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Nonlocal carrier's double phase problem

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We study the existence and multiplicity of positive bounded solutions for a class of nonlocal, non-variational elliptic problems governed by a nonhomogeneous operator with unbalanced growth, specifically the double phase operator. To tackle these challenges, we employ a combination of analytical techniques, including the sub-super solution method, variational and truncation approaches, and set-valued analysis. Furthermore, we examine a one-dimensional fixed-point problem. To the best of our knowledge, this is the first work addressing nonlocal double phase problems using these methods.

Keywords: many positive solutions; nonlocal term; set-valued analysis; double phase problem

2020 *Mathematics Subject Classification:* 35A01; 35B09; 35J60

1. Introduction

The study of the double phase problem gained widespread attention in the last century, primarily due to the seminal works of Marcellini [62, 63] and Zhikov [77, 79]. Interest in this type of unbalanced growth problem,

$$\mathcal{G}(u) = \operatorname{div}\left(\mu(x)|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\right), \quad \forall u \in W_0^{1,\mathcal{H}}(\Omega), \tag{1}$$

is driven by its broad applicability in physical, biological, and engineering models. Notable examples include transonic flow models studied by Bahrouni–Rădulescu–Repovš [4], applications in quantum physics explored by Benci–D'Avenia–Fortunato–Pisani [8], and models in electromagnetism (specifically the Born–Infeld equation) investigated by Bonheure–D'Avenia–Pomponio [9] Further applications include image restoration and denoising (see Charkaoui–Ben

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Loghfyry–Zeng [19] and Harjulehto–Hästö [55]), reaction-diffusion system models (see Cherfils–Il'yasov [20]), and elasticity theory to describe the behaviour of composite materials under deformation (see Zhikov [77]).

In recent years, research on the double phase problem has expanded significantly, resulting in a rich body of literature. For further references, we direct interested readers to Bai–Papageorgiou–Zeng [5], Crespo Blanco–Papageorgiou–Winkert [35], D'Aguì–Sciammetta–Tornatore–Winkert [37], Gambera–Guarnotta–Papageorgiou [43], Gasiński–Papageorgiou [48, 49], Gasiński–Winkert [50, 51], Gou–Rădulescu [52], Guarnotta–Livrea–Winkert [53], Jing–Liu–Papageorgiou [58], Marino–Winkert [64], Papageorgiou–Pudełko–Rădulescu [68], Papageorgiou–Rădulescu–Repovš [69], Papageorgiou–Rădulescu–Zhang [71], Papageorgiou–Rădulescu–Zhang [70], Papageorgiou–Vetro–Vetro [72], Sciammetta–Tornatore–Winkert [76], and the reference therein.

Moreover, the double phase operator holds significant mathematical interest in the study of Lavrentiev's phenomenon (see Borowski–Chlebicka–De Filippis–Miasojedow [10], De Filippis–Mingione [38], and Zhikov [78]). For regularity results, we refer to the seminal works of Baroni–Colombo–Mingione [7], Colombo–Mingione [28], Ho–Winkert [56], Ragusa–Tachikawa [75] and the related references. Recently, Perera–Squassina [74] have approached this operator using Morse theory. For a comprehensive overview, we refer to the survey by Papageorgiou [67] and Chapter 4, Section 9 of the book by Papageorgiou–Winkert [73].

Our problem is nonlocal in the sense introduced by Carrier [17] in the study of beam deformation in lower dimensions:

$$\left[1+\frac{\alpha^{-2}}{2\pi}\int_0^{\pi}\varphi^2(\xi,\eta)d\xi\right]\frac{\partial^2\varphi}{\partial\xi^2}=\frac{\partial^2\varphi}{\partial\eta^2},$$

where α is a positive constant. These types of nonlocal models have been extensively used in biological applications, such as population dynamics:

$$-a\left(\int_\Omega |u|^p dx\right)\Delta u=f(u),$$

where the dispersion velocity depends on the population density gradient:

$$v=a\left(\int_\Omega u^q dx\right)\nabla u. \tag{2}$$

Relevant studies in this context include those by Chang–Chipot [18], Chipot–Lovat [21, 22], Chipot–Molinet [23], Chipot–Rodrigues [24], Chipot–Siegwart [25], Corrêa [29], Corrêa–Menezes–Fereira [30], Furter–Grinfeld [42], and the reference therein. Bifurcation arguments for these nonlocal problems can be found in Figueiredo Sousa–Morales Rodrigo–Suárez [41] and Schauder's fixed point theorem application in Bueno–Ercole–Ferreira–Zumpano [12].

We point out that, unlike Kirchhoff type models,

$$-a\left(\int_\Omega\left(\mu(x)\frac{|\nabla u|^p}{p}+\frac{|\nabla u|^q}{q}\right)dx\right)\mathcal{G}(u)=f(u),$$

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which permit a variational formulation (see Arora–Fiscella–Mukherjee–Winkert [2, 3], Colasuonno–Perera [26], Crespo Blanco–Gasński–Winkert [34], Santos Júnior–Siciliano [57]), our nonlocal problem is non-variational due to the presence of the nonlocal weight function on the left-hand side. Consequently, we employ a combination of techniques to obtain our results.

In particular, we study a variational problem with a "frozen" nonlocal weight function. Furthermore, we approach the problem through a one-dimensional fixed-point method, utilizing the sub-super solution technique and set-valued analysis to construct a well-defined fixed-point map. Early works in this direction include Gasiński–Santos Júnior [44, 45] for the Laplacian case, Gasiński–Santos Júnior–Siciliano [46] for the Laplacian with possible singularities, Candito–Gasiński–Livrea–Santos Júnior [15] for the p-Laplacian problem, Crespo Blanco–Failla–Vassallo [32] for the p-Laplacian with Neumann boundary conditions, and Candito–Failla–Livrea [14] for variable exponent $p(x)$-Laplacian.

In previous studies, a crucial assumption involved the monotonicity of the map $t \mapsto \frac{f(t)}{t^{p-1}}$. However, in a recent work by Candito–Failla–Gasiński–Livrea [13], the authors removed this assumption by introducing a novel sub-solution construction combined with a set-valued map. Building on these ideas, our objective is to establish the existence and multiplicity of solutions for the following non-variational nonlocal elliptic problem driven by an unbalanced growth operator, namely, the double phase operator.

To the best of our knowledge, this is the first study addressing the nonlocal double phase problem of the form:

$$\begin{cases} -a\left(\displaystyle\int_\Omega (u^r + \nu(x)u^s)\,dx\right)\mathcal{G}(u) = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ has a Lipschitz boundary $\partial\Omega$ and $1 < q < p$, $r, s \geq 1$.

We now present the precise hypotheses required for our main multiplicity result. First on the exponents and weight functions we require that

- (H_1) $1 < q < N$, $q < p$, $\frac{p}{q} < 1 + \frac{1}{N}$, $r, s \geq 1$, and $\mu \in C^{0,1}(\overline{\Omega})$, $\mu : \Omega \to [0, +\infty)$, $\nu \in L^\infty(\Omega)$, $\nu : \Omega \to [0, +\infty)$.

Notice that, the above assumption implies that $q < p < q^*$, where $q^* = \frac{Nq}{N-q}$ is the Sobolev critical exponent. Next, we impose an "oscillatory" behaviour on a, characterized by multiple zeros and strictly positive values in alternating intervals.

- (H_2) The function $a : [0, +\infty) \to \mathbb{R}$ is continuous and there exist positive numbers $0 \leq \alpha_0 \leq \alpha_1 < \alpha_2 \leq \alpha_3 < \alpha_4 \leq \ldots \leq \alpha_{2k-1} < \alpha_{2K}$ $(K \geq 1)$ such that $a > 0$ in $(\alpha_{2k-1}, \alpha_{2k})$ and $a(\alpha_{2k-1}) = a(\alpha_{2k}) = 0$ for all $k \in \{1, \ldots, K\}$.

For the right hand side nonlinearity we assume the following regularity conditions.

