European Journal of Applied Mathematics (2026), **37**, pp. 882–896
doi:10.1017/S0956792525100065

CAMBRIDGE UNIVERSITY PRESS

PAPER

Periodic dynamics of a general switching dynamical system

Qian Ding[1,2], Jianshe Yu[1], Zhiming Guo[3], Yuming Chen[4] and Yunfeng Liu[1,3]

[1]Center for Applied Mathematics, Guangzhou University, Guangzhou, Guangdong, China
[2]College of Science, Hunan City University, Yiyang, China
[3]School of Mathematics and Information Sciences, Guangzhou University, Guangzhou, Guangdong, China
[4]Department of Mathematics, Wilfrid Laurier University, Waterloo, ON, Canada
Corresponding author: Yunfeng Liu; Email: lyfwl1986316@163.com

Received: 04 October 2024; **Revised:** 05 May 2025; **Accepted:** 08 May 2025;
First published online: 08 August 2025

Keywords: seasonal change; cyclical human activity; switching dynamical system; periodic solution

2020 Mathematics Subject Classification: 34K13, 92D25 (Primary); 34K09, 34K20, 92D30, 92D40 (Secondary)

Abstract
Seasonal changes and cyclical human activities (such as periodic fishing bans, *Wolbachia*-based mosquito population control, and school term breaks) have significant impacts on population dynamics. We propose a general switching dynamical model to describe these periodic changes. The existence, uniqueness and stability of positive periodic solutions are thoroughly investigated. The results are stated in terms of an introduced threshold value. To demonstrate their practicability, the obtained results are applied to two biological situations.

1 Introduction

Many organisms experience periodic environmental variations influenced by seasonal changes and cyclical human activities. These fluctuations often affect the availability of resources (such as food, water, nutrients, and energy) as well as territories and habitats. For instance, seasonal succession (such as the succession of phytoplankton species [9]) and cyclical human behaviours (such as periodic open-closed fishery management policies [6, 11–13, 23] and the periodic release of *Wolbachia*-infected mosquitoes [24, 25, 26]) can cause the living environments of the populations to undergo regular changes. These periodic variations have profound effects on population growth dynamics and the success of species invasions.

The growth of organisms requires specific environmental conditions and hence not all the times are suitable for the growth of certain species. As a result, growth seasons alternate with non-growth seasons, during which biomass increases continuously and decreases gradually, respectively. For example, climate patterns repeat annually, birds migrate according to seasonal successions, and the opening and closing of schools are almost always cyclical. The incidence and contact rates of many infectious diseases are often influenced by seasonal changes. Autumn and winter are typically the high-incidence seasons for influenza. Childhood diseases such as measles, rubella, and whooping cough are highly sensitive to the seasonal pressures caused by the alternation between school terms and holidays. Seasonal factors significantly affect the survival of pathogens in the environment, host behaviours, and the abundance of vectors and non-human hosts. Existing extensive studies [1, 2, 4, 20] suggest that seasonality plays a crucial role in the outbreak and evolution of epidemics and the transmission of diseases.

© The Author(s), 2025. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

https://doi.org/10.1017/S0956792525100065 Published online by Cambridge University Press

Pan et al. [15] analysed the following switching dynamical system,

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = xf(x), & t \in (nT, nT + \overline{T}], \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = xg(x), & t \in (nT + \overline{T}, (n+1)T]. \end{cases} \tag{1.1}$$

It is worth noting that the functions f and g in (1.1) are autonomous functions, namely, they are independent of the time t. However, in practical applications to biological systems, it has been observed that f and g actually depend on t, with specific details to be provided in the examples in Section 3. As a result, in this paper, we consider the following switching dynamical system,

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = xf(t, x, \alpha), & t \in (nT, nT + \overline{T}], \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = xg(t, x, \alpha), & t \in (nT + \overline{T}, (n+1)T], \end{cases} \tag{1.2}$$

with $x(0) \in \mathbb{R}_+ = [0, +\infty)$ (Here and in the sequel, without mentioning, $n = 0, 1, 2, \ldots$). T and $\overline{T}$ are two important time points satisfying $T > \overline{T}$. The parameter α is of particular interest in this study and hence it is highlighted here.

We assume that both $f(t, x, \alpha)$ and $g(t, x, \alpha)$ from $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{R}$ are continuous, T-periodic in t, and continuously differentiable in both x and α. Then it is easy to show the existence and uniqueness of solutions to system (1.2). In fact, taking $n = 0$, by a standard application of the existence theory of ordinary differential equations, one can obtain the existence and uniqueness of solutions on $(0, \overline{T}]$. Then, similarly, we can prove the existence and uniqueness of solutions in $(\overline{T}, T]$ with the initial data $x(\overline{T})$. Continuing this way, we have the existence and uniqueness of solutions to system (1.2) on $\mathbb{R}_+$.

The rest of the paper is organised as follows. In Section 2, we provide the detailed analysis on the dynamics of (1.2) under certain reasonable assumptions. Then the obtained theoretical results are illustrated with two applications in Section 3. The paper concludes with a brief summary.

2 The dynamics of system (1.2)

In this section, we develop the existence and stability of positive periodic solutions to system (1.2) under certain assumptions.

We first state some assumptions on f and g.

(H1) $f(t, 0, \alpha) \geq f(t, x, \alpha)$ and $g(t, 0, \alpha) \geq g(t, x, \alpha)$ for $x > 0$.

(H2) There exist M_1 and M_2 such that $f(t, x, \alpha) \leq 0$ for all $x \geq M_1 > 0$ and $g(t, x, \alpha) \leq 0$ for $x \geq M_2 > 0$.

(H3) $\begin{cases} f_x(t, x, \alpha) < 0 & \text{for } t \in (nT, nT + \overline{T}] \text{ and } x > 0, \\ g_x(t, x, \alpha) < 0 & \text{for } t \in (nT + \overline{T}, (n+1)T] \text{ and } x > 0. \end{cases}$

Denote $F(T, \overline{T}, \alpha) = \int_0^{\overline{T}} f(t, 0, \alpha)\,\mathrm{d}t + \int_{\overline{T}}^{T} g(t, 0, \alpha)\,\mathrm{d}t$. We also have the following assumption on F.

(H) The function $F(T, \overline{T}, \alpha)$ satisfies at least one of the following three assumptions about parameters $T > 0$, $\overline{T} > 0$, and $\alpha > 0$.

- (a) For any fixed $T > 0$ and $\alpha > 0$, $F(T, \overline{T}, \alpha)$ is strictly monotonic in $\overline{T}$ and $F(T, \overline{T}, \alpha) = 0$ has a unique non-negative root with respect to $\overline{T}$.
- (b) For any fixed $\overline{T} > 0$ and $\alpha > 0$, $F(T, \overline{T}, \alpha)$ is strictly monotonic in T and $F(T, \overline{T}, \alpha) = 0$ has a unique non-negative root with respect to T.
- (c) For fixed $T > 0$ and $\overline{T} > 0$, $F(T, \overline{T}, \alpha)$ is strictly monotonic in α and $F(T, \overline{T}, \alpha) = 0$ has a unique non-negative root with respect to α.

