Combinatorics, Probability and Computing (2006) 15, 753–763. © 2006 Cambridge University Press
doi:10.1017/S0963548306007577 Printed in the United Kingdom

Best Odds for Finding a Perfect Matching in a Bipartite Graph

W. PAULSEN

Department of Mathematics and Statistics, Arkansas State University,
PO Box 70, State University, AR 72432, USA
(e-mail: wpaulsen@math.astate.edu)

Received 15 September 2003; revised 1 September 2004

We will study the best way to reveal a hidden perfect matching in a balanced bipartite graph by eliminating edges, one by one, in the hope that the eliminated edge is not part of the mystery perfect matching. We will look for the strategy that maximizes the odds of finding the perfect matching without revealing a fixed number of the edges in that perfect matching. For a complete bipartite graph, this is equivalent to finding a mystery permutation via negative guesses with only a fixed number of incorrect negative guesses.

1. Introduction

Suppose one is given a 6 by 6 array of face-down cards, with 6 of the cards, 1 in each row and 1 in each column, being a joker. The goal is to turn over the cards, one at a time, to reveal the remaining 30 cards, but revealing at most 2 of the jokers. Note that if a joker is revealed, then all remaining cards in either the same row or the same column can safely be revealed.

The natural questions are: 'What is the best card to reveal at each stage in the game?' and 'What is the probability of success, given that one plays the best strategy?' Algebraic graph theory can help us with both questions. We can generalize the problem to an n by n array of cards with j jokers allowed to be revealed, but when n gets very large, the problem becomes intractable.

To make the connection between this game and graph theory, we form a 6 by 6 binary matrix M for each stage of the game, with a 0 if the corresponding card is revealed and is not a joker, and 1 otherwise. This matrix also represents a balanced bipartite graph Γ of order $2n$. A bipartite graph is *balanced* if the partitioning of the vertex set V_1 and V_2 can be chosen so that $|V_1| = |V_2|$. If we order the vertices so that those in V_1 come first, then the adjacency matrix of a balanced bipartite graph is of the form

$$A = \begin{pmatrix} \mathbf{0} & M \\ M^t & \mathbf{0} \end{pmatrix}.$$

https://doi.org/10.1017/S0963548306007577 Published online by Cambridge University Press

A *perfect matching*, or a 1-*factor*, is a set of edges such that every vertex is contained in exactly one of the edges in the set.

We can now state the original problem in the language of bipartite graphs. We start with a complete bipartite graph $K_{n,n}$, and one hidden perfect matching chosen at random from the $n!$ possible. We remove the edges, one at a time, hoping not to hit one of the edges of the perfect matching. If an edge from the perfect matching is removed, then the two vertices which the edge joined are also removed, forming a graph with 2 fewer vertices. The goal is to remove the edges in such a way as to maximize the chances of forming the 1-regular graph with at least $2(n-j)$ vertices, which would indicate that at most j edges of the perfect matching were accidentally revealed.

2. The winning function $W_j(\Gamma)$

For an n by n matrix M, the *permanent* of the matrix, denoted pet M, is given by

$$\text{pet } M = \sum a_{1,i_1} \cdot a_{2,i_2} \cdot a_{3,i_3} \cdots a_{n,i_n},$$

where the sum is taken over all permutations $\{i_1, i_2, i_3, \ldots, i_n\}$. As with determinants, we can evaluate permanents using the Laplace development:

$$\text{pet } M = \sum_{i=1}^{n} a_{i,j}\text{pet } (\text{sub}_{i,j}M) = \sum_{j=1}^{n} a_{i,j}\text{pet } (\text{sub}_{i,j}M),$$

where $\text{sub}_{i,j}M$ denotes the $n-1$ by $n-1$ submatrix formed by deleting the ith row and jth column of M. In spite of the similarity to the determinant, it is NP-hard to compute the permanent [3]. There are algorithms that are more efficient than the Laplace development with running time $O(n!)$, but the best-known algorithm is $O(n^2 2^n)$ [4].

If M is a binary matrix, then pet M counts the number of perfect matchings in the corresponding graph Γ. Because of this correspondence, we will use pet M and pet Γ interchangeably. Since initially all $n!$ perfect matchings of $K_{n,n}$ occur with equal probability, all pet Γ perfect matchings of the subgraph Γ will occur with equal probability. For an edge e in the graph Γ, we will let $\Gamma \backslash e$ be the subgraph of Γ obtained by deleting the edge e, and we let $\Gamma \backslash\backslash e$ be the subgraph formed by deleting both vertices of e, and all edges connecting to those two vertices. Thus, the Laplace development can be written

$$\text{pet } \Gamma = \sum_{e} \text{pet } (\Gamma \backslash\backslash e),$$

where the sum is taken over all edges which go through a single vertex.