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- (H_3) There exists $t_* > 0$, such that function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f \in C([0, t_*])$, $f(t) > 0$ for all $t \in (0, t_*)$, $f(t_*) = 0$ and

$$\liminf_{s \to 0^+} f(s) > \max_{k \in \{1,\dots,K\}} \max_{\alpha \in [\alpha_{2k-1}, \alpha_{2k}]} a(\alpha).$$

In our proofs, we will make use of the solution to the following torsion problem:

$$\begin{cases} -\mathcal{G}(u) = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{T}$$

By the Minty–Browder Theorem (see Brézis [11, Theorem 5.16]), since $-\mathcal{G}(u)$ is strictly monotone (see Gasiński–Papageorgiou [49, Proposition 2.4]) and $1 \in \left(W_0^{1,\mathcal{H}}(\Omega)\right)^*$, there exists a unique positive solution $e_1 \in W_0^{1,\mathcal{H}}(\Omega)$. Moreover, by results from Ho–Winkert [56, Theorem 4.2] and Crespo Blanco–Winkert [36, Theorem 3.1], it follows that $e_1 \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$.

The following conditions impose further constraints on the domains of f and a.

- (H_4) $t_* \leq \|e_1\|_\infty$ and $\alpha_{2K} < t_*^r \|e\|_r^r + t_*^s \|e\|_s^s \min_\Omega \nu$, where $e = e_1/\|e_1\|_\infty$.

Finally, we impose some assumptions on the height of each "even bump" of a.

- (H_5) For each $k \in \{1, ..., K\}$ there exists $\tau_k \in (\alpha_{2k-1}, \alpha_{2k})$ such that

$$\tau_k^q a(\tau_k) > \frac{|\Omega|^q}{\lambda_{1,q}} \left(t_*^{r-1} + t_*^{s-1} \max_\Omega \nu \right)^q \max_{t \in [0, t_*]} [t f(t)],$$

where $\lambda_{1,q}$ is the first eigenvalue of the operator $\left(-\Delta_q, W_0^{1,q}(\Omega)\right)$.

Our goal is to establish the following multiplicity theorem (see Section 2 for the definition of the spaces).

THEOREM 1.1. *Assume that* (H_1), (H_2), (H_3), (H_4) *and* (H_5) *hold. Then, problem* (P) *admits at least* $2K$ *positive solutions in* $W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$.

The rest of the paper is organized as follows. Section 2 introduces the functional setting for our solutions, specifically the Sobolev–Musielak–Orlicz spaces, and recalls relevant results from set-valued analysis. Section 3 establishes the existence of bounded solutions for a frozen problem using a combination of variational and truncation methods with the sub-super solution approach. Section 4 constructs a set-valued map to build the minimal positive solution of the frozen problem. Section 5 formulates a one-dimensional fixed-point problem and derives geometric constraints on the map to obtain multiple solutions for the original problem.

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2. Preliminaries

2.1. Setting spaces

We introduce the natural setting for our problem, see Adams–Fournier [1] for a complete overview on this topic. Let us begin by considering the Lebesgue space, set $1 \leq r < +\infty$,

$$L^r(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u|^r dx < \infty \right\}$$

with the classical r-norm

$$\|u\|_r = \left(\int_\Omega |u|^r dx \right)^{\frac{1}{r}}.$$

For $r = +\infty$, we consider

$$L^\infty(\Omega) = \{ u : \Omega \to \mathbb{R} \text{ measurable} : \operatorname{ess\,sup}_{x\in\Omega} |u(x)| < +\infty \}$$

with the sup-norm

$$\|u\|_\infty = \operatorname{ess\,sup}_{x\in\Omega} |u(x)|.$$

Moreover, for $1 \leq r < +\infty$, we use the Sobolev space

$$W^{1,r}(\Omega) = \{ u \in L^r(\Omega) : \nabla u \in L^r(\Omega) \}$$

with the norm

$$\|u\|_{1,r} = \|u\|_r + \|\nabla u\|_r.$$

In particular, we introduce the Sobolev space

$$W_0^{1,r}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,r}} \tag{3}$$

endowed with the equivalent norm

$$\|u\| = \|\nabla u\|_r.$$

For our purpose, we introduce the notion of Musielak–Orlicz space and Musielak–Orlicz–Sobolev space, see Harjulehto–Hästö [54] for details. Let us define the nonlinear function $\mathcal{H}\colon \Omega \times [0, +\infty) \to [0, +\infty)$

$$\mathcal{H}(x,t) = \mu(x)t^p + t^q$$

which in fact has an unbalance growth, that is,

$$b_1|\xi|^q \leq \mathcal{H}(x,\xi) \leq b_2\,(1 + |\xi|^p) \quad \text{for a.a. } x \in \Omega, \forall \xi \in \mathbb{R}^N,$$

for some $b_1, b_2 > 0$ and let us consider the related modular function

$$\rho_{\mathcal{H}}(u) = \int_\Omega \mathcal{H}(x, |u|)dx. \tag{4}$$

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Then, we introduce the so called Musielak–Orlicz space as follows

$$L^{\mathcal{H}}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable} : \rho_{\mathcal{H}}(u) < +\infty\}$$

endowed with the Luxemburg norm

$$\|u\|_{L^{\mathcal{H}}(\Omega)} = \inf\left\{\lambda > 0 : \rho_{\mathcal{H}}\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Consider the following space

$$L^p_\mu(\Omega) = \left\{u : \Omega \to \mathbb{R}, \text{measurable } : \int_\Omega \mu(x)|u|^p dx < +\infty\right\}, \tag{5}$$

endowed with the semi-norm

$$\|u\|_{p,\mu} = \left(\int_\Omega \mu(x)|u|^p dx\right)^{\frac{1}{p}}. \tag{6}$$

By Colasuonno–Squassina [27, Proposition 2.15], it follows that the embeddings

$$L^p(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega) \hookrightarrow L^p_\mu(\Omega) \cap L^q(\Omega), \tag{7}$$

are continuous. Analogously, we introduce the Musielak–Orlicz–Sobolev space as

$$W^{1,\mathcal{H}}(\Omega) = \left\{u \in L^{\mathcal{H}}(\Omega) : |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\},$$

with the norm

$$\|u\|_{W^{1,\mathcal{H}}(\Omega)} = \|u\|_{L^{\mathcal{H}}(\Omega)} + \|\nabla u\|_{L^{\mathcal{H}}(\Omega)}.$$

Starting from Colombo–Mingione [28] and Crespo Blanco–Gasiński–Harjulehto–Winkert [33, Theorem 2.23], assumption (H_1) allows us to consider the space

$$W^{1,\mathcal{H}}_0(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{W^{1,\mathcal{H}}(\Omega)}}, \tag{8}$$

and by the Musielak–Orlicz version of the Poincaré inequality (see e.g., Harjulehto–Hästö [54, Theorem 6.2.8, pag. 130]) we can equip this space with the equivalent norm

$$\|u\| = \|\nabla u\|_{L^{\mathcal{H}}(\Omega)}.$$

Note that, by Colasuonno–Squassina [27, Proposition 2.18] (see also, Crespo Blanco–Gasiński–Harjulehto–Winkert [33, Proposition 2.12] and Gasiński–Winkert [50, Section 2]), in our assumptions, the norm $\|\cdot\|_{L^{\mathcal{H}}(\Omega)}$ is uniformly convex, and then, $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}_0(\Omega)$ are reflexive Banach spaces.

Remark 2.1. We point out that the following relation holds, see e.g., [50, eq. (2.2)]

$$\min\left\{\|u\|^p_{L^{\mathcal{H}}(\Omega)}, \|u\|^q_{L^{\mathcal{H}}(\Omega)}\right\} \leq \|u\|^p_{p,\mu} + \|u\|^q_q \leq \max\left\{\|u\|^p_{L^{\mathcal{H}}(\Omega)}, \|u\|^q_{L^{\mathcal{H}}(\Omega)}\right\} \quad \forall u \in L^{\mathcal{H}}(\Omega). \tag{9}$$

For readers convenience, we recall the following properties of double phase operator, see Liu–Dai [60, Proposition 3.1].