In the following, we assume that the monotonicity in **(H)** is strictly increasing. For results in the case of monotonically decreasing, just reverse the directions of inequalities in conditions. The non-negative

https://doi.org/10.1017/S0956792525100065 Published online by Cambridge University Press

root in **(H)** will serve as the threshold value. In fact, the linearisation of (1.2) at 0 is

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = yf(t,0,\alpha), & t \in (nT, nT+\overline{T}], \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = yg(t,0,\alpha), & t \in (nT+\overline{T}, (n+1)T]. \end{cases} \tag{2.1}$$

By the Floquet theory, any fundamental solution of equation (2.1) has the form $y(t) = e^{-\lambda t}\psi(t)$, where $\lambda \in \mathbb{C}$ is a characteristic exponent and $\psi(t+T) = \psi(t)$ for all $t \in \mathbb{R}$. Thus, we have

$$\begin{cases} \dfrac{\mathrm{d}\psi}{\mathrm{d}t} = f(t,0,\alpha)\psi + \lambda\psi, & t \in (0, \overline{T}], \\ \dfrac{\mathrm{d}\psi}{\mathrm{d}t} = g(t,0,\alpha)\psi + \lambda\psi, & t \in (\overline{T}, T], \\ \psi(0) = \psi(T) \end{cases} \tag{2.2}$$

or equivalently,

$$\begin{cases} \dfrac{1}{\psi(t)}\dfrac{d\psi(t)}{dt} = f(t,0,\alpha) + \lambda, & t \in (0, \overline{T}], \\ \dfrac{1}{\psi(t)}\dfrac{d\psi(t)}{dt} = g(t,0,\alpha) + \lambda, & t \in (\overline{T}, T], \\ \psi(0) = \psi(T). \end{cases} \tag{2.3}$$

Integrating the first equation of (2.3) with respect to t from 0 to $\overline{T}$ and the second from $\overline{T}$ to T, we obtain

$$\ln \psi(\overline{T}) - \ln \psi(0) = \int_0^{\overline{T}} (f(t,0,\alpha) + \lambda)\,\mathrm{d}t$$

and

$$\ln \psi(T) - \ln \psi(\overline{T}) = \int_{\overline{T}}^{T} (g(t,0,\alpha) + \lambda)\,\mathrm{d}t,$$

respectively. Adding the above two equations and using $\psi(0) = \psi(T)$, we get

$$\lambda T + \int_0^{\overline{T}} f(t,0,\alpha)\,\mathrm{d}t + \int_{\overline{T}}^{T} g(t,0,\alpha)\,\mathrm{d}t = 0. \tag{2.4}$$

By **(H)**, let ω^* be the unique root of $F(T, \overline{T}, \alpha) = 0$, where ω is one of T, $\overline{T}$, and α and correspondingly, ω^* can be one of T^*, $\overline{T}^*$, and α^*, respectively. Moreover, we have that λ satisfying (2.4) has the property,

$$\lambda \begin{cases} > 0 & \text{if } \omega < \omega^*, \\ = 0 & \text{if } \omega = \omega^*, \\ < 0 & \text{if } \omega > \omega^*. \end{cases}$$

Now, we are ready to give the main results of this paper.

We first present a result on the existence and attractivity of positive T-periodic solutions to system (1.2), that is,

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = xf(t,x,\alpha), & t \in (0, \overline{T}], \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = xg(t,x,\alpha), & t \in (\overline{T}, T], \\ x(0) = x(T). \end{cases} \tag{2.5}$$

Theorem 2.1. *Suppose that* ***(H2)*** *holds. If $\omega > \omega^*$, then system (1.2) has at least one positive T-periodic solution. Furthermore, if* ***(H3)*** *holds, then system (1.2) has a unique globally attractive positive T-periodic solution.*

https://doi.org/10.1017/S0956792525100065 Published online by Cambridge University Press

Proof. Let $\psi(t)$ be a positive eigenfunction of (2.2) associated with the eigenvalue λ satisfying (2.4). Denote $\underline{x}(t) = \epsilon\psi(t)$. Since $\omega > \omega^*$, we have $\lambda < 0$. For small enough $\epsilon > 0$, we obtain

$$\begin{cases} \dfrac{\mathrm{d}\underline{x}}{\mathrm{d}t} - \underline{x}f(t,\underline{x},\alpha) - \underline{x}f(t,0,\alpha) + \underline{x}f(t,0,\alpha) \\ = (-f(t,\epsilon\psi(t),\alpha) + f(t,0,\alpha) + \lambda)\epsilon\psi(t) \le 0 \quad \text{for } t \in (0,\overline{T}], \\ \dfrac{\mathrm{d}\underline{x}}{\mathrm{d}t} - \underline{x}g(t,\underline{x},\alpha) - \underline{x}g(t,0,\alpha) + \underline{x}g(t,0,\alpha) \\ = (-g(t,\epsilon\psi(t),\alpha) + g(t,0,\alpha) + \lambda)\epsilon\psi(t) \le 0 \quad \text{for } t \in (\overline{T},T], \\ \psi(0) = \psi(T). \end{cases}$$

Thus, $\epsilon\psi$ is a sub-solution of the periodic system (2.5). On the other hand, with the help of **(H2)**, we can deduce that

$$\begin{cases} \dfrac{\mathrm{d}\bar{M}}{\mathrm{d}t} - \bar{M}f(t,\bar{M},\alpha) \\ = -\bar{M}f(t,\bar{M},\alpha) \ge 0 \quad \text{for } t \in (0,\overline{T}], \\ \dfrac{\mathrm{d}\bar{M}}{\mathrm{d}t} - \bar{M}g(t,\bar{M},\alpha) \\ = -\bar{M}g(t,\bar{M},\alpha) \ge 0 \quad \text{for } t \in (\overline{T},T] \end{cases}$$

for $\bar{M} \ge \max\{M_1, M_2\}$. Therefore, $(\bar{M}, \epsilon\psi(t))$ forms a pair of upper and lower solutions for the periodic system (2.5). This implies that (2.5) has a positive T-periodic solution $x_1(t)$ initiated from the minimal point u_1.

Next, we prove that the periodic system (2.5) has a unique positive T-periodic solution when **(H3)** also holds.

Suppose, for the sake of contradiction, that $x_2(t)$ is also a positive T-periodic solution to system (2.5) initiated from u_2 with $u_1 < u_2$. Since x_i $(i = 1, 2)$ is a positive solution to the following equation

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = xf(t,x,\alpha), \quad t \in (0,\overline{T}], \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = xg(t,x,\alpha), \quad t \in (\overline{T},T], \\ x(0) = (T), \end{cases}$$

we have

$$\begin{cases} \dfrac{1}{x_i}\dfrac{dx_i}{dt} = f(t,x_i,\alpha), \quad t \in (0,\overline{T}], \\ \dfrac{1}{x_i}\dfrac{dx_i}{dt} = g(t,x_i,\alpha), \quad t \in (\overline{T},T]. \end{cases} \tag{2.6}$$

Integrating the first equation of (2.6) in t from 0 to $\overline{T}$ and the second one from $\overline{T}$ to T, and then summing the resultants, we obtain

$$\int_0^{\overline{T}} f(t,x_i,\alpha)\,\mathrm{d}t + \int_{\overline{T}}^{T} g(t,x_i,\alpha)\,\mathrm{d}t = 0.$$

According to **(H3)**, this is impossible. Therefore, system (2.5) has a unique positive T-periodic solution. Using a standard monotone iteration scheme and the instability of the trivial equilibrium point 0 of system (1.2), we see that the unique positive T-periodic solution of system (1.2) is globally attractive. This completes the proof. □

Next, we consider the extinction of system (1.2).