We can now define

$$W_j(\Gamma) = \text{pet } \Gamma \cdot (\text{the probability of winning with at most } j \text{ more jokers revealed}).$$

This defines $W_j(\Gamma)$ for all bipartite graphs which have a perfect matching. We define $W_j(\Gamma) = 0$ for all j whenever Γ does not have a perfect matching.

There are some values of j for which $W_j(\Gamma)$ are easy to find. For example,

$$W_0(\Gamma) = \begin{cases} 1 & \text{if pet } \Gamma > 0, \\ 0 & \text{if pet } \Gamma = 0. \end{cases} \tag{2.1}$$

https://doi.org/10.1017/S0963548306007577 Published online by Cambridge University Press

The reason is that if no jokers can be revealed, then regardless of which strategy we use, the only way to win would be to essentially correctly guess which of the pet Γ perfect matchings is the correct one, making the odds of winning $1/\text{pet}\,\Gamma$. Also, we are assured of a win if $j \geqslant n-1$ jokers are allowed, since after $n-1$ jokers are found, there will only be 1 card remaining, which must be the final joker. Thus, $W_j(\Gamma) = \text{pet}\,\Gamma$ for $j \geqslant n-1$.

In fact, we can show that $W_{n-2}(\Gamma) = \text{pet}\,\Gamma$ provided that Γ is not a complete bipartite graph. The winning strategy will be to always remove an edge such that the graph will not be complete even if we have to remove the 2 vertices with the edge. (For non-complete graphs there will always be such an edge.) After all but 2 jokers are revealed, there would be only 4 cards in question. But one of the 4 would be revealed since we avoided complete graphs, so we know the location of the remaining 2 jokers. However, if Γ is a complete bipartite graph, then there is a small chance of losing if $j = n-2$, namely, if one is so unfortunate as to always pick a joker on the first $n-1$ guesses. Since this can only happen if we essentially guessed the perfect matching, we find that $W_{n-2}(K_{n,n}) = n! - 1$.

For other values of j we can define $W_j(\Gamma)$ recursively. Let $E\Gamma$ be the set of edges in Γ. For $e \in E\Gamma$, the number of perfect matchings which include the edge e is pet $(\Gamma\backslash\backslash e)$. Hence, there will be a pet $(\Gamma\backslash\backslash e)/\text{pet}\,\Gamma$ chance of revealing a joker. In this case, one must proceed with only $j-1$ jokers allowed to be revealed. Hence, the probability of winning from the position Γ, given that edge e is chosen, is

$$\frac{\text{pet}\,\Gamma - \text{pet}\,(\Gamma\backslash\backslash e)}{\text{pet}\,\Gamma} \cdot \frac{W_j(\Gamma\backslash e)}{\text{pet}\,(\Gamma\backslash e)} + \frac{\text{pet}\,(\Gamma\backslash\backslash e)}{\text{pet}\,\Gamma} \cdot \frac{W_{j-1}(\Gamma\backslash\backslash e)}{\text{pet}\,(\Gamma\backslash\backslash e)}.$$

However, since it is clear that pet $(\Gamma\backslash e) + \text{pet}\,(\Gamma\backslash\backslash e) = \text{pet}\,\Gamma$, this simplifies to just $(W_j(\Gamma\backslash e) + W_{j-1}(\Gamma\backslash\backslash e))/\text{pet}\,\Gamma$. Thus, if we consider the best strategy over all possible choices of e, we have

$$W_j(\Gamma) = \max_{e \in E\Gamma}\{W_j(\Gamma\backslash e) + W_{j-1}(\Gamma\backslash\backslash e)\}, \qquad j > 0. \tag{2.2}$$

From this recursion formula, we see that $W_j(P)$ will always be an integer.

The function $W_j(\Gamma)$ is defined completely from (2.1) and (2.2). However, unlike the similar looking definitions for the chromatic polynomial and the Tutte polynomial [1, pp. 65, 97], $W_j(\Gamma\backslash e)$ must be known for *all* the edges of Γ in order to determine $W_j(\Gamma)$. Hence, using only equations (2.1) and (2.2), finding $W_j(\Gamma)$ is even more intractable than computing the Tutte polynomial.

