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A class of pairwise-independent joinings

ÉLISE JANVRESSE and THIERRY DE LA RUE

Laboratoire de Mathématiques Raphaël Salem, UMR 6085 CNRS – Université de Rouen, Avenue de l'Université, B.P. 12, F76801 Saint-Étienne-du-Rouvray Cedex, France
(e-mail: Elise.Janvresse@univ-rouen.fr, Thierry.de-la-Rue@univ-rouen.fr)

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Dedicated to the memory of Professor José de Sam Lazaro

Abstract. We introduce a special class of pairwise-independent self-joinings for a stationary process: those for which one coordinate is a continuous function of the two others. We investigate which properties on the process the existence of such a joining entails. In particular, we prove that if the process is aperiodic, then it has positive entropy. Our other results suggest that such pairwise independent, non-independent self-joinings exist only in very specific situations: essentially when the process is a subshift of finite type topologically conjugate to a full-shift. This provides an argument in favor of the conjecture that two-fold mixing implies three-fold-mixing.

1. *Introduction*

The purpose of this paper is to study in which cases it is possible to construct some special self-joinings for a stationary process. Throughout the text, $\xi = (\xi_i)_{i\in\mathbb{Z}}$ is a stationary process taking values in the finite alphabet $\mathbb{A}$, and μ is its probability distribution on $\mathbb{A}^{\mathbb{Z}}$. For any $i < j$, we denote by ξ_i^j the sequence of random variables $(\xi_i, \xi_{i+1}, \ldots, \xi_j)$.

Definition 1.1. Let $k \geq 2$. A *k-fold self-joining* of ξ is a stationary probability distribution on $(\mathbb{A}^k)^{\mathbb{Z}}$ (canonically identified with $(\mathbb{A}^{\mathbb{Z}})^k$), whose k marginals on $\mathbb{A}^{\mathbb{Z}}$ are all equal to μ.

In what follows, we will mostly concentrate on two-fold and three-fold self-joinings. A three-fold self-joining of ξ can be viewed as the joint distribution of three processes, ξ, ξ' and ξ'', with the same distribution μ, provided this joint distribution is stationary.

1.1. *Pairwise-independent joinings.*

Definition 1.2. We define a *pairwise-independent joining of* ξ to be a three-fold self-joining of ξ such that the three two-dimensional marginals are equal to $\mu \otimes \mu$.

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An obvious example of a pairwise-independent joining of ξ is the product measure $\mu \otimes \mu \otimes \mu$, corresponding to the case where ξ, ξ' and ξ'' are independent.

Two other classical examples arise for particular processes ξ.

Example 1.3. When $\mathbb{A} = \mathbb{Z}/n\mathbb{Z}$ and ξ is a uniform Bernoulli process (i.e. the random variables ξ_i are independent and identically distributed (i.i.d.) and uniformly distributed in $\mathbb{A}$), we can take ξ' as an independent copy of ξ, and set, for all $i \in \mathbb{Z}$,

$$\xi''_i := \xi_i + \xi'_i \mod n.$$

Then ξ'' follows the same distribution as ξ and ξ', and is independent from ξ and from ξ', but the joint distribution is clearly not the product measure, since ξ'' is determined by ξ and ξ'.

Example 1.4. When $\mathbb{A} = \mathbb{Z}/n\mathbb{Z}$ for an odd integer n, and ξ is a periodic process satisfying $\xi_{i+1} = \xi_i + 1 \mod n$ for all i, we can also take an independent copy ξ' of ξ, and set, for all $i \in \mathbb{Z}$,

$$\xi''_i := 2\xi_i - \xi'_i \mod n.$$

The same conclusion follows for the joint distribution of ξ, ξ' and ξ''.

Whether there exist examples of a different nature is an outstanding problem in ergodic theory, which can be stated as follows.

QUESTION 1.5. *Does there exist a zero-entropy, weakly mixing, stationary process ξ with a pairwise-independent joining λ, $\lambda \neq \mu \otimes \mu \otimes \mu$?*

Observe that the two preceding examples do not belong to the category of zero-entropy, weakly mixing processes. The above open question is related to two important problems in ergodic theory: whether two-fold mixing implies three-fold mixing, and whether two-fold minimal self-joinings implies three-fold minimal self-joinings. Indeed, a negative answer to Question 1.5 would prove both implications, since any counterexample to one of these problems would automatically lead to a non-trivial pairwise-independent joining.

Many important results have already been obtained on this question. Most of them consider some special category of stationary processes (e.g. finite-rank processes [**3, 10**], or processes with singular spectrum [**2**]), and prove that in this category a pairwise-independent three-fold self-joining has to be the product measure. In the present work, we have a slightly different approach: consider a stationary process which admits some kind of pairwise-independent three-fold self-joining which is not the product measure, and see which other properties on the process this assumption entails.

1.2. *The PIJ property.* The special kind of pairwise-independent joining we want to study is the case where the third coordinate ξ'' is determined in a continuous way by the first two.

