{k+1} > q_k^n$.

DEFINITION 2.1. Let $x \in \mathbb{R} \setminus \mathbb{Q}$ and let p_k/q_k be the kth convergent of the continued fraction expansion of x. Then, x is called a *strong Liouville number* if for every $n \in \mathbb{N}$, there exists an index $N(n)$ such that $q_{k+1} > q_k^n$ for all $k \geq N(n)$.

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REMARK 2.2. We shall also use the following equivalent definition given by Petruska [9]. Let $x \in \mathbb{R} \setminus \mathbb{Q}$ and let p_k/q_k be the kth convergent of the continued fraction expansion of x. Then, x is a *strong Liouville number* if and only if

$$\frac{\log q_{n+1}}{\log q_n} \to \infty \quad \text{as } n \to \infty.$$

PROPOSITION 2.3. *If $\ell = \sum_{n=1}^{\infty} 10^{-n!}$ is the Liouville constant, then it is not a strong Liouville number.*

PROOF. We use Petruska's criterion [9]: if $a \geq 2$ is an integer and $\xi = \sum_{n=1}^{\infty} a^{-v_n}$, where v_n is a strictly increasing sequence of positive integers satisfying $v_{n+1} \geq 2v_n + 2$ for all sufficiently large n, then ξ is not a strong Liouville number.

For the Liouville constant, take $a = 10$ and $v_n = n!$. Then, $\ell = \sum_{n=1}^{\infty} 10^{-v_n}$. Moreover, $v_{n+1} = (n+1)! = (n+1)n!$. For every $n \geq 2$, we have $(n+1)n! \geq 2n! + 2$, because $(n-1)n! \geq 2$. Thus, $v_{n+1} \geq 2v_n + 2$ for all $n \geq 2$. Therefore, Petruska's criterion applies and ℓ is not a strong Liouville number. □

The following is an example of a strong Liouville number.

EXAMPLE 2.4. Let $q_{-1} = 0$, $q_0 = 1$ and define the continued fraction

$$\alpha = [0; 1, a_2, a_3, \ldots]$$

recursively by $a_{n+1} := q_n^n$ $(n \geq 1)$, where q_n denotes the denominator of the nth convergent. Thus,

$$\alpha = [0; 1, 1, 4, 729, \ldots].$$

Since $q_{n+1} = a_{n+1}q_n + q_{n-1} = q_n^{n+1} + q_{n-1} > q_n^{n+1}$, we have

$$\frac{\log q_{n+1}}{\log q_n} > n + 1.$$

Hence,

$$\frac{\log q_{n+1}}{\log q_n} \to \infty.$$

Therefore, by Petruska's definition from Remark 2.2, α is a strong Liouville number.

We shall need the standard identity relating a real number to the successive differences of its convergents.

LEMMA 2.5. *Let $\mu = [a_0; a_1, a_2, \ldots]$ be irrational and let p_k/q_k be its convergents. Then,*

$$\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k q_{k+1}} \ (k \geq 0) \quad \textit{and, consequently,} \quad \mu = a_0 + \sum_{k=0}^{\infty} \frac{(-1)^k}{q_k q_{k+1}}.$$

PROOF. The determinant identity $p_{k+1}q_k - p_k q_{k+1} = (-1)^k$ is standard in continued fraction theory and immediately yields the first formula. Since $p_k/q_k \to \mu$, summing

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from $k = 0$ to N and letting $N \to \infty$ gives

$$\mu - \frac{p_0}{q_0} = \sum_{k=0}^{\infty} \left(\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} \right).$$

As $p_0/q_0 = a_0$, the result follows. □

Petruska proved the following fundamental closure property.

THEOREM 2.6 (Petruska [9]). *Let x be a strong Liouville number and y a Liouville number. Then, $x + y$ is either rational or Liouville. The same is true for xy.*

LEMMA 2.7. *Let x be a strong Liouville number and let $r \in \mathbb{Q} \setminus \{0\}$. Then, rx is a Liouville number.*

PROOF. By definition, a strong Liouville number is in particular a Liouville number. Hence, x is Liouville.