It should be noted that (2.2) remains valid even if pet $(\Gamma\backslash\backslash e) = 0$. In this case, we defined $W_{j-1}(\Gamma\backslash\backslash e)$ to be 0, and so by (2.2) we have $W_j(\Gamma) = W_j(\Gamma\backslash e)$. This merely says that since e is not part of any perfect matching, we can safely remove that edge from the graph. This is one of the times that the correct strategy is obvious.

Define the *filter* of a graph Γ, denoted filt Γ, to be the graph Γ in which all edges which are not part of a perfect matching are removed. From the previous paragraph, we see that $W_j(\Gamma) = W_j(\text{filt}\,\Gamma)$. It is also clear that filt (filt Γ) = filt Γ. Note that if pet $\Gamma = 0$, then filt Γ is the empty graph.

Lemma 2.1. *If* pet $\Gamma < 4$, *then* $W_1(\Gamma) = \text{pet}\,\Gamma$.

https://doi.org/10.1017/S0963548306007577 Published online by Cambridge University Press

Proof. We must show that one can always win from Γ if one joker is allowed to be revealed. This is obvious if filt (Γ) is 1-regular, so let v_1 be a vertex of filt (Γ) with valency at least 2. Then there will be one edge e on this vertex with pet $(\Gamma\backslash\backslash e) = 1$ since the sum of two or more of these positive numbers must sum to less than 4. Hence, we can safely select edge e, for if it is part of the hidden perfect matching, we will know the perfect matching. Otherwise, we have reduced the number of edges in the graph. By repeating this strategy, we find that indeed $W_1(\Gamma) = \text{pet } \Gamma$. □

We now proceed to show that $W_1(\Gamma)$ is related to the *co-rank* of Γ. For a balanced bipartite graph of order $2n$, with m edges and c connected components, we define [1, p. 25]

$$\text{the rank of } \Gamma = r(\Gamma) = 2n - c,$$
$$\text{the co-rank of } \Gamma = s(\Gamma) = m - 2n + c.$$

Lemma 2.2. *Let* $e \in E\Gamma$ *and suppose that* pet $(\Gamma\backslash\backslash e) > 0$. *Then* $s(\text{filt } (\Gamma\backslash e)) < s(\text{filt } (\Gamma))$.

Proof. Note that it is trivial that $s(\text{filt } (\Gamma\backslash e)) \leqslant s(\text{filt } (\Gamma))$. The crux of this lemma is to show that the co-rank *strictly* decreases.

We can assume that pet $(\Gamma\backslash e) > 0$, otherwise filt $(\Gamma\backslash e)$ becomes the empty graph, which has co-rank 0. We can also assume that filt Γ has only one connected component, since we could apply the result to the connected component of filt Γ containing e, whereas the other components would be unaffected. Finally, we can assume that filt $(\Gamma\backslash e)$ has $c > 1$ components, otherwise the result is trivial. Hence, we need to show that filt $(\Gamma\backslash e)$ has at least c fewer edges than filt Γ.

Let us call an edge b in filt Γ a *bridge* if this edge, when added to the graph filt $(\Gamma\backslash e)$, reduces its number of connected components. (Note that this is not the same as a cut-edge, since we are adding the edge to filt $(\Gamma\backslash e)$, not deleting it from filt Γ. Also, e itself might be a bridge.) Since pet $(\Gamma\backslash\backslash e) > 0$, there is a perfect matching of Γ which uses the edge e. If all such perfect matchings avoided bridges, then filt Γ would also have c connected components, so we can assume that there is a perfect matching of Γ which includes both e and a bridge b (which might be e) which connects v_1 to v_2.

Consider the connected component of filt $(\Gamma\backslash e)$ containing v_2. Because each component is balanced (and hence contains an even number of vertices), there must be another bridge in the perfect matching connecting v_3 to v_4, where v_3 is in the same connected component of filt $(\Gamma\backslash e)$ as v_2.

Now we repeat the process, using the connected component of filt $(\Gamma\backslash e)$ containing v_4. We find yet another bridge connecting v_5 to v_6, with v_4 and v_5 in the same connected component of filt $(\Gamma\backslash e)$. Eventually we will form a cycle (not necessarily using the first component) which has p components connected by p bridges.

If $p \neq c$, then there will still be $c - p$ other components of filt $(\Gamma\backslash e)$, which will require at least $c - p$ additional bridges to connect together. Thus, there are at least c bridges, which proves the lemma. □

https://doi.org/10.1017/S0963548306007577 Published online by Cambridge University Press

Remark. The argument in this lemma actually proves a slightly stronger statement, that filt Γ contains a 'cycle', whose vertices are components of filt $(\Gamma\backslash e)$, and the edges are the bridges.

