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Fixed points of imaginary reflections on hyperbolic handlebodies

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Abstract

A Klein–Schottky group is an extended Kleinian group, containing no reflections and whose orientation-preserving half is a Schottky group. A dihedral-Klein–Schottky group is an extended Kleinian group generated by two different Klein–Schottky groups, both with the same orientation-preserving half. We provide a structural description of the dihedral-Klein–Schottky groups.

Let M be a handlebody of genus g, with a Schottky structure. An imaginary reflection τ of M is an orientation-reversing homeomorphism of M, of order two, whose restriction to its interior is an hyperbolic isometry having at most isolated fixed points. It is known that the number of fixed points of τ is at most $g + 1$; τ is called a maximal imaginary reflection if it has $g + 1$ fixed points. As a consequence of the structural description of the dihedral-Klein–Schottky groups, we are able to provide upper bounds for the cardinality of the set of fixed points of two or three different imaginary reflections acting on a handlebody with a Schottky structure. In particular, we show that maximal imaginary reflections are unique.

1. *Introduction*

A *uniformization* of a Riemann surface S is a triple $(\Delta, G, P{:}\Delta \to S)$ where G is a Kleinian group, Δ is a connected component of the region of discontinuity of G which is invariant under the action of G and $P{:}\Delta \to S$ is a regular covering map with $G = \mathrm{Deck}(P)$. The collection of uniformizations of S is partially ordered and the highest ones are produced when Δ is simply-connected. For instance, if S is a closed Riemann surface of genus $g \geqslant 2$, then highest uniformizations are produced by Fuchsian groups (*Fuchsian uniformizations*). In recent years a great deal of attention in the theory of closed Riemann surfaces has been given to their study by means of Fuchsian uniformizations. If S is a closed Riemann surface of genus g, then the lowest ones are produced by the Schottky groups of rank g (*Schottky uniformizations*). Schottky uniformizations have not been exploited in this context as for the case of Fuchsian ones.

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An interesting class of closed Riemann surfaces, called *symmetric Riemann surfaces*, are those admitting an order two anticonformal automorphism, called a *symmetry*. If the symmetry has fixed points, then it is a *reflection*; otherwise, it is an *imaginary reflection*. As a consequence of the known topological actions of symmetries, if S is a symmetric Riemann surface and $\tau:S \to S$ is a symmetry, then there is a Schottky uniformization $(\Delta, G, P:\Delta \to S)$ so that τ lifts, that is, there is an anticonformal automorphism $\eta:\Delta \to \Delta$ so that $\tau P = P\eta$. The automorphism η is an extended Möbius transformation satisfying $\eta^2 \in G$ and $\eta G \eta^{-1} = G$. The group $K = \langle G, \eta \rangle$ is an extended Kleinian group whose orientation-preserving half is the Schottky group G.

An *extended Schottky group* of rank g is an extended Kleinian group whose orientation-preserving half is a Schottky group of rank g. A *Klein–Schottky group* is an extended Schottky group for which every orientation-reversing transformation has no fixed points on its region of discontinuity. The structural description of Klein–Schottky groups is provided in [**7**]; the structure of general extended Schottky groups will appear elsewhere [**5**]. As a consequence, each symmetry on a closed Riemann surface of genus g can be realized by an extended Schottky group; moreover, if the symmetry is an imaginary reflection, the corresponding extended Schottky group is a Klein–Schottky group.

An extended Kleinian group generated by two different Klein–Schottky groups, both with the same orientation-preserving half is called a *dihedral-Klein–Schottky group*. In the above context, each dihedral-Klein–Schottky group contains a Schottky group as a normal subgroup uniformizing a closed Riemann surface admitting two different imaginary reflections. In this paper, using the Klein–Maskit combination theorem [**9**], we provide a structural description of the dihedral-Klein–Schottky groups.