Write $r = a/b$ with $a \in \mathbb{Z} \setminus \{0\}$ and $b \in \mathbb{N}$. We prove that rx is Liouville. Let $N \geq 1$ be arbitrary. Since x is Liouville, there exist infinitely many rational numbers p/q with $q \geq 1$ such that $0 < |x - p/q| < 1/q^{N+1}$. Then, $rx - ap/bq = r(x - p/q)$ and, therefore, $0 < |rx - ap/bq| = |r||x - p/q| < |r|/q^{N+1}$.

Given p and q, set $P = ap$ and $Q = bq$. Then, $Q \geq q$ and

$$0 < \left| rx - \frac{P}{Q} \right| < \frac{|r|}{q^{N+1}} = \frac{|r| b^{N+1}}{Q^{N+1}}.$$

Since there are infinitely many such q, we may choose q sufficiently large so that $|r|b^{N+1} \leq q$. For such q, we have $|r|b^{N+1}/Q^{N+1} \leq 1/Q^N$. Hence, for every $N \geq 1$, there exist integers P, Q with $Q \geq 1$ such that

$$0 < \left| rx - \frac{P}{Q} \right| < \frac{1}{Q^N}.$$

Since there are infinitely many choices for p/q, there are infinitely many such P/Q. This shows that rx is a Liouville number. □

REMARK 2.8. In fact, more is true. Maillet [6, Ch. III] shows that given a rational function f with rational coefficients, if ξ is a Liouville number, then so is $f(\xi)$.

LEMMA 2.9. *Let x be a strong Liouville number. Then, $-x$ is also a strong Liouville number.*

PROOF. Let p_n/q_n be the sequence of convergents of x. Then, the convergents of $-x$ are given by $-p_n/q_n$. In particular, the sequence of denominators (q_n) is unchanged. Since x is a strong Liouville number, its convergents satisfy the defining growth condition required in Definition 2.1 (or the equivalent characterisation in Remark 2.2). Because the denominators and approximation quality are preserved under negation, the same condition holds for $-x$. Hence, $-x$ is a strong Liouville number. □

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COROLLARY 2.10. *Let $x_1, \ldots, x_n$ be strong Liouville numbers and let $m_1, \ldots, m_n \in \mathbb{Z}$. Then, $m_1x_1 + \cdots + m_nx_n$ is either rational or Liouville.*

PROOF. Set $S := m_1x_1 + \cdots + m_nx_n$ and $M := |m_1| + \cdots + |m_n|$. We argue by induction on M.

If $M = 0$, then all $m_i = 0$, so $S = 0$, which is rational.

Assume the result holds for all integer coefficients with total weight strictly less than M and suppose $M \geq 1$. Choose an index j such that $m_j \neq 0$. Write $m_j = \varepsilon + m_j'$, where

$$\varepsilon = \begin{cases} 1 & m_j > 0, \\ -1 & m_j < 0, \end{cases} \qquad m_j' = m_j - \varepsilon.$$

Then, $|m_j'| = |m_j| - 1$ and, hence, $|m_1| + \cdots + |m_j'| + \cdots + |m_n| = M - 1$. Thus, we can write $S = \varepsilon x_j + T$, where

$$T := m_1x_1 + \cdots + m_{j-1}x_{j-1} + m_j'x_j + m_{j+1}x_{j+1} + \cdots + m_nx_n.$$

By the induction hypothesis, T is either rational or Liouville. By Lemma 2.9, εx_j is a strong Liouville number. We now consider two cases.

Case 1: T is Liouville. Theorem 2.6 implies that the sum of a strong Liouville number and a Liouville number is either rational or Liouville. Hence, $S = \varepsilon x_j + T$ is rational or Liouville.

Case 2: T is rational. Then, $S = \varepsilon x_j + T$ is irrational, since εx_j is irrational. Moreover, S is Liouville because the sum of a Liouville number and a rational number is Liouville. Indeed, if x is Liouville, then for every $N \geq 1$, there exist infinitely many p/q such that $0 < |x - p/q| < 1/q^N$. Thus, for $r \in \mathbb{Q}$,

$$0 < \left|(x + r) - \left(\frac{p}{q} + r\right)\right| < \frac{1}{q^N}$$

and $p/q + r \in \mathbb{Q}$, proving that $x + r$ is Liouville.