This lemma shows that the co-rank of the filtered graphs must decrease if we first delete an edge, and then re-filter it. However, in certain circumstances we can show that the co-rank goes down by exactly 1.

Lemma 2.3. *Let* $e \in E\Gamma$ *be such that* pet $(\Gamma\backslash\backslash e)$ *has the smallest nonzero value over all edges in* Γ. *Then* $s(\text{filt } (\Gamma\backslash e)) = s(\text{filt } \Gamma) - 1$.

Proof. We may assume that filt Γ has only one connected component, since we could apply the result to the component containing e, whereas the other components would be unaffected. From the previous lemma, we need only show that $s(\text{filt } (\Gamma\backslash e)) \geqslant s(\text{filt } \Gamma) - 1$. Let B be the set of edges in filt Γ which are not in filt $(\Gamma\backslash e)$. (In particular, $e \in B$.) To prove the lemma, we must show that filt $(\Gamma\backslash e)$ has at least $|B|$ connected components.

If $b \in B$ and $b \neq e$, then every perfect matching of filt Γ which contains b must also contain e. This tells us that pet $(\Gamma\backslash\backslash b) \leqslant$ pet $(\Gamma\backslash\backslash e)$. However, e was chosen so that pet $(\Gamma\backslash\backslash e)$ is minimal, so we have pet $(\Gamma\backslash\backslash b) =$ pet $(\Gamma\backslash\backslash e)$. We can now reverse the argument, showing that every perfect matching of filt Γ which contains e must also contain b. Thus, every perfect matching of Γ containing one edge of B in fact contains all the edges in B.

If there were fewer than $|B|$ connected components of filt $(\Gamma\backslash e)$, we could find $b_1, b_2 \in B$, with b_1 connecting v_1 and v_2, b_2 connecting v_3 and v_4, with v_1 and v_3 in the same component of filt $(\Gamma\backslash e)$, and coloured the same in the 2-colouring of Γ. Since there is no perfect matching of Γ containing b_1 but not b_2, we have

$$\text{pet } ((\Gamma\backslash b_1)\backslash\backslash b_2) = 0. \tag{2.3}$$

Now Hall's theorem [5] states that, for a balanced bipartite graph Γ, pet $\Gamma = 0$ if, and only if, there is a bottleneck formed by a collection of r vertices in V_1 which connect to fewer than r vertices in V_2. Since $(\Gamma\backslash b_1)\backslash\backslash b_2$ has $|V_1| = |V_2| = n - 1$, then (2.3) implies that there are r vertices of V_1 and $n - r$ vertices in V_2 that are mutually non-adjacent. This same set of vertices will be mutually non-adjacent in $\Gamma\backslash b_1$. All perfect matchings of $\Gamma\backslash b_1$ must match these n vertices to the remaining n vertices. Thus, filt $(\Gamma\backslash b_1)$ is disconnected. Furthermore, b_1 must connect 2 of the n mutually non-adjacent vertices, or else by the same reasoning filt Γ would be disconnected.

Now we see that filt $(\Gamma\backslash b_1)$ has one component containing v_1 as one of the n mutually non-adjacent vertices. However, v_3 has the same colour in the 2-colouring, and is not one of the n mutually non-adjacent vertices, so it is in another component of filt $(\Gamma\backslash b_1)$. However, every perfect matching of Γ which contains b_1 also contains e, and *vice versa*, so filt $(\Gamma\backslash b_1) =$ filt $(\Gamma\backslash e)$. So b_1 and b_2 are not in the same connected component of filt $(\Gamma\backslash e)$, forming a contradiction. Thus, there are at least $|B|$ connected components of filt $(\Gamma\backslash e)$, and the lemma is proved. □

We are now ready to find a formula for $W_1(\Gamma)$.

https://doi.org/10.1017/S0963548306007577 Published online by Cambridge University Press

Proposition 2.4. *Let Γ be a balanced bipartite graph on $2n$ vertices. Then*

$$W_1(\Gamma) = \begin{cases} s(\text{filt } \Gamma) + 1 & \text{if pet } (\Gamma) > 0, \\ 0 & \text{if pet } (\Gamma) = 0. \end{cases} \tag{2.4}$$

□

Proof. We have already defined $W_j(\Gamma) = 0$ when pet $(\Gamma) = 0$, so we will suppose that pet $(\Gamma) > 0$. If Γ has only n edges, then each edge would be its own component, so pet $(\Gamma) = 1$ and $s(\text{filt } \Gamma) = 0$. Since the game is trivially won, we have $W_1(\Gamma) = 1$, so (2.4) is satisfied in this case.