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PROPOSITION 2.2. *Let* $L : W_0^{1,\mathcal{H}}(\Omega) \to \left(W_0^{1,\mathcal{H}}(\Omega)\right)^*$ *be such that*

$$\langle Lu, v\rangle = \int_\Omega \left(\mu(x)|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}.\nabla u\right)\nabla v dx.$$

Then, L *is a continuous, bounded, strictly monotone operator. Moreover,* L *is type* (S_+)*, i.e., if* $u_n \rightharpoonup u$ *in* $W_0^{1,\mathcal{H}}(\Omega)$ *and* $\limsup_{n\to+\infty}\langle L(u_n) - L(u), u_n - u\rangle \leq 0$*, then* $u_n \to u$ *in* $W_0^{1,\mathcal{H}}(\Omega)$*.*

2.2. Set-valued analysis

Next, let us recall some notions concerning partially ordered sets and set-valued maps.

DEFINITION 2.3. *Let* $(P, \leq)$ *be a partially ordered set and let* $A \subseteq P$*. An element* b *is called lower bound of* A *if* $b \leq x$ *for each* $x \in A$*. If* $b \in A$*, we say that* b *is the smallest element of* A*. An element* $x \in A$*, is called minimal element, if there is no* $y \neq x$ *in* A *for which* $y \leq x$*, and every smallest element of* A *is minimal element of* A*. A subset* C *of* P *is said downward direct if for each pair* $x, y \in C$ *there is* $w \in C$ *such that* $w \leq x$ *and* $w \leq y$*.*

The following Kuratowski–Zorn Lemma can be found in Carl–Le–Motreanu [16, Theorem 2.62]

LEMMA 2.4. *If in a partially ordered set* P*, every chain has a lower bound, then* P *possesses a minimal element.*

DEFINITION 2.5. *We say that a function* $G : X \to Y$ *is compact if* G *is continuous and maps bounded sets into relatively compact sets (i.e., sets whose closure is compact).*

For more information concerning set-valued functions, we refer to Denkowski–Migórski–Papageorgiou [39, Chapter 4], Gasiński–Papageorgiou [47, Chapters 3 and 7] or Migórski–Ochal–Sofonea [65, Chapter 3].

DEFINITION 2.6. *Let* X*,* Y *be topological spaces. Consider a set-valued map* $F : X \to 2^Y$*. We say that*

- F *is upper semi-continuous (briefly u.s.c.), if for every closed subset* $C \subset Y$*, the set*

$$F^-(C) = \{x \in X : F(x) \cap C \neq \varnothing\}$$

is closed in X*. Alternatively,* F *is u.s.c., if for every open subset* $O \subset Y$*, the set*

$$F^+(O) = \{x \in X : F(x) \subset O\}$$

is open in X*.*

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- *F is lower semi-continuous (briefly l.s.c.), if for every closed subset $C \subset Y$, the set*
$$F^+(C) = \{x \in X : F(x) \subset C\}$$
is closed in X. Alternatively, F is l.s.c., for every open subset $O \subset Y$, the set
$$F^-(O) = \{x \in X : F(x) \cap O \neq \varnothing\}$$
is open in X.
- *We say that F is continuous if it is u.s.c. and l.s.c.*

The following characterization of l.s.c. set-valued function holds (see e.g., Migórski–Ochal–Sofonea [65, Proposition 3.9], Liu–Motreanu–Zeng [61, Proposition 5] or Baldelli–Guarnotta [6, Lemma 4.4] and Klein–Thompson [59, Theorem 7.1.7] for the proof).

LEMMA 2.7. *Let X, Y be two topological spaces. Let $F : X \to 2^Y$ be a set-valued function. Then, F is l.s.c. if and only if the following hold: if $u \in X$, $\{u_\gamma\}_{\gamma \in I} \subset X$ is a net such that $u_\gamma \to u$ and $u^* \in F(u)$, then for each $\gamma \in I$ there is $u^*_\gamma \in F(u_\gamma)$ with $u^*_\gamma \to u^*$ in Y.*

3. Frozen problem

In this section we fix $k \in \{1, ..., K\}$ and $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$. Consider the following auxiliary frozen problem

$$\begin{cases} -a(\alpha)\mathcal{G}(u) = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{F_α}$$

We recall the following definitions (see e.g., Guarnotta–Livrea–Winkert [53, Definition 5.2] or Motreanu–Motreanu–Papageorgiou [66, Definition 11.5]) of sub- and super-solutions of the above problem. Consider

$$L^{\mathcal{H}}_+(\Omega) = \left\{u \in L^{\mathcal{H}}(\Omega) : u \geq 0\right\}.$$

DEFINITION 3.1. *We say that $\underline{u} \in W^{1,\mathcal{H}}(\Omega)$ is a sub-solution for (F_α), if $\underline{u} \leq 0$ on $\partial\Omega$ and*

$$a(\alpha) \int_\Omega \left(\mu(x)|\nabla \underline{u}|^{p-2}\nabla \underline{u} + |\nabla \underline{u}|^{q-2}\nabla \underline{u}\right) \nabla v dx \leq \int_\Omega f(x, \underline{u}) v dx \quad \forall v \in W^{1,\mathcal{H}}_0(\Omega) \cap L^{\mathcal{H}}_+(\Omega).$$

Analogously, $\bar{u} \in W^{1,\mathcal{H}}(\Omega)$ is called super-solution for (F_α), if $\bar{u} \geq 0$ on $\partial\Omega$ and

$$a(\alpha) \int_\Omega \left(\mu(x)|\nabla \bar{u}|^{p-2}\nabla \bar{u} + |\nabla \bar{u}|^{q-2}\nabla \bar{u}\right) \nabla v dx \geq \int_\Omega f(x, \bar{u}) v dx \quad \forall v \in W^{1,\mathcal{H}}_0(\Omega) \cap L^{\mathcal{H}}_+(\Omega).$$

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Similarly, $u \in W_0^{1,\mathcal{H}}(\Omega)$ is called (weak) solution for (F_α), if $u = 0$ on $\partial\Omega$ and

$$a(\alpha)\int_\Omega \left(\mu(x)|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\right)\nabla v dx = \int_\Omega f(x,u)v dx \quad \forall v \in W_0^{1,\mathcal{H}}(\Omega).$$

In all three cases, we assume that the right-hand side is well defined, i.e., $f(\cdot, u(\cdot)) \in L^{m'}(\Omega)$ for some $m \in [1, q^*)$.

LEMMA 3.2. *Assume that (H_1), (H_2), (H_3) and (H_4) hold. Then (F_α) admits a positive bounded sub-solution $\underline{u} \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$.*

Proof. To obtain a positive sub-solution for (F_α), consider the torsion problem (T). It admits a unique, positive solution $e_1 \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$. Define

$$t(\alpha) = \min\left\{t \in [0, t_*] : f(t) = a(\alpha)\right\}, \tag{10}$$

and consider $\underline{u} = t(\alpha)e$ where $e = e_1/\|e_1\|_\infty$. By the definition of $t(\alpha)$, we have $f(t(\alpha)) \le f(s)$ for all $s \in (0, t(\alpha))$ and by (H_4), $t(\alpha) \le t_* \le \|e_1\|_\infty$. So

$$\begin{aligned}
&- a(\alpha)\,\mathrm{div}\left(\mu(x)|\nabla \underline{u}|^{p-2}\nabla \underline{u} + |\nabla \underline{u}|^{q-2}\nabla \underline{u}\right)\\
\le\ &- a(\alpha)\min\left\{\left(\frac{t(\alpha)}{\|e_1\|_\infty}\right)^{p-1}, \left(\frac{t(\alpha)}{\|e_1\|_\infty}\right)^{q-1}\right\}\\
&\times \mathrm{div}\left(\mu(x)|\nabla e_1|^{p-2}\nabla e_1 + |\nabla e_1|^{q-2}\nabla e_1\right)\\
\le\ & a(\alpha) = f(t(\alpha)) \le f(\underline{u}),
\end{aligned}$$

thus $\underline{u} \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ is a sub-solution of problem (F_α). □

REMARK 3.3. Note that $\overline{u} = t_*$ is a super-solution for (F_α) and $\underline{u} \le \overline{u}$.