In both cases, S is either rational or Liouville. This completes the induction. □

To build many distinct groups, we need a large algebraically independent family of strong Liouville numbers.

LEMMA 2.11. *The set $\mathcal{L}_{\text{strong}}$ of strong Liouville numbers has cardinality* $\mathfrak{c}$.

PROOF. Since $\mathcal{L}_{\text{strong}} \subseteq \mathbb{R}$, we immediately have $|\mathcal{L}_{\text{strong}}| \leq |\mathbb{R}| = \mathfrak{c}$. It therefore suffices to prove that $|\mathcal{L}_{\text{strong}}| \geq \mathfrak{c}$.

We construct an injective map from $\{0, 1\}^{\mathbb{N}}$ into $\mathcal{L}_{\text{strong}}$.

Fix $\varepsilon = (\varepsilon_n)_{n\geq 1} \in \{0, 1\}^{\mathbb{N}}$. Define a continued fraction $x_\varepsilon = [0; a_1, a_2, a_3, \ldots]$ recursively as follows. Set $a_1 := 1$. For each $n \geq 1$, let q_n denote the denominator of the nth convergent of $[0; a_1, \ldots, a_n]$ and define $a_{n+1} := q_n^n + 1 + \varepsilon_n$. Since $q_n \geq 1$ and $\varepsilon_n \in \{0, 1\}$, each a_{n+1} is a positive integer, so this indeed defines an irrational real number x_ε.

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Let p_n/q_n be the nth convergent of x_ε. By the standard recurrence relation for continued fractions, $q_{n+1} = a_{n+1}q_n + q_{n-1}$ $(n \geq 1)$, where $q_0 = 1$ and $q_{-1} = 0$. Hence, $q_{n+1} \geq a_{n+1}q_n = (q_n^n + 1 + \varepsilon_n)q_n > q_n^{n+1}$. Therefore, for every $n \geq 1$, $q_{n+1} > q_n^{n+1}$.

We now show that x_ε is a strong Liouville number. Let $N \geq 1$ be arbitrary. If $n \geq N$, then $q_{n+1} > q_n^{n+1} \geq q_n^N$. Thus, the convergents of x_ε satisfy $q_{n+1} > q_n^N$ for all sufficiently large n. By the definition of a strong Liouville number, $x_\varepsilon \in \mathcal{L}_{\text{strong}}$.

It remains to prove injectivity. Assume $\varepsilon, \delta \in \{0,1\}^{\mathbb{N}}$ with $\varepsilon \neq \delta$. Let m be the least index such that $\varepsilon_m \neq \delta_m$. Then, the recursive construction gives

$$a_1(\varepsilon) = a_1(\delta), \ldots, a_m(\varepsilon) = a_m(\delta),$$

so the first m convergents, and in particular the denominator q_m, are the same for both sequences. Hence, $a_{m+1}(\varepsilon) = q_m^m + 1 + \varepsilon_m \neq q_m^m + 1 + \delta_m = a_{m+1}(\delta)$. Therefore, the continued fraction expansions of x_ε and x_δ differ, so $x_\varepsilon \neq x_\delta$. Thus, the map

$$\{0,1\}^{\mathbb{N}} \to \mathcal{L}_{\text{strong}}, \quad \varepsilon \mapsto x_\varepsilon,$$

is injective. Since $|\{0,1\}^{\mathbb{N}}| = \mathfrak{c}$, we obtain $|\mathcal{L}_{\text{strong}}| \geq \mathfrak{c}$. Combining this with the upper bound $|\mathcal{L}_{\text{strong}}| \leq \mathfrak{c}$, we conclude that $|\mathcal{L}_{\text{strong}}| = \mathfrak{c}$. □

LEMMA 2.12. *There exists a subset $B \subset \mathcal{L}_{\text{strong}}$ of cardinality $\mathfrak{c}$ that is linearly independent over the field $\mathbb{A}$ of algebraic numbers.*

PROOF. We first establish a cardinality bound.