Definition 1.6. We say that the process ξ has the *PIJ property* if there exist a pairwise-independent joining λ of ξ and a continuous function $\varphi : \mathbb{A}^{\mathbb{Z}} \times \mathbb{A}^{\mathbb{Z}} \to \mathbb{A}$ satisfying

$$\xi''_0 = \varphi(\xi, \xi') \quad (\lambda\text{-almost everywhere}). \tag{1}$$

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This amounts to saying that there exists some $p \in \mathbb{N}$ such that

$$\xi''_0 = \varphi(\xi^{\,p}_{-p}, \xi'^{\,p}_{-p}) \quad (\lambda\text{-almost everywhere}). \tag{2}$$

We will sometimes refer to this integer p as the *width* of the joining λ.

The two examples given above (uniform Bernoulli and periodic process) obviously have the PIJ property, with $p = 0$. It is straightforward to check that the PIJ property is preserved by topological conjugacy and independent product. Therefore, we get from these examples a whole family of PIJ processes, which are all subshifts of finite type. This leads to the following question.

QUESTION 1.7. *Do there exist processes enjoying the PIJ property other than the processes arising from Examples 1.3 and 1.4?*

A connected problem was raised by Ryzhikov [**9**] who asked under which conditions it is possible to find a process $\zeta = \varphi(\xi, \xi')$ independent from ξ and from ξ' (where ξ and ξ' are two independent copies of the same process).

1.3. *Results.* Our first result, presented in §2, states that a process which has the PIJ property is either periodic or has positive entropy. Therefore, no process possessing the PIJ property can answer positively to Question 1.5. This generalizes the results obtained in [**7, 8**] in the case where the width of the joining is zero. Note that the first result in this direction was obtained by Ryzhikov in 1993: de Sam Lazaro, Thouvenot and Ryzhikov noticed that [**11**, Assertion 4.4] could be reinterpreted in the following elegant form: if ξ, ξ' and $\xi + \xi'$ are pairwise-independent two-valued processes, then ξ is Bernoulli or periodic.

In §3, we study irreducible aperiodic subshifts of finite type having the PIJ property. We show that their Perron value is an integer, which implies that they are measure-theoretically isomorphic to a uniform Bernoulli shift (cf. Example 1.3). We even prove a stronger result: any irreducible aperiodic subshift of finite type with the PIJ property is shift-equivalent† to a full shift.

Finally, we ask whether any process with the PIJ property is a subshift of finite type. We prove in §4 that a slightly stronger property than PIJ is sufficient: if, in the definition of the PIJ property, we further require that the first coordinate be a continuous function of ξ' and ξ'', then the process is a subshift of finite type.

All these results lead to the conjecture that there are no other processes with the PIJ property than those which arise from our two examples by topological conjugacy or direct product.

2. *General consequences of the PIJ property*

2.1. *Zero-entropy implies periodicity.*

THEOREM 2.1. *If ξ has the PIJ property, then:*

- *either ξ has positive entropy;*
- *or ξ is periodic.*

† Williams [**12**] conjectured that shift equivalence was the same as topological conjugacy, which was disproved by Kim and Roush [**4**]. Even so, Williams' conjecture could be true when one of the subshift is the full shift.

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One of the main ingredients for the proof of this theorem is the property of *quasi-uniformity* of the measure μ which is a consequence of the PIJ property.

We define a *cylinder of length* ℓ to be any event of the form $(\xi_i = a_i, \dots, \xi_{i+\ell-1} = a_{i+\ell-1})$, provided that the sequence $(a_i, \dots, a_{i+\ell-1})$ is seen with positive probability. This cylinder will be denoted by $[a_i^{i+\ell-1}]$. The set of all words of length ℓ appearing with positive probability will be denoted by $\mathcal{L}_\ell$.

Definition 2.2. The probability distribution μ is said to be *quasi-uniform* if there exists a constant $K > 0$ such that, if B and C are two cylinders of the same length,

$$\frac{1}{K}\mu(B) \le \mu(C) \le K\mu(B). \tag{3}$$

LEMMA 2.3. *Assume that ξ has the PIJ property with some joining λ of width p.*

(i) *If A and B are two cylinders with respective lengths $\ell \ge 1$ and $\ell + 2p$, then $\mu(A) \ge \mu(B)$.*

(ii) *μ is quasi-uniform.*

(iii) *μ has maximal entropy among all invariant probability measures with the same support.*

Proof. Since μ is shift-invariant, there is no loss of generality in assuming that A is of the form $[a_0^{\ell-1}]$ and $B = [b_{-p}^{p+\ell-1}]$. Under λ, ξ' and ξ'' are independent, so

$$\lambda(\xi' \in B, \xi'' \in A) > 0.$$

We can thus find some $\theta > 0$ small enough such that

$$C_\theta := \{\xi \in \mathbb{A}^{\mathbb{Z}} \mid \lambda(\xi' \in B, \xi'' \in A|\xi) > \theta\}$$

has positive probability. By (2) and by definition of C_θ,

$$(\xi' \in B, \xi \in C_\theta) \subset (\xi'' \in A, \xi \in C_\theta).$$

Using pairwise independence, we get

$$\mu(B)\mu(C_\theta) = \lambda(\xi' \in B, \xi \in C_\theta) \le \lambda(\xi'' \in A, \xi \in C_\theta) = \mu(A)\mu(C_\theta).$$

Hence $\mu(B) \le \mu(A)$.