Consider the following truncation on the reaction term

$$\hat{f}(x,s) = \begin{cases} f(\underline{u}(x)) & \text{if} \quad s < \underline{u}(x),\\ f(s) & \text{if} \quad \underline{u}(x) \le s \le \overline{u}(x),\\ f(\overline{u}(x)) & \text{if} \quad \overline{u}(x) < s, \end{cases} \tag{11}$$

and consider the associated truncated problem

$$\begin{cases} -a(\alpha)\mathcal{G}(u) = \hat{f}(x,u) & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{FT_α}$$

The following existence result for problem (FT_α) holds.

PROPOSITION 3.4. *Assume that (H_1), (H_2), (H_3) and (H_4) hold. Then (FT_α) admits at least one positive solution $u \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$. Moreover, each solution of (FT_α) is such that $\underline{u} < u < \overline{u}$.*

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Proof. The associated energy function is

$$J(u) = \frac{a(\alpha)}{p}\|\nabla u\|_{p,\mu}^{p} + \frac{a(\alpha)}{q}\|\nabla u\|_{q}^{q} - \int_{\Omega} \hat{F}(u)dx, \tag{12}$$

where $\hat{F}(t) = \displaystyle\int_0^t \hat{f}(s)ds$. Clearly, J is Gâteaux differentiable, coercive and l.s.c., then, by Tonelli–Weiestrass Theorem (see e.g., Costa [31, Theorem 1.2]) J has a global minimum u, which is obviously a weak solution of (FT_{α}). As in Lemma 3.2, $u \in W_0^{1,\mathcal{H}}(\Omega) \cap L^{\infty}(\Omega)$. Now, we prove that every solution u of (FT_{α}) lies in $[\underline{u}, \overline{u}]$, that is, u is a solution of (F_{α}) by (11). Since $\underline{u}$ is a sub-solution, then

$$a(\alpha)\int_{\Omega}\left(\mu(x)|\nabla \underline{u}|^{p-2}\nabla \underline{u} + |\nabla \underline{u}|^{q-2}\nabla \underline{u}\right)\nabla v dx \leq \int_{\Omega} \hat{f}(x,\underline{u})v dx \quad \forall v \in W_0^{1,\mathcal{H}}(\Omega) \cap L_{+}^{\mathcal{H}}(\Omega). \tag{13}$$

Moreover, $\overline{u}$ is a super-solution, then

$$a(\alpha)\int_{\Omega}\left(\mu(x)|\nabla \overline{u}|^{p-2}\nabla \overline{u} + |\nabla \overline{u}|^{q-2}\nabla \overline{u}\right)\nabla v dx \geq \int_{\Omega} \hat{f}(x,\overline{u})v dx \quad \forall v \in W_0^{1,\mathcal{H}}(\Omega) \cap L_{+}^{\mathcal{H}}(\Omega). \tag{14}$$

Finally, u is a solution of (FT_{α}), then

$$a(\alpha)\int_{\Omega}\left(\mu(x)|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\right)\nabla v dx = \int_{\Omega} \widehat{f}(x,u)v dx \quad \forall v \in W_0^{1,\mathcal{H}}(\Omega). \tag{15}$$

Next, subtratting (13) and (15) and choosing $v = (\underline{u} - u)^{+}$, we get

$$a(\alpha)\int_{\Omega}\left(\mu(x)|\nabla \underline{u}|^{p-2}\nabla \underline{u} + |\nabla \underline{u}|^{q-2}\nabla \underline{u} - \mu(x)|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\right) \times \nabla(\underline{u} - u)^{+} dx \leq 0. \tag{16}$$

Recalling (9) and strictly monotonicity of double phase operator (Proposition 2.2), we infer

$$0 \leq \min\left\{\|(\underline{u} - u)^{+}\|^{p}, \|(\underline{u} - u)^{+}\|^{q}\right\} \leq 0. \tag{17}$$

Then, we obtain that the sub-level $\{\underline{u} > u\}$ has null Lebesgue measure, that is $u \geq \underline{u}$. Similarly, subtracting (15) and (14) and choosing $v = (u - \overline{u})^{+}$, we deduce that $u \leq \overline{u}$. □

4. Minimal solution

In this section we fix $k \in \{1, \ldots, K\}$. Set $X = W_0^{1,\mathcal{H}}(\Omega) \cap L^{\infty}(\Omega)$. Following the ideas of Baldelli–Guarnotta [6], Faraci–Motreanu–Puglisi [40], and Liu–Motreanu–Zeng [61], we consider the following set-valued map $\mathscr{S}_k : (\alpha_{2k-1}, \alpha_{2k}) \to 2^X$,

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$$\mathscr{S}_k(\alpha) = \{u \in X : u \text{ solves } (FT_\alpha) \text{ and } u \geq \underline{u}\}. \tag{18}$$

Note that, by Proposition 3.4, $\mathscr{S}_k$ is well defined, i.e., it has nonempty values.

PROPOSITION 4.1. *Assume that* (H_1), (H_2), (H_3) *and* (H_4) *hold. Then,* $\mathscr{S}_k$ *maps convergent sequences in* $(\alpha_{2k-1}, \alpha_{2k})$ *into sequences with convergent subsequences in* X.

Proof. Let $\{\alpha_n\} \subset (\alpha_{2k-1}, \alpha_{2k})$ be a sequence such that $\alpha_n \to \alpha \in (\alpha_{2k-1}, \alpha_{2k})$ and $u_n \in \mathscr{S}_k(\alpha_n)$. We prove that, up to a subsequence $u_n \to u \in \mathscr{S}_k(\alpha)$. First, by definition of $\mathscr{S}_k$, we have that, for all $n \in \mathbb{N}$, u_n is a solution of

$$\begin{cases} -a(\alpha_n)\mathcal{G}(u_n) = \hat{f}(x, u_n) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

such that $u_n \geq \underline{u}$. Note that, $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence. Indeed

$$\begin{aligned} a(\alpha_n) \int_\Omega \left(\mu(x)|\nabla u_n|^{p-2}\nabla u_n + |\nabla u_n|^{q-2}\nabla u_n\right) \nabla v dx \\ = \int_\Omega \hat{f}(x, u_n) v dx \quad \forall v \in W_0^{1,\mathcal{H}}(\Omega). \end{aligned}$$

That is, by (9) and (H_3),

$$a(\alpha_n) \int_\Omega \left(\mu(x)|\nabla u_n|^{p-2}\nabla u_n + |\nabla u_n|^{q-2}\nabla u_n\right) \nabla v dx \leq |\Omega| \max_{t\in[0,t_*]} [tf(t)] \tag{20}$$

and $\|u_n\| < +\infty$ $\forall n \in \mathbb{N}$. Then, up to a subsequence, we have $u_n \rightharpoonup u$ in $W_0^{1,\mathcal{H}}(\Omega)$ and $u_n \to u$ in $L^{\mathcal{H}}(\Omega)$ (remember that $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^q(\Omega)$ compactly for $q < p^*$ and $L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ continuously, see (9) and Colasuonno–Squassina [27, Proposition 2.18]). Then, we have

$$\lim_{n\to+\infty} \langle \mathcal{G}(u_n) - \mathcal{G}(u), u_n - u\rangle = 0, \tag{21}$$

and by the (S_+)-property of $\mathcal{G}$ (see Lemma 2.2), $u_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$. Moreover, by the Lebesgue Dominate Convergence Theorem (see e.g., Brézis [11, Theorem 4.2]), passing to the limit in (19), u is a solution of

$$\begin{cases} -a(\alpha)\mathcal{G}(u) = \hat{f}(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{22}$$

such that $u \geq \underline{u}$, i.e., $u \in \mathscr{S}_k(\alpha)$. □

PROPOSITION 4.2. *Assume that* (H_1), (H_2), (H_3) *and* (H_4) *hold. Then,* $\mathscr{S}_k$ *is l.s.c.*