Theorem 2.2. *Suppose that* ***(H1)*** *holds. If $\omega < \omega^*$, then the trivial equilibrium point* 0 *of system (1.2) is globally asymptotically stable.*

https://doi.org/10.1017/S0956792525100065 Published online by Cambridge University Press

Proof. Let $x(t)$ be a solution of (1.2). Define $\bar{x} = Me^{-\lambda t}\psi(t)$, where M is chosen large enough such that $x(0) \le \bar{x}(0)$ and ψ is the one chosen in the proof of Theorem 2.1. Then we have

$$\begin{cases} \dfrac{\mathrm{d}\bar{x}}{\mathrm{d}t} - \bar{x}f(t,\bar{x},\alpha) - \bar{x}f(t,0,\alpha) + \bar{x}f(t,0,\alpha) \\ = (-f(t,\bar{x},\alpha) + f(t,0,\alpha))\bar{x} \ge 0, & t \in (nT, nT+\overline{T}], \\ \dfrac{\mathrm{d}\bar{x}}{\mathrm{d}t} - \bar{x}g(t,\bar{x},\alpha) - \bar{x}g(t,0,\alpha) + \bar{x}g(t,0,\alpha) \\ = (-g(t,\bar{x},\alpha) + g(t,0,\alpha))\bar{x} \ge 0, & t \in (nT+\overline{T}, (n+1)T], \\ \bar{x}(0) \ge x(0), \end{cases}$$

where we have used **(H1)**. By the comparison principle, we can obtain that $x(t) \le \bar{x}(t)$ for all $t > 0$. Note that $\omega < \omega^*$ implies $\lambda > 0$. It follows that $\bar{x}(t) \to 0$ exponentially as $t \to +\infty$. Therefore, $x(t)$ must decay toward zero exponentially as well. This demonstrates that the trivial equilibrium point 0 is globally asymptotically stable. □

The result below gives the asymptotical stability of 0 when $\omega = \omega^*$ under a little stronger condition.

Theorem 2.3. *Suppose that **(H1)** holds and the inequalities there are strict. If $\omega = \omega^*$, then the trivial equilibrium point* 0 *of system (1.2) is globally asymptotically stable.*

Proof. We first show that there is no positive T-periodic solution to system (1.2). Assume, by contradiction, that there exists $\bar{x}(t) > 0$ for $t \in [0, T]$ satisfying system (2.5), that is,

$$\begin{cases} \dfrac{\mathrm{d}\bar{x}}{\mathrm{d}t} = \bar{x}f(t,\bar{x},\alpha), & t \in (0,\overline{T}], \\ \dfrac{\mathrm{d}\bar{x}}{\mathrm{d}t} = \bar{x}g(t,\bar{x},\alpha), & t \in (\overline{T}, T], \\ \bar{x}(0) = \bar{x}(T). \end{cases}$$

or, equivalently

$$\begin{cases} \dfrac{1}{\bar{x}}\dfrac{d\bar{x}}{dt} = f(t,\bar{x},\alpha), & t \in (0,\overline{T}], \\ \dfrac{1}{\bar{x}}\dfrac{d\bar{x}}{dt} = g(t,\bar{x},\alpha), & t \in (\overline{T}, T]. \end{cases}$$

It follows that

$$\int_0^{\overline{T}} f(t,\bar{x},\alpha)\,\mathrm{d}t + \int_0^{T} g(t,\bar{x},\alpha)\,\mathrm{d}t = 0.$$

This combined with (2.4) implies that

$$\lambda T = \int_0^{\overline{T}} [f(t,\bar{x},\alpha) - f(t,0,\alpha)]\,\mathrm{d}t + \int_0^{T} [g(t,\bar{x},\alpha) - g(t,0,\alpha)]\,\mathrm{d}t.$$

By the assumption that $f(t,0,\alpha) > f(t,x,\alpha)$ and $g(t,0,\alpha) > g(t,x,\alpha)$ for any $x > 0$, we obtain that $\lambda < 0$. However, since $\omega = \omega^*$, we have $\lambda = 0$. This leads to a contradiction, and hence (1.2) has no positive T-periodic solution.

By a standard monotone iteration scheme and the nonexistence of positive T-periodic solution of system (1.2), the trivial equilibrium point 0 of system (1.2) is globally asymptotically stable. This completes the proof. □

https://doi.org/10.1017/S0956792525100065 Published online by Cambridge University Press

3 Two illustrating examples

In this section, we apply the results established in Section 2 to two switching systems from population dynamics.

3.1 A harvesting model

Due to overfishing, many fish resources are facing depletion worldwide, which has broken the balance of ecosystems. This has led to changes in the populations of other species and a weakening of overall ecosystem functions. As a response, many countries have implemented fishing bans to restore and protect marine biological resources, ensure ecosystem stability and diversity, and safeguard the livelihoods of fishermen as well as future food for humans.

However, a complete ban on all forms of fishing, though potentially beneficial for resource recovery, also presents several potential problems and negative impacts. Such a ban could lead to significant socioeconomic issues, particularly in communities that rely heavily on fishing as it is their primary economic activity. These communities may face severe livelihood losses. Additionally, fish are a critical source of protein for millions of people, especially in some poor countries. A total fishing ban could reduce their nutritional intake, potentially leading to malnutrition and associated health problems.

Therefore, developing reasonable fishing ban strategies is crucial for policymakers. The works [6, 11, 13, 23] have explored how to determine appropriate fishing ban thresholds based on varying fishing intensities. Following the idea in [6, 11], the main objective of this subsection is to understand the dynamics of the following problem,

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = ru\left(1-\dfrac{u}{G}\right), & t\in(nT, nT+\overline{T}], \\ \dfrac{\mathrm{d}u}{\mathrm{d}t} = ru\left(1-\dfrac{u}{G}\right) - \dfrac{q(t)E(t)u}{cE(t)+lu}, & t\in(nT+\overline{T}, (n+1)T], \end{cases} \tag{3.1}$$

where $u(t)$ represents the population size of a single species at time t. T is the period of the seasonal fluctuation environment where the single species lives; $\overline{T}$ measures the length of the closed season (non-harvesting season) and hence $T-\overline{T}$ is the length of the open season (harvesting season); the parameter r is the intrinsic growth rate of the species; and G is the carrying capacity of the environment. The term $\frac{q(t)E(t)u}{cE(t)+lu}$ is the harvesting rate, which is the Michaelis–Menten type functional form for the catch rate. Here $q(t)$ is the catch-ability coefficient at time t and $E(t)$ is the external effort devoted to harvesting at time t. We further assume that $q(t)$ and $E(t)$ are continuous and T-periodic in t. c and l are two constants. All parameters are assumed to be non-negative.