Let $A_1 = [a_0^{\ell-1}]$ and A_2 be two cylinder sets of the same length ℓ. Applying the first part of the lemma, we obtain

$$\mu(A_1) = \sum_{a_{-p}^{-1}, a_\ell^{\ell+p-1} :\, a_{-p}^{\ell+p-1} \in \mathcal{L}_{\ell+2p}} \mu([a_{-p}^{\ell+p-1}]) \le |\mathbb{A}|^{2p}\mu(A_2),$$

which proves (ii).

It follows that the entropy of ξ is

$$h(\xi) = \lim_{\ell\to\infty} \frac{1}{\ell} \log |\mathcal{L}_\ell|,$$

which clearly achieves the maximum possible entropy for invariant measures with the same support as μ. □

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Proof of Theorem 2.1. Let us assume that ξ is a zero-entropy process satisfying the PIJ property, with the three-fold self-joining λ of width p. We have to prove that ξ is a periodic process. Without loss of generality, we can assume that $p > 0$.

We know that ξ is quasi-uniform. Let $K > 0$ be a constant satisfying (3) for all cylinders B and C of the same length. Observe that, if X is a random variable taking values in $\mathbb{A}$ such that, for all $a, b \in \mathbb{A}$,

$$\mathbb{P}(X = a)\mathbb{P}(X = b) > 0 \Longrightarrow \frac{1}{K}\mathbb{P}(X = a) \leq \mathbb{P}(X = b) \leq K\mathbb{P}(X = a), \tag{4}$$

then there exists $\delta > 0$ satisfying

$$H(X) < \delta \Longrightarrow \text{there exists } a \in \mathbb{A}, \quad \mathbb{P}(X = a) = 1. \tag{5}$$

We now consider the distribution of ξ_0 conditioned on $\xi_{-1}, \ldots, \xi_{-k}$, $k \geq 1$. It obviously satisfies (4) for any choice of $\xi_{-1}, \ldots, \xi_{-k}$. Moreover, since ξ has zero entropy, the measure of the set of sequences $(\xi_{-1}, \ldots, \xi_{-k})$ such that $H(\xi_0|\xi_{-1}, \ldots, \xi_{-k}) < \delta$ tends to 1 as $k \to \infty$:

$$\mu(H(\xi_0|\xi_{-1}, \ldots, \xi_{-k}) < \delta) \xrightarrow[k\to\infty]{} 1.$$

Therefore,

$$\mu(\{[a_{-k}^{-1}] \mid \xi_0 \text{ is determined by } (\xi_{-k}^{-1} = a_{-k}^{-1})\}) \xrightarrow[k\to\infty]{} 1,$$

where we say that a random variable X is *determined* by $(Y = y)$ if there exists a value a such that $\mathbb{P}(X = a|Y = y) = 1$. Let us define the sets of cylinders

$$\mathscr{C}_k := \{[a_{-k}^k] \mid \xi_{-k-p}^{k+p} \text{ is determined by } (\xi_{-k}^k = a_{-k}^k)\}.$$

Observe that all cylinders in $\mathscr{C}_k$ have the same measure μ_k: indeed, let A and B be two cylinders in $\mathscr{C}_k$ and A' be the cylinder of length $(2k + 1) + 2p$ such that

$$\xi_{-k}^k \in A \Longleftrightarrow \xi_{-k-p}^{k+p} \in A'.$$

By Lemma 2.3, we have $\mu(A) = \mu(A') \leq \mu(B)$. By symmetry, we conclude that $\mu(A) = \mu(B)$.

Moreover, $\mu(\mathscr{C}_k) \to 1$. Hence, for k large enough, there exists a cylinder $[a_{-k}^k] \in \mathscr{C}_k$ such that the cylinder $[a_{-k-1}^{k+1}]$ determined by $[a_{-k}^k]$ is in $\mathscr{C}_{k+1}$. This implies that $\mu_k = \mu_{k+1}$. Let us denote by $\mu_\infty > 0$ the ultimate value of the μ_k. If k is large enough so that $\mu(\mathscr{C}_k) > 1 - \mu_\infty$, then $\mu(\mathscr{C}_k) = 1$.