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Proof. We use the characterization of l.s.c. contained in Lemma 2.7. Let $\{\alpha_n\} \subset (\alpha_{2k-1}, \alpha_{2k})$ be a sequence such that $\alpha_n \to \alpha \in (\alpha_{2k-1}, \alpha_{2k})$. Let $v \in \mathscr{S}_k(\alpha)$ and for all $n \in \mathbb{N}$, consider the problem

$$\begin{cases} -a(\alpha_n)\mathcal{G}(u) = \hat{f}(x, v) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (FT_{\alpha n}{}^{v})$$

Clearly $v \geq \underline{u}$ by definition of $\mathscr{S}_k(\alpha)$. Moreover, by Minty–Browder's Theorem (see e.g., [11, Theorem 5.16]), $(FT_{\alpha n}{}^{v})$ has a unique positive solution u_n^0 for all $n \in \mathbb{N}$. Note that, as in Proposition 4.1, since $\alpha_n \to \alpha \in (\alpha_{2k-1}, \alpha_{2k})$, then u_n^0 is relatively compact in X and, up to a sub-sequence, $u_{n_k}^0 \to u$ in X as $k \to +\infty$, being u the unique solution of

$$\begin{cases} -a(\alpha)\mathcal{G}(u) = \hat{f}(x, v) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $v \in \mathscr{S}_k(\alpha)$, by the definition of $\mathscr{S}_k$, v is a solution of

$$\begin{cases} -a(\alpha)\mathcal{G}(v) = \hat{f}(x, v) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, it follows that $u = v$. Since each subsequence $\{u_{n_k}^0\}$ of $\{u_n^0\}$ converges to the same limit v, then in fact

$$\lim_{n \to +\infty} u_n^0 = v.$$

Again, for each $n \in \mathbb{N}$ consider the problem

$$\begin{cases} -a(\alpha_n)\mathcal{G}(u) = \hat{f}(x, u_n^0) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (FT_{\alpha n}u_n{}^{0})$$

As above, there exists a unique solution u_n^1 of $(FT_{\alpha n}u_n{}^{0})$, for each $n \in \mathbb{N}$, such that

$$\lim_{n \to +\infty} u_n^1 = v.$$

By iterating this process, for any $k \geq 1$, we get a sequence $\{u_n^k\}$, such that u_n^k is a solution of

$$\begin{cases} -a(\alpha_n)\mathcal{G}(u_n^k) = \hat{f}(x, u_n^{k-1}) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (FT_{\alpha n}u_n{}^{k-1})$$

and

$$\lim_{n \to +\infty} u_n^k = v \quad \forall k \in \mathbb{N}. \tag{23}$$

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Fix $n \geq 1$, since $\{u_n^k\}$ is relatively compact in X, by Proposition 4.1, we have that

$$u_n^k \to u_n \text{as } k \to +\infty,$$

that is, it solves

$$\begin{cases} -a(\alpha_n)\mathcal{G}(u_n) = \hat{f}(x, u_n) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (FT_{\alpha n}{}^{u_n})$$

and $u_n \geq \underline{u}$, i.e., $u_n \in \mathscr{S}_k(\alpha_n)$. By compact embedding (as in Proposition 4.1), (23), and double limit lemma (see e.g., Gasiński–Papageorgiou [47, Proposition A.2.35]), we have that $u_n \to v$ in X. Thus, by Lemma 2.7 we conclude that $\mathscr{S}_k$ is l.s.c. □

The following is a technical lemma that will allow us to obtain the downward directness of $\mathscr{S}_k$.

LEMMA 4.3. *If $\overline{u}_1, \overline{u}_2 \in W^{1,\mathcal{H}}(\Omega)$ are two super-solutions of (F_α), then $\min\{\overline{u}_1, \overline{u}_2\} \in W^{1,\mathcal{H}}(\Omega)$ is also a super-solution for (F_α).*

Proof. Let $\overline{u}_1, \overline{u}_2 \in W^{1,\mathcal{H}}(\Omega)$ be two super-solutions for (F_α). Let $\varepsilon > 0$ and consider the following Lipschitz continuous truncation function $\eta_\varepsilon : \mathbb{R} \to \mathbb{R}$, defined by

$$\eta_\varepsilon(t) = \begin{cases} 0 & \text{if} \quad t \leq 0; \\ \frac{t}{\varepsilon} & \text{if} \quad 0 < t < \varepsilon; \\ 1 & \text{if} \quad t \geq \varepsilon. \end{cases} \qquad (24)$$

Note that, by Adams–Fournier [1, Lemma 8.34] or Ziemer [80, Theorem 2.1.11], a direct computation and having in mind that $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,q}(\Omega)$, we can shows that, $\eta_\varepsilon(\overline{u}_2 - \overline{u}_1) \in W^{1,\mathcal{H}}(\Omega)$, and, by the Lebesgue Dominate Convergence Theorem (see e.g., Brézis [11, Theorem 4.2]), we get

$$\eta_\varepsilon(\overline{u}_2 - \overline{u}_1)(x) \to \chi_{\{\overline{u}_1 < \overline{u}_2\}}(x) \text{ for a.a. } x \in \Omega \text{ as } \varepsilon \to 0^+, \qquad (25)$$

where χ_E is the characteristic function of the set E. Moreover, by chain rule, we observe that

$$\nabla\eta_\varepsilon(\overline{u}_2 - \overline{u}_1)v = v\nabla\eta_\varepsilon(\overline{u}_2 - \overline{u}_1) + \eta_\varepsilon(\overline{u}_2 - \overline{u}_1)\nabla v \quad \forall v \in C_0^\infty(\Omega)_+.$$

Now, by Definition 3.1, we have

$$a(\alpha)\int_\Omega \left(\mu(x)|\nabla\overline{u}_1|^{p-2}\nabla\overline{u}_1 + |\nabla\overline{u}_1|^{q-2}\nabla\overline{u}_1\right)\nabla\varphi_1 dx \geq \int_\Omega f(x, \overline{u}_1)\varphi_1 dx$$
$$\forall \varphi_1 \in W_0^{1,\mathcal{H}}(\Omega) \cap L_+^{\mathcal{H}}(\Omega),$$

and

$$a(\alpha)\int_\Omega \left(\mu(x)|\nabla\overline{u}_2|^{p-2}\nabla\overline{u}_2 + |\nabla\overline{u}_2|^{q-2}\nabla\overline{u}_2\right)\nabla\varphi_2 dx \geq \int_\Omega f(x, \overline{u}_2)\varphi_2 dx$$
$$\forall \varphi_2 \in W_0^{1,\mathcal{H}}(\Omega) \cap L_+^{\mathcal{H}}(\Omega),$$

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Choosing $\varphi_1 = \eta_\varepsilon(\overline{u}_2 - \overline{u}_1)v$ and $\varphi_2 = (1 - \eta_\varepsilon(\overline{u}_2 - \overline{u}_1))v$ for $v \in C_0^\infty(\Omega)_+$, and adding the two previous inequalities, we get

$$\begin{aligned}
&a(\alpha)\int_\Omega \left(\mu(x)|\nabla\overline{u}_1|^{p-2}\nabla\overline{u}_1 + |\nabla\overline{u}_1|^{q-2}\nabla\overline{u}_1\right)\nabla\left[\eta_\varepsilon(\overline{u}_2 - \overline{u}_1)v\right]dx \\
&+ a(\alpha)\int_\Omega \left(\mu(x)|\nabla\overline{u}_2|^{p-2}\nabla\overline{u}_2 + |\nabla\overline{u}_2|^{q-2}\nabla\overline{u}_2\right)\nabla\left[(1 - \eta_\varepsilon(\overline{u}_2 - \overline{u}_1))v\right]dx \\
\geq &\int_\Omega f(x,\overline{u}_1)\eta_\varepsilon(\overline{u}_2 - \overline{u}_1)vdx + \int_\Omega f(x,\overline{u}_2)(1 - \eta_\varepsilon(\overline{u}_2 - \overline{u}_1))vdx.
\end{aligned}$$