THEOREM 1. *A dihedral-Klein–Schottky group $\widetilde{G}$ is the combination theorem free product of a finite number of elementary groups of the following forms*:

(i) *imaginary reflection cyclic groups (groups of type $\mathcal{Q}$)*;
(ii) *elliptic cyclic groups (groups of type $\mathcal{R}$)*;
(iii) *loxodromic (including hyperbolic) cyclic groups (groups of type $\mathcal{S}$)*;
(iv) *cyclic groups generated by a semi-hyperbolic transformation; i.e., an orientation-reversing transformation whose square is hyperbolic (groups of type $\mathcal{T}$)*;
(v) *abelian groups generated by an elliptic element and by a loxodromic (including hyperbolic) element (groups of type $\mathcal{U}$)*;
(vi) *groups generated by an elliptic element a, and a semi-hyperbolic transformation b, satisfying the relation, $bab^{-1}a = 1$ (groups of type $\mathcal{V}$).*

Let G be a combination theorem free product constructed with q groups of type $\mathcal{Q}$, r groups of type $\mathcal{R}$, s groups of type $\mathcal{S}$, t groups of type $\mathcal{T}$, u groups of type $\mathcal{U}$ and v groups of type $\mathcal{V}$. Then, G is a dihedral-Klein–Schottky group if and only if (i) *and* (ii) *below both hold*:

(i) $q + t + v > 0$;
(ii) *either* $v > 0$ *or* $q + r + s + t + u > 1$.

At the level of 3-dimensional hyperbolic manifolds, each Schottky group of rank g induces a complete geometrically finite hyperbolic structure, with injectivity radius bounded away from zero, in the interior of a handlebody of genus g; the conformal boundary being a closed Riemann surface. In this case we talk of a *Schottky structure on the handlebody*. An *imaginary reflection* on a handlebody with Schottky structure is an orientation-reversing

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self-homeomorphism of order 2, whose restriction to the interior is an isometry for the corresponding hyperbolic structure, and having at most finitely many fixed points. Each Klein–Schottky group produces a handlebody with Schottky structure admitting an imaginary reflection. Similarly, a dihedral-Klein–Schottky group produces a handlebody with Schottky structure admitting two imaginary reflections. As an application of Theorem 1, in Section 2, we provide some information on pairs and triples of imaginary reflections acting on handlebodies with a Schottky structure that generalize known results for symmetries on closed Riemann surfaces.

2. *Application to handlebodies*

Let M be a handlebody of genus g with interior M^0 and topological boundary S. A complete hyperbolic structure on M^0 is of the form $\mathbb{H}^3/G$, where G is a Kleinian group isomorphic to a free group of rank g; the conformal boundary is Ω/G, where $\Omega \subset \widehat{\mathbb{C}}$ is the region of discontinuity of G (it may be empty). In general, Ω/G is topologically equivalent to some portion of S. Conversely, due to the Marden conjecture (or tame ends conjecture) proved recently by Agol [**1**] and Calegari–Gabai [**4**], each Kleinian group isomorphic to a free group produces a complete hyperbolic Riemannian metric on M^0. Moreover, complete hyperbolic structures which are geometrically finite and have injectivity radius bounded away from zero are exactly those provided by *Schottky groups*, that is, a geometrically finite purely loxodromic Kleinian group isomorphic to a free group of rank g (see for instance [**9**]). In the former case we will say that the Schottky group provides a *Schottky structure*.

An *automorphism* of a handlebody M, with a Schottky structure, is a self-homeomorphism whose restriction to its interior is an hyperbolic isometry. We denote by $Aut(M)$ (respectively, $Aut^+(M)$) the group of automorphisms (respectively, orientation-preserving automorphisms) of M. A *symmetry* of a handlebody, with a Schottky structure, is an orientation-reversing automorphism of order two. If the symmetry has only isolated fixed points, then we say that it is an *imaginary reflection*; otherwise, it is called a *reflection*.

Let M be a handlebody with Schottky structure and let S be its conformal boundary. Every conformal automorphism (respectively, anticonformal automorphism) τ of M induces (by restriction) a conformal automorphism (respectively, anticonformal automorphism) τ_S on S. If τ is a reflection, then τ_S is also a reflection (the fixed points of τ_S being the boundary loops of the connected components of fixed points of τ). If τ is an imaginary reflection, then τ_S is also an imaginary reflection.

Remark 2. The topological type of a symmetry τ of a Riemann surface S is determined by its type (k, e), where k is the number of its ovals and e is the orientability type of the quotient orbifold $S/\langle\tau\rangle$. It follows that any two imaginary reflections (on Riemann surfaces of the same genus) are topologically conjugated by a suitable orientation-preserving homeomorphism. In the case of handlebodies with Schottky structures, the structural description of Klein–Schottky groups in [**7**] asserts that any two imaginary reflections on handlebodies of the same genus are topologically conjugated by a suitable orientation-preserving homeomorphism if and only if they have the same number of fixed points.

