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Pseudo-orbit tracing and algebraic actions of countable amenable groups

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Abstract. Consider a countable amenable group acting by homeomorphisms on a compact metrizable space. Chung and Li asked if expansiveness and positive entropy of the action imply existence of an off-diagonal asymptotic pair. For algebraic actions of polycyclic-by-finite groups, Chung and Li proved that they do. We provide examples showing that Chung and Li's result is near-optimal in the sense that the conclusion fails for some non-algebraic action generated by a single homeomorphism, and for some algebraic actions of non-finitely generated abelian groups. On the other hand, we prove that every expansive action of an amenable group with positive entropy that has the pseudo-orbit tracing property must admit off-diagonal asymptotic pairs. Using Chung and Li's algebraic characterization of expansiveness, we prove the pseudo-orbit tracing property for a class of expansive algebraic actions. This class includes every expansive principal algebraic action of an arbitrary countable group.

1. *Introduction*

This paper is partly motivated by relatively recent work of Chung and Li [**3**] about the dynamics of countable subgroups of automorphisms for compact groups, and algebraic actions in particular. Part of the paper [**3**] is an investigation of the relation between topological entropy and asymptotic behavior of orbits for actions $\Gamma \curvearrowright X$, where X is a compact metrizable group and Γ acts by group automorphisms [**3**]. The following question was posed and left open by Chung and Li [**3**].

Question 1.1. Let a countable amenable group Γ act by homeomorphisms on a compact metric space X. Suppose the action $\Gamma \curvearrowright X$ is expansive and has positive topological entropy. Must there be an off-diagonal asymptotic pair in X?

Under the additional assumptions that Γ is a polycyclic-by-finite group, that X is a compact abelian group and that $\Gamma \curvearrowright X$ is an expansive action of Γ by automorphisms, Chung and Li obtained an affirmative answer to Question 1.1 [**3**, Theorem 1.2]. On the

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other hand, as remarked in [**3**], there is a much older example due to Lind and Schmidt of non-expansive $\mathbb{Z}$-actions (in fact, toral automorphisms) with positive entropy where all asymptotic pairs are on the diagonal [**7**, Example 3.4].

In addition to expansiveness, the two main assumptions in [**3**, Theorem 1.2] concern the algebraic nature of the action (action by automorphisms on a compact group) and the group-theoretic condition on Γ (polycyclic-by-finite). We show that these assumptions are not just artifacts of the proof method. Specifically, we prove the following theorems.

THEOREM 1.2. *There exist a totally disconnected compact abelian group X and a countable amenable subgroup $\Gamma \subset \mathrm{Aut}(X)$ so that $\Gamma \curvearrowright X$ is expansive, has positive topological entropy and no off-diagonal asymptotic pairs. In particular, the group Γ can be an abelian group (for instance, $\bigoplus_{n\in\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$).*

THEOREM 1.3. *There exists an expansive homeomorphism $T : X \to X$ of a totally disconnected compact metrizable space X with positive topological entropy and no off-diagonal asymptotic pair.*

Theorem 1.3 might be considered a distant cousin of a striking result due to Ornstein and Weiss stating that every transformation is bilaterally deterministic [**9**].

In attempt to understand when Chung and Li's question has an affirmative answer, we are led to explore the pseudo-orbit tracing property. This fundamental dynamical property turns out to ensure an affirmative answer to Question 1.1. The pseudo-orbit tracing property was first introduced and studied by Bowen [**1**] for $\mathbb{Z}$-actions, motivated by the study of Axiom A maps. Walters and others continued this study and obtained further consequences of pseudo-orbit tracing [**13**]. Chung and Lee recently considered the pseudo-orbit tracing property for actions of (finitely generated) countable amenable groups and showed that topological stability and other important consequences of the pseudo-orbit tracing property hold in this more general setting [**2**]. In relation with Question 1.1 above we have the following theorem.

THEOREM 1.4. *Let Γ be a countable amenable group. Every expansive Γ-action on a compact metrizable space that satisfies the pseudo-orbit tracing property and has positive topological entropy admits an off-diagonal asymptotic pair.*

The pseudo-orbit tracing property is of interest in the context of algebraic actions. A particular instance of one of our results applies to *principal algebraic actions*, a well-known class of algebraic actions.

THEOREM 1.5. *Let Γ be a countable group. Every expansive principal algebraic Γ-action satisfies the pseudo-orbit tracing property.*

Chung and Li [**3**] also provided an affirmative answer to Question 1.1 for principal algebraic actions for any countable amenable group Γ (in fact they proved much more; see Remark 3.6 below). This is also a conclusion of Theorem 1.5 combined with Theorem 1.4.

The organization of the paper is as follows. In §2 we recall the pseudo-orbit tracing property, subshifts, and subshifts of finite type in particular. We also prove Theorem 1.4. In §3 we discuss the pseudo-orbit tracing for algebraic actions and derive

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some consequences, in particular the proof of Theorem 1.5. In §4 we prove Theorem 1.2, providing a negative answer to Question 1.1 within the class of algebraic actions. In §5 we prove Theorem 1.3.

2. *The pseudo-orbit tracing property for actions of countable groups*

Throughout this paper, Γ will be a countable group and X will be a compact metrizable space, equipped with a metric $d : X \times X \to \mathbb{R}_+$. We denote by $\mathrm{Act}(\Gamma, X)$ the space of all continuous actions $\Gamma \curvearrowright X$. The space $\mathrm{Act}(\Gamma, X)$ inherits a Polish topology from $\mathrm{Homeo}(X)^\Gamma$. Specifically, given an enumeration of $\Gamma = \{g_1, g_2, \ldots\}$, we have the following metric on $\mathrm{Act}(\Gamma, X)$:

$$\rho(\alpha, \beta) := \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{x \in X} d(\alpha_{g_n}(x), \beta_{g_n}(x)).$$

In this paper when the action $\alpha \in \mathrm{Act}(\Gamma, X)$ is clear from the context we will sometimes write $g \cdot x$ instead of $\alpha_g(x)$.

Definition 2.1. (Compare with [**2**, Definition 2.5]) Fix $S \subset \Gamma$ and $\delta > 0$. An (S, δ) *pseudo-orbit* for $\alpha \in \mathrm{Act}(\Gamma, X)$ is a Γ-sequence $(x_g)_{g \in \Gamma}$ in X such that $d(\alpha_s(x_g), x_{sg}) < \delta$ for all $s \in S$ and $g \in \Gamma$. We say that a pseudo-orbit $(x_g)_{g \in \Gamma}$ is ϵ-*traced* by $x \in X$ if $d(\alpha_g(x), x_g) < \epsilon$ for all $g \in \Gamma$.

Definition 2.2. An action $\alpha \in \mathrm{Act}(\Gamma, X)$ has the *pseudo-orbit tracing property (p.o.t.)* if for every $\epsilon > 0$ there exist $\delta > 0$ and a finite set $S \subset \Gamma$ such that every (S, δ) pseudo-orbit is ϵ-traced by some point $x \in X$.

Definition 2.3. (Compare with [**2**, Definition 2.1]) We say that $\alpha \in \mathrm{Act}(\Gamma, X)$ is *topologically stable* if for every $\epsilon > 0$ there exists an open neighborhood $\mathcal{U} \subset \mathrm{Act}(\Gamma, X)$ of α such that for every $\beta \in \mathcal{U}$ there exists a continuous $f : X \to X$ so that $\alpha_g \circ f = f \circ \beta_g$ for every $g \in \Gamma$ and

$$\sup_{x \in X} d(f(x), x) \leq \epsilon.$$

Chung and Lee [**2**, Theorem 2.8] proved the following theorem.

THEOREM 2.4. *If an action $\alpha \in \mathrm{Act}(\Gamma, X)$ is expansive and satisfies p.o.t., then it is topologically stable. Moreover, if $\eta > 0$ is an expansive constant for α then:*

(1) *for every $0 < \epsilon < \eta$ there exists an open neighborhood $\alpha \in \mathcal{U} \subset \mathrm{Act}(\Gamma, X)$ so that for every $\beta \in \mathcal{U}$ there exists a unique map $f : X \to X$ so that $\alpha_g \circ f = f \circ \beta_g$ for every $g \in \Gamma$ and $\sup_{x \in X} d(f(x), x) \leq \epsilon$;*
(2) *if, furthermore, β as above is expansive with expansive constant 2ϵ, then the conjugating map f above is injective.*

Here is a quick sketch of the proof for Theorem 2.4. If $\beta \in \mathrm{Act}(\Gamma, X)$ is sufficiently close to α then every β-orbit is an (S, δ) pseudo-orbit for α. By p.o.t. this pseudo-orbit is ϵ-traced by the α-orbit of some point y. If ϵ is sufficiently small, expansiveness of α implies that y as above is unique. Furthermore, expansiveness implies that the function $f : X \to X$ sending a point $x \in X$ to the unique point y whose α-orbit traces the β-orbit of

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x is continuous. Uniqueness of the map f implies it is Γ-equivariant. If 2ϵ is an expansive constant for β, then it is impossible for a single y to ϵ-trace the β-orbits of two distinct points, so f is injective.

Remark 2.5. Strictly speaking, Chung and Lee restricted attention to finitely generated groups in [**2**]. The extension to general countable groups does not require any new ideas. Note that the following simplification occurs in the finitely generated case. If Γ is generated by a finite set S, $\alpha \in \mathrm{Act}(\Gamma, X)$ satisfies p.o.t. if for every $\epsilon > 0$ there exists $\delta > 0$ such that every (S, δ) pseudo-orbit is ϵ-traced by some point $x \in X$.

Definition 2.6. Given $\alpha \in \mathrm{Act}(\Gamma, X)$, $(x, y) \in X \times X$ is called an *α-asymptotic pair* if $\lim_{\Gamma \ni g \to \infty} d(\alpha_g(x), \alpha_g(y)) = 0$—in other words, if for every $\epsilon > 0$ there are at most finitely many $g \in \Gamma$ so that $d(\alpha_g(x), \alpha_g(y)) > \epsilon$.