By direct computation, we have

$$\begin{aligned}
&a(\alpha)\int_\Omega \left(\mu(x)|\nabla\overline{u}_1|^{p-2}\nabla\overline{u}_1 + |\nabla\overline{u}_1|^{q-2}\nabla\overline{u}_1\right)\nabla\left[\eta_\varepsilon(\overline{u}_2 - \overline{u}_1)v\right]dx \\
= &\, a(\alpha)\int_\Omega \left(\mu(x)|\nabla\overline{u}_1|^{p-2}\nabla\overline{u}_1 + |\nabla\overline{u}_1|^{q-2}\nabla\overline{u}_1\right)\nabla\eta_\varepsilon(\overline{u}_2 - \overline{u}_1)\cdot vdx \\
&+ a(\alpha)\int_\Omega \left(\mu(x)|\nabla\overline{u}_1|^{p-2}\nabla\overline{u}_1 + |\nabla\overline{u}_1|^{q-2}\nabla\overline{u}_1\right)\nabla v\cdot\eta_\varepsilon(\overline{u}_2 - \overline{u}_1)dx \\
= &\, \frac{a(\alpha)}{\varepsilon}\int\limits_{\{0<\overline{u}_2-\overline{u}_1<\varepsilon\}} \left(\mu(x)|\nabla\overline{u}_1|^{p-2}\nabla\overline{u}_1 + |\nabla\overline{u}_1|^{q-2}\nabla\overline{u}_1\right)\nabla(\overline{u}_2 - \overline{u}_1)\cdot vdx \\
&+ a(\alpha)\int_\Omega \left(\mu(x)|\nabla\overline{u}_1|^{p-2}\nabla\overline{u}_1 + |\nabla\overline{u}_1|^{q-2}\nabla\overline{u}_1\right)\nabla v\cdot\eta_\varepsilon(\overline{u}_2 - \overline{u}_1)dx,
\end{aligned}$$

and

$$\begin{aligned}
&a(\alpha)\int_\Omega \left(\mu(x)|\nabla\overline{u}_2|^{p-2}\nabla\overline{u}_2 + |\nabla\overline{u}_2|^{q-2}\nabla\overline{u}_2\right)\nabla\left[(1 - \eta_\varepsilon(\overline{u}_2 - \overline{u}_1))v\right]dx \\
= &\, a(\alpha)\int_\Omega \left(\mu(x)|\nabla\overline{u}_2|^{p-2}\nabla\overline{u}_2 + |\nabla\overline{u}_2|^{q-2}\nabla\overline{u}_2\right)\nabla(1 - \eta_\varepsilon(\overline{u}_2 - \overline{u}_1))\cdot vdx \\
&+ a(\alpha)\int_\Omega \left(\mu(x)|\nabla\overline{u}_2|^{p-2}\nabla\overline{u}_2 + |\nabla\overline{u}_2|^{q-2}\nabla\overline{u}_2\right)\nabla v\cdot(1 - \eta_\varepsilon(\overline{u}_2 - \overline{u}_1))dx \\
= &\, -\frac{a(\alpha)}{\varepsilon}\int\limits_{\{0<\overline{u}_2-\overline{u}_1<\varepsilon\}} \left(\mu(x)|\nabla\overline{u}_2|^{p-2}\nabla\overline{u}_2 + |\nabla\overline{u}_2|^{q-2}\nabla\overline{u}_2\right)\nabla(\overline{u}_2 - \overline{u}_1)\cdot vdx \\
&+ a(\alpha)\int_\Omega \left(\mu(x)|\nabla\overline{u}_2|^{p-2}\nabla\overline{u}_2 + |\nabla\overline{u}_2|^{q-2}\nabla\overline{u}_2\right)\nabla v\cdot(1 - \eta_\varepsilon(\overline{u}_2 - \overline{u}_1))dx.
\end{aligned}$$

Then, we infer

$$\begin{aligned}
&a(\alpha)\int_\Omega \left(\mu(x)|\nabla\overline{u}_1|^{p-2}\nabla\overline{u}_1 + |\nabla\overline{u}_1|^{q-2}\nabla\overline{u}_1\right)\nabla v\cdot\eta_\varepsilon(\overline{u}_2 - \overline{u}_1)dx \\
&+ a(\alpha)\int_\Omega \left(\mu(x)|\nabla\overline{u}_2|^{p-2}\nabla\overline{u}_2 + |\nabla\overline{u}_2|^{q-2}\nabla\overline{u}_2\right)\nabla v\cdot(1 - \eta_\varepsilon(\overline{u}_2 - \overline{u}_1))dx \\
\geq &\int_\Omega f(x,\overline{u}_1)\eta_\varepsilon(\overline{u}_2 - \overline{u}_1)vdx + \int_\Omega f(x,\overline{u}_2)(1 - \eta_\varepsilon(\overline{u}_2 - \overline{u}_1))vdx.
\end{aligned}$$

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By (25), passing to the limit as $\varepsilon \to 0^+$, we get

$$\begin{aligned} & a(\alpha) \int\limits_{\{\overline{u}_1<\overline{u}_2\}} \left(\mu(x)|\nabla \overline{u}_1|^{p-2}\nabla \overline{u}_1 + |\nabla \overline{u}_1|^{q-2}\nabla \overline{u}_1\right)\nabla v dx \\ & + a(\alpha) \int\limits_{\{\overline{u}_1\geq\overline{u}_2\}} \left(\mu(x)|\nabla \overline{u}_2|^{p-2}\nabla \overline{u}_2 + |\nabla \overline{u}_2|^{q-2}\nabla \overline{u}_2\right)\nabla v dx \\ \geq & \int\limits_{\{\overline{u}_1<\overline{u}_2\}} f(x,\overline{u}_1)v dx + \int\limits_{\{\overline{u}_1\geq\overline{u}_2\}} f(x,\overline{u}_2)v dx \end{aligned}$$

for all $v \in C_0^\infty(\Omega)_+$. Recall that, since $\overline{u}_1, \overline{u}_2 \in W^{1,\mathcal{H}}(\Omega)$, then $\overline{u} = \min\{\overline{u}_1, \overline{u}_2\} \in W^{1,\mathcal{H}}(\Omega)$, indeed, since if $\overline{u}_1, \overline{u}_2 \in W^{1,\mathcal{H}}(\Omega)$, then, $\overline{u}_1, \overline{u}_2 \in W^{1,q}(\Omega)$, so, bearing in mind relation (7), we have

$$\rho_{\mathcal{H}}(u) = \int_\Omega \left(\mu(x)|\min\{\overline{u}_1, \overline{u}_2\}|^p + |\min\{\overline{u}_1, \overline{u}_2\}|^q\right) dx < +\infty, \tag{26}$$

and

$$\rho_{\mathcal{H}}(\nabla u) = \int_\Omega \left(\mu(x)|\nabla \min\{\overline{u}_1, \overline{u}_2\}|^p + |\nabla \min\{\overline{u}_1, \overline{u}_2\}|^q\right) dx < +\infty.$$

Moreover, $\overline{u} \geq 0$ on $\partial\Omega$, and

$$\nabla \overline{u}(x) = \begin{cases} \nabla \overline{u}_1(x) & \text{for a.a. } x \in \{\overline{u}_1 < \overline{u}_2\}, \\ \nabla \overline{u}_2(x) & \text{for a.a. } x \in \{\overline{u}_1 \geq \overline{u}_2\}. \end{cases} \tag{27}$$

Thus, we obtain that

$$a(\alpha)\int_\Omega \left(\mu(x)|\nabla \overline{u}|^{p-2}\nabla \overline{u} + |\nabla \overline{u}|^{q-2}\nabla \overline{u}\right)\nabla v dx \geq \int_\Omega f(x,\overline{u})v dx \quad \forall v \in C_0^\infty(\Omega)_+.$$

Finally, since $C_0^\infty(\Omega)_+$ is dense in $W_0^{1,\mathcal{H}}(\Omega)_+$ (see (8)), then the last inequality holds for any $v \in W_0^{1,\mathcal{H}}(\Omega) \cap L_+^{\mathcal{H}}(\Omega)$ and we conclude the proof. □

PROPOSITION 4.4. *Assume that* (H_1), (H_2), (H_3) *and* (H_4) *hold. Then, for any* $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$, *the set* $\mathscr{S}_k(\alpha)$ *is downward direct.*