We conclude that, for k large enough, ξ_{-k-p}^{k+p} is always determined by ξ_{-k}^k. This means that ξ is periodic. □

2.2. *An algebraic lemma and its consequence on PIJ processes.* We present in this section an algebraic lemma which can be stated in a very simple context. Let F and F' be two finite sets, and let C be a subset of $F \times F'$. We are interested in the situation where we can find two probability measures m on F and m' on F' such that, under the product distribution $m \otimes m'$, 1_C is independent of each coordinate. In other words, we demand

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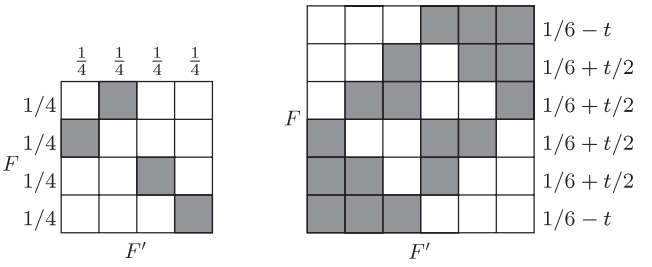


FIGURE 1. Let C be the colored elements in $F \times F'$. We provide examples of pairs of probability measures satisfying (6). On the left, $m \otimes m'(C) = 1/4$. On the right, there is a continuous family of probability measures satisfying (6) (the measure on F' is defined symmetrically, with a possibly different parameter), but for each pair of them, $m \otimes m'(C) = 1/2$.

that there exist two finite families of non-negative numbers $(m(x))_{x\in F}$ and $(m'(x'))_{x'\in F'}$ such that

$$\begin{cases} \displaystyle\sum_{x\in F} m(x) = \sum_{x'\in F'} m'(x') = 1 \\ \text{for all } x_0 \in F, \text{ for all } x_0' \in F', \quad \displaystyle\sum_{x:(x,x_0')\in C} m(x) = \sum_{x':(x_0,x')\in C} m'(x'). \end{cases} \tag{6}$$

It may happen that many pairs (m, m') satisfy this requirement (see Figure 1), but, as the following lemma shows, $m \otimes m'(C)$ does not depend on the particular choice of the pair (m, m'). Moreover, the upshot is that this quantity is always a rational number†!

LEMMA 2.4. *Let (m_1, m_1') and (m_2, m_2') be two pairs of probability measures which satisfy (6). Then, $m_1 \otimes m_1'(C) = m_2 \otimes m_2'(C) \in \mathbb{Q}$.*

Proof. We fix $x_0 \in F$. We have

$$m_1 \otimes m_1'(C) = \sum_{x\in F} m_1(x) \sum_{x':(x,x')\in C} m_1'(x') = \sum_{x':(x_0,x')\in C} m_1'(x') = m_2 \otimes m_1'(C).$$

The same argument shows that $m_2 \otimes m_1'(C) = m_2 \otimes m_2'(C)$.

Notice that (6) is a system of linear equations with rational coefficients. Therefore, if it has a solution in $\mathbb{R}$, it also has a solution in $\mathbb{Q}$. □

The preceding lemma has a direct consequence on the process ξ if we assume that it has the PIJ property. Let $\ell \geq 1$ be an integer, and denote by F (respectively F') the set of all cylinders $[x_{-p}^{\ell+p-1}]$ of length $\ell + 2p$ for ξ (respectively ξ'). Then any cylinder $C = [c_0^{\ell-1}]$ of length ℓ for ξ'' can be identified *via* the special pairwise-independent joining λ with a subset of $F \times F'$ which is independent of ξ and independent of ξ'. Therefore, we get the following corollary.

COROLLARY 2.5. *If ξ has the PIJ property, then, for any cylinder C,*

$$\mu(C) \in \mathbb{Q}.$$

† It would be interesting to bound the denominator of $m \otimes m'(C)$ by a (possibly linear) function of the sizes of F and F'.

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3. *Subshifts of finite type with the PIJ property*

We recall that a *subshift of finite type* is a subset Σ of $\mathbb{A}^{\mathbb{Z}}$ whose elements are all sequences in which a finite number of forbidden patterns never appear. We may always assume that all forbidden patterns have length 2: replace if necessary $\mathbb{A}$ by $\mathbb{A}^{\ell-1}$, where ℓ is the largest length of a forbidden word. Then the subshift of finite type can be defined by an $|\mathbb{A}|$-by-$|\mathbb{A}|$ matrix M, with 0–1 entries, called the *adjacency matrix*: $M_{ab} = 1$ if and only if ab is an allowed pattern.

The subshift of finite type is *irreducible* if, for all $a, b \in \mathbb{A}$, we can find $k \geq 1$ such that $M^k_{ab} \geq 1$. In this case, there exists on Σ a unique shift-invariant measure of maximal entropy, and the stationary process it defines is quasi-uniform (see [**5**, p. 166]). This and the fact that all the examples of processes having the PIJ property that we know arise from subshifts of finite type justify our interest in the links between the PIJ property and subshifts of finite type.

Observe that there can exist only one quasi-uniform measure whose support is a given irreducible subshift of finite type. Indeed, any such measure would be equivalent to the maximal-entropy measure. Since the latter is ergodic, these measures are equal. Note also that the motivations for the study of the PIJ property allow us to restrict our interest to weakly mixing processes. Therefore, we can always assume that the subshift of finite type we consider is aperiodic.

In what follows, we only consider subshifts of finite type which are irreducible and aperiodic, endowed with their probability measure of maximal entropy, and use the abbreviation IASFT to denote them.