Proof of Theorem 1.4. Suppose $\alpha \in \mathrm{Act}(\Gamma, X)$ is expansive, satisfies p.o.t. and has positive topological entropy.

Fix $\epsilon > 0$ so that 2ϵ is an expansive constant for α. Because α satisfies p.o.t. there exist $\delta > 0$ and a finite set $S \subset \Gamma$ so that every (δ, S) pseudo-orbit is $\epsilon/2$-traced by some $x \in X$. By further increasing S we can safely assume that $S = S^{-1}$ and that S contains the identity.

Let $(F_n)_{n=1}^{\infty}$ be a left-Følner sequence in Γ. Denote

$$\partial_S F_n := \{g \in \Gamma : Sg \cap F_n \neq \emptyset \text{ and } Sg \cap F_n^c \neq \emptyset\}.$$

Because $(F_n)_{n=1}^{\infty}$ is a left-Følner sequence in Γ, it follows that

$$\lim_{n\to\infty} \frac{|\partial_S F_n|}{|F_n|} = 0. \tag{1}$$

Given a finite $F \subset \Gamma$ and $\delta > 0$, a set $Y \subset X$ is (F, δ)-*separated* if

$$\max_{g\in F} d(\alpha_g(x), \alpha_g(y)) \geq \delta \quad \text{for every distinct } x, y \in Y.$$

Let $\mathrm{sep}_{\delta,F}(X, d)$ denote the maximal cardinality of an (F, δ)-separated set in X.

A standard argument gives that for every finite $F \subset \Gamma$ and every $\delta > 0$ the following holds:

$$\log \mathrm{sep}_{\delta,F}(X, d) \leq |F| \log \mathrm{sep}_{\delta/2,\{1\}}(X, d).$$

Thus by (1),

$$\log \mathrm{sep}_{\delta,\partial_S F_n}(X, d) = o(|F_n|). \tag{2}$$

For every $n > 0$, let $X_n \subset X$ be a $(2\epsilon, F_n)$-separated set of maximal cardinality. Because 2ϵ is an expansive constant for α, the topological entropy of α is equal to

$$h(\alpha) = \lim_{n\to\infty} \frac{1}{|F_n|} \log |X_n| > 0. \tag{3}$$

By (2) and (3),

$$\mathrm{sep}_{\delta,\partial_S F_n}(X, d) = o(|X_n|).$$

In particular, for large enough n there exist distinct $x, x' \in X_n$ so that

$$\max_{g\in\partial_S F_n} d(\alpha_g(x), \alpha_g(x')) < \delta. \tag{4}$$

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Define $(y_g)_{g\in\Gamma} \in X^\Gamma$ as follows:

$$y_g = \begin{cases} \alpha_g(x'), & g \in F_n, \\ \alpha_g(x), & g \in \Gamma \setminus F_n. \end{cases} \tag{5}$$

Then $(y_g)_{g\in\Gamma}$ is a (S, δ) pseudo-orbit for α, and by p.o.t. it is ϵ-traced by some $y \in X$. Thus $d(\alpha_g(x), \alpha_g(y)) < \epsilon$ for every $g \in \Gamma \setminus F_n$. This implies that (x, y) is an α-asymptotic pair. Also

$$\max_{F\in F_n} d(\alpha_g(x'), \alpha_g(y)) < \epsilon.$$

Because $\{x, x'\}$ are $(F_n, 2\epsilon)$-separated, it follows that there exists $g \in F_n$ so that $d(\alpha_g(x'), \alpha_g(x)) > 2\epsilon$. By the triangle inequality, there exists $g \in \Gamma$ so that $d(\alpha_g(x), \alpha_g(y)) > \epsilon$, and in particular $x \neq y$. □

We now recall a class of Γ-actions called Γ-*subshifts*. These are also referred to as *symbolic dynamical systems*.

Definition 2.7. Let $\mathcal{A}$ be a discrete finite set. Consider $\mathcal{A}^\Gamma$ as a (metrizable) topological space with the product topology. The (left) shift action $\sigma \in \mathrm{Act}(\Gamma, \mathcal{A}^\Gamma)$ is given by

$$\sigma_g \cdot (x)_h = x_{g^{-1}h}. \tag{6}$$

The pair $(\mathcal{A}^\Gamma, \sigma)$ is called the *full shift* with alphabet $\mathcal{A}$ over the group Γ. A Γ-*subshift* is a subsystem of a full shift. In other words, the dynamical systems (X, σ_X) where action $\sigma_X := \sigma\,|_X \in \mathrm{Act}(\Gamma, X)$, where $X \subset \mathcal{A}^\Gamma$ is a closed σ-invariant subset of $\mathcal{A}^\Gamma$.

Evidently, every Γ-subshift is expansive. It is also well known that every expansive action $\Gamma \curvearrowright X$ on a totally disconnected compact metrizable space X is isomorphic to a Γ-subshift. An action $\Gamma \curvearrowright X$ that is isomorphic to a Γ-subshift is sometimes called a *symbolic dynamical system*. From this abstract point of view, symbolic dynamics is the study of expansive actions on a totally disconnected compact metrizable space.

Let us recall a class of systems called *subshifts of finite type*, arguably the most important class of systems in symbolic dynamics.

Definition 2.8. A Γ-*subshift of finite type* (Γ-*SFT*) is a subshift (X, σ_x) of the form

$$X = \{x \in \mathcal{A}^\Gamma : (g \cdot x)\,|_F \in L \text{ for every } g \in \Gamma\}, \tag{7}$$

where $F \subset \Gamma$ is a finite set and $L \subset \mathcal{A}^F$.

Subshifts of finite type over a countable group Γ can be characterized as the set of expansive $\alpha \in \mathrm{Act}(\Gamma, X)$ that satisfy p.o.t., where X is a totally disconnected compact metrizable space [**2**, **8**, **13**]. Another characterization of subshifts of finite type is given in terms of a certain descending chain condition that is reminiscent of the definition of Noetherian rings [**12**]. A variant of this descending chain condition appeared back in the early work of Kitchens and Schmidt on automorphisms of compact groups [**6**].

Remark 2.9. Theorem 1.4 is a direct extension of the classical observation that every subshift of finite type with positive entropy has an off-diagonal asymptotic pair [**11**, Proposition 2.1].

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As we will now see, in many respects, expansive actions on a compact metrizable space X that satisfy p.o.t. can be thought of as 'systems of finite type', even when X is not totally disconnected.

Definition 2.10. Suppose X is a compact metric space and $\alpha \in \mathrm{Act}(\Gamma, X)$. Let

$$\mathrm{Sub}_\alpha(X) := \{Y \subset X : Y \text{ is closed and } \alpha_g(Y) = Y \text{ for every } g \in \Gamma\}. \tag{8}$$

The Hausdorff metric on the closed subsets of X induces a topology on $\mathrm{Sub}_\alpha(X)$ that makes it a compact metric space. We say that $Y \in \mathrm{Sub}_\alpha(X)$ is a *local maximum* if there exists an open neighborhood $\mathcal{U} \subset \mathrm{Sub}_\alpha(X)$ of Y such that $Z \subseteq Y$ for every $Z \in \mathcal{U}$.

PROPOSITION 2.11. *Let $\alpha \in \mathrm{Act}(\Gamma, X)$ be expansive and satisfy p.o.t. If $\beta \in \mathrm{Act}(\Gamma, Y)$ is expansive and $\Phi : X \to Y$ is continuous, Γ-equivariant and injective then $\Phi(X) \in \mathrm{Sub}_\beta(Y)$ is a local maximum in $\mathrm{Sub}_\beta(Y)$.*

Proof. Let $\alpha \in \mathrm{Act}(\Gamma, X)$, $\beta \in \mathrm{Act}(\Gamma, Y)$ and $\Phi : X \to Y$ be as above. Then $\beta|_{\Phi(X)} \in \mathrm{Act}(\Gamma, \Phi(X))$ is isomorphic to α and, in particular, satisfies p.o.t. Let ϵ be an expansive constant for β. Choose a finite $S \subset \Gamma$ and $\delta \in (0, \epsilon)$ so that every (S, δ) pseudo-orbit for $\beta|_{\Phi(X)}$ is $\epsilon/2$-traced by some $y \in \Phi(X)$. Assume without loss of generality that $1 \in S$. Now let

$$\mathcal{U} := \left\{ Z \in \mathrm{Sub}_\beta(Y) : \sup_{z \in Z} \inf_{x \in X} \max_{g \in S} d(\beta_g(z), \beta_g(\Phi(x))) < \delta/2 \right\}. \tag{9}$$

Then $\mathcal{U}$ is an open neighborhood of $\Phi(X)$ in $\mathrm{Sub}_\beta(Y)$. Now if $Z \in \mathcal{U}$, then by definition of $\mathcal{U}$, for every $z \in Z$ there is $(y_g)_{g \in \Gamma} \in \Phi(X)^\Gamma$ so that

$$d(\beta_{hg}(z), \beta_h(y_g)) < \delta/2 \quad \text{for every } g \in \Gamma,\ h \in S.$$

It follows that $d(\beta_s(y_g), y_{sg}) < \delta$ for all $s \in S$ and $g \in \Gamma$, so $(y_g)_{g \in \Gamma}$ is an (S, δ) pseudo-orbit for $\beta|_{\Phi(X)}$. Thus it is $\epsilon/2$-traced by some $y \in \Phi(X)$. But for every $g \in \Gamma$ we have

$$d(\beta_g(y), \beta_g(z)) < d(\beta_g(y), y_g) + d(y_g, \beta_g(z)) < \epsilon.$$

It follows that $z = y$, so $z \in \Phi(X)$. It follows that $Z \subseteq \Phi(X)$. □

Remark 2.12. A subsystem $Z \in \mathrm{Sub}_\alpha(Y)$ is a local maximum if and only if it satisfies the following *stable intersection property*. For every decreasing sequence $(Y_n)_{n=1}^\infty \in \mathrm{Sub}_\alpha(Y)^\mathbb{N}$,

$$Y \supseteq Y_1 \cdots \supseteq Y_n \supseteq Y_{n+1} \supseteq \cdots \tag{10}$$

such that $Z = \bigcap_{n=1}^\infty Y_n$, there exists $N \in \mathbb{N}$ so that $Y_n = Y_N$ for all $n \geq N$. The equivalence of these two conditions follows because the relation $\subseteq$ is closed in $\mathrm{Sub}_\alpha(Y) \times \mathrm{Sub}_\alpha(Y)$. The observation that the stable intersection property characterizes subshifts of finite type is due to Schmidt [**12**].