CLAIM. *Let F be a countable field and let S be a set with $|S| = \kappa < \mathfrak{c}$. Then, $|\text{span}_F(S)| < \mathfrak{c}$.* □

PROOF OF THE CLAIM. Every element of $\text{span}_F(S)$ can be written as a finite linear combination $\sum_{i=1}^n a_i s_i$ $(a_i \in F,\ s_i \in S,\ n \in \mathbb{N})$. Hence, $\text{span}_F(S) \subseteq \bigcup_{n=0}^\infty F^n \times S^n$, where we interpret $F^0 \times S^0$ as a singleton corresponding to 0. Thus,

$$|\text{span}_F(S)| \leq \sum_{n=0}^{\infty} |F|^n \cdot |S|^n.$$

Since F is countable, $|F| = \aleph_0$. Hence, for each $n \geq 1$, $|F|^n \cdot |S|^n = \aleph_0 \cdot \kappa$, which is equal to κ when κ is infinite and to $\aleph_0$ when κ is finite. Therefore, $|\text{span}_F(S)| \leq \aleph_0 \cdot \kappa < \mathfrak{c}$. This proves the claim. □

We now construct the desired set B by transfinite recursion. Since $|\mathcal{L}_{\text{strong}}| = \mathfrak{c}$ by Lemma 2.11, we may well-order $\mathcal{L}_{\text{strong}}$ and construct a family $B = \{b_\alpha : \alpha < \mathfrak{c}\} \subset \mathcal{L}_{\text{strong}}$ as follows.

Assume that for some ordinal $\alpha < \mathfrak{c}$, we have already chosen $\{b_\beta : \beta < \alpha\} \subset \mathcal{L}_{\text{strong}}$ to be linearly independent over $\mathbb{A}$. Let $S_\alpha := \{b_\beta : \beta < \alpha\}$. Then, $|S_\alpha| < \mathfrak{c}$. Since $\mathbb{A}$ is countable, the claim above shows that $|\text{span}_{\mathbb{A}}(S_\alpha)| < \mathfrak{c}$. However, $|\mathcal{L}_{\text{strong}}| = \mathfrak{c}$, so $\mathcal{L}_{\text{strong}} \setminus \text{span}_{\mathbb{A}}(S_\alpha) \neq \emptyset$. Choose $b_\alpha \in \mathcal{L}_{\text{strong}} \setminus \text{span}_{\mathbb{A}}(S_\alpha)$. By construction, b_α is not in the $\mathbb{A}$-linear span of the previously chosen elements, so the set $\{b_\beta : \beta \leq \alpha\}$ remains linearly independent over $\mathbb{A}$.

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Continuing this process for all $\alpha < \mathfrak{c}$, we obtain a set $B = \{b_\alpha : \alpha < \mathfrak{c}\} \subset \mathcal{L}_{\text{strong}}$ that is linearly independent over $\mathbb{A}$. Finally, since the index set has cardinality $\mathfrak{c}$, we have $|B| = \mathfrak{c}$, as required.

We can now obtain many Liouville number groups.

THEOREM 2.13. *There exist $2^{\mathfrak{c}}$ distinct Liouville number groups in $\mathbb{R}$.*

PROOF. By Lemma 2.11, the set $\mathcal{L}_{\text{strong}}$ of strong Liouville numbers has cardinality $\mathfrak{c}$. We first construct a subset $B \subset \mathcal{L}_{\text{strong}}$ such that $|B| = \mathfrak{c}$ and $\{1\} \cup B$ is linearly independent over the field $\mathbb{A}$ of algebraic numbers.

We do this by transfinite recursion. Assume that for some ordinal $\alpha < \mathfrak{c}$, we have already chosen $\{b_\beta : \beta < \alpha\} \subset \mathcal{L}_{\text{strong}}$ so that $\{1\} \cup \{b_\beta : \beta < \alpha\}$ is linearly independent over $\mathbb{A}$. Set $S_\alpha := \{1\} \cup \{b_\beta : \beta < \alpha\}$. Then, $|S_\alpha| < \mathfrak{c}$. Since $\mathbb{A}$ is countable, its $\mathbb{A}$-linear span also has cardinality $< \mathfrak{c}$. Indeed, every element of $\text{span}_{\mathbb{A}}(S_\alpha)$ is a finite $\mathbb{A}$-linear combination of elements of S_α, so $\text{span}_{\mathbb{A}}(S_\alpha) \subseteq \bigcup_{n=0}^{\infty} \mathbb{A}^n \times S_\alpha^n$.