Corresponding to (1.2), here for (3.1), we have

$$f(t,u,\alpha) = r\left(1-\frac{u}{G}\right) \quad \text{and} \quad g(t,u,\alpha) = r\left(1-\frac{u}{G}\right) - \frac{q(t)E(t)}{cE(t)+lu}.$$

Then

$$\begin{aligned} F(T,\overline{T},\alpha) &= \int_0^{\overline{T}} f(t,0,\alpha)\,\mathrm{d}t + \int_{\overline{T}}^{T} g(t,0,\alpha)\,\mathrm{d}t \\ &= \int_0^{\overline{T}} r\,\mathrm{d}t + \int_{\overline{T}}^{T}\left(r-\frac{q(t)}{c}\right)\mathrm{d}t \\ &= rT - \frac{1}{c}\int_{\overline{T}}^{T} q(t)\,\mathrm{d}t. \end{aligned}$$

We first make some reasonable assumptions on the parameters. On the one hand, in the absence of fishing, that is, $T=\overline{T}$, system (3.1) reduces to

$$\frac{\mathrm{d}u}{\mathrm{d}t} = ru\left(1-\frac{u}{G}\right), \quad t>0.$$

https://doi.org/10.1017/S0956792525100065 Published online by Cambridge University Press

It is necessary to assume that the population is persistent in the absence of fishing and hence $r > 0$. On the other hand, when there is no fishing ban, that is, $\overline{T} = 0$, system (3.1) reduces to

$$\frac{\mathrm{d}u}{\mathrm{d}t} = ru\left(1 - \frac{u}{G}\right) - \frac{q(t)E(t)u}{cE(t) + lu}, \quad t > 0.$$

Let $\inf_{t\in[\overline{T},T]} E(t) = E_0$ and $\inf_{t\in[\overline{T},T]} q(t) = q_0$. Then

$$\frac{\mathrm{d}u}{\mathrm{d}t} = ru\left(1 - \frac{u}{G}\right) - \frac{q(t)E(t)u}{cE(t) + lu} \le ru\left(1 - \frac{u}{G}\right) - \frac{q_0E_0u}{cE_0 + lu}.$$

Consider the system

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}t} = r\bar{u}\left(1 - \frac{\bar{u}}{G}\right) - \frac{q_0E_0\bar{u}}{cE_0 + l\bar{u}}. \tag{3.2}$$

By comparison principle, $u(t) \le \bar{u}(t)$, where u is the solution of system (3.1) and $\bar{u}(t)$ is the solution of (3.2) with $u(0) = \bar{u}(0)$. Clearly, $\bar{u} = 0$ is the trivial equilibrium of (3.2) and its other positive equilibria are determined by the quadratic equation

$$\frac{rl}{G}\bar{u}^2 + \left(\frac{rcE_0}{G} - rl\right)\bar{u} + (q_0 - cr)E_0 = 0. \tag{3.3}$$

It is clear that equation (3.3) has no positive solution if

$$E_0 \ge \frac{Gl}{c} \quad \text{and} \quad q_0 \ge cr. \tag{3.4}$$

Thus, if (3.4) holds, then $\bar{u}(t) \to 0$ and hence $u(t) \to 0$ as $t \to +\infty$. This means that the population will become extinct without fishing ban. Based on the above discussion, throughout the remainder of this subsection, for system (3.1), we always assume that (3.4) holds.

Clearly, $F(T, \overline{T}, \alpha)$ is monotonically increasing in $\overline{T}$ for fixed T and α. Moreover,

$$F(T, 0, \alpha) = rT - \frac{1}{c}\int_0^T q(t)\,\mathrm{d}t \le rT - \frac{q_0}{c}T \le 0$$

by (3.4) and

$$F(T, T, \alpha) = rT > 0$$

Thus **(H)**(b) is satisfied. Let $\overline{T}^*$ be the unique non-negative root of $F(T, \overline{T}, \alpha) = 0$ with fixed T and α. Actually, $\overline{T}^* = \frac{\bar{q}-cr}{\bar{q}}$, where $\bar{q} = \frac{\int_{\overline{T}}^T q(t)\,\mathrm{d}t}{T-\overline{T}}$.

Viewed from different perspectives, the fishing ban policy in high seas is usually a pre-established norm by countries, mainly aimed at protecting the breeding of marine fish during the breeding season. This means that the closing period $\overline{T}$ is fixed. Hence, from **(H)**(c), we have $r^* = \frac{(T-\overline{T})\bar{q}}{T}$ for fixed T and $\overline{T}$. In this case, the parameter α in system (1.2) is the parameter r. However, in this subsection, we only considered the case of the threshold $\overline{T}^*$. The case of threshold r^* can be handled in a similar manner, and hence it is omitted here.

Now, we are ready to apply the results in Section 2 to system (3.1).

Theorem 3.1. *Suppose that $\overline{T} > \overline{T}^*$. Then system (3.1) has at least one positive T-periodic solution. Furthermore, if $E_0 > \frac{q^0lG}{rc^2}$, then system (3.1) has a unique positive T-periodic solution, which is globally asymptotically stable, where $q^0 := \sup_{t\in[\overline{T},T]} q(t)$.*

Proof. Clearly **(H2)** holds for $M_1 = M_2 = G$ and $f_u(t, u, \alpha) < 0$ for $t \in (nT, nT + \overline{T}]$ and $u > 0$. Moreover, when $E_0 > \frac{q^0lG}{rc^2}$, for $t \in (nT + \overline{T}, (n + 1)T]$ and $u > 0$, we have

$$g_u(t, u, \alpha) = -\frac{r}{G} + \frac{q(t)E(t)l}{(cE(t) + lu)^2} \le -\frac{r}{G} + \frac{q(t)E(t)l}{(cE(t))^2} \le -\frac{r}{G} + \frac{q^0l}{c^2E_0} < 0.$$

Thus **(H3)** holds if $E_0 > \frac{q^0lG}{rc^2}$. According to Theorem 2.1, we are left to show that if $E_0 > \frac{q^0lG}{rc^2}$ then the unique globally attractive positive T-periodic solution $\bar{u}(t)$ is locally stable.

https://doi.org/10.1017/S0956792525100065 Published online by Cambridge University Press

Since $\bar{u}(t)$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}\bar{u}}{\mathrm{d}t} = r\bar{u}\left(1-\frac{\bar{u}}{G}\right), & t\in(0,\overline{T}], \\ \dfrac{\mathrm{d}\bar{u}}{\mathrm{d}t} = r\bar{u}\left(1-\frac{\bar{u}}{G}\right) - \dfrac{q(t)E(t)\bar{u}}{cE(t)+l\bar{u}}, & t\in(\overline{T},T], \\ \bar{u}(0)=\bar{u}(T), \end{cases} \tag{3.5}$$

dividing the first two equations of (3.5) by $\bar{u}$, then integrating the first resultant over $(0,\overline{T}]$ and the latter over $(\overline{T},T]$, and adding up the integrals, we obtain

$$\int_0^T \frac{1}{\bar{u}}\frac{d\bar{u}}{dt}\,\mathrm{d}t = rT - r\int_0^T \frac{\bar{u}}{G}\,\mathrm{d}t - \int_{\overline{T}}^T \frac{q(t)E(t)}{cE(t)+l\bar{u}}\,\mathrm{d}t.$$

Since $\bar{u}$ is T-periodic, we have $\int_0^T \frac{1}{\bar{u}}\frac{d\bar{u}}{dt}\mathrm{d}t = 0$ and hence

$$rT - r\int_0^T \frac{\bar{u}}{G}\,\mathrm{d}t - \int_{\overline{T}}^T \frac{q(t)E(t)}{cE(t)+l\bar{u}}\,\mathrm{d}t = 0. \tag{3.6}$$

Let $v = u-\bar{u}$. Then the linearised system of (3.1) about $\bar{u}$ is

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}t} = rv\left(1-\dfrac{2\bar{u}}{G}\right), & t\in(0,\overline{T}], \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = rv\left(1-\dfrac{2\bar{u}}{G}\right) - \dfrac{cE^2(t)q(t)v}{(cE(t)+l\bar{u})^2}, & t\in(\overline{T},T]. \end{cases}$$

Let $v = e^{-\tau t}\varphi(t)$ with $\varphi(0)=\varphi(T)$. This leads to the following eigenvalue problem,