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3. *Pseudo-orbit tracing for algebraic actions*

In this section we discuss the pseudo-orbit tracing for algebraic actions and some consequences. We derive Theorem 1.5 as a particular case of Theorem 3.4 below, thus establishing pseudo-orbit tracing for a class of algebraic actions.

Let X be a compact metrizable *abelian* group. We denote by $\mathrm{Act}_{\mathrm{alg}}(\Gamma, X) \subset \mathrm{Act}(\Gamma, X)$ the collection of Γ-actions on X by continuous group automorphisms. Every $\alpha \in \mathrm{Act}_{\mathrm{alg}}(\Gamma, X)$ is called an *algebraic action*. We recall some notation, essentially following [**3**].

Denote by $\mathbb{Z}\Gamma$ the group ring of Γ. Denote by $\ell^\infty(\Gamma)$ the Banach space of all bounded $\mathbb{R}$-valued functions on Γ, equipped with the $\|\cdot\|_\infty$-norm. Also, denote by $\ell^1(\Gamma)$ the Banach algebra of all absolutely summable $\mathbb{R}$-valued functions on Γ, equipped with the ℓ^1-norm $\|\cdot\|_1$ and the involution $f \mapsto f^*$ defined by $(\sum_{s\in\Gamma} f_s s)^* := \sum_{s\in\Gamma} f_s s^{-1}$.

For $k \in \mathbb{N}$ and $p \in [1, \infty]$, we write $\ell^p(\Gamma, \mathbb{R}^k) := (\ell^p(\Gamma))^k$, equipped with the suitable $\|\cdot\|_p$-norm. We denote by $\ell^p(\Gamma, \mathbb{Z}^k)$ the integer-valued elements of $\ell^p(\Gamma, \mathbb{R}^k)$. For $k \in \mathbb{N}$, let $M_k(\ell^1(\Gamma))$ denote the Banach algebra of $k \times k$ matrices with $\ell^1(\Gamma)$-entries, with the norm

$$\|(f_{i,j})_{1\le i,j\le k}\|_1 := \sum_{1\le i,j\le k} \|f_{i,j}\|_1.$$

The involution on $\ell^1(\Gamma)$ also extends naturally to an isometric linear involution on $M_k(\ell^1(\Gamma))$ given by

$$(f_{i,j})^*_{1\le i,j\le k} := (f^*_{j,i})_{1\le i,j\le k}.$$

The following is a classical and crucial fact in the theory of algebraic actions: Pontryagin duality yields a natural one-to-one correspondence between algebraic actions of Γ and (discrete, countable) $\mathbb{Z}\Gamma$-modules. Thus, to each $\mathbb{Z}\Gamma$-module $\mathcal{M}$ there corresponds an algebraic action $\alpha^{(\mathcal{M})} \in \mathrm{Act}_{\mathrm{alg}}(\Gamma, \widehat{\mathcal{M}})$. The dynamics of an algebraic action $\alpha^{(\mathcal{M})}$ are completely determined by the algebraic properties of the dual as a $\mathbb{Z}\Gamma$-module. Over the years, a significant number of important dynamical properties have found beautiful algebraic interpretations in terms of the dual module. For $f \in \mathbb{Z}\Gamma$, we let X_f denote the dual group of the $\mathbb{Z}\Gamma$-module $\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$. The corresponding algebraic action $\alpha^{(f)} \in \mathrm{Act}_{\mathrm{alg}}(\Gamma, X_f)$ is called the *principal algebraic action* associated with f.

Definition 3.1. If X is a compact group with identity element $1 \in X$, and $\alpha \in \mathrm{Act}_{\mathrm{alg}}(\Gamma, X)$, a point $x \in X$ is called *homoclinic* with respect to the action α if $(x, 1)$ is an α-asymptotic pair. In this case the *homoclinic group*, denoted by $\Delta(X)$, is the set of all homoclinic points in X.

It is straightforward to check that $\Delta(X)$ is a Γ-invariant subgroup and (x, y) is an α-asymptotic pair if and only if $xy^{-1} \in \Delta(X)$.

We set up some more notation. Let

$$P : (\ell^\infty(\Gamma))^k \to ((\mathbb{R}/\mathbb{Z})^k)^\Gamma \tag{11}$$

denote the canonical projection map, and let ρ_∞ be the metric on $(\mathbb{R}/\mathbb{Z})^k$ given by

$$\rho_\infty(v + \mathbb{Z}^k, w + \mathbb{Z}^k) := \min_{m\in\mathbb{Z}^k} \|v - w - m\|_\infty, \quad v, w \in \mathbb{R}^k. \tag{12}$$

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LEMMA 3.2. *For any $\delta < 1/2$ and $\tilde{a}, \tilde{b} \in (\mathbb{R}/\mathbb{Z})^k$ satisfying $\rho_\infty(\tilde{a}, \tilde{b}) < \delta$, the following holds. Let $a \in \mathbb{R}^k$ be the unique element of $[-\frac{1}{2}, \frac{1}{2})^k$ such that $\tilde{a} = a + \mathbb{Z}^k$. Then there exists a unique $b \in [-1, 1]^k$ so that $\tilde{b} = b + \mathbb{Z}^k$ and $\|a - b\|_\infty < \delta$.*

Proof. Existence and uniqueness of a as above follow from the fact that $\mathbb{R}^k = \biguplus_{n\in\mathbb{Z}^k}([-\frac{1}{2}, \frac{1}{2})^k + n)$. Similarly, there exists a unique $b \in [-\frac{1}{2}, \frac{1}{2})^k + a$ so that $b = \tilde{b} + \mathbb{Z}^k$. Note that $[-\frac{1}{2}, \frac{1}{2})^k + a \subseteq [-1, 1]^k$ because $a \in [-\frac{1}{2}, \frac{1}{2})^k$, so $b \in [-1, 1]^k$. From the definition of ρ_∞ in (12), there exists $n \in \mathbb{Z}^k$ so that $\|a - (b + n)\|_\infty < \delta$. It follows that

$$\|n\|_\infty = \|b - (b - n)\|_\infty < \|b - a\|_\infty + \|a - (b + n)\|_\infty \le \tfrac{1}{2} + \delta < 1.$$

It follows in turn that $n = 0$, so $\|b - a\|_\infty < \delta$. □

LEMMA 3.3. *Suppose $\delta \in (0, 1/2)$ and that $K, W \subset \Gamma$ are finite subsets that contain the identity element of Γ. Assume $(x^{(g)})_{g\in\Gamma} \in ((\mathbb{R}/\mathbb{Z})^k)^\Gamma$ satisfy*

$$\rho_\infty(x^{(g)}_{w^{-1}f}, x^{(wg)}_f) < \delta \quad \text{for every } g \in \Gamma,\ f \in K \text{ and } w \in K^{-1}W. \tag{13}$$

Then there exists $y^{(g)} \in ([-1, 1]^k)^\Gamma \subset (\ell^\infty(\Gamma))^k$ with $P(y^{(g)}) = x^{(g)}$ for every $g \in \Gamma$ so that

$$\|y^{(g)}_{h^{-1}f} - y^{(hg)}_f\|_\infty < 2\delta \quad \text{for every } g \in \Gamma,\ f \in K \text{ and } h \in W. \tag{14}$$

Proof. Let $(x^{(g)})_{g\in\Gamma} \in ((\mathbb{R}/\mathbb{Z})^k)^\Gamma$ satisfy (13). Define $y^{(g)} \in ([-1, 1]^k)^\Gamma \subset (\ell^\infty(\Gamma))^k$ as follows. First, for every $g \in \Gamma$ let $y^{(g)}_1$ be the unique $a \in [-\frac{1}{2}, \frac{1}{2})^k$ so that $P(a) = x^{(g)}_1$. Next, for every $g \in \Gamma$, $h \in W$ and $f \in K \setminus \{h\}$, let $y^{(g)}_{h^{-1}f}$ be the unique element of $[-1, 1]^k$ satisfying

$$\|y^{(g)}_{h^{-1}f} - y^{(f^{-1}hg)}_1\|_\infty < \delta.$$

Existence and uniqueness of such elements follow by applying Lemma 3.2 above with $\tilde{b} = x^{(g)}_{h^{-1}f}$ and $\tilde{a} = x^{(f^{-1}hg)}_1$, noting that $\rho_\infty(x^{(g)}_{h^{-1}f}, x^{(f^{-1}hg)}_1) < \delta$ by (13) because $1 \in K$. Finally, for every $g \in \Gamma$, $h \in \Gamma \setminus W^{-1}K$, let $y^{(g)}_h$ be the unique $a \in [-\frac{1}{2}, \frac{1}{2})^k$ so that $P(a) = x^{(g)}_h$. It now follows that for every $g \in \Gamma$, $f \in K$ and $h \in W$,

$$\|y^{(g)}_{h^{-1}f} - y^{(hg)}_f\| \le \|y^{(g)}_{h^{-1}f} - y^{(f^{-1}hg)}_1\| + \|y^{(f^{-1}hg)}_1 - y^{(hg)}_f\| < 2\delta.$$ □

For $A \in M_k(\mathbb{Z}\Gamma)$, denote $X_A := \widehat{(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^k A}$. Explicitly,

$$X_A := \{x \in ((\mathbb{R}/\mathbb{Z})^k)^\Gamma : (xA^*)_g = 0_{(\mathbb{R}/\mathbb{Z})^k} \text{ for every } g \in \Gamma\}. \tag{15}$$

Note that the identity element $0_{(\mathbb{R}/\mathbb{Z})^k}$ in $(\mathbb{R}/\mathbb{Z})^k$ is equal to the coset $\mathbb{Z}^k$, so the above formula for X_A can be written as

$$X_A := \{x \in ((\mathbb{R}/\mathbb{Z})^k)^\Gamma : (xA^*)_g = \mathbb{Z}^k \text{ for every } g \in \Gamma\}. \tag{16}$$

Let $\alpha^{(A)} \in \mathrm{Act}_{\mathrm{alg}}(\Gamma, X_A)$ denote the canonical algebraic action on X_A. Theorem 1.5 is a particular instance of the following more general result, inspired by [**3**].