Proof. Fix $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$ and let $u_1, u_2 \in \mathscr{S}_k(\alpha)$. Consider $u = \min\{u_1, u_2\}$. Since u_1, u_2 are solution of (FT_α) they are also super-solutions of (FT_α). Then, by Lemma 4.3, u is a super-solution for (FT_α). Let us consider the following truncation

$$f_*(x,s) = \begin{cases} \hat{f}(x,\underline{u}) & \text{if} \quad s < \underline{u}(x); \\ \hat{f}(x,s) & \text{if} \quad \underline{u}(x) \leq s \leq u(x); \\ \hat{f}(x,u) & \text{if} \quad s > u(x), \end{cases} \tag{28}$$

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and the associated problem

$$\begin{cases} -a(\alpha)\mathcal{G}(u) = f_*(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{FTT_α}$$

Now, as in Proposition 3.4, there exists $\tilde{u}$ solution of (FTT_α), such that $\underline{u} \leq \tilde{u} \leq u$ (choose $v = (\tilde{u} - u)^+$ in the weak formulation). Then, $\tilde{u}$ is obviously a solution of (FT_α) and so $\tilde{u} \in \mathscr{S}_k(\alpha)$. □

THEOREM 4.5. *Assume that* (H_1), (H_2), (H_3) *and* (H_4) *hold. Then, for any* $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$, *the set* $\mathscr{S}_k(\alpha)$ *has the smallest element* u^*_α.

Proof. By Proposition 4.4, for each $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$, we have that $\mathscr{S}_k(\alpha)$ is partially ordered and downward direct. Let $\mathcal{C} \subset \mathscr{S}_k(\alpha)$ a chain. We can find a sequence $\{u_n\} \subset \mathcal{C}$ such that

$$\lim_{n\to+\infty} u_n = \inf \mathcal{C}.$$

Moreover, since $u_n \geq \underline{u}$ and it is a solution of (FT_α), arguing as in Proposition 4.1, up to a sub-sequence, there exists $v \in X$ such that

$$u_n \to v \text{in } X,$$

and $v \geq \underline{u}$. Therefore, $v = \inf \mathcal{C}$, so by Kuratowski–Zorn Lemma (Lemma 2.4), there exists a minimal element u^*_α for $\mathscr{S}_k(\alpha)$. Now, we show that u^*_α is the smallest solution of (FT_α) such that $u^*_\alpha \geq \underline{u}$. Let $u \in \mathscr{S}_k(\alpha)$. By Proposition 4.4, there exists $\tilde{u} \in \mathscr{S}_k(\alpha)$ such that $\tilde{u} \leq \min\{u^*_\alpha, u\}$. From the minimality of $u^*_\alpha \in \mathscr{S}_k(\alpha)$, we get $\underline{u} \leq u^*_\alpha \leq \tilde{u} \leq u$, that is, u^*_α is the smallest solution of (FT_α) such that $u^*_\alpha \geq \underline{u}$. □

By the use of Theorem 4.5, we can define the function $\Gamma : (\alpha_{2k-1}, \alpha_{2k}) \subset \mathbb{R} \to X$, by

$$\Gamma(\alpha) = u^*_\alpha, \tag{29}$$

where u^*_α is the smallest solution of (FT_α) such that $u^*_\alpha \geq \underline{u}$ (its uniqueness comes from the definition of the smallest element in a partially ordered set).

PROPOSITION 4.6. *Assume that* (H_1), (H_2), (H_3) *and* (H_4) *hold. Then, the function* Γ *given by* (29) *is continuous.*

Proof. Consider a sequence $\{\alpha_n\} \subset (\alpha_{2k-1}, \alpha_{2k})$ such that $\alpha_n \to \alpha \in (\alpha_{2k-1}, \alpha_{2k})$. Then, we have

$$\begin{cases} -a(\alpha_n)\mathcal{G}(u_n) = \hat{f}(x,u_n) & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{$FT_{\alpha n}$}$$

where $u_n = \Gamma(\alpha_n)$. By compactness of $\mathscr{S}_k$ (see Proposition 4.1), we have that the sequence $\{u_n\}$ is relatively compact in X. Thus, passing to a subsequence, $u_n \to u$

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in X for some $u \in X$, and, since $u_n \geq \underline{u}$ for all $n \in \mathbb{N}$, we get $u \geq \underline{u}$. Passing to the limit in ($(FT_{\alpha n}{}^{u_n})$), we get

$$\begin{cases} -a(\alpha)\mathcal{G}(u) = \hat{f}(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{30}$$

so $u \in \mathscr{S}_k(\alpha)$. Now, by Lemma 2.7, there exists a sequence $\{v_n\} \subset X$ such that $v_n \in \mathscr{S}_k(\alpha_n)$ for all $n \in \mathbb{N}$ and $v_n \to \Gamma(\alpha) \in \mathscr{S}_k(\alpha)$. Then, by the minimality of $\Gamma(\alpha)$, we conclude that

$$\Gamma(\alpha) = u_\alpha^* \leq u = \lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \Gamma(\alpha_n) \leq \lim_{n\to+\infty} v_n = \Gamma(\alpha).$$

Thus $\Gamma(\alpha) = u_\alpha^* = \lim_{n\to+\infty} \Gamma(\alpha_n)$. □

5. Fixed Point

We associate with (P) an auxiliary one dimension fixed-point problem. By the results of the previous section, for each $k \in \{1, ..., K\}$ and $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$ there exists u_α^*, the (unique) smallest solution of (F_α) in $[\underline{u}, \overline{u}]$. Thus, we can define the map $\mathcal{P}_k : (\alpha_{2k-1}, \alpha_{2k}) \to \mathbb{R}$ by

$$\mathcal{P}_k(\alpha) = \int_\Omega ((u_\alpha^*)^r + \nu(x)(u_\alpha^*)^s)\, dx \quad \forall \alpha \in (\alpha_{2k-1}, \alpha_{2k}).$$

It plays a key role in our approach since we have that

$$\text{if } \alpha \in \mathrm{Fix}(\mathcal{P}_k), \text{ then } u_\alpha \text{ is a solution of problem } (P), \tag{31}$$

where $\mathrm{Fix}(\mathcal{P}_k) = \{\alpha \in (\alpha_{2k-1}, \alpha_{2k}) : \mathcal{P}_k(\alpha) = \alpha\}$. Indeed, if $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$ is such that $\mathcal{P}_k(\alpha) = \alpha$, then

$$-a\left(\int_\Omega ((u_\alpha^*)^r + \nu(x)(u_\alpha^*)^s)\, dx\right)\mathcal{G}(u_\alpha^*) = -a(\alpha)\mathcal{G}(u_\alpha^*) = f(u_\alpha^*) \quad \text{in} \quad \Omega.$$

Proposition 5.1. *Assume that (H_1), (H_2), (H_3) and (H_4) hold. Then $\mathcal{P}_k$ is a continuous function.*

Proof. Note that $\mathcal{P}_k(\alpha) = \|\Gamma(\alpha)\|_r^r + \|\Gamma(\alpha)\|_{s,\nu}^s$. Let $\{\alpha_n\} \subset (\alpha_{2k-1}, \alpha_{2k})$ be a sequence such that $\alpha_n \to \alpha \in (\alpha_{2k-1}, \alpha_{2k})$.