$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}t} = r\left(1-\dfrac{2\bar{u}}{G}\right)\varphi + \tau\varphi, & t\in(0,\overline{T}], \\ \dfrac{\mathrm{d}\varphi}{\mathrm{d}t} = r\left(1-\dfrac{2\bar{u}}{G}\right)\varphi - \dfrac{cE^2(t)q(t)\varphi}{(cE(t)+l\bar{u})^2} + \tau\varphi, & t\in(\overline{T},T], \\ \varphi(0)=\varphi(T). \end{cases}$$

Similar to obtain (3.6), we can get

$$\tau T - \int_{\overline{T}}^T \frac{cE^2(t)q(t)}{(cE(t)+l\bar{u})^2}\,\mathrm{d}t + rT - 2\frac{r}{G}\int_0^T \bar{u}\,\mathrm{d}t = 0. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\begin{aligned} \tau T &= \int_{\overline{T}}^T \frac{cE^2(t)q(t)}{(cE(t)+l\bar{u})^2}\,\mathrm{d}t + rT - 2\int_{\overline{T}}^T \frac{q(t)E(t)}{cE(t)+l\bar{u}}\,\mathrm{d}t, \\ &= rT - \int_{\overline{T}}^T \frac{cE^2(t)q(t)+2E(t)q(t)l\bar{u}}{(cE(t)+l\bar{u})^2}\,\mathrm{d}t, \\ &\geq rT - \int_{\overline{T}}^T \frac{q(t)}{c}\,\mathrm{d}t \\ &= \frac{\bar{q}}{c}(\overline{T}-\overline{T}^*). \end{aligned}$$

Thus, $\tau > 0$ due to $\overline{T} > \overline{T}^*$, which implies that $\bar{u}$ is locally stable. This completes the proof. □

Note that, for $u > 0$,

$$f(t,0,\alpha) = r > r\left(1-\frac{u}{G}\right) = f(t,u,\alpha)$$

https://doi.org/10.1017/S0956792525100065 Published online by Cambridge University Press

and

$$g(t,0,\alpha)-g(t,u,\alpha)=u\left(\frac{r}{G}-\frac{q(t)l}{c(cE(t)+lu)}\right)$$
$$\geq u\left(\frac{r}{G}-\frac{q(t)l}{c^2E(t)}\right)$$
$$\geq u\left(\frac{r}{G}-\frac{q^0l}{c^2E_0}\right).$$

These combined with Theorem 2.2 and Theorem 2.3 produce the following result on the extinction of system (3.1).

Theorem 3.2. *The trivial equilibrium point* 0 *of system (3.1) is globally asymptotically stable if either*

$$\overline{T}<\overline{T}^* \quad \textit{and} \quad E_0\geq\frac{q^0lG}{rc^2}$$

or

$$\overline{T}=\overline{T}^* \quad \textit{and} \quad E_0>\frac{q^0lG}{rc^2}.$$

3.2 A switching mosquito population suppression model

Mosquito-borne diseases are caused by bacteria, viruses or parasites transmitted to people by mosquitoes. Typical mosquito-borne diseases include Zika, West Nile, Chikungunya, dengue, yellow fever, and malaria. These diseases have become a serious threat to public health worldwide, which cause over 725,000 deaths and nearly 700 million infections each year [18]. Due to the lack of vaccines and effective clinical treatments for most of the above-mentioned mosquito-borne diseases, and the additional risks associated with inadequate preventive measures, mosquito vector control has become the most important strategy [3, 7, 19]. The traditional measure is the application of chemical insecticides. This not only brings some environment problems and public health issues but also has limited effects on depressing the mosquito population as mosquitoes can quickly develop insecticide resistance [10, 14]. Therefore, scientists have started to develop environment-friendly insect control measures, including Sterile insect technique (SIT) and Incompatible insect technology (IIT).

SIT is an environment-friendly insect control technology that aimed to reduce the population of target pests by releasing radiation-sterilised males into the field to mate with wild females so that these wild females cannot produce viable offsprings. In practice, SIT has been successfully applied in the control of several important pests in agriculture and animal husbandry [5].

IIT is an approach that releases male mosquitoes infected with the maternally inherited endosymbiotic bacteria *Wolbachia* into the field, resulting in cytoplasmic incompatibility so that the matings with wild females that are not infected with the same *Wolbachia* strain become sterile – thereby suppressing the mosquito population [21, 22]. *Wolbachia* is an endosymbiotic bacterium that is parasitised exclusively in invertebrates and can be transmitted through eggs [17]. It is estimated that about 65% insect species and 28% mosquito species naturally carry *Wolbachia* [8, 16]. IIT is very promising since it is not only environment friendly but also very efficient with strong specificity and low cost. In 2005, a group led by Xi successfully transplanted *Wolbachia* carried by fruit flies into the germ cells of *Aedes aegypti* and made it spread stably in *Aedes aegypti* [22]. This made the mass release of millions of factory-reared incompatible adult *Aedes aegypti* technically feasible.

The basic principles of mosquito population suppression based on SIT and IIT are the same, both involving the release of sterile male mosquitoes in the target area to suppress the target mosquito population. In a recent work [24], Yu and Li formulated and analysed the following interactive model of wild

https://doi.org/10.1017/S0956792525100065 Published online by Cambridge University Press

and sterile mosquitoes,

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \left[\frac{aw}{w+g(t)} - (\mu + \xi(w+g(t)))\right] w. \tag{3.8}$$

Here w is the number of wild male mosquitoes at time t, a represents the birth rate of wild mosquitoes, μ denotes the density-independent death rate and ξ is the density-dependent death coefficient. In this context, g is treated as a given function rather than an independent variable governed by a dynamic equation. They assumed that a fixed number c of mosquitoes is released at regular intervals of length T. Additionally, their research highlights the importance of a crucial parameter – the sexual lifespan, denoted by $\overline{T}$ – which plays a significant role in mosquito suppression strategies. It is assumed that after the sexually active period, mosquitoes are no longer vigorous, and their competitive impact on wild mosquitoes becomes negligible, though alive. Under the assumption that $g(t)$ becomes a piecewise constant T-periodic function defined by

$$g(t) = \begin{cases} c, & t \in (nT, nT+\overline{T}], \\ 0, & t \in (nT+\overline{T}, (n+1)T], \end{cases} \tag{3.9}$$

for $n = 0, 1, 2, \ldots$, system (3.8) becomes

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}t} = w\left(\dfrac{aw}{w+c} - (\mu + \xi(w+c))\right), & t \in (nT, nT+\overline{T}], \\ \dfrac{\mathrm{d}w}{\mathrm{d}t} = w\,(a - \mu - \xi w)\,, & t \in (nT+\overline{T}, (n+1)T]. \end{cases}$$

One assumption that cannot be overlooked in [24] is that mosquitoes released during the period $(nT, nT+\overline{T}]$ will not die. In this subsection, we propose a more reasonable assumption: mosquitoes do die during their sexual lifespan. As a result, the constant c in (3.9) should be a monotonically decreasing function of t with period T. This leads to the following model on mosquito suppression,

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}t} = w\left(\frac{aw}{w+c(t)} - (\mu + \xi(w+c(t)))\right), & t \in (nT, nT+\overline{T}], \\ \dfrac{\mathrm{d}w}{\mathrm{d}t} = w\,(a - \mu - \xi w)\,, & t \in (nT+\overline{T}, (n+1)T]. \end{cases} \tag{3.10}$$