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THEOREM 3.4. *Let $k \in \mathbb{N}$ and $A \in M_k(\mathbb{Z}\Gamma)$ be invertible in $M_k(\ell^1(\Gamma))$. Then the canonical action $\alpha_A \in \mathrm{Act}_{\mathrm{alg}}(\Gamma, X_A)$ is expansive and satisfies p.o.t.*

Proof. To avoid some subscripts, in this proof we let

$$X := X_A \quad \text{and} \quad \alpha := \alpha_A \in \mathrm{Act}_{\mathrm{alg}}(\Gamma, X). \tag{17}$$

By [**3**, Lemma 3.7], $\alpha \in \mathrm{Act}_{\mathrm{alg}}(\Gamma, X)$ is expansive. Fix a metric d on X.

Let $\epsilon > 0$ be arbitrary. By definition of p.o.t., we need to show that there exist $\delta' > 0$ and a finite set $W \subset \Gamma$ so that every (W, δ') pseudo-orbit is ϵ-traced by some point $x \in X$.

Choose $\delta > 0$ small enough so that

$$\delta \leq \min\left\{\tfrac{1}{4}\|A\|_1^{-1}, \tfrac{1}{4}, \epsilon\right\}. \tag{18}$$

From the fact that α is expansive it follows that, by possibly making δ even smaller,

$$\sup_{g\in\Gamma} \rho_\infty(x_g, \tilde{x}_g) < 2\delta \text{ implies } x = \tilde{x}. \tag{19}$$

In fact, the proof of [**3**, Lemma 3.7] shows that (18) already implies (19), but we will not need this.

Since A is invertible in $M_k(\ell^1(\Gamma))$, so is A^*. Let $B \in M_k(\ell^1(\Gamma)) \cong \ell^1(\Gamma, M_k)$ denote the inverse of A^*. Using the natural identification of $M_k(\ell^1(\Gamma))$ as a subset of the $M_k(\mathbb{R})$-valued functions on Γ, write $B = \sum_{g\in\Gamma} B_g$ with $B_g \in M_k(\mathbb{R})$. There exists a finite symmetric set $F \subset \Gamma$ containing the identity with the property that

$$\sum_{g\in\Gamma\setminus F} \|B_g\| < \frac{\delta}{2}\|A^*\|_1^{-1}. \tag{20}$$

Choose a finite symmetric set $S \subset \Gamma$ containing the identity that supports A^* in the sense that there exists $(A_s^*)_{s\in S} \in (M_k(\mathbb{Z}))^S$ so that

$$(yA^*)_g = \sum_{s\in S} y_{gs^{-1}} A_s^* \quad \text{for every } y \in (\ell^\infty(\Gamma))^k \text{ and } g \in \Gamma,$$

and let

$$K := FS^{-1} \subset \Gamma. \tag{21}$$

Choose $\delta' > 0$ small enough so that

$$d(x, y) < \delta' \text{ implies } \max_{f\in K} \rho_\infty(x_f, y_f) < \delta. \tag{22}$$

By compactness of X and (19) it follows that there exists a finite set $W \subset \Gamma$ with the property that

$$\max_{g\in W} \rho_\infty(x_{g^{-1}}, \tilde{x}_{g^{-1}}) < 2\delta \text{ implies } d(x, \tilde{x}) < \min\{\delta', \epsilon\} \quad \text{for all } x, \tilde{x} \in X. \tag{23}$$

Let $W \subset \Gamma$ be such a finite set,so that, in addition, $F^{-1} \subset W$.

Suppose $(x^{(g)})_{g\in\Gamma} \in X^\Gamma$ is a $(K^{-1}W, \delta')$ pseudo-orbit. Then by (22), it follows that (13) holds.

Let $y^{(g)} \in ([-1, 1]^k)^\Gamma \subset (\ell^\infty(\Gamma))^k$ be as in the conclusion of Lemma 3.3. Let

$$z^{(g)} := y^{(g)}A^* \in \ell^\infty(\Gamma, \mathbb{Z}^k). \tag{24}$$

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Then (14) and (18) together imply

$$\|z^{(g)}_{h^{-1}f} - z^{(hg)}_{f}\|_\infty < 1 \quad \text{for every } g \in \Gamma,\ f \in K \text{ and } h \in W. \tag{25}$$

But $z^{(g)}_{h^{-1}f}, z^{(hg)}_{f} \in \mathbb{Z}^k$ so

$$z^{(g)}_{h^{-1}f} = z^{(hg)}_{f} \quad \text{for every } g \in \Gamma,\ f \in K \text{ and } h \in W. \tag{26}$$

Also note that

$$\|z^{(g)}\|_\infty \le \|y^{(g)}\|_\infty \|A^*\|_1 \le \|A^*\|_1. \tag{27}$$

Define $z \in \ell^\infty(\Gamma, \mathbb{Z}^k)$ by

$$z_{g^{-1}} := z^{(g)}_1 \quad \text{for every } g \in \Gamma. \tag{28}$$

Using (26) and the fact that $F^{-1} \subset W$, it follows that

$$\alpha_g(z)|_F = z^{(g)}|_F. \tag{29}$$

By (20), (27) and (29),

$$\|(\alpha_g(z)B)_1 - (z^{(g)}B)_1\| < \delta \quad \text{for every } g \in \Gamma. \tag{30}$$

Let

$$y := zB \in (\ell^\infty(\Gamma))^k \quad \text{and} \quad x := P(y). \tag{31}$$

It follows that

$$yA^* = z \in \ell^\infty(G, \mathbb{Z}^k).$$

Thus $x \in X_A$, recalling the definition of X_A in (16).

It follows from (30) and (13) that

$$\rho_\infty((\alpha_g(x))_{f^{-1}}, x^{(g)}_{f^{-1}}) < 2\delta \quad \text{for every } g \in \Gamma \text{ and } f \in W.$$

By (23) this implies

$$d(\alpha_g(x), x^{(g)}) < \epsilon \quad \text{for every } g \in \Gamma.$$

We have found $x \in X_A$ that ϵ-traces $(x^{(g)})_{g\in\Gamma}$, so the proof is complete. □

Theorem 1.5 follows from Theorem 3.4 above by letting $k = 1$, so $A \in M_k(\mathbb{Z}\Gamma) \cong \mathbb{Z}\Gamma$.

Combining Theorem 1.5 with Theorem 1.4 we recover the following result.

COROLLARY 3.5. *Let Γ be a countable amenable group. Every expansive principal algebraic Γ-action with positive topological entropy admits a non-diagonal asymptotic pair. Equivalently, it has a non-trivial homoclinic group.*

Remark 3.6. By [**3**, Corollary 7.9] every non-trivial expansive principal algebraic action of an amenable group has positive entropy. By [**3**, Lemma 5.4] algebraic actions of the form $\alpha_A \in \mathrm{Act}(\Gamma, X_A)$ as in the statement of Theorem 3.4 have a dense set of homoclinic points. In particular, this proves that Corollary 3.5 holds. I thank Nhan-Phu Chung for pointing this out to me.

Combining Theorem 1.5 with Theorem 2.4 we get the following corollary.

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COROLLARY 3.7. *Every expansive principal algebraic* Γ*-action is topologically stable.*

Using an algebraic characterization of expansive algebraic actions due to Chung and Li we have the following result.

COROLLARY 3.8. *Let* Γ *be a countable group. Suppose* $\alpha \in \mathrm{Act}_{\mathrm{alg}}(\Gamma, X)$ *is expansive. Then* α *is algebraically conjugate to a subsystem of an algebraic action that is expansive and satisfies p.o.t. In other words, there exist an expansive* $\beta \in \mathrm{Act}_{\mathrm{alg}}(\Gamma, Y)$ *that satisfies p.o.t. and a* Γ*-equivariant continuous monomorphism* $\Phi : X \to Y$.

Proof. Suppose $\alpha \in \mathrm{Act}_{\mathrm{alg}}(\Gamma, X)$ is expansive. As in [**3**], we can identify the left $\mathbb{Z}\Gamma$-module $\widehat{X}$ with $(\mathbb{Z}\Gamma)^k/J$ for some $k \in \mathbb{N}$ and some left $\mathbb{Z}\Gamma$-submodule J. By [**3**, Theorem 3.1] there exists $A \in M_k(\mathbb{Z}\Gamma)$ invertible in $M_k(\ell^1(\Gamma))$ so that the rows of A are contained in J. Then α is algebraically conjugate to a subsystem of the natural algebraic Γ-action on the dual of $(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^k A$. By Theorem 1.5 this Γ-action satisfies p.o.t. □

As we mentioned, for algebraic actions every dynamical property corresponds to some algebraic property of $\mathbb{Z}\Gamma$-modules. Is there a natural algebraic interpretation for p.o.t. in terms of the Pontryagin dual? Theorem 3.4 is a sufficient condition for algebraic actions to satisfy p.o.t. The following gives a natural necessary condition for expansive actions to satisfy p.o.t. Given a ring $\mathcal{R}$, recall that a left $\mathcal{R}$-module $\mathcal{M}$ is *finitely presented* if it is isomorphic to $\mathcal{R}^k/J$, where $J \subset \mathcal{R}^k$ is a finitely generated left ideal.