Hence, $|\text{span}_{\mathbb{A}}(S_\alpha)| \leq \sum_{n=0}^{\infty} |\mathbb{A}|^n |S_\alpha|^n < \mathfrak{c}$. Because $|\mathcal{L}_{\text{strong}}| = \mathfrak{c}$, there exists $b_\alpha \in \mathcal{L}_{\text{strong}} \setminus \text{span}_{\mathbb{A}}(S_\alpha)$. Then, $\{1\} \cup \{b_\beta : \beta \leq \alpha\}$ remains linearly independent over $\mathbb{A}$.

Continuing this construction for all $\alpha < \mathfrak{c}$ yields a set $B = \{b_\alpha : \alpha < \mathfrak{c}\} \subset \mathcal{L}_{\text{strong}}$ such that $|B| = \mathfrak{c}$ and $\{1\} \cup B$ is linearly independent over $\mathbb{A}$.

Now, let $\mathcal{P}_\infty(B) := \{\Delta \subseteq B : \Delta \text{ is infinite}\}$. Since $|B| = \mathfrak{c}$, we have $|\mathcal{P}_\infty(B)| = 2^{\mathfrak{c}}$. For each $\Delta \in \mathcal{P}_\infty(B)$, define

$$G_\Delta := \langle \Delta \rangle = \left\{ \sum_{i=1}^{n} m_i x_i : n \geq 1,\ m_i \in \mathbb{Z},\ x_i \in \Delta \right\} \cup \{0\}.$$

We claim that each G_Δ is a Liouville number group.

First, we show that $G_\Delta \cap \mathbb{Q} = \{0\}$. Let $g = m_1 x_1 + \cdots + m_n x_n \in G_\Delta$ with $m_i \in \mathbb{Z}$ and $x_i \in \Delta$, and suppose that $g \in \mathbb{Q} \setminus \{0\}$. Since $\mathbb{Q} \subseteq \mathbb{A}$,

$$m_1 x_1 + \cdots + m_n x_n - g \cdot 1 = 0.$$

This is a nontrivial $\mathbb{A}$-linear relation among elements of $\{1, x_1, \ldots, x_n\} \subseteq \{1\} \cup B$, because the coefficient of 1 is $-g \neq 0$. This contradicts the $\mathbb{A}$-linear independence of $\{1\} \cup B$. Hence, $G_\Delta \cap \mathbb{Q} = \{0\}$.

Next, let $g \in G_\Delta$ be nonzero. Then, g is a finite integer linear combination of elements of Δ, and every element of Δ is a strong Liouville number. By Corollary 2.10, it follows that g is either rational or Liouville. Since $g \neq 0$ and $G_\Delta \cap \mathbb{Q} = \{0\}$, the rational alternative is impossible. Therefore, g is a Liouville number. Thus, every nonzero element of G_Δ is Liouville, so G_Δ is a Liouville number group.

Finally, we show that distinct infinite subsets yield distinct groups. Let Δ_1, $\Delta_2 \in \mathcal{P}_\infty(B)$ with $\Delta_1 \neq \Delta_2$. Choose $b \in \Delta_2 \setminus \Delta_1$ without loss of generality. Then, $b \in G_{\Delta_2}$. We claim that $b \notin G_{\Delta_1}$. Otherwise, there would exist distinct elements $x_1, \ldots, x_n \in \Delta_1$ and integers $m_1, \ldots, m_n$ such that $b = m_1 x_1 + \cdots + m_n x_n$. Rearranging gives

$$b - m_1 x_1 - \cdots - m_n x_n = 0,$$

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which is a nontrivial $\mathbb{A}$-linear relation among elements of B, which contradicts the $\mathbb{A}$-linear independence of B. Hence, $b \in G_{\Delta_2} \setminus G_{\Delta_1}$, so $G_{\Delta_1} \neq G_{\Delta_2}$.

Therefore, the family $\{G_\Delta : \Delta \in \mathcal{P}_\infty(B)\}$ consists of $2^{\mathfrak{c}}$ pairwise distinct Liouville number groups in $\mathbb{R}$. □

THEOREM 2.14. *There exist $2^{\mathfrak{c}}$ pairwise nonhomeomorphic Liouville number groups.*

PROOF. By Theorem 2.13, there exists a family $\mathcal{G}$ of $2^{\mathfrak{c}}$ pairwise distinct Liouville number groups, each of which is a subspace of $\mathbb{R} \subseteq \mathbb{C}$.