Since $\Gamma(\alpha_n) = u_n$ is a solution of (FT_α), then $u_n \leq t_*$ for each $n \in \mathbb{N}$. Now, combining the Lebesgue Dominate Convergence Theorem (see e.g., Brézis [11, Theorem 4.2]) and the continuity of Γ (see Proposition 4.6), we get

$$\begin{aligned} \lim_{n\to+\infty} \mathcal{P}_k(\alpha_n) &= \lim_{n\to+\infty} \int_\Omega (|\Gamma(\alpha_n)|^r + \nu(x)|\Gamma(\alpha_n)|^s)\, dx \\ &= \int_\Omega (|\Gamma(\alpha)|^r + \nu(x)|\Gamma(\alpha)|^s)\, dx = \mathcal{P}_k(\alpha). \end{aligned} \tag{32}$$

□

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PROPOSITION 5.2. *Assume that* (H_1), (H_2), (H_3), *and* (H_4) *hold. Then,*

$$\lim_{\alpha\to\alpha_{2k-1}^+} \mathcal{P}_k(\alpha) > \alpha_{2k-1} \text{ and } \lim_{\alpha\to\alpha_{2k}^-} \mathcal{P}_k(\alpha) > \alpha_{2k}. \tag{33}$$

Proof. We show the first estimate (the second being analogous). First, by Lemma 3.2, for all $k \in \{1, ..., K\}$ and $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$, we have

$$\begin{aligned}\mathcal{P}_k(\alpha) &= \int_\Omega ((u_\alpha^*)^r + \nu(x)(u_\alpha^*)^s)\,dx \geq \int_\Omega \underline{u}^r dx + \min_\Omega \nu \int_\Omega \underline{u}^s dx \\ &= t(\alpha)^r \|e\|_r^r + t(\alpha)^s \min_\Omega \nu \|e\|_s^s. \end{aligned} \tag{34}$$

By the definition of $t(\alpha)$ (see (10)) and by hypothesis (H_f), we have that

$$\lim_{\alpha\to\alpha_{2k-1}^+} t(\alpha) = \lim_{\alpha\to\alpha_{2k}^-} t(\alpha) = t_*. \tag{35}$$

Indeed, by hypothesis (H_f), the map $t \mapsto f(t)$ is strictly positive in $(0, t_*)$, it possesses a zero value at $t = t_*$ and

$$\liminf_{t\to 0^+} f(t) > \max_{t\in[\alpha_{2k-1},\alpha_{2k}]} a(t) \geq a(\alpha).$$

Arguing by contradiction, assume that $t(\alpha) \nrightarrow t_*$ as $\alpha \to \alpha_{2k-1}^+$. Recall that $a(\alpha) \to 0$ as $\alpha \to \alpha_{2k-1}^+$ and so, there exists an accumulation point $\bar{t} < t_*$ of $t(\alpha)$, i.e., there exists a sequence $\{\alpha_n\} \subseteq [\alpha_{2k-1}, \alpha_{2k}]$, such that $\alpha_n \to \alpha_{2k-1}^+$, $t(\alpha_n) \to \bar{t}$ (obviously $\bar{t} > 0$). But, by the continuity of f and a, this means that $f(\bar{t}) = \lim_{n\to+\infty} a(\alpha_n) = 0$ (see (10)), so $f(\bar{t}) = 0$, a contradiction to hypotheses (H_f). This prove (35).

Finally, passing to the limit in (34) as $\alpha \to \alpha_{2k-1}^+$, by (H_4), we get

$$\lim_{\alpha\to\alpha_{2k-1}^+} \mathcal{P}_k(\alpha) \geq t_*^r \|e\|_r^r + t_*^s \min_\Omega \nu \|e\|_s^s > \alpha_{2K}.$$

□

PROPOSITION 5.3. *Assume that* (H_1), (H_2), (H_3), (H_4) *and* (H_5) *hold. Then, for each* $k \in \{1, ..., K\}$, *there exists* $\tilde{\alpha} \in (\alpha_{2k-1}, \alpha_{2k})$ *such that*

$$\mathcal{P}_k(\tilde{\alpha}) < \tilde{\alpha}.$$

Proof. Let us start with the following auxiliary problem

$$\begin{cases} -\Delta_q w = (u_\alpha^*)^{r-1} + \nu(x)(u_\alpha^*)^{s-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{36}$$

where $u_\alpha^* \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ the unique minimal solution of (F_α). By Minty–Browder's Theorem (see e.g., Brézis [11, Theorem 5.16]), there exists a

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unique positive solution $w \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ of (36). Recalling that the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$ is continuous, choosing $v = u_\alpha$ as a test function in weak formulation of (36) and applying Hölder's inequality, we have

$$\begin{aligned}
\mathcal{P}_k(\alpha) &= \int_\Omega \left((u_\alpha^*)^r + \nu(x)(u_\alpha^*)^s\right) dx \\
&= \int_\Omega |\nabla w|^{q-2}\nabla w \nabla u_\alpha^* \, dx \\
&\leq \int_\Omega |\nabla w|^{q-1}|\nabla u_\alpha^*| \, dx \\
&\leq \left(\int_\Omega |\nabla w|^{(q-1)q'} \, dx\right)^{\frac{1}{q'}} \left(\int_\Omega |\nabla u_\alpha^*|^q \, dx\right)^{\frac{1}{q}} \\
&= \|\nabla w\|_q^{q-1} \|\nabla u_\alpha^*\|_q
\end{aligned} \tag{37}$$

Note that, since u_α^* is solution of (F_α), then, for $v = u_\alpha^*$,

$$a(\alpha) \int_\Omega \left(\mu(x)|\nabla u_\alpha^*|^p + |\nabla u_\alpha^*|^q\right) dx = \int_\Omega f(u_\alpha^*) u_\alpha^* dx.$$

Then,

$$\|\nabla u_\alpha^*\|_q^q \leq \|\nabla u_\alpha^*\|_{p,\mu}^p + \|\nabla u_\alpha^*\|_q^q \leq \frac{|\Omega|}{a(\alpha)} \max_{t\in[0,t_*]} [tf(t)] \tag{38}$$

and so

$$\|\nabla u_\alpha^*\|_q \leq \left(\frac{|\Omega|}{a(\alpha)} \max_{t\in[0,t_*]} [tf(t)]\right)^{\frac{1}{q}}. \tag{39}$$

On the other hand, since w is a weak solution of (36), choosing $v = w$ as test function and by Hölder's inequality, we have

$$\begin{aligned}
\|\nabla w\|_q^q &= \int_\Omega |\nabla w|^q dx \\
&= \int_\Omega \left((u_\alpha^*)^{r-1} + \nu(x)(u_\alpha^*)^{s-1}\right) w dx \\
&\leq \left(\int_\Omega \left((u_\alpha^*)^{r-1} + \nu(x)(u_\alpha^*)^{s-1}\right)^{q'} dx\right)^{\frac{1}{q'}} \left(\int_\Omega |w|^q dx\right)^{\frac{1}{q}} \\
&\leq \left(t_*^{r-1} + t_*^{s-1} \max_\Omega \nu\right) |\Omega|^{\frac{1}{q'}} \|w\|_q \\
&\leq \frac{|\Omega|^{1/q'}}{\lambda_{1,q}^{1/q}} (t_*^{r-1} + t_*^{s-1} \max_\Omega \nu) \|\nabla w\|_q,
\end{aligned}$$

that is,

$$\|\nabla w\|_q^{q-1} \leq \frac{|\Omega|^{1/q'}}{\lambda_{1,q}^{1/q}} \left(t_*^{r-1} + t_*^{s-1} \max_\Omega \nu\right). \tag{40}$$

Finally, putting together (39) and (40) in (37), we have

$$\mathcal{P}_k(\alpha) \leq \|\nabla w\|_q^{q-1} \|\nabla u_\alpha^*\|_q$$

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$$\leq |\Omega|^{\frac{1}{q'}} \left(t_*^{r-1} + t_*^{s-1} \max_{\Omega} \nu \right) \left(\frac{|\Omega|}{a(\alpha)\lambda_{1,q}} \max_{t\in[0,t_*]} [tf(t)] \right)^{\frac{1}{q}}$$

$$= |\Omega| \left(t_*^{r-1} + t_*^{s-1} \max_{\Omega} \nu \right) \left(\frac{\max\limits_{t\in[0,t_*]} [tf(t)]}{a(\alpha)\lambda_{1,q}} \right)^{\frac{1}{q}}.$$

By assumption (H_5), we conclude the proof. □

Now we are in position to conclude the proof of the main result.

Proof of Theorem 1.1. By (31), it is enough to prove that the map $\mathcal{P}_k$ possesses at least two fixed points in the interval (α_{k-1}, α_k) (for each $k \in \{1, \dots, K\}$). Propositions 5.2 and 5.3 combined with the continuity of $\mathcal{P}_k$ (Proposition 5.1) permit us to apply the Intermediate Value Theorem and archive the proof. □

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