PROPOSITION 3.9. *Let* Γ *be a countable group and* $\Gamma \curvearrowright X$ *an algebraic action. If* $\alpha \in \mathrm{Act}_{\mathrm{alg}}(\Gamma, X)$ *is expansive and satisfies p.o.t., then the Pontryagin dual* $\widehat{X}$ *of* X *is a finitely presented left* $\mathbb{Z}\Gamma$*-module.*

Proof. Let $\alpha \in \mathrm{Act}_{\mathrm{alg}}(\Gamma, X)$ be expansive. In particular, it follows that the dual module $\widehat{X}$ is finitely generated as a $\mathbb{Z}\Gamma$-module [**6**]. Furthermore, by a characterization of expansive algebraic actions given in [**3**], there exist $k \in \mathbb{N}$, a left $\mathbb{Z}\Gamma$-submodule J of $(\mathbb{Z}\Gamma)^k$, and an $A \in M_k(\mathbb{Z}\Gamma)$ invertible in $M_k(\ell^1(\Gamma))$ such that the rows of A are contained in J so that $\widehat{X}$ is isomorphic as a $\mathbb{Z}\Gamma$-module to $(\mathbb{Z}\Gamma)^k/J$. Suppose J is not a finitely generated $\mathbb{Z}\Gamma$-module. Let J_1 be the $\mathbb{Z}\Gamma$-module generated by the rows of A. Because J is not finitely generated, there exists a strictly increasing sequence of left $\mathbb{Z}\Gamma$-modules

$$J_1 \subset J_2 \subset J_3 \subset \cdots,$$

so that the row of A are contained in J_1 and $J = \bigcup_{n=1}^{\infty} J_n$. It follows that X_A is expansive and that $X \in \mathrm{Sub}_\alpha(X_A)$ is a strictly decreasing intersection of the systems $\widehat{(\mathbb{Z}\Gamma)^k/J_n}$. By Proposition 2.11 and Remark 2.12, it follows that α does not satisfy p.o.t. □

If X is a disconnected compact metrizable abelian group then every expansive $\alpha \in \mathrm{Act}_{\mathrm{alg}}(\Gamma, X)$ is isomorphic to an abelian *group shift*. In this case it was shown in [**10**] that if Γ is polycyclic-by-finite then α as above is topologically conjugate to a shift of finite type. In fact, the arguments above and those of [**10**] easily imply that in the totally disconnected case an expansive algebraic action satisfies p.o.t. if and only if the dual module is finitely presented.

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In view of the above, one might try to guess that an expansive algebraic Γ-action satisfies p.o.t. if and only if the Pontryagin is a finitely presented left $\mathbb{Z}\Gamma$-module.

The following example shows that for some expansive algebraic actions, having a finitely presented Pontryagin dual does not imply p.o.t.

Example 3.10. Let $\Gamma = \langle a, b\rangle$ be the free group generated by two elements a, b. Consider the natural algebraic Γ-action on the Pontryagin dual of the following finitely presented left $\mathbb{Z}\Gamma$-module:

$$\widehat{X} = \mathbb{Z}\Gamma/\langle a - 2, b - 2\rangle.$$

So

$$X = \{x \in (\mathbb{R}/\mathbb{Z})^{\Gamma} : x_{ga} = 2x_g \text{ and } x_{gb} = 2x_g \text{ for every } g \in \Gamma\}.$$

Let

$$Y = \{x \in (\mathbb{R}/\mathbb{Z})^{\Gamma} : x_{ga} = 2x_g\}.$$

Then $\Gamma \curvearrowright Y$ is expansive. Now let

$$X_n = \left\{x \in (\mathbb{R}/\mathbb{Z})^{\Gamma} : x_{ga} = 2x_g \text{ and } \rho_{\infty}(x_{gb}, 2x_g) \leq \frac{1}{n} \text{ for every } g \in \Gamma\right\}.$$

Then $X = \bigcap_{n=1}^{\infty} X_n$ is a strictly decreasing intersection of expansive Γ-systems, so it does not satisfy p.o.t., by Proposition 2.11 and Remark 2.12.

Question 3.11. Let Γ be a countable group and suppose $\mathbb{Z}\Gamma$ is left Noetherian (suppose, for example, that Γ is polycyclic-by-finite). Does every expansive $\alpha \in \mathrm{Act}_{\mathrm{alg}}(\Gamma, X)$ satisfy p.o.t.?

An affirmative answer to Question 3.11 would, in particular, recover [**3**, Corollary 9.12], an affirmative answer to Question 1.1 for expansive algebraic Γ-actions when $\mathbb{Z}\Gamma$ is left Noetherian.

4. *An expansive algebraic action with positive entropy and trivial homoclinic group*

In this section we prove Theorem 1.2, providing a negative answer to Question 1.1 within the class of algebraic actions. By [**3**], if $\alpha \in \mathrm{Act}_{\mathrm{alg}}(\Gamma, X)$ is an algebraic counterexample, the acting group Γ cannot be polycyclic-by-finite. The construction makes crucial use of the fact that $\mathbb{Z}\Gamma$ is not left Noetherian for the group Γ we use.

Let $(\Gamma_n)_{n=1}^{\infty}$ be a sequence of finite groups, and let $\Gamma := \bigoplus_{n=1}^{\infty} \Gamma_n$ be the direct sum of these groups. Namely, Γ is the countable subgroup of $\prod_{n=1}^{\infty} \Gamma_n$,

$$\Gamma = \bigcup_{N=1}^{\infty} \left\{(g_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \Gamma_n : g_n = 1_{\Gamma_n} \text{ for all } n \geq N\right\}. \tag{32}$$

For every $n \in \mathbb{N}$, we naturally identify Γ_n with the following subgroup of Γ:

$$\Gamma_n \cong \{(g_k)_{k=1}^{\infty} \in \Gamma : g_k = 1_{\Gamma_k} \text{ for all } k \neq n\}.$$

The reader can keep in mind the case $\Gamma_n = (\mathbb{Z}/2\mathbb{Z})^{a_n}$, where a_n is some rapidly increasing sequence of integers. In this case Γ will be isomorphic to the countable abelian 2-torsion group $\bigoplus_{\mathbb{N}}(\mathbb{Z}/2\mathbb{Z})^{a_n} \cong \bigoplus_{\mathbb{N}}(\mathbb{Z}/2\mathbb{Z})$, also isomorphic to the additive group of polynomials over the finite field of size 2.

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For every $n \in \mathbb{N}$ let

$$\tilde{\Gamma}_n := \{(g_n)_{k=1}^{\infty} \in \Gamma : g_k = 1_{\Gamma_k} \text{ for all } k > n\}. \tag{33}$$

Clearly $\tilde{\Gamma}_n \cong \bigoplus_{k=1}^{n} \Gamma_k$ is a finite subgroup of Γ. Also, the sequence $(\tilde{\Gamma}_n)_{n=1}^{\infty}$ is a left-Følner sequence for Γ.

For $n \geq 1$ let $f_n \in \mathbb{Z}\Gamma$ be given by

$$f_n := \sum_{g \in \Gamma_n} g. \tag{34}$$

Let J be the left $\mathbb{Z}\Gamma$-ideal generated by the element $2 \in \mathbb{Z}\Gamma$ and by $\{f_n\}_{n=1}^{\infty}$, and let

$$X := \widehat{\mathbb{Z}\Gamma/J}.$$

Then X is a totally disconnected compact abelian group. We will identify X with the following Γ-subshift:

$$\begin{aligned} X &= \{x \in (\mathbb{Z}/2\mathbb{Z})^{\Gamma} : x \cdot f_n = 0 \text{ for every } n \in \mathbb{N}\} \\ &= \left\{x \in (\mathbb{Z}/2\mathbb{Z})^{\Gamma} : \sum_{h \in \Gamma_n} x_{gh} = 0 \text{ for every } n \in \mathbb{N} \text{ and } g \in \Gamma\right\}. \end{aligned}$$

For every $n \in \mathbb{N}$, choose $\gamma_n \in \Gamma_n$ so that $\gamma_n \neq 1$ for all but finitely many values of n. Let

$$E = \{(g_n)_{n=1}^{\infty} \in \Gamma : g_n \neq \gamma_n \text{ for every } n \geq 1\}. \tag{35}$$

LEMMA 4.1. *For every $w \in \{0, 1\}^E$ there exists $x \in X$ such that $x|_E = w$.*

Proof. For $g \in \Gamma$ let

$$I(g) := \{n \in \mathbb{N} : g_n = \gamma_n\} \tag{36}$$

and

$$T(g) := \{(h_n)_{n \in I(g)} : h_n \in \Gamma_n \setminus \{1\}\}. \tag{37}$$

Fix $w \in \{0, 1\}^E$. We define $x \in \{0, 1\}^{\Gamma}$ as follows:

$$x_g := \begin{cases} \omega_g & g \in E, \\ \displaystyle\sum_{(h_n)_{n \in I(g)} \in T(g)} \omega_{g \prod_{n \in I(g)} h_n} & \text{otherwise.} \end{cases} \tag{38}$$

Note that the definition is independent of the order of the product $\prod_{n \in I(g)} h_n$, because Γ_n and Γ_m are commuting subgroups of Γ for $n \neq m$. Also note that $g \prod_{n \in I(g)} h_n \in E$ for every $(h_n)_{n \in I(g)}$, so (38) is well defined. It is clear that $x|_E = \omega$. We will now verify that $x \in X$. Equivalently, we need to show that for every $g \in \Gamma$ and every $n \in \mathbb{N}$,

$$\sum_{h \in \Gamma_n} x_{gh} = 0 \mod 2. \tag{39}$$

Fix $g \in \Gamma$ and $n \in \mathbb{N}$. We have

$$\sum_{h \in \Gamma_n} x_{gh} = \sum_{h \in \Gamma_n} x_{gg_n^{-1}h} = \sum_{h \in \Gamma_n \setminus \{1\}} x_{gg_n^{-1}\gamma_n h} + x_{gg_n^{-1}\gamma_n}.$$

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The first equality follows because $g_n \in \Gamma_n$, so $g_n\Gamma_n = \Gamma_n$, so the summands in the sum on the left-hand side of the equality are a permutation of the summands on the right.