Similarly as in Section 3.1, on the one hand, we assume persistence of system (3.10) in the absence of sterile male mosquitoes, which leads to the condition that $a > \mu$. On the other hand, we assume that extinction occurs when $T = \overline{T}$ (meaning the release frequency equals the mosquito's sexual lifespan). Under this assumption, system (3.10) becomes

$$\frac{\mathrm{d}w}{\mathrm{d}t} = w\left(\frac{aw}{w+c(t)} - (\mu + \xi(w+c(t)))\right), \quad t > 0.$$

Since $\inf_{t\in[0,\overline{T}]} c(t) = c(\overline{T})$, we have

$$\frac{\mathrm{d}w}{\mathrm{d}t} = w\left(\frac{aw}{w+c(t)} - (\mu + \xi(w+c(t)))\right) \le w\left(\frac{aw}{w+c(\overline{T})} - (\mu + \xi(w+c(\overline{T})))\right).$$

Now, consider the following auxiliary equation,

$$\frac{\mathrm{d}w}{\mathrm{d}t} = w\left(\frac{aw}{w+c(\overline{T})} - (\mu + \xi(w+c(\overline{T})))\right), \quad t > 0. \tag{3.11}$$

When $c(\overline{T}) > g^* := \frac{(a-\mu)^2}{4\xi a}$, 0 is the only equilibrium of (3.11) and it is globally asymptotically stable. By the comparison principle, under this assumption, the mosquito population will be extinct. As a result, we assume that

$$a > \mu \quad \text{and} \quad c(\overline{T}) > g^*.$$

https://doi.org/10.1017/S0956792525100065 Published online by Cambridge University Press

In terms of (1.2), for (3.10), we have

$$f(t,w,\alpha)=\frac{aw}{w+c(t)}-(\mu+\xi(w+c(t))),\quad g(t,w,\alpha)=a-\mu-\xi w.$$

Thus,

$$\begin{aligned}F(T,\overline{T},\alpha)&=\int_0^{\overline{T}}f(t,0,\alpha)\,\mathrm{d}t+\int_{\overline{T}}^{T}g(t,0,\alpha)\,\mathrm{d}t\\&=\int_0^{\overline{T}}[-(\mu+\xi c(t))]\,\mathrm{d}t+\int_{\overline{T}}^{T}(a-\mu)\,\mathrm{d}t\\&=-\mu T+a(T-\overline{T})-\xi\int_0^{\overline{T}}c(t)\,\mathrm{d}t.\end{aligned}$$

Clearly, for fixed $\overline{T}>0$ and $\alpha>0$, $F(T,\overline{T},\alpha)$ is strictly monotonically increasing in T,

$$F(\overline{T},\overline{T},\alpha)=-\mu\overline{T}-\xi\int_0^{\overline{T}}c(t)\,\mathrm{d}t<0,$$

and $\lim_{T\to+\infty}F(T,\overline{T},\alpha)=+\infty$ as $a>\mu$. Thus, **(H)**(b) holds. Let T^* be the unique positive root of $F(T,\overline{T},\alpha)=0$, that is,

$$T^*=\frac{a\overline{T}+\xi\int_0^{\overline{T}}c(t)\,\mathrm{d}t}{a-\mu}.$$

One can easily check that

$$f(t,w,\alpha)\le 0\quad\text{for } w\ge M_1:=\frac{a}{\xi c(\overline{T})}$$

and

$$g(t,w,\alpha)\le 0\quad\text{for } w\ge M_2:=\frac{a}{\xi},$$

that is, **(H2)** holds. Moreover, for $w>0$,

$$f_w(t,w,\alpha)=\frac{ac(t)}{(w+c(t))^2}-\xi<\frac{a}{c(t)}-\xi\le\frac{a}{c(\overline{T})}-\xi$$

and

$$g_w(t,w,\alpha)=-\xi<0.$$

It follows that **(H3)** holds if $c(\overline{T})\ge\frac{a}{\xi}$. According to Theorem 2.1 and arguing similarly as in the proof of Theorem 3.1 for local stability, we have the following result on the existence and stability of T-periodic solutions to (3.10).

Theorem 3.3. *Suppose $T>T^*$. Then system 3.10) has at least one positive T-periodic solution. Furthermore, if $c(\overline{T})\ge\frac{a}{\xi}$, then it has a unique positive T-periodic solution, which is globally asymptotically stable.*

Now, for $w>0$, we observe that

$$\begin{aligned}f(t,0,\alpha)-f(t,w,\alpha)&=w\left(-\frac{a}{w+c(t)}+\xi\right)\\&>w\left(-\frac{a}{c(t)}+\xi\right)\\&\ge w\left(-\frac{a}{c(\overline{T})}+\xi\right)\\&>0\quad\left(\text{if } c(\overline{T})\ge\frac{a}{\xi}\right)\end{aligned}$$

https://doi.org/10.1017/S0956792525100065 Published online by Cambridge University Press

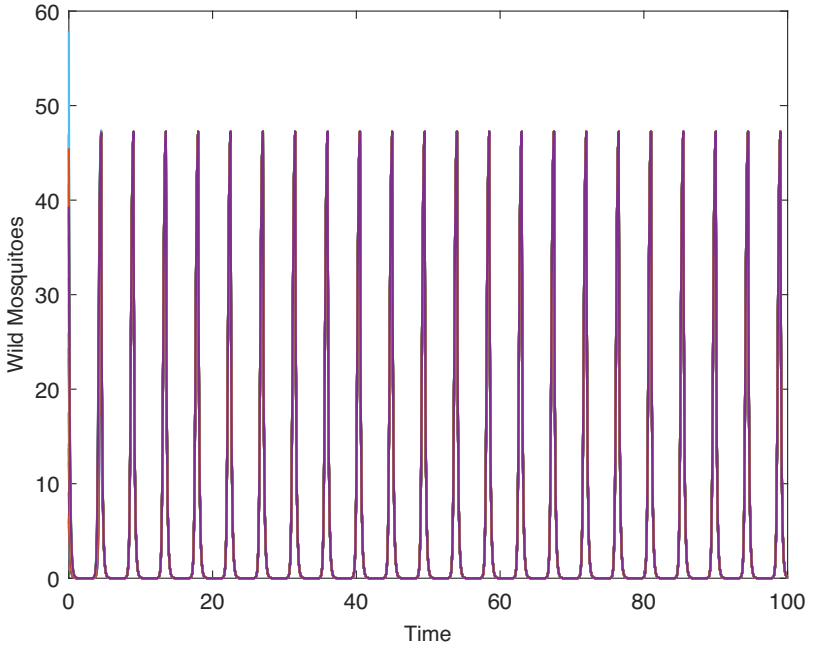


Figure 1 *When $T > T^*$ and $c(\overline{T}) \geq \frac{a}{\xi}$, (3.10) has a globally asymptotically stable positive T-periodic solution.*

and

$$g(t, 0, \alpha) = a - \mu > a - \mu - \xi w = g(t, w, \alpha).$$

Therefore, if $c(\overline{T}) \geq \frac{a}{\xi}$ then **(H1)** holds and the inequalities there are strict ones. Applying Theorems 2.2 and 2.3 to (3.10) produces the following result on extinction.