3.1. *The Perron value.* Let β be the largest eigenvalue of the adjacency matrix M, called its *Perron value*. Let $\ell = (\ell_1, \ldots, \ell_{|\mathbb{A}|})$ (respectively $r = (r_1, \ldots, r_{|\mathbb{A}|})$) be a left (respectively right) Perron eigenvector normalized such that $\sum \ell_i r_i = 1$. We recall (see [**5**, p. 166]) that the measure of maximal entropy satisfies, for any cylinder $[a_0^k]$,

$$\mu([a_0^k]) = \frac{\ell_{a_0} r_{a_k}}{\beta^k}. \tag{7}$$

THEOREM 3.1. *Let ξ be an IASFT with the PIJ property. Then the Perron value of the adjacency matrix is an integer.*

Proof. Fix $a \in \mathbb{A}$. We know from Corollary 2.5 that all cylinders have rational probabilities. Hence, $\ell_a r_a \in \mathbb{Q}$ and $\ell_a r_a / \beta^k \in \mathbb{Q}$ as soon as $M^k_{aa} > 0$.

Since the subshift is aperiodic, $\gcd(\{k \mid M^k_{aa} > 0\}) = 1$. Therefore, there exist integers $t_1, \ldots, t_j$ and $k_1, \ldots, k_j$ such that $M^{k_i}_{aa} > 0$ and $\sum_{i=1}^{j} t_i k_i = 1$. It follows that $\beta^{k_i} \in \mathbb{Q}$ for all $1 \leq i \leq j$, and $\beta^{\sum t_i k_i} = \beta \in \mathbb{Q}$.

We let the reader check that if a square matrix with integer coefficients has a rational eigenvalue β, then $\beta \in \mathbb{Z}$. Hence, the Perron value of M is a positive integer. □

Note that if we drop the aperiodicity assumption, the fact that all cylinders have rational probabilities does not imply that the Perron value is an integer: consider the subshift of finite type whose adjacency matrix is

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$$M := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

with Perron value $\sqrt{2}$.

Theorem 3.1 means that, if ξ is an IASFT with the PIJ property, then there exists a positive integer n such that $h(\xi) = \log n$. Since the IASFT we consider is a mixing Markov process, it satisfies the weak-Bernoulli property, and thus is measure-theoretically isomorphic to the full n-shift described in Example 1.3 (see [**1**]).

The conclusion of Theorem 3.1 is also equivalent to the following property (see [**6**, p. 156]): the IASFT is topologically conjugate to another IASFT for which each vertex has n incoming edges and n outcoming edges. We will call such a subshift a *uniform subshift*.

We may ask whether any IASFT with integer Perron value satisfies the PIJ property. The following section shows that it is not true. From now on, we will denote by n instead of β the Perron value of the adjacency matrix M.

3.2. *Independence property.*

THEOREM 3.2. *Let ξ be an IASFT with the PIJ property. For k large enough, $\xi_{-\infty}^{0}$ and $\xi_{k}^{+\infty}$ are independent.*

Proof. Without loss of generality, we assume that ξ is a uniform subshift. Indeed, the independence property is preserved by topological conjugacy (with a different k). Therefore, for any cylinder $[a_0^k]$, $\mu([a_0^k]) = (|\mathbb{A}|n^k)^{-1}$.

We consider a three-fold, pairwise-independent self-joining λ of width p such that $\xi''_0{}^k = \varphi(\xi_{-p}^{k+p}, \xi'_{-p}{}^{k+p})$ λ-almost surely for any $k \geq 0$.

Let us fix ξ_{-p}^{k+p}. Then $\xi''_0{}^k$ is a function of $\xi'_{-p}{}^{k+p}$. We will first compute the number of preimages of $\xi''_0{}^k$:

$$\text{for all } \xi_{-p}^{k+p}, \text{ for all } a_0^k, \quad |\{\xi'_{-p}{}^{k+p} \mid \varphi(\xi_{-p}^{k+p}, \xi'_{-p}{}^{k+p}) = a_0^k\}| = n^{2p}. \tag{8}$$

Indeed, for all cylinders $[a_0^k]$, by independence of ξ and ξ'',

$$\lambda(\xi''_0{}^k = a_0^k | \xi_{-p}^{k+p}) = \mu(\xi''_0{}^k = a_0^k) = \frac{1}{|\mathbb{A}|n^k}.$$

On the other hand, this is also equal to

$$\sum_{\xi'_{-p}{}^{k+p} : \varphi(\xi_{-p}^{k+p}, \xi'_{-p}{}^{k+p}) = a_0^k} \mu([\xi'_{-p}{}^{k+p}]) = \frac{1}{|\mathbb{A}|n^{k+2p}} |\{\xi'_{-p}{}^{k+p} \mid \varphi(\xi_{-p}^{k+p}, \xi'_{-p}{}^{k+p}) = a_0^k\}|,$$

which proves (8).

Recall that M_{ab}^k is the number of cylinders of length $k+1$ starting in a and ending in b. Since the IASFT is irreducible aperiodic, there exists k large enough such that $M_{ab}^{k-2p} > 0$ for any $a, b \in \mathbb{A}$.