Let

$$I(g) \setminus \{n\} = \{n_1, \ldots, n_k\}.$$

Then we have

$$I(gg_n^{-1}\gamma_n) = \{n\} \uplus \{n_1, \ldots, n_k\}.$$

By definition of x it follows that

$$x_{gg_n^{-1}\gamma_n} = \sum_{h\in\Gamma_n\setminus\{1\}} \sum_{h_1\in\Gamma_{n_1}\setminus\{1\}} \cdots \sum_{h_k\in\Gamma_{n_k}\setminus\{1\}} \omega_{gg_n^{-1}\gamma_n h h_1 \ldots h_k}.$$

Similarly, for $h \in \Gamma_n \setminus \{1\}$, we have

$$I(gg_n^{-1}\gamma_n h) = \{n_1, \ldots, n_k\}.$$

Thus, for every $h \in \Gamma_n \setminus \{1\}$, we have

$$x_{gg_n^{-1}\gamma_n h} = \sum_{h_1\in\Gamma_{n_1}\setminus\{1\}} \cdots \sum_{h_k\in\Gamma_{n_k}\setminus\{1\}} \omega_{gg_n^{-1}\gamma_n h h_1 \ldots h_k}.$$

Thus,

$$x_{gg_n^{-1}\gamma_n} = \sum_{h\in\Gamma_n\setminus\{\gamma_n\}} x_{gg_n^{-1}h}.$$

We have thus shown that (39) holds. So $x \in X$ and $x\,|_E = \omega$. □

LEMMA 4.2.

$$h(\Gamma \curvearrowright X) = \prod_{n=1}^{\infty} (1 - |\Gamma_n|^{-1}) \log 2. \tag{40}$$

Proof. Let

$$N_n := \#\{x\,|_{\tilde{\Gamma}_n} : x \in X\}.$$

From Lemma 4.1, it follows that

$$N_n = 2^{\prod_{k=1}^{n}(\Gamma_k\setminus\{\gamma_k\})}.$$

Because $\{\tilde{\Gamma}_n\}$ is a left-Følner sequence, it follows that

$$h(\Gamma \curvearrowright x) = \lim_{n\to\infty} \frac{\log N_n}{|\tilde{\Gamma}_n|}.$$

Thus,

$$h(\Gamma \curvearrowright x) = \lim_{n\to\infty} \frac{\prod_{k=1}^{n} |\Gamma_k \setminus \{\gamma_k\}|}{\prod_{k=1}^{n} |\Gamma_k|} \log 2.$$

From this we immediately get (40). □

LEMMA 4.3. *The action $\Gamma \curvearrowright X$ has no off-diagonal asymptotic pairs.*

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Proof. We will show that the only homoclinic point $x \in X$ is the identity element. In other words, we will show that if $x \in X$ and $F \subset \Gamma$ is a finite set such that $x_g = 0$ for all $g \in \Gamma \setminus F$, then $x_g = 0$ for all $g \in \Gamma$. Indeed, because F is finite, there exists $n \in \mathbb{N}$ such that $g_n = 1$ for all $g \in F$. Suppose $g \in F$. Then for every $h \in \Gamma_n \setminus \{1\}$, $gh \notin F$. It follows that

$$x_g = \sum_{h \in \Gamma_n} x_{gh} = 0.$$

This proves that $x_g = 0$ for all $g \in F$, thus x is the identity element of X. □

The above construction with $\Gamma_n = (\mathbb{Z}/2\mathbb{Z})^{a_n}$ completes the proof of Theorem 1.2.

Using the characterization of completely positive entropy from [**3**], we can further show that the construction above has completely positive entropy. Let $\mathcal{B}_X$ denote the Borel σ-algebra on X, and let μ_X denote Haar measure on X, normalized to be a probability measure. Then $\Gamma \curvearrowright (X, \mathcal{B}_X, \mu_X)$ is a probability preserving Γ-action.

PROPOSITION 4.4. *If* $\prod_{n=1}^{\infty}(1 - |\Gamma_n|^{-1}) > 0$ *then the measure-preserving action* $\Gamma \curvearrowright (X, \mathcal{B}_X, \mu_X)$ *has completely positive entropy.*

Proof. By [**3**, Corollary 8.4] we need to show that $IE(X) = X$. Equivalently (see [**3**, Definition 2.3]) we need to show that for any pair of non-empty open sets U_0, U_1 with $0 \in U_0$ there exist a finite $K \subset \Gamma$ and $c, \epsilon > 0$ such that for any finite set $F \subset \Gamma$ with $|KF \setminus F| < \epsilon|F|$, the pair (U_0, U_1) has an independence set $F' \subset F$ with $|F'| \geq c|F|$. In our case it suffices to prove the above holds for open sets U_0, U_1 of the form

$$U_0 = [0]_{\tilde{\Gamma}_n} \quad \text{and} \quad U_1 = [x]_{\tilde{\Gamma}_n}, \quad x \in X, \ n \in \mathbb{N}.$$

The choice of K and ϵ will not be relevant for us. Let

$$c = \frac{1}{2}|\tilde{\Gamma}_n|^{-1} \prod_{k=n+1}^{\infty}(1 - |\Gamma_k|^{-1}).$$

Choose any finite set $F \subset \Gamma$. For any choice of $\tilde{\gamma}_k \in \Gamma_k$ for $1 \leq k \leq n$, let

$$F_{\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n} := \{(g_k)_{k=1}^{\infty} \in F : g_k = \gamma_n \text{ for all } k \leq n\}.$$

It follows that the sets $F_{\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n}$ are a partition of F into $|\tilde{\Gamma}_n|$ sets, so there exists a choice of $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$ as above so that $|F_{\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n}| \geq |\tilde{\Gamma}_n|^{-1}|F|$. Next, for every $k > n$ define $F_k \subset \Gamma$ and $\gamma_k \in \Gamma_k$ by induction as follows. To start the induction, set $F_n := F_{\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n}$. Assume F_k has been defined. Choose $\gamma_{k+1} \in \Gamma_{k+1}$ so that

$$|\{(g_j)_{j=1}^{\infty} \in F_k : g_{k+1} = \gamma_{k+1}\}| \leq |\Gamma_{k+1}|^{-1}|F_k|,$$

and let

$$F_{k+1} := \{(g_j)_{j=1}^{\infty} \in F_k : g_{k+1} \neq \gamma_{k+1}\}.$$

Let $F' = \bigcap_{k=n+1}^{\infty} F_k$. Note that the decreasing sequence $\cdots \subset F_{k+1} \subset F_k$ stabilizes because F is finite. Also note that $\gamma_n \neq 1$ for all but finitely many values of n. From the construction and choice of c it is clear that $|F'| \geq c|F|$. An application of Lemma 4.1 with the corresponding γ_ks so that $\gamma_k \neq \tilde{\gamma}_k$ for $k \leq n$ gives that F' is an independence set for (U_0, U_1). □

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5. *Positive entropy subshifts without off-diagonal asymptotic pairs*

In this section we prove Theorem 1.3. We actually prove a slightly more general result. This extra generalization does not seem to complicate the construction much and we hope it helps isolate the main idea. Throughout this section Γ will be a countably infinite amenable group that is *residually finite*.

Recall that a group Γ is residually finite if and only if there exist a decreasing sequence of normal subgroups so that

$$\cdots \Gamma_{n+1} \lhd \Gamma_n \lhd \cdots \Gamma_1 \lhd \Gamma_0 = \Gamma, \quad \bigcap_{n=1}^{\infty} \Gamma_n = \{1\}, \tag{41}$$

and

$$[\Gamma : \Gamma_n] < \infty \quad \text{for every } n \geq 1. \tag{42}$$

Let

$$\mathcal{A} = \{0, 1, 2\} \cong \mathbb{Z}/3\mathbb{Z}. \tag{43}$$

Fix a strictly decreasing sequence of finite-index normal subgroups $(\Gamma_n)_{n=1}^{\infty}$ with trivial intersection as in (41). For every $n \geq 1$, let

$$b_n := [\Gamma : \Gamma_n] \quad \text{and} \quad a_n := [\Gamma_{n-1} : \Gamma_n]. \tag{44}$$

For every $n \geq 1$, choose a set of representatives $T_n \subset \Gamma_{n-1}$ for Γ_n-cosets in Γ_{n-1}. Namely, the following result holds:

$$\Gamma_{n-1} = \biguplus_{t \in T_n} \Gamma_n t = \biguplus_{t \in T_n} t \Gamma_n. \tag{45}$$

Furthermore, assume that $1 \in T_n$ for every $n \geq 1$. Note that $|a_n| > 1$ for every n, and so by passing to a subsequence we can make sure the sequence $(a_n)_{n=1}^{\infty}$ grows sufficiently fast in order to satisfy some conditions that will be stated later.

It follows that $|T_n| = [\Gamma_{n-1} : \Gamma_n] = a_n$. Also, let

$$E_n := T_n \cdots T_1 = \{t_n \cdots t_1 : t_i \in T_i i = 1, \ldots, n\}. \tag{46}$$

Note that

$$\Gamma = \biguplus_{g \in E_n} \Gamma_n g = \biguplus_{g \in E_n} g \Gamma_n, \tag{47}$$

and $|E_n| = [\Gamma : \Gamma_n] = b_n$. It will be useful to define

$$\Gamma_0 := \Gamma, \quad T_0 := \{1\} \quad \text{and} \quad a_0 := b_0 := 1. \tag{48}$$

Note that $b_n = a_n \cdot b_{n-1}$.

For $E \subset \Gamma$, $w \in \mathcal{A}^E$ and $g \in \Gamma$, let $g \cdot w \in \mathcal{A}^{gE}$ be given by

$$(g \cdot w)_{gf} := w_f \quad \text{for } f \in E. \tag{49}$$

The first step in our construction is to inductively define sequences $(w^{(n)})_{n=0}^{\infty}$ and $(s^{(n)})_{n=1}^{\infty}$ with $w^{(n)}, s^{(n)} \in \mathcal{A}^{E_n}$ together with sequences $A_n \subset \mathcal{A}^{E_n}$ so that $w^{(n)} \in A_n$ as follows.