Theorem 3.4. *Assume that $T \leq T^*$ and $c(\overline{T}) \geq \frac{a}{\xi}$. Then the trivial equilibrium 0 of system (3.10) is globally asymptotically stable.*

Finally, we provide some numerical simulations to support the theoretical results obtained for system (3.10). We take

$$a = 10, \quad \mu = 0.3, \quad \xi = 0.2, \quad \overline{T} = 2.5.$$

With $c(t) = 32.5 - 2t$ for $t \in [0, \overline{T}]$, we have $c(\overline{T}) = 27.5 > \frac{a}{\xi} = 11.7612$. Corresponding to this release strategy, the threshold of release waiting period is $T^* = \frac{a\overline{T} + \xi \int_0^{\overline{T}} c(t)\,\mathrm{d}t}{(a-\mu)} = 400/97 \approx 4.124$. If the release waiting period $T = 4.5 > T^*$, it follows from Theorem 3.3 that system (3.10) has a unique globally asymptotically stable positive T-periodic solution as shown in Figure 1. If $T \leq T^*$, then Theorem 3.4 tells us that the origin is globally asymptotically stable for (3.10). Figure 2 is for the case where $T = T^*$ whereas Figure 3 shows the case $T = 4.024 < T^*$. The initial values for the figures in Figure 2 are chosen from the intervals [10, 30], [5, 10), [1, 4], and (0, 1], respectively. Though the rate approaching 0 is somewhat slow, they do indicate the convergence to 0.

4 Conclusion and discussion

The evolution of many biological processes are affected by seasonal factors. In this paper, we proposed a general periodic switching system to describe such phenomena. Under some reasonable assumptions, we

https://doi.org/10.1017/S0956792525100065 Published online by Cambridge University Press

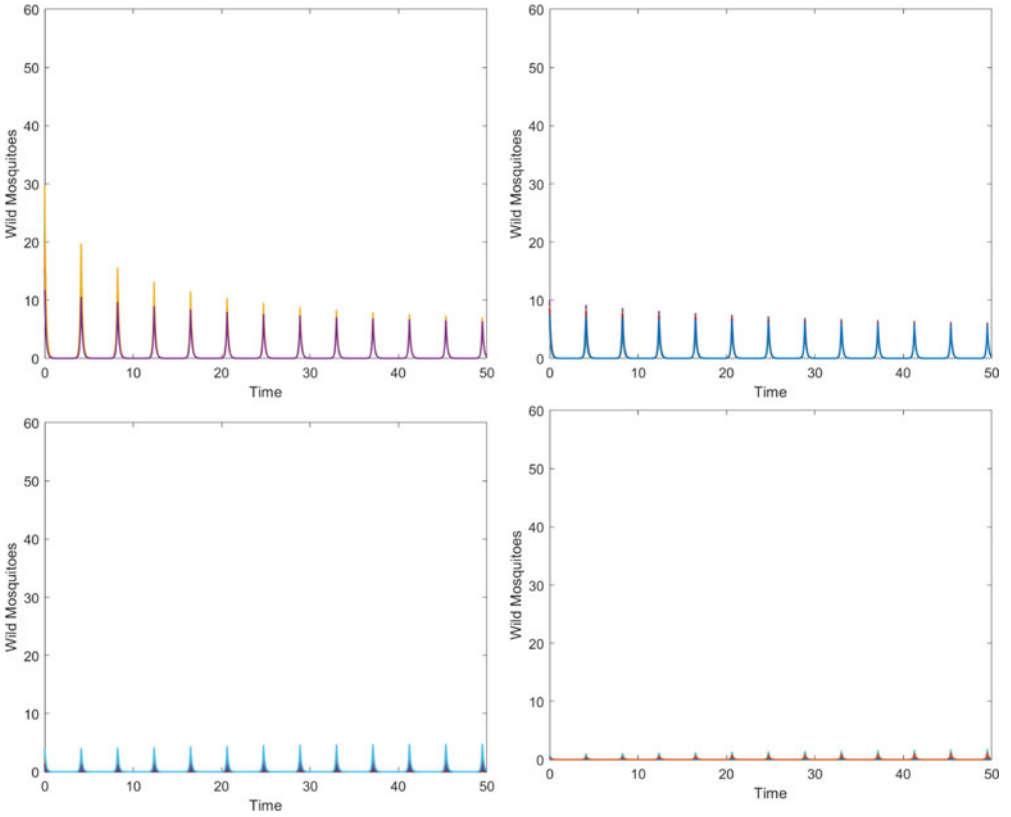


Figure 2 *When $T = T^*$ and $c(\overline{T}) \geq \frac{a}{\xi}$, the origin is globally asymptotically stable for (3.10). The initial values for the four figures are chosen form the four intervals,* $[10, 30]$, $[5, 10)$, $(1, 4]$, *and* $(0, 1]$. *From the four figures, it can be seen that the origin is stable, although its approach to the origin is somewhat slow.*

established a threshold dynamics, which is completely characterised by a threshold value ω^*. Roughly speaking, when $\omega > \omega^*$, there is a positive periodic solution, which is globally attractive under further conditions; when $\omega \leq \omega^*$, the origin is globally asymptotically stable.

We also applied the theoretical results to two situations. One is for a population model with harvesting and the other is for mosquito control. For the harvesting model, we obtained a globally asymptotically stable positive periodic solution when $\bar{T} > \bar{T}^*$. This implies that for resource sustainability, the harvesting should be banned at least for a certain time during a cycle. While for the mosquito suppression model, for a better control result, the waiting period for releasing *Wolbachia*-infected male mosquitoes should not be too large.

We mention that the obtained theoretical results for the general system (1.2) is applicable to other situations. To end this paper, we briefly introduce an SIS model without conducting the detailed analysis. The classical SIS model is described by the system of ordinary differential equations,

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = -\beta f(S, I) + \gamma I, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \beta f(S, I) - \gamma I, \end{cases}$$

where $S(t)$ and $I(t)$ are the numbers of susceptible and infectious individuals at time t, respectively. Here $f(S, I)$ is the incidence, β is the infection rate, and γ is the recovery rate. Seasonal succession, the start of the school season and holiday breaks periodically alter behaviour patterns of people whereas

https://doi.org/10.1017/S0956792525100065 Published online by Cambridge University Press

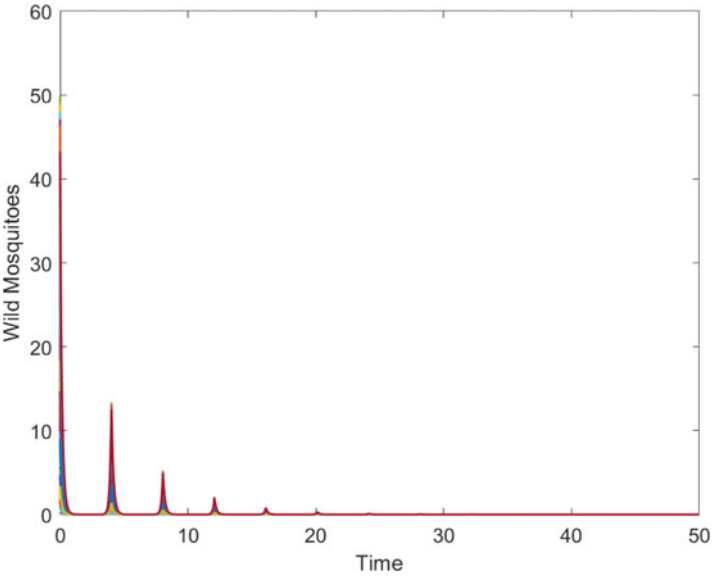


Figure 3 *When $T < T^*$ and $c(\overline{T}) \geq \frac{a}{\xi}$, the origin is globally asymptotically stable for (3.10).*

environmental conditions and medical resources affect the infection and recovery rates [20]. With these factors in consideration, we introduce the following modified periodic SIS model,