Let us fix a_0 and a_k in $\mathbb{A}$. Since there are $M_{a_0 a_k}^k$ cylinders of length $k+1$ starting in a_0 and ending in a_k, we deduce from (8) that, once ξ_{-p}^{k+p} is fixed, there are exactly $M_{a_0 a_k}^k n^{2p}$

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cylinders $[\xi''^{k+p}_{-p}]$ yielding $\xi''_0 = a_0$ and $\xi''_k = a_k$. But these cylinders are characterized by $\varphi(\xi^p_{-p}, \xi'^p_{-p}) = a_0$ and $\varphi(\xi^{k+p}_{k-p}, \xi'^{k+p}_{k-p}) = a_k$. Once ξ'^p_{-p} and ξ'^{k+p}_{k-p} are chosen, there are $M^{k-2p}_{\xi'_p \xi'_{k-p}}$ ways of completing the cylinders. We thus obtain

$$M^k_{a_0 a_k} n^{2p} = \sum_{\xi'^p_{-p}} \sum_{\xi'^{k+p}_{k-p}} M^{k-2p}_{\xi'_p \xi'_{k-p}} 1_{\{\varphi(\xi^p_{-p}, \xi'^p_{-p}) = a_0\}} 1_{\{\varphi(\xi^{k+p}_{k-p}, \xi'^{k+p}_{k-p}) = a_k\}}.$$

Observe that there are $|\mathbb{A}|^2 n^{4p}$ ways of choosing ξ^p_{-p} and ξ^{k+p}_{k-p}, because $M^{k-2p}_{ab} > 0$ for any $a, b \in \mathbb{A}$. Therefore, summing over all these possible choices, we get that $M^k_{a_0 a_k} |\mathbb{A}|^2 n^{6p}$ is equal to

$$\sum_{\xi'^p_{-p}} \sum_{\xi'^{k+p}_{k-p}} M^{k-2p}_{\xi'_p \xi'_{k-p}} \sum_{\xi^p_{-p}} 1_{\{\varphi(\xi^p_{-p}, \xi'^p_{-p}) = a_0\}} \sum_{\xi^{k+p}_{k-p}} 1_{\{\varphi(\xi^{k+p}_{k-p}, \xi'^{k+p}_{k-p}) = a_k\}}.$$

Exchanging the roles of ξ and ξ' in (8), we have

$$\sum_{\xi^p_{-p}} 1_{\{\varphi(\xi^p_{-p}, \xi'^p_{-p}) = a_0\}} = \sum_{\xi^{k+p}_{k-p}} 1_{\{\varphi(\xi^{k+p}_{k-p}, \xi'^{k+p}_{k-p}) = a_k\}} = n^{2p}.$$

We conclude that

$$M^k_{a_0 a_k} = \frac{1}{|\mathbb{A}|^2 n^{2p}} \sum_{\xi'^p_{-p}} \sum_{\xi'^{k+p}_{k-p}} M^{k-2p}_{\xi'_p \xi'_{k-p}},$$

which does not depend on a_0 and a_k. Let us denote by C_k the common value of all coefficients in M^k. We can compute C_k by recalling that all cylinders of length $k+1$ have the same mass $(|\mathbb{A}| n^k)^{-1}$. Since there are exactly $C_k |\mathbb{A}|^2$ such cylinders, $C_k = n^k / |\mathbb{A}|$.

Let us see why this implies the independence property for ξ. For any $j \geq 0$,

$$\begin{aligned} \mu(\xi^0_{-j} = a^0_{-j}, \xi^{k+j}_k = a^{k+j}_k) &= C_k \frac{1}{|\mathbb{A}| n^{k+2j}} = \left(\frac{1}{|\mathbb{A}| n^j}\right)^2 \\ &= \mu(\xi^0_{-j} = a^0_{-j}) \mu(\xi^{k+j}_k = a^{k+j}_k). \end{aligned}$$

□

Note that since $C_k = n^k / |\mathbb{A}|$ is an integer we get the following corollary.

COROLLARY 3.3. *Let ξ be a uniform IASFT with the PIJ property. Then, for k large enough, $|\mathbb{A}|$ divides n^k, where n is the Perron value of ξ.*

Example 3.4. (The triangle subshift, see Figure 2) Consider the IASFT on the alphabet $\mathbb{A} = \{0, 1, 2\}$ with adjacency matrix

$$M := \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

This is a uniform IASFT whose Perron value is $n = 2$. In view of Corollary 3.3, it does not satisfy the PIJ property.

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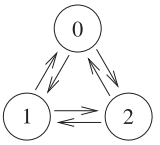


FIGURE 2. A graph defining the triangle subshift of finite type.

3.3. *Shift-equivalence to a full shift.* We now wish to investigate whether any IASFT with the PIJ property is topologically conjugate to the full shift on n symbols (where n is the Perron value of the adjacency matrix). Deciding whether two IASFTs are topologically conjugate is a difficult problem, for which the work of Williams [**12**] gives partial answers. Williams introduced purely algebraic properties for square matrices A and B with entries in $\mathbb{Z}^+$. The matrices A and B are said to be *strong shift equivalent* if there exist non-negative integer matrices $R_1, \ldots, R_m$ and $S_1, \ldots, S_m$ with

$$A = R_1 S_1, \quad S_1 R_1 = R_2 S_2, \ldots, \quad S_m R_m = B.$$

Two IASFTs are topologically conjugate if and only if their adjacency matrices are strong shift equivalent. However, strong shift equivalence is not easy to check. That is why the weaker notion of shift equivalence has been introduced: A is *shift equivalent* to B if there are two non-negative integer matrices R and S and an integer m such that $RA = BR$, $SB = AS$, $SR = A^m$ and $RS = B^m$.