To start the induction define $w^{(0)} \in \mathcal{A}^{T_0} \cong \mathcal{A}$ by $w_1^{(0)} := 0$, and define $A_0 := \mathcal{A}^{T_0}$.

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Suppose that $n \geq 1$, and that $w^{(n-1)}$ and A_{n-1} have been defined.

Let

$$B'_n := \{w \in \mathcal{A}^{E_n} : t^{-1} \cdot w \mid_{E_{n-1}} \in A_{n-1} \setminus \{w^{(n-1)}\} \ \forall t \in T_n \setminus \{1\}\} \tag{50}$$

and

$$B_n = \{w \in B'_n : w \mid_{E_{n-1}} = w^{(n-1)}\}. \tag{51}$$

For $s \in \mathcal{A}^{E_{n-1}}$ let

$$C_{s,n} := \left\{ w \in B_n : \sum_{t \in T_n} (t^{-1} \cdot w) \mid_{E_{n-1}} = s \right\}. \tag{52}$$

In (52) and elsewhere we sum elements of $\mathcal{A}^{E_{n-1}}$ pointwise, as $\mathcal{A}$-valued functions, recalling that $\mathcal{A} = \mathbb{Z}/3\mathbb{Z}$. Thus $B_n = \biguplus_{s \in \mathcal{A}^{E_{n-1}}} C_{s,n}$. Choose $s^{(n-1)} \in \mathcal{A}^{E_{n-1}}$ so that

$$|C_{s^{(n-1)},n}| = \max\{|C_{s,n}| : s \in \mathcal{A}^{E_{n-1}}\}. \tag{53}$$

Now define

$$A_n := C_{s^{(n-1)},n},$$

and choose w_n to be an arbitrary element of A_n.

Example 5.1. Consider the case $\Gamma = \mathbb{Z}$. Let $\Gamma_1 = 4\mathbb{Z}$, $\Gamma_2 = 12\mathbb{Z}$, $E_1 = T_1 = \{0, 1, 2, 3\}$, $T_2 = \{0, 4, 8\}$, $E_2 = \{0, 1, 2, \ldots, 11\}$. We have $a_1 = 4$ and $a_2 = 3$. Because $w^{(0)} = 0$, it follows that $A_1 = \{1, 2\}^{\{0,1,2,3\}}$. Considering elements of A_1 as strings of length 4 over the alphabet A_1, we have

$$C_{0,1} = \{0111, 0222\}, \quad C_{1,1} = \{0121, 0211, 0112\}, \quad C_{2,1} = \{0221, 0122, 0212\}.$$

So at this point the procedure above allows us to choose either $s^{(1)} = 1$ or $s^{(1)} = 2$. Suppose we choose $s^{(1)} = 1$. Then

$$A_1 = C_{1,1} = \{0121, 0211, 0112\},$$

and we can choose, for example, $w^{(1)} = 0121$.

We note that if for some n it happens that $|A_n| \leq 2$, then $|A_{n+1}| = 1$ and $|A_{n+2}| = \emptyset$, so we will not be able to continue the construction. The following lemma shows that this does not happen if the sequence $(a_n)_{n=1}^{\infty}$ grows rapidly.

LEMMA 5.2. *If $A_n \neq \emptyset$ then*

$$|A_{n+1}| \geq \frac{1}{3^{b_n}} (|A_n| - 1)^{a_{n+1}-1}. \tag{54}$$

Proof. Because $B_{n+1} = \biguplus_{s \in \mathcal{A}^{E_n}} C_{s,n+1}$, it follows that

$$|B_{n+1}| = \sum_{s \in \mathcal{A}^{E_n}} |C_{s,n+1}| \leq 3^{b_n} \max\{C_{s,n+1} : s \in \mathcal{A}^{E_n}\} = 3^{b_n} |A_{n+1}|. \tag{55}$$

It is clear that B_{n+1} is in bijection with $(A_n \setminus \{w_n\})^{a_{n+1}-1}$, so

$$|B_{n+1}| = (|A_n| - 1)^{a_{n+1}-1}. \tag{56}$$

Inequality (54) follows from (55) and (56). □

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LEMMA 5.3. *Suppose*

$$a_{n+1} \geq 2b_n + 3 \quad \textit{for every } n \geq 1. \tag{57}$$

Then for every $n \geq 1$,

$$|A_n| \geq 3^{b_{n-1}} + 1. \tag{58}$$

In particular, $A_n \neq \emptyset$ *for every* $n \geq 0$.

Proof. Proceed by induction. For $n = 0$, we have $b_0 = 1$ and $A_0 = \mathcal{A}$, so $|A_0| = 3 > 3^0 + 1 = 3^{b_0} + 1$, so (58) is true for $n = 0$. Assume by induction that (58) holds for fixed n. In particular, $|A_n| \geq 3$. Thus by Lemma 5.2 and (57),

$$|A_{n+1}| \geq \frac{1}{3^{b_n}} 3^{b_{n-1}(a_{n+1}-1)} \geq 3^{-b_n} \cdot 3^{b_{n-1}(2b_n+2)} \geq 3^{(2b_{n-1}-1)b_n} \cdot 3^{2n} \geq 3^{b_n} + 1. \qquad \square$$

From now on assume (57) holds, and so by Lemma 5.3, $A_n \neq \emptyset$. For $n \in \mathbb{N}$, define $R_n \subset \mathcal{A}^\Gamma$ as follows:

$$R_n := \{x \in \mathcal{A}^\Gamma : (g \cdot x)\,|_{E_n} \in A_n \text{ for every } g \in \Gamma_n\}. \tag{59}$$

Clearly R_n is a compact Γ_n-invariant subset of $\mathcal{A}^\Gamma$. By (47), the action $\Gamma_n \curvearrowright R_n$ is trivially isomorphic to the shift action $\Gamma_n \curvearrowright A_n^{\Gamma_n}$.

LEMMA 5.4. *For every* $n \geq 0$,

$$R_{n+1} \subset R_n.$$

Proof. Suppose $x \in R_{n+1}$. Choose $g \in \Gamma_n$. By (45), $\Gamma_n = \Gamma_{n+1} T_{n+1}$. Thus there exist $g' \in \Gamma_{n+1}$ and $t \in T_{n+1}$ so that $g = g't$. By definition of R_{n+1}, $w := ((g')^{-1} \cdot x)\,|_{E_{n+1}} \in A_{n+1}$. By definition of A_{n+1}, it follows that $(t^{-1} \cdot w)\,|_{E_n} \in A_n$. But $(t^{-1} \cdot w)\,|_{E_n} = (g^{-1} \cdot x)\,|_{E_n}$, so $x \in R_n$. $\square$

LEMMA 5.5. *For every* $n \geq 1$, $x \in R_n$, *we have*

$$(g \cdot x)\,|_{E_n} \in A_n \quad \textit{if and only if } g \in \Gamma_n. \tag{60}$$

Proof. We prove this by induction on n. To start the induction, note that if $x \in R_1$ then $x_g = 0$ if and only if $g \in \Gamma_1$, so $g \cdot x \in R_1$ if and only if $g \in \Gamma_1$.

For the induction step, fix $x \in R_n$. From the definition of R_n it is clear that if $g \in \Gamma_n$ then $(g \cdot x)\,|_{E_n} \in A_n$. By Lemma 5.4, $R_n \subset R_{n-1}$, so by the induction hypothesis, for every $g \in \Gamma \setminus \Gamma_{n-1}$ we have $(g \cdot x)\,|_{E_{n-1}} \notin A_{n-1}$.

It remains to show that $g^{-1} \cdot x \notin R_n$ for every $g \in \Gamma_{n-1} \setminus \Gamma_n$. Choose $g \in \Gamma_{n-1} \setminus \Gamma_n$. Because $\Gamma_{n-1} = \biguplus_{t \in T_{n-1}} \Gamma_n t$, there exist $g' \in \Gamma_n$ and $t \in T_{n-1} \setminus \{1\}$ so that $g = g't$. Let $v := ((g')^{-1} \cdot x)\,|_{E_n}$. Because $x \in R_n$, it follows from the definition of R_n that $v \in A_n$, so $(t^{-1} \cdot v)\,|_{E_{n-1}} \in A_{n-1} \setminus \{w^{(n-1)}\}$. But $(t^{-1} \cdot v)\,|_{E_{n-1}} = (g^{-1} \cdot x)\,|_{E_{n-1}}$, so $g^{-1} \cdot x \notin R_n$. $\square$

Define

$$X_n := \bigcup_{g \in E_n} g \cdot R_n. \tag{61}$$

X_n is a finite union of compact subsets so it is compact. Also, because R_n is Γ_n-invariant, it follows that X_n is Γ-invariant.

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LEMMA 5.6. *For every $n \geq 1$, $X_{n+1} \subset X_n$.*

Proof. By Lemma 5.4, $R_{n+1} \subset R_n$ so

$$X_{n+1} = \bigcup_{g \in E_{n+1}} g \cdot R_{n+1} \subset \bigcup_{g \in E_{n+1}} g \cdot R_n.$$

Because $E_{n+1} = T_{n+1} E_n$,

$$\bigcup_{g \in E_{n+1}} g \cdot R_n = \bigcup_{t \in T_{n+1}} t \cdot \left(\bigcup_{g \in E_n} g \cdot R_n \right) = \bigcup_{t \in T_{n+1}} t \cdot X_n.$$

Note that $t \cdot X_n = X_n$ for $t \in T_{n+1}$. We conclude that indeed $X_{n+1} \subset X_n$. □

We now define

$$X := \bigcap_{n=1}^{\infty} X_n. \tag{62}$$

LEMMA 5.7. *Suppose $(x, y) \in X_{n+1} \times X_{n+1}$, and $F \subset \Gamma$ is a finite set such that*

$$x_g = y_g \quad \text{for all } g \in \Gamma \setminus F. \tag{63}$$

In addition, suppose that $g_1 \Gamma_n \neq g_2 \Gamma_n$ for every pair of distinct elements $g_1, g_2 \in F$. Then $x = y$.