We now use a consequence of Lavrentiev's theorem (see, for example, [10, Theorem A8.5]): if $X \subseteq \mathbb{C}$ is a subspace, then the collection of all subspaces of $\mathbb{C}$ that are homeomorphic to X has cardinality at most $\mathfrak{c}$.

Partition $\mathcal{G}$ into homeomorphism classes. Suppose, towards a contradiction, that fewer than $2^{\mathfrak{c}}$ homeomorphism classes are represented. Then, there exists a cardinal $\lambda < 2^{\mathfrak{c}}$ such that $\mathcal{G}$ is the union of λ homeomorphism classes.

Since each homeomorphism class has cardinality at most $\mathfrak{c}$, it follows that

$$|\mathcal{G}| \leq \lambda \cdot \mathfrak{c} = \max(\lambda, \mathfrak{c}).$$

Because $\lambda < 2^{\mathfrak{c}}$ and $\mathfrak{c} < 2^{\mathfrak{c}}$, we have $\max(\lambda, \mathfrak{c}) < 2^{\mathfrak{c}}$, and, hence, $|\mathcal{G}| < 2^{\mathfrak{c}}$, which contradicts $|\mathcal{G}| = 2^{\mathfrak{c}}$. Therefore, there must be at least $2^{\mathfrak{c}}$ distinct homeomorphism classes among the groups in $\mathcal{G}$. In particular, there exist $2^{\mathfrak{c}}$ pairwise nonhomeomorphic Liouville number groups. □

We next show that there are many countable Liouville number groups homeomorphic, as spaces, to $\mathbb{Q}$.

LEMMA 2.15. *The set $\mathcal{L}_{\text{strong}}$ is dense in $\mathbb{R}$.*

PROOF. Let (u, v) be a nonempty open interval in $\mathbb{R}$. Since irrational numbers are dense in $\mathbb{R}$ and continued-fraction cylinders form a base for the topology on $\mathbb{R} \setminus \mathbb{Q}$, there exists a finite continued-fraction block $[a_0; a_1, \ldots, a_N]$ such that every irrational number whose continued fraction begins with this block lies in (u, v).

We now extend this block recursively. Assume that $a_0, \ldots, a_k$ have been chosen and let q_k be the denominator of the kth convergent determined by these partial quotients. Define $a_{k+1} := q_k^k + 1$ $(k \geq N)$. This produces an irrational number $x = [a_0; a_1, \ldots, a_N, a_{N+1}, a_{N+2}, \ldots]$ whose continued fraction begins with the prescribed block; hence, $x \in (u, v)$.

Let p_k/q_k denote the convergents of x. For $k \geq N$, the standard recurrence gives

$$q_{k+1} = a_{k+1}q_k + q_{k-1} \geq (q_k^k + 1)q_k > q_k^{k+1}.$$

Now, let $n \in \mathbb{N}$ be arbitrary. If $k \geq \max\{N, n\}$, then $q_{k+1} > q_k^{k+1} \geq q_k^n$. Therefore, there exists $N(n) = \max\{N, n\}$ such that $q_{k+1} > q_k^n$ for all $k \geq N(n)$. By Definition 2.1, x is a strong Liouville number.

Since (u, v) was arbitrary, every nonempty open interval contains a strong Liouville number. Hence, $\mathcal{L}_{\text{strong}}$ is dense in $\mathbb{R}$. □

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LEMMA 2.16. *There exist a sequence $(x_n)_{n\geq 1}$ in $\mathcal{L}_{\text{strong}}$ and a set $T \subset \mathcal{L}_{\text{strong}}$ such that:*

(1) $x_n \to 0$;
(2) $|T| = \mathfrak{c}$;
(3) $\{1\} \cup \{x_n : n \in \mathbb{N}\} \cup T$ *is linearly independent over* $\mathbb{A}$.