THEOREM 3.5. *Let ξ be an IASFT with the PIJ property. Then it is shift-equivalent to the full shift on n symbols, where n is the Perron value of ξ.*

Proof. Without loss of generality, we may still assume that ξ is a uniform subshift. Let M be the adjacency matrix of ξ. We have shown in the proof of Theorem 3.2 that there exists a large enough integer k such that all the entries of M^k are equal. Hence the eigenvalues of M^k are 0 (of order $|\mathbb{A}| - 1$) and n^k (of order 1), and the characteristic polynomial of M is $X^{|\mathbb{A}|-1}(X - n)$. Kitchens [**5**, Lemma 2.2.6, p. 49] then ensures that M is shift equivalent to the 1-by-1 matrix (n), which itself is shift-equivalent to the n-by-n matrix whose entries are all equal to 1. □

Williams had conjectured in 1973 that shift equivalence was the same as topological conjugacy. Kim and Roush have found a counterexample in 1997 [**4**] which does not involve a full shift. If Williams' conjecture is true when one of the IASFTs is the full shift, then any IASFT with the PIJ property is topologically conjugate to the full shift.

Figure 3 shows a graph defining a uniform subshift of finite type which is shift-equivalent to the full 2-shift. This example was proposed by Ashley in 1989. It is not known whether it is topologically conjugate to the full 2-shift (see [**5**, p. 50]). Neither do we know whether this IASFT has the PIJ property.

4. *Does PIJ imply subshift of finite type?*

We do not know yet if the PIJ property implies that the process is a subshift of finite type. However, it is true for a slight strengthening of the PIJ property.

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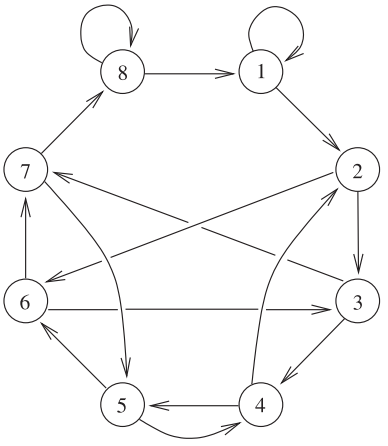


FIGURE 3. The Ashley subshift.

Definition 4.1. We say that the process ξ has the *PIJ* property* if there exist a pairwise-independent joining λ of ξ and two continuous functions $\varphi_1, \varphi_2 : \mathbb{A}^{\mathbb{Z}} \times \mathbb{A}^{\mathbb{Z}} \to \mathbb{A}$ satisfying

$$\xi_0'' = \varphi_1(\xi, \xi') \quad \text{and} \quad \xi_0 = \varphi_2(\xi', \xi'') \quad (\lambda\text{-almost everywhere}). \tag{9}$$

It is easy to find a pairwise-independent joining λ satisfying (1) but not (9): take ξ to be the uniform Bernoulli process on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$, ξ' an independent copy of ξ, and set

$$\xi_0'' = \xi_0 + \xi_1 + \xi_0' + \xi_1'.$$

However, for this process, we can easily construct another pairwise-independent joining satisfying (9). We do not know whether PIJ* is strictly stronger than PIJ.

THEOREM 4.2. *Any process ξ enjoying the* PIJ* property *is a subshift of finite type, equipped with an invariant measure of maximal entropy.*

Proof. We denote by $\mathcal{L} = \bigcup_\ell \mathcal{L}_\ell$ the set of authorized words in the process ξ, that is to say all finite words that occur with positive probability. Observe that ξ is a subshift of finite type if and only if there exists $\ell \in \mathbb{N}$ such that for any word W of length larger than ℓ, for any $a, b \in \mathbb{A}$,

$$aW \in \mathcal{L} \quad \text{and} \quad Wb \in \mathcal{L} \Longrightarrow aWb \in \mathcal{L}.$$

By hypothesis, there exist a pairwise independent joining λ of ξ and an integer p such that, λ-almost everywhere,