Proof. Let $(x, y) \in X_{n+1} \times X_{n+1}$ and $F \subset \Gamma$ be as above. By (61) there exist $f, \tilde{f} \in E_{n+1}$ so that $f \cdot x \in R_{n+1}$ and $\tilde{f} \cdot y \in R_{n+1}$. Because $f \cdot x \in R_{n+1}$, it follows that $(g \cdot f \cdot x)\,|_{E_{n+1}} \in A_{n+1}$ for all $g \in \Gamma_{n+1}$.

By (63),

$$(f \cdot x)_g = (f \cdot y)_g \quad \text{for all } g \in \Gamma \setminus (fF). \tag{64}$$

Because Γ_{n+1} is infinite and F is finite, by (64) there are infinitely many $g \in \Gamma_{n+1}$ so that

$$(g^{-1} \cdot f \cdot y)\,|_{E_{n+1}} = (g^{-1} \cdot f \cdot x)\,|_{E_{n+1}} \in A_{n+1}.$$

By Lemma 5.5 it follows that $g^{-1} \cdot f \cdot \tilde{f}^{-1} \in \Gamma_{n+1}$, so $f \cdot \tilde{f}^{-1} \in \Gamma_{n+1}$ and so $f = \tilde{f}$. Thus $f \cdot y \in R_{n+1}$.

Because $f \cdot x,\ f \cdot y \in R_{n+1}$ it follows that for every $g \in \Gamma_{n+1}$,

$$(g^{-1} \cdot f \cdot x)\,|_{E_{n+1}},\ (g^{-1} \cdot f \cdot y)\,|_{E_{n+1}} \in A_{n+1}.$$

Thus for every $r \in E_{n+1}$,

$$\sum_{t \in T_{n+1}} (t^{-1} \cdot g^{-1} \cdot f \cdot x)_r = \sum_{t \in T_{n+1}} (t^{-1} \cdot g^{-1} \cdot f \cdot y)_r = s_r^{(n)}. \tag{65}$$

Equivalently,

$$\sum_{t \in T_{n+1}} x_{f^{-1} \cdot g \cdot t \cdot r} = \sum_{t \in T_{n+1}} y_{f^{-1} \cdot g \cdot t \cdot r} = s_r^{(n)}. \tag{66}$$

We claim that there exists at most one $t \in T_{n+1}$ so that $f^{-1} \cdot g \cdot t \cdot r \in F$. Indeed, because $T_{n+1} \subset \Gamma_n$, and because Γ_n is normal, the set $\{f^{-1} \cdot g \cdot t \cdot r : t \in T_{n+1}\}$ is contained in a single Γ_n-coset.

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It follows that there exists at most one $t \in T_{n+1}$ so that $x_{f^{-1}\cdot g\cdot t\cdot r} \neq y_{f^{-1}\cdot g\cdot t\cdot r}$. By (66), it follows that in fact

$$x_{f^{-1}\cdot g\cdot t\cdot r} = y_{f^{-1}\cdot g\cdot t\cdot r} \quad \text{for every } g \in \Gamma_{n+1},\ t \in T_{n+1} \text{ and } r \in E_{n+1}, \tag{67}$$

Every $\tilde{g} \in \Gamma$ can be written as $\tilde{g} = f^{-1}gtr$ for some $r \in E_{n+1}$, $t \in T_{n+1}$ and $g \in \Gamma_{n+1}$, so $x_{\tilde{g}} = y_{\tilde{g}}$ for every $\tilde{g} \in \Gamma$. □

Our next goal is to show that if the sequence $(a_n)_{n=1}^{\infty}$ grows sufficiently fast, then $h(X) > 0$. To be more specific, we prove the following lemma.

LEMMA 5.8. *If $(a_n)_{n=1}^{\infty}$ satisfies the assumption in the statement of Lemma 5.3 then*

$$h(\Gamma \curvearrowright X_n) \geq \frac{a_1 - 1}{a_1}\log(2) - \frac{1}{a_1}\log(3) - \sum_{k=2}^{n}\left(\frac{2}{3^{b_{k-2}}+1} + \frac{2}{a_k}\right)\cdot\log(3). \tag{68}$$

Proof. Because $\Gamma_n \curvearrowright tR_n$ is isomorphic to the full shift over the alphabet A_n for every $n \geq 1$ and every $t \in E_n$, it follows that

$$h(\Gamma_n \curvearrowright tR_n) = \log|A_n|.$$

Because $X_n = \bigcup_{t\in E_n} tR_n$ is a finite union of Γ_n-invariant sets it follows that

$$h(\Gamma_n \curvearrowright X_n) = \max_{t\in E_n} h(\Gamma_n \curvearrowright tR_n) = \log|A_n|.$$

Thus, by the subgroup formula for entropy (see [**4**, Theorem 2.16] for a stronger result),

$$h(\Gamma \curvearrowright X_n) = [\Gamma : \Gamma_n]^{-1}\log|A_n| = b_n^{-1}\log|A_n|.$$

Taking logs in (54), we get

$$\log|A_n| \geq (a_n - 1)\log(|A_{n-1}| - 1) - b_{n-1}\log 3. \tag{69}$$

Denote

$$h_n := h(\Gamma \curvearrowright X_n) = b_n^{-1}\log|A_n|. \tag{70}$$

Recall that $|A_0| = 3$ and $b_1 = a_1$. So substituting $n = 1$ in (69), we get

$$h_1 = \frac{1}{a_1}\log(|A_1|) \geq \frac{a_1 - 1}{a_1}\log(2) - \frac{1}{a_1}\log(3). \tag{71}$$

Dividing (69) by b_n, we get

$$\frac{1}{b_n}\log|A_n| \geq \frac{(a_n - 1)\cdot b_{n-1}}{b_n}\cdot\frac{\log(|A_{n-1}| - 1)}{b_{n-1}} - \frac{b_{n-1}}{b_n}\log 3.$$

Using the relation $b_n = a_n b_{n-1}$, it follows that

$$h_n \geq \left(1 - \frac{1}{a_n}\right)\left[h_{n-1}\cdot\frac{\log(|A_{n-1}| - 1)}{\log|A_{n-1}|}\right] - \frac{1}{a_n}\log 3.$$

Apply the estimate

$$\frac{\log(x-1)}{\log(x)} = 1 - \frac{\log(x) - \log(x-1)}{\log(x)} \geq 1 - \frac{1}{(x-1)\log(x)} \geq 1 - \frac{2}{x} \quad \text{for } x \geq 2,$$

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with $x = |A_{n-1}|$, to get

$$h_n \geq \left(1 - \frac{1}{a_n}\right)\left[h_{n-1}\left(1 - \frac{2}{|A_{n-1}|}\right)\right] - \frac{1}{a_n}\log 3.$$

Equivalently,

$$h_n - h_{n-1} \geq -\left(\frac{1}{a_n} + \frac{2}{|A_{n-1}|} - \frac{1}{a_n|A_{n-1}|}\right)h_{n-1} - \frac{1}{a_n}\log 3.$$

By Lemma 5.3, for every $n \geq 2$,

$$|A_{n-1}| \geq 3^{b_{n-2}} + 1.$$

Note that $X_0 = (\mathbb{Z}/3\mathbb{Z})^\Gamma$ so $h_0 = h(\Gamma \curvearrowright X_0) = \log 3$. Because the sequence $\{h_n\}_{n=1}^\infty$ is monotone non-increasing, $h_n \leq \log 3$ for every $n \geq 1$. We thus have

$$h_k - h_{k-1} \geq -\left(\frac{2}{3^{b_{k-2}} + 1} + \frac{2}{a_k}\right) \cdot \log(3) \quad \text{for every } k \geq 2. \tag{72}$$

Summing (72) over $2 \leq k \leq n$, we get

$$h_n \geq h_1 - \sum_{k=2}^{n}\left(\frac{2}{3^{b_{k-2}} + 1} + \frac{2}{a_k}\right) \cdot \log(3).$$

The estimate (68) follows using the estimate (71) for h_1. □

To conclude our construction, we state the following proposition.

PROPOSITION 5.9. *If $(a_n)_{n=1}^\infty$ satisfies the assumption in the statement of Lemma 5.3 and $a_1, a_2, \ldots$ are sufficiently big then $h(\Gamma \curvearrowright X) > 0$ and X has no off-diagonal asymptotic pairs.*

Proof. Because $X = \bigcap_{n=1}^\infty X_n$, and $X_{n+1} \subset X_n$ it follows from upper semi-continuity of topological entropy (see [**5**, Proposition 1.3]) that

$$h(\Gamma \curvearrowright X) = \inf_n h(\Gamma \curvearrowright X_n).$$

By Lemma 5.8,

$$\inf_n h(\Gamma \curvearrowright X_n) \geq \frac{a_1 - 1}{a_1}\log(2) - \frac{1}{a_1}\log(3) - \sum_{k=2}^{\infty}\left(\frac{2}{3^{b_{k-1}} + 1} + \frac{2}{a_k}\right) \cdot \log(3).$$

If a_1 is sufficiently big we have $(a_1 - 1/a_1)\log(2) - (1/a_1)\log(3) > \log(2) - \epsilon$. Also, by choosing $a_2, a_3, \ldots$ to grow sufficiently fast the series $\sum_{k=2}^\infty (2/(3^{b_{k-2}} + 1) + 2/a_k)$ converges and the sum can be made smaller than $\frac{1}{2}\log(3) + \epsilon$ for any $\epsilon > 0$. This proves $\Gamma \curvearrowright X$ has positive entropy as soon as $a_1, a_2, \ldots$ are sufficiently big.

Let us show that X has no off-diagonal asymptotic pairs. Suppose $(x, y) \in X \times X$ is an asymptotic pair. If follows that (x, y) satisfy (63) for some finite $F \subset \Gamma$. Because $\bigcap_n \Gamma_n = \{1\}$ it follows that there exists n so that the map $g \mapsto g\Gamma_n$ is injective of F. Because $(x, y) \in X \times X \subset X_{n+1} \times X_{n+1}$ it follows from Lemma 5.7 that $x = y$. □

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