We now proceed by induction on the number of edges in Γ. Since

$$W_1(\Gamma) = \max_{e \in E\Gamma} \{W_1(\Gamma \backslash e) + W_0(\Gamma \backslash\backslash e)\},$$

we can use the induction argument on $W_1(\Gamma \backslash e)$. Clearly if pet $(\Gamma \backslash e) = 0$, then this edge could not produce the maximum, since $W_1(\Gamma \backslash e) + W_0(\Gamma \backslash\backslash e) \leqslant 1$. Thus, we can assume that pet $(\Gamma \backslash e) > 0$, and so

$$W_1(\Gamma) = \max_{e \in E\Gamma} \{s(\text{filt } (\Gamma \backslash e)) + 1 + W_0(\Gamma \backslash\backslash e)\}.$$

If $W_0(\Gamma \backslash\backslash e) = 0$, then pet $(\Gamma \backslash\backslash e) = 0$, and so filt $(\Gamma \backslash e) = \text{filt } \Gamma$, making $s(\text{filt } (\Gamma \backslash e)) + 1 + W_0(\Gamma \backslash\backslash e) = s(\text{filt } \Gamma) + 1$. On the other hand, if $W_0(\Gamma \backslash\backslash e) = 1$, then by Lemma 2.2,

$$s(\text{filt } (\Gamma \backslash e)) + 1 + W_0(\Gamma \backslash\backslash e) \leqslant s(\text{filt } \Gamma) + 1.$$

Since we have the inequality for all edges e, we find that

$$W_1(\Gamma) \leqslant s(\text{filt } \Gamma) + 1.$$

To prove the inequality in the other direction, let e be an edge such that pet $(\Gamma \backslash\backslash e)$ is as small as possible. If pet $(\Gamma \backslash\backslash e) > 0$, we can use Lemma 2.3 to get

$$s(\text{filt } (\Gamma \backslash e)) + 1 + W_0(\Gamma \backslash\backslash e) = s(\text{filt } \Gamma) + 1.$$

We have already shown that this equation is true if pet $(\Gamma \backslash\backslash e) = 0$, so in either case, the maximum is realized, and hence $W_1(\Gamma) = s(\text{filt } \Gamma) + 1$. □

3. Globally strategic edges

Although there is a formula for $W_0(\Gamma)$ and $W_1(\Gamma)$, finding a formula for $W_2(\Gamma)$ is still an open problem. However, the key to finding $W_j(\Gamma)$ for any $j > 1$ is to find a j-strategic edge. If pet $(\Gamma) > 0$, we say that an edge e of Γ is *j-strategic* if e is one of the edges for which

$$W_j(\Gamma) = W_j(\Gamma \backslash e) + W_{j-1}(\Gamma \backslash\backslash e).$$

The proof of Proposition 2.4 shows that all edges with minimum pet $(\Gamma \backslash\backslash e)$ are 1-strategic. However, not all of these edges will be j-strategic for other values of j. For example, the

https://doi.org/10.1017/S0963548306007577 Published online by Cambridge University Press

graph given by the matrix

$$\Gamma = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & 1 & \mathbf{1} & \mathbf{1} & 0 \end{pmatrix}$$

has 8 edges for which pet $(\Gamma \backslash\backslash e) = 6$, which is the minimum value for pet $(\Gamma \backslash\backslash e)$. These edges are represented by the bold elements of the matrix. If e is one of the boldface entries not along the main diagonal, we find that $W_2(\Gamma \backslash e) = 20$ and $W_1(\Gamma \backslash\backslash e) = 5$, so that $W_2(\Gamma) \geqslant 25$. By checking all 18 of the edges, we find that indeed $W_2(\Gamma) = 25$. However, if we let e be a boldface entry along the main diagonal, we find that $W_2(\Gamma \backslash e) = 19$, and $W_1(\Gamma \backslash\backslash e) = 5$. Hence, this edge is not 2-strategic even though it has minimum pet $(\Gamma \backslash\backslash e)$.