PROOF. We first note that every nonempty open interval contains $\mathfrak{c}$ many strong Liouville numbers. Indeed, fix a nonempty open interval (u, v). Choose a finite continued-fraction block $[a_0; a_1, \ldots, a_N]$ whose cylinder is contained in (u, v). For each binary sequence $\varepsilon = (\varepsilon_k)_{k\geq N} \in \{0, 1\}^{\mathbb{N}}$, extend this block recursively by

$$a_{k+1} := q_k^k + 1 + \varepsilon_k \qquad (k \geq N),$$

where q_k is the denominator of the kth convergent determined by the initial segment. Exactly as in Lemma 2.11, this defines a strong Liouville number in (u, v), and different binary sequences give different continued fractions. Hence, $|\mathcal{L}_{\text{strong}} \cap (u, v)| = \mathfrak{c}$.

We construct (x_n) recursively. Suppose that $x_1, \ldots, x_{n-1}$ have already been chosen so that $\{1, x_1, \ldots, x_{n-1}\}$ is linearly independent over $\mathbb{A}$. Let

$$V_{n-1} := \text{span}_{\mathbb{A}}(\{1, x_1, \ldots, x_{n-1}\}).$$

Since $\mathbb{A}$ is countable and the generating set is finite, V_{n-1} is countable. However, $|\mathcal{L}_{\text{strong}} \cap (0, 2^{-n})| = \mathfrak{c}$, so $(\mathcal{L}_{\text{strong}} \cap (0, 2^{-n})) \setminus V_{n-1} \neq \emptyset$. Choose

$$x_n \in (\mathcal{L}_{\text{strong}} \cap (0, 2^{-n})) \setminus V_{n-1}.$$

Then, $0 < x_n < 2^{-n}$, so $x_n \to 0$ and, by construction, $\{1, x_1, x_2, \ldots\}$ is linearly independent over $\mathbb{A}$. Let $W := \text{span}_{\mathbb{A}}(\{1\} \cup \{x_n : n \in \mathbb{N}\})$. Since $\mathbb{A}$ is countable and the set $\{1\} \cup \{x_n : n \in \mathbb{N}\}$ is countable, the space W is countable.

We now construct T by transfinite recursion. Since $|\mathcal{L}_{\text{strong}}| = \mathfrak{c}$ by Lemma 2.11, we may choose $T = \{t_\alpha : \alpha < \mathfrak{c}\} \subset \mathcal{L}_{\text{strong}}$ so that for each $\alpha < \mathfrak{c}$,

$$t_\alpha \notin \text{span}_{\mathbb{A}}(W \cup \{t_\beta : \beta < \alpha\}).$$

This is possible because for each $\alpha < \mathfrak{c}$, the set $\text{span}_{\mathbb{A}}(W \cup \{t_\beta : \beta < \alpha\})$ has cardinality $< \mathfrak{c}$, whereas $\mathcal{L}_{\text{strong}}$ has cardinality $\mathfrak{c}$. Then, it follows that $|T| = \mathfrak{c}$ and, by construction, $\{1\} \cup \{x_n : n \in \mathbb{N}\} \cup T$ is linearly independent over $\mathbb{A}$. □

THEOREM 2.17. *There exist $\mathfrak{c}$ pairwise distinct countably infinite Liouville number groups $\{G_\tau : \tau \in T\}$ such that each G_τ is generated by countably infinitely many strong Liouville numbers and is homeomorphic, as a topological space, to $\mathbb{Q}$.*

PROOF. Choose (x_n) and T as in Lemma 2.16. For each $\tau \in T$, define the group $G_\tau := \text{gp}(\{x_n : n \in \mathbb{N}\} \cup \{\tau\})$. Each G_τ is countable, being generated by a countable set. Moreover, G_τ is infinite because it contains the infinitely many distinct nonzero elements x_n.

We claim that G_τ is a Liouville number group. Let $g \in G_\tau \setminus \{0\}$. Then, $g = m\tau + \sum_{j=1}^{n} m_j x_j$ for some integers $m, m_1, \ldots, m_n$. By Corollary 2.10, g is either rational or Liouville. If g were rational, then $m\tau + \sum_{j=1}^{n} m_j x_j - g \cdot 1 = 0$ would be a nontrivial

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$\mathbb{A}$-linear relation among elements of $\{1, \tau, x_1, \ldots, x_n\} \subseteq \{1\} \cup \{x_n : n \in \mathbb{N}\} \cup T$, contrary to Lemma 2.16. Hence, g is Liouville. Therefore, every nonzero element of G_τ is Liouville, so G_τ is a Liouville number group.