$$\xi_0'' = \varphi_1(\xi_{-p}^{p}, \xi'^{p}_{-p}), \tag{10}$$

$$\xi_0 = \varphi_2(\xi'^{p}_{-p}, \xi''^{p}_{-p}). \tag{11}$$

For $U \in \mathcal{L}$ and $d \geq 1$, we define

$$\mathrm{Ext}(d, U) = \{U' \in \mathcal{L}_d \mid UU' \in \mathcal{L}\}.$$

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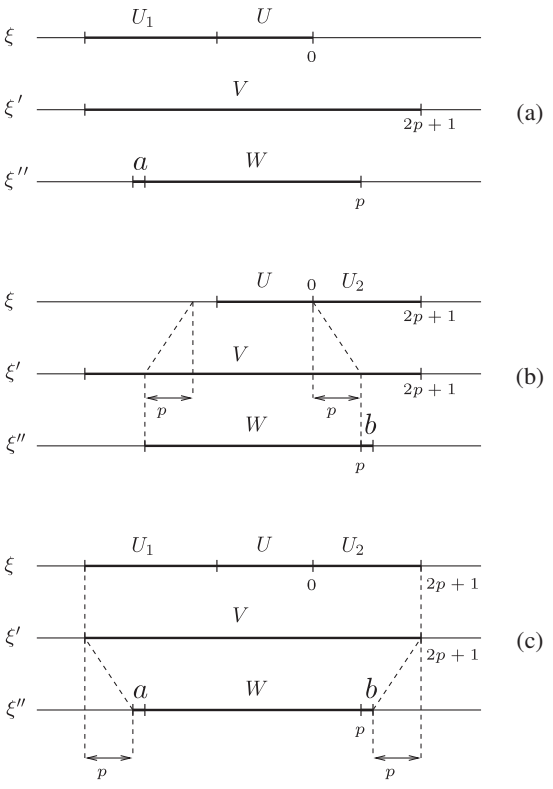


FIGURE 4. The three situations used in the proof of Theorem 4.2. In (a), we put together U and aW. In (b), we put together V and Wb, which forces U. In (c), we put together U_1UU_2 and V, which forces aWb.

Observe that there exists a word $U \in \mathcal{L}$ of length $|U| > p$ such that, for any word U_1 with $U_1U \in \mathcal{L}$, $\mathrm{Ext}(2p+1, U) = \mathrm{Ext}(2p+1, U_1U)$. Indeed, it is sufficient to take U such that the cardinal of $\mathrm{Ext}(2p+1, U)$ is minimum.

We consider a word W of length $|W| \geq |U| + 2p$, and $a, b \in \mathbb{A}$ such that $aW \in \mathcal{L}$ and $Wb \in \mathcal{L}$. We have to prove that $aWb \in \mathcal{L}$.

Since ξ and ξ'' are independent under λ, we can observe, with positive probability, the following situation: ${\xi}^{0}_{-|U|+1} = U$ and ${\xi''}^{p}_{p-|W|} = aW$. Therefore, there exist U_1 of length $|U_1| = |W| + 1 - |U|$ and V of length $|V| = |W| + 2 + 2p$ such that, with positive probability, we can observe that

$$\xi^0_{-|W|} = U_1U, \quad {\xi'}^{2p+1}_{-|W|} = V, \quad {\xi''}^{p}_{p-|W|} = aW,$$

all together (see Figure 4(a)).

On the other hand, since ξ' and ξ'' are independent under λ, we can observe, with positive probability, that ${\xi'}^{2p+1}_{-|W|} = V$ and ${\xi''}^{p+1}_{p-|W|+1} = Wb$. By (11), this forces ${\xi}^{0}_{-|U|+1} = U$. Hence, there exists $U_2 \in \mathrm{Ext}(2p+1, U)$ such that, with positive probability, we can observe that (see Figure 4(b))

$$\xi^{2p+1}_{-|U|+1} = UU_2, \quad {\xi'}^{2p+1}_{-|W|} = V, \quad {\xi''}^{p+1}_{p-|W|+1} = Wb,$$

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all together. But $\mathrm{Ext}(2p+1, U) = \mathrm{Ext}(2p+1, U_1U)$. Therefore, $U_1UU_2 \in \mathcal{L}$. Since ξ and ξ' are independent under λ, we can observe, with positive probability, the following situation (see Figure 4(c)): $\xi_{-|W|}^{2p+1} = U_1UU_2$ and $\xi'^{2p+1}_{-|W|} = V$. By (10), this forces $\xi''^{p+1}_{p-|W|} = aWb$. Hence, $aWb \in \mathcal{L}$.

Now, using Lemma 2.3(iii), we conclude the proof. □

Observe that possessing the PIJ* property does not imply that the subshift of finite type is irreducible: indeed, let $\mathbb{A} = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, and define the subshift Σ as the set of sequences in $\mathbb{A}^{\mathbb{Z}}$ for which the first coordinate is constant. Let μ_0 and μ_1 be the two ergodic invariant measures on Σ. This is not an irreducible subshift of finite type, and any convex combination of μ_0 and μ_1 has maximal entropy. Let $\mu = (\mu_0 + \mu_1)/2$ be the measure giving the same mass to all cylinders of the same length in Σ. The process defined by μ enjoys the PIJ* property with $\xi''_0 = \xi_0 + \xi'_0$ and $\xi_0 = \xi'_0 + \xi''_0$.

5. *Further questions*

Our results lead to the conjecture that, if ξ has the PIJ property, then ξ is a continuous coding of one of the two examples presented in §1.1. If we weaken the requirements of the PIJ property, by replacing the continuity of φ with the assumption that ξ'' be a *measurable* function of ξ and ξ', we may ask whether ξ is a measurable coding of one of these examples.

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