Lemma 3.1. *If* $W_{j-1}(\Gamma \backslash\backslash e) = \text{pet}\ (\Gamma \backslash\backslash e)$*, then* e *is a* j*-strategic edge.*

Proof. We must show that the odds of winning cannot be increased by choosing an edge other than e. If the edge e is picked, and it turns out to be part of the hidden perfect matching, then the game will be won, since the remaining graph $\Gamma \backslash\backslash e$ can be won by revealing at most $j - 1$ more jokers. (This is because $W_{j-1}(\Gamma \backslash\backslash e) = \text{pet}\ (\Gamma \backslash\backslash e)$.) On the other hand, if e is not part of a perfect matching, we are no worse off than where we started. Either way, we cannot improve our odds by picking a different edge, and so e is j-strategic. □

We say that an edge e is *globally strategic* if it is j-strategic for all $j > 0$. Finding a globally strategic edge is key to reducing the problem of finding $W_j(\Gamma)$ in terms of a simpler problem. We can use Proposition 2.4 to help locate globally strategic edges.

Corollary 3.2. *Let* pet $(\Gamma) > 0$*, and let* e *be an edge with minimum* pet $(\Gamma \backslash\backslash e)$*. If* $W_1(\Gamma \backslash\backslash e) =$ pet $(\Gamma \backslash\backslash e)$*, then* e *is a globally strategic edge.*

Proof. Proposition 2.4 shows us that e is 1-strategic. Since W_j is non-decreasing as j increases, $W_j(\Gamma \backslash\backslash e) = \text{pet}\ (\Gamma \backslash\backslash e)$ for all $j \geqslant 1$. Hence, Lemma 3.1 shows us that e is j-strategic for $j > 1$, hence e is globally strategic. □

Corollary 3.2 suggests that an edge with minimum pet $(\Gamma \backslash\backslash e)$ is often a globally strategic edge, and this is in fact usually true. It is tempting, then, to simplify the search for a globally strategic edge by looking only at the edges with minimum pet $(\Gamma \backslash\backslash e)$. But there are graphs for which *none* of the edges with minimum pet $(\Gamma \backslash\backslash e)$ are globally strategic. For example, the graph with the matrix

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \mathbf{1} & \mathbf{1} \\ 1 & 1 & 1 & \mathbf{1} & 0 & \mathbf{1} \\ 1 & 1 & 1 & \mathbf{1} & \mathbf{1} & 0 \end{pmatrix}$$

https://doi.org/10.1017/S0963548306007577 Published online by Cambridge University Press

has a minimum pet $(\Gamma\backslash\backslash e)$ of 12, produced by the edges corresponding to the boldface entries of the matrix. For these edges, $W_2(\Gamma\backslash\backslash e) = 11$. However, if e is an edge represented by a non-bold element in one of the last three rows, then $W_2(\Gamma\backslash\backslash e) = \text{pet}\,(\Gamma\backslash\backslash e) = 16$. By Lemma 3.1 these non-bold elements are 3-strategic, so by calculating $W_3(\Gamma\backslash e) = 56$, we find that $W_3(\Gamma) = \text{pet}\,(\Gamma) = 72$. Hence, if we are allowed to reveal 3 jokers, we are guaranteed success if we choose one of the 6 non-bold elements in the lower rows, even though pet $(\Gamma\backslash\backslash e)$ is not the minimum for these edges. However, if we choose a boldface element with minimum pet $(\Gamma\backslash\backslash e)$, there is a small chance of failure.

There are similar counter-examples for which the edge with minimum pet $(\Gamma\backslash\backslash e)$ is not a globally strategic edge, even with pet $(\Gamma\backslash\backslash e) = 4$. However, if there is an edge for which pet $(\Gamma\backslash\backslash e)$ is less than 4, we can quickly find a globally strategic edge.

Corollary 3.3. *Let* pet $(\Gamma) > 0$*, and let e be an edge with minimum* pet $(\Gamma\backslash\backslash e)$*. If* pet $(\Gamma\backslash\backslash e) < 4$*, then e is a globally strategic edge.*

Proof. From Corollary 3.2, we only need to prove that $W_1(\Gamma\backslash\backslash e) = \text{pet}\,(\Gamma\backslash\backslash e)$. But this true by Lemma 2.1. □

Most graphs have a globally strategic edge. In fact, all graphs of order $\leqslant 12$ contain at least 1 globally strategic edge. However, the graph of order 14 produced by the matrix

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.1}$$

does *not* have a globally strategic edge. This graph has 148 perfect matchings, $W_2(\Gamma) = 80$, and $W_3(\Gamma) = 143$. To maximize the odds of finding the perfect matching via revealing at most 3 jokers, one must begin by choosing the edge corresponding to the 1 in the upper left corner. Choosing any other edge reduces the chances to 142/148 or less. However, if the goal is to find the perfect matching via revealing at most 2 jokers, choosing the upper left 1 gives the odds 77/148, whereas choosing any other edge improves the odds to 80/148.