Since $x_n \in G_\tau$ for all n and $x_n \to 0$, the point 0 is not isolated in G_τ. Because G_τ is a topological group, translation by any element is a homeomorphism, so no point of G_τ is isolated. Thus, G_τ is a countable metrisable space with no isolated points. By the Sierpiński–Alexandroff–Urysohn theorem, G_τ is homeomorphic to $\mathbb{Q}$.

Finally, if $\tau \neq \sigma$ in T and $G_\tau = G_\sigma$, then $\tau \in G_\sigma = \mathrm{gp}(\{x_n : n \in \mathbb{N}\} \cup \{\sigma\})$, so $\tau = m\sigma + \sum_{j=1}^{n} m_j x_j$ for some integers $m, m_1, \ldots, m_n$. Hence, $\tau - m\sigma - \sum_{j=1}^{n} m_j x_j = 0$, which is a nontrivial $\mathbb{A}$-linear relation among elements of $\{1\} \cup \{x_n : n \in \mathbb{N}\} \cup T$, contrary to Lemma 2.16. Therefore, $G_\tau \neq G_\sigma$ whenever $\tau \neq \sigma$.

Since $|T| = \mathfrak{c}$, the family $\{G_\tau : \tau \in T\}$ consists of $\mathfrak{c}$ pairwise distinct countably infinite Liouville number groups, each generated by countably infinitely many strong Liouville numbers and each homeomorphic to $\mathbb{Q}$. □

REMARK 2.18. The groups in Theorem 2.17 are homeomorphic to $\mathbb{Q}$ as topological spaces, but they are not isomorphic to $\mathbb{Q}$ as groups (and hence, not as topological groups). Indeed, each G_τ is a free abelian group on countably many generators, whereas $\mathbb{Q}$ is divisible.

We now answer a natural topological question.

THEOREM 2.19. *There is no subgroup of $\mathbb{R}$ homeomorphic to $\mathcal{L}$ or to $\mathbb{P}$. In particular, no Liouville number group is homeomorphic to $\mathcal{L}$.*

PROOF. Let G be a subgroup of $\mathbb{R}$. By [2], the space $\mathcal{L}$ of Liouville numbers is homeomorphic to the space $\mathbb{P}$ of irrational real numbers. Thus, it suffices to rule out subgroups of $\mathbb{R}$ homeomorphic to $\mathcal{L}$.

If $G = \mathbb{R}$, then G is connected, whereas $\mathcal{L}$ is totally disconnected, so G is not homeomorphic to $\mathcal{L}$.

Assume therefore that G is a proper subgroup of $\mathbb{R}$. By [7, Proposition 19], every subgroup of $\mathbb{R}$ is either dense or closed. If G is closed, then by [7, Proposition 20], it is discrete; hence, not homeomorphic to $\mathcal{L}$.

Assume finally that G is dense and homeomorphic to $\mathcal{L}$. Since $\mathcal{L}$ is homeomorphic to $\mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{P}$ is a G_δ subset of $\mathbb{R}$, the space $\mathbb{P}$ is completely metrisable. Hence, $\mathcal{L}$ is completely metrisable and, therefore, so is G. Now, G is a completely metrisable subspace of the complete metric space $\mathbb{R}$. By the classical Alexandrov theorem, a subspace of a complete metric space is completely metrisable if and only if it is a G_δ subset. Thus, G is a dense G_δ subset of $\mathbb{R}$. By Erdős [3], every dense G_δ subset of $\mathbb{R}$ has the Erdős property: every real number is the sum of two of its elements. However, no proper subgroup of $\mathbb{R}$ has this property, because if $x \notin G$, then x cannot be a sum of two elements of G. This contradiction shows that no proper subgroup of $\mathbb{R}$ is homeomorphic to $\mathcal{L}$.

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Therefore, there is no subgroup of $\mathbb{R}$ homeomorphic to $\mathcal{L}$ and, hence, none homeomorphic to $\mathbb{P}$. In particular, no Liouville number group is homeomorphic to $\mathcal{L}$. □

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