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = -\beta(t)f(S,I) + \gamma I, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \beta(t)f(S,I) - \gamma I. \end{cases} \tag{4.1}$$

It is easy to see that $\frac{\mathrm{d}(S+I)}{\mathrm{d}t} = 0$. This means that the total population size $S(t) + I(t)$ is a constant and is denoted by N. Then (4.1) is reduced to

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta(t)f(N-I,I) - \gamma I. \tag{4.2}$$

Now, we assume that the infection rate fluctuate periodically due to seasonal changes (such as temperature variations) as well as the effects of the school season and holiday breaks. Consequently, the infection rate exhibits cyclical alternations, that is, we assume

$$\beta(t) = \begin{cases} \beta_1(t), & t \in (nT, nT + \overline{T}], \\ \beta_2(t), & t \in (nT + \overline{T}, (n+1)T] \end{cases}$$

for $n = 0, 1, 2, \ldots$. Here T is the period of the disease transmissions. Due to variations in weather temperature and timing factors such as the school season and holidays, the parameter $\overline{T} \in (0, T)$ is used to describe the transition points between different seasons. Similar to the analysis in the preceding sections, under certain assumptions, we can derive the threshold value of the recovery rate γ based on assumption **(H)**.

Financial support. This research was supported partially by the National Natural Science Foundation of China [12471458, 12171110, 12331017, 12071095], the Natural Sciences and Engineering Research Council of Canada [RGPIN-2024-05593], and the Basic and Applied Basic Research Foundation of Guangdong Province [2023A1515011110].

Competing interests. Authors state no conflict of interest.

Data availability statement. Our manuscript has no associated data.

https://doi.org/10.1017/S0956792525100065 Published online by Cambridge University Press

References

[1] Augeraud-Véron, E. & Sari, N. (2014) Seasonal dynamics in an SIR epidemic system. *J. Math. Biol.* **68**, 701–725.
[2] Barrientos, P. G., Rodríguez, J.Á. & Ruiz-Herrera, A. (2017) Chaotic dynamics in the seasonally forced SIR epidemic model. *J. Math. Biol.* **75**, 1655–1668.
[3] Daily, J. P. (2012) Malaria vaccine trials–beyond efficacy end points. *N. Engl. J. Med.* **367**, 2349–2351.
[4] Duarte, J., Januário, C. & Martins, N. (2019) Chaos analysis and explicit series solutions to the seasonally forced SIR epidemic model. *J. Math. Biol.* **78**, 2235–2258.
[5] Dyck, V. A., Hendrichs, J. & Robinson, A. S. (2005). *Sterile Insect Technique Principles and Practice in Area-Wide Integrated Pest Management*. New Agriculturist.
[6] Feng, X., Liu, Y., Ruan, S. & Yu, J. (2023) Periodic dynamics of a single species model with seasonal Michaelis-Menten type harvesting. *J. Differ. Equ.* **354**, 237–263.
[7] Godói, I. P., Lemos, L. L. P., de Araújo, Vânia E., Bonoto, B. C., Godman, B. & Guerra Júnior, A. A. (2017) CYD-TDV dengue vaccine: systematic review and meta-analysis of efficacy, immunogenicity and safety. *J. Comput. Eff. Res*. **6**, 165–180.
[8] Hilgenboecker, K., Hammerstein, P., Schlattmann, P., et al. (2008) How many species are infected with *Wolbachia*?-A statistical analysis of current data. *FEMS Microbiol. Lett.* **281**, 215–220.
[9] Hsu, S. & Zhao, X. (2012) A Lotka-Volterra competition model with seasonal succession. *J. Math. Biol.* **64**, 109–130.
[10] Kyle, J. L. & Harris, E. (2008) Global spread and persistence of dengue. *Annu. Rev. Microbiol.* **62**, 71–92.
[11] Liu, Y., Feng, X., Ruan, S. & Yu, J. (2024) Periodic dynamics of a single species model with seasonal Michaaelis-Menten type harvesting, II: Existence of two periodic solutions. *J. Differ. Equ.* **388**, 253–285.
[12] Liu, Y., Yu, J., Chen, Y. & Guo, Z. (2024) Periodic dynamics of a reaction-diffusion-advection model with Michaelis-Menten type harvesting in heterogeneous environments. *SIAM J. Appl. Math.* **84**, 1891–1909.
[13] Liu, Y., Yu, J. & Li, J. (2024) Global dynamics of a competitive system with seasonal succession and different harvesting strategies. *J. Differ. Equ.* **382**, 211–245.
[14] Ooi, E. E., Goh, K. T. & Gubler, D. J. (2006) Dengue prevention and 35 years of vector control in Singapore. *Emerg. Infect. Diseases* **12**, 887–893.
[15] Pan, X., Shu, H., Wang, L., Wang, X. & Yu, J. (2025) On the periodic solutions of switching scalar dynamical systems. *J. Differ. Equ.* **415**, 365–382.
[16] Pattamaporn, K., Baisley, K. J., Visut, B. & O'Neill, S. L. (2000) Distribution and diversity of *wolbachia* infections in southeast asian mosquitoes (Diptera: Culicidae). *J. Med. Entomol.* **37**, 340–345.
[17] Pfarr, K., Foster, J. & Slatko, B. (2007) On the taxonomic status of the intracellular bacterium *wolbachia* pipientis: Should this species name include the intracellular bacteria of filarial nematodes? *Int. J. Syst. Evol. Microbiol.* **57**, 1677–1678.
[18] Pfizer, Mosquito as deadly menace. https://www.pfizer.com/news/articles/mosquito_as_deadly_menace#.Yweo2_DM88I.link
[19] Schwartz, L. M., Halloran, M. E., Durbin, A. P. & Longini, I. M., Jr (2015) The dengue vaccine pipeline: Implications for the future of dengue control. *Vaccine* **33**, 3293–3298.
[20] Tang, Y., Xiao, D., Zhang, W. & Zhu, D. (2017) Dynamics of epidemic models with asymptomatic infection and seasonal succession. *Math. Biosci. Eng.* **14**, 1407–1424.
[21] Turelli, M. & Hoffmann, A. A. (1991) Rapid spread of an inherited incompatibility factor in California drosophila. *Nature* **353**, 440–442.
[22] Xi, Z., Khoo, C. & Dobson, S. (2005) *Wolbachia* establishment and invasion in an aedes aegypti laboratory population. *Science* **310**, 326–328.
[23] Xiao, D. (2017) Dynamics and bifurcations on a class of population model with seasonal constant-yield harvesting. *Discrete Contin. Dyn. Syst., Ser. B* **21**, 699–719.
[24] Yu, J. & Li, J. (2020) Global asymptotic stability in an interactive wild and sterile mosquito model. *J. Differ. Equ.* **269**, 6193–6215.
[25] Zheng, B., Zhou, H. & Yu, J. (2025) Periodic dynamics of a single-species population model based on the discrete Beverton-Holt equation. *Discret. Contin. Dyn. Syst.* **45**, 2126–2144.
[26] Zheng, B., Yang, H., Elaydi, S. & Yu, J. (2025) *w*Stri spread dynamics in *Nilaparvata lugens* via discrete mathematical models. *J. Math. Biol.* **90**, 38.

Cite this article: Ding Q., Yu J., Guo Z., Chen Y. and Liu Y. (2026). Periodic dynamics of a general switching dynamical system. *European Journal of Applied Mathematics*, **37**, 882–896. https://doi.org/10.1017/S0956792525100065

https://doi.org/10.1017/S0956792525100065 Published online by Cambridge University Press