We say that a graph Γ is *standard* if all subgraphs of Γ contain a globally strategic edge. The example given in (3.1) is the smallest graph which is not standard.

It makes sense to form a generating polynomial for Γ that would encode $W_j(\Gamma)$ for all j. Since $W_{j-1}(\Gamma) = W_j(\Gamma)$ for $j \geqslant n$, a natural choice would be

$$P(\Gamma; x) = W_0(\Gamma) + \sum_{j=1}^{n} (W_j(\Gamma) - W_{j-1}(\Gamma))x^j. \tag{3.2}$$

For example, $P(K_{4,4}; x) = 1 + 9x + 13x^2 + x^3$. From this polynomial, we can easily determine $W_j(\Gamma)$ for all j. It is clear that $P(\Gamma; 1) = \text{pet}\,(\Gamma)$, and the x coefficient is the co-rank of Γ, but one reason for the definition of $P(\Gamma; x)$ comes from the following lemma.

https://doi.org/10.1017/S0963548306007577 Published online by Cambridge University Press

Lemma 3.4. *Let e be a globally strategic edge of Γ. Then*

$$P(\Gamma;x) = P(\Gamma\backslash e;x) + x \cdot P(\Gamma\backslash\backslash e;x).$$

Proof. We can assume that pet $(\Gamma) \neq 0$, since otherwise both sides of the equation are 0. We can also assume that pet $(\Gamma\backslash\backslash e) > 0$, since otherwise this would mean that there are no perfect matchings which use e, and so $P(\Gamma;x) = P(\Gamma\backslash e;x)$. Finally, we can assume that pet $(\Gamma\backslash e) > 0$, otherwise *all* perfect matchings use the edge e, and e could not possibly be globally strategic.

Since e is globally strategic, we have that

$$W_j(\Gamma) = W_j(\Gamma\backslash e) + W_{j-1}(\Gamma\backslash\backslash e)$$

for all $j > 0$, and $W_0(\Gamma) = 1$. Hence, equation (3.2) gives us

$$\begin{aligned}
P(\Gamma;x) &= 1 + \sum_{j=1}^{n}(W_j(\Gamma) - W_{j-1}(\Gamma))x^j \\
&= 1 + W_1(\Gamma\backslash e)x + \sum_{j=2}^{n}(W_j(\Gamma\backslash e) + W_{j-1}(\Gamma\backslash\backslash e) - W_{j-1}(\Gamma\backslash e) - W_{j-2}(\Gamma\backslash\backslash e))x^j \\
&= 1 + W_1(\Gamma\backslash e)x + \sum_{j=2}^{n}((W_j(\Gamma\backslash e) - W_{j-1}(\Gamma\backslash e))x^j \\
&\quad + \sum_{j=2}^{n}(W_{j-1}(\Gamma\backslash\backslash e) - W_{j-2}(\Gamma\backslash\backslash e))x^j \\
&= 1 + (W_1(\Gamma\backslash e) - 1)x + \sum_{j=2}^{n}((W_j(\Gamma\backslash e) - W_{j-1}(\Gamma\backslash e))x^j \\
&\quad + x + \sum_{j=1}^{n-1}(W_j(\Gamma\backslash\backslash e) - W_{j-1}(\Gamma\backslash\backslash e))x^{j+1} \\
&= P(\Gamma\backslash e;x) + x \cdot P(\Gamma\backslash\backslash e;x).
\end{aligned}$$

□

Although this lemma justifies the definition of the polynomial $P(\Gamma;x)$, the next proposition shows the real motivation.

Proposition 3.5. *Let Γ be a balanced, bipartite graph with two components, Γ_1 and Γ_2, either of which is standard. Then*

$$P(\Gamma;x) = P(\Gamma_1;x) \cdot P(\Gamma_2;x). \tag{3.3}$$

Proof. Suppose that Γ_1 is standard. Then we will always be able to find a globally strategic edge on the Γ_1 component, and so we do not need to know the value of j to determine the best edge to remove. Hence, we can safely reduce Γ_1 first, and then, knowing how many jokers were revealed, find the best strategy for reducing Γ_2.

https://doi.org/10.1017/S0963548306007577 Published online by Cambridge University Press

Using this approach, we find that, for a fixed $j > 0$,

$$W_j(\Gamma) = W_j(\Gamma_2) + \sum_{i=1}^{j}(W_i(\Gamma_1) - W_{i-1}(\Gamma_1))W_{j-i}(\Gamma_2).$$

From this, we find that

$$\begin{aligned} P(\Gamma;x) &= 1 + \sum_{j=1}^{n}\Bigg(W_j(\Gamma_2) - W_{j-1}(\Gamma_2) + W_j(\Gamma_1) - W_{j-1}(\Gamma_1) \\ &\quad + \sum_{i=1}^{j-1}(W_i(\Gamma_1) - W_{i-1}(\Gamma_1))(W_{j-i}(\Gamma_2) - W_{j-i-1}(\Gamma_2))\Bigg)x^j \\ &= P(\Gamma_1;x)\cdot P(\Gamma_2;x). \end{aligned}$$

□

It should be noted that if *neither* Γ_1 or Γ_2 has a globally strategic edge, then $P(\Gamma;x) \neq P(\Gamma_1;x)\cdot P(\Gamma_2;x)$. For example, we can create a graph 2Γ of order 28 which contains two disjoint copies of Γ from equation (3.1). We can calculate $P(\Gamma;x) = 1 + 15x + 64x^2 + 63x^3 + 5x^4$, and

$$P(2\Gamma;x) = 1 + 30x + 353x^2 + 2045x^3 + 5982x^4 + 8165x^5 + 4658x^6 + 645x^7 + 25x^8.$$

It is easy to see that $P(2\Gamma;x) \neq P(\Gamma;x)^2$.

Proposition 3.5 is a great help in finding $P(\Gamma;x)$, since rather than removing edges one at a time until the edges are gone, we only have to remove edges until the graph splits into two components, at which point we have reduced the problem to two much simpler problems.

4. Computational results

We are now ready to solve the problem posed in the Introduction of this paper. Using *Mathematica*, we can recursively define $P(\Gamma;x)$ for a matrix Γ. The routine takes into account Corollary 3.2, Corollary 3.3, Lemma 3.4, and Proposition 3.5. (For Proposition 3.5, the matrices will be sufficiently small that we can assume they are standard.) *Mathematica* can quickly find that $P(K_{5,5};x) = 1 + 16x + 61x^2 + 41x^3 + x^4$, but it takes almost a full day to compute

$$P(K_{6,6};x) = 1 + 25x + 181x^2 + 381x^3 + 131x^4 + x^5.$$

Thus, $W_2(K_{6,6}) = 207$, so there is a 28.75% chance of winning the game. Ironically, if we only considered removing edges which have minimum pet $(\Gamma\backslash\backslash e)$, we get the same answer, only much more quickly. Hence, at least for the 6 by 6 case, there is an easy way of determining which card to reveal.

The 7 by 7 case is much harder. If we only check edges which have minimum pet $(\Gamma\backslash\backslash e)$, we find that

$$P(K_{7,7};x) \approx 1 + 36x + 421x^2 + 1819x^3 + 2333x^4 + 429x^5 + x^6.$$

https://doi.org/10.1017/S0963548306007577 Published online by Cambridge University Press

It is conjectured that this is the correct value of $P(K_{7,7}; x)$, but using the algorithm that checks *every* edge takes too long to verify this result.

Although this paper refers only to bipartite graphs, the definitions in (2.1), (2.2), and (3.2) could be applied to *any* graph: pet (Γ) would be the number of perfect matchings in the graph. For example, one can calculate $P(K_6; x) = 1 + 9x + 5x^2$. Proposition 2.4 would not hold for general graphs, as this example illustrates. Perhaps in another paper the applications of $P(\Gamma; x)$ for general graphs can be explored.

References

[1] Biggs, N. (1993) *Algebraic Graph Theory*, 2nd edn, Cambridge University Press.
[2] Godsil, C. and Royle, G. (2001) *Algebraic Graph Theory*, Springer.
[3] Minc, H. (1978) Permanents. In *Encyclopedia of Mathematics and its Applications*, Vol. 6, Addison-Wesley.
[4] Nijenhuis, A. and Wilf, H. (1978) *Combinatorial Algorithms for Computers and Calculators*, 2nd edn, Academic Press.
[5] Sun, Z. (2001) Hall's theorem revisited. *Proc. Amer. Math. Soc.* **129** 3129–3131.

https://doi.org/10.1017/S0963548306007577 Published online by Cambridge University Press