Let M be a handlebody of genus g with a Schottky structure provided by a Schottky group G, $\pi^0\colon \mathbb{H}^3 \to M^0$ be a universal covering induced by G, $\Omega \subset \widehat{\mathbb{C}}$ be the region of discontinuity of G and $S = \Omega/G$ be the conformal boundary of M. Clearly, π^0 extends to a regular covering $\pi\colon \mathbb{H}^3 \cup \Omega \to M$ with G as covering group. Let $\tau \in Aut(M)$ be a

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symmetry. If we lift τ to the universal cover $\mathbb{H}^3$, then we obtain an orientation-reversing hyperbolic isometry $\widehat{\tau}$ that normalizes G and so that $\widehat{\tau}^2 \in G$. The extended Kleinian group $\widehat{G} = \langle G, \widehat{\tau} \rangle$ contains G as its index two orientation-preserving half, that is, $\widehat{G}$ is an extended Schottky group. Let us assume $\tau \in Aut(M)$ is an imaginary reflection, in particular, τ necessarily acts on S as an imaginary reflection. As $S/\langle \tau \rangle$ is a closed non-orientable surface, the orientation-reversing elements of $\widehat{G}$ are either semi-hyperbolics or imaginary reflections (i.e. conjugates of $z \mapsto -1/\overline{z}$). It follows that $\widehat{G}$ is a Klein–Schottky group of rank g. The structural description of Klein–Schottky groups of [**7**] asserts that $\widehat{G}$ is the Klein–Maskit free product of m semi-hyperbolics and n imaginary reflections so that $g = 2m + n - 1$ (we say a *Klein–Schottky group of type* (n, m)); in this case, n is exactly the number of fixed points of the imaginary reflection τ. It follows that (i) $0 \leqslant n \leqslant g + 1$ (this is also a consequence of the Lefschetz fixed-point theorem) and

$$n \in N_g = \begin{cases} \{1, 3, 5, ..., g-3, g-1, g+1\}, \text{ if } g \text{ is even;} \\ \{0, 2, 4, ..., g-3, g-1, g+1\}, \text{ if } g \text{ is odd.} \end{cases}$$

The classification in [**7**] ensures that for every $n \in N_g$ there is a handlebody of genus $g \geqslant 0$ admitting an imaginary reflection with exactly n fixed points. An imaginary reflection on a handlebody having the maximal number of fixed points, that is $g+1$, is called a *maximal imaginary reflection*.

Remark 3. At the level of closed Riemann surfaces, Harnack's theorem [**2**, **6**] asserts that the number of connected components of the locus of fixed points of a symmetry $\tau: S \to S$ is at most $(g + 1)$, where g is the genus of S, each component being a simple closed curve (called an *oval* or a *mirror*). The above states a similar fact for imaginary reflections on handlebodies.

Consider two different imaginary reflections $\tau_1, \tau_2 \in Aut(M)$. By lifting both of them to the universal cover $\mathbb{H}^3$, we obtain an extended Kleinian group $\widetilde{G}$. If we denote by $\widehat{G}_j$ the Klein–Schottky group obtained by lifting τ_j, then $\widetilde{G} = \langle \widehat{G}_1, \widehat{G}_2 \rangle$ and $\widetilde{G}/G$ is a dihedral group of order $2p$, where p is the order of the product $\tau_1\tau_2$. It follows that $\widetilde{G}$ is a dihedral-Klein–Schottky group of rank g. We then obtain from Theorem 1 a catalog of the types of orbifolds $M/\langle \tau_1, \tau_2 \rangle$ that can be obtained as the quotient of a handlebody M, with Schottky structure, by a dihedral group generated by two different imaginary reflections τ_1 and τ_2. Each group of type $\mathcal{Q}$ produces an isolated conical point (locally not a manifold), each group of type $\mathcal{R}$ produces an arc of conical points with distinct end points on the conformal boundary, each group of type $\mathcal{S}$ produces a genus 1 orientable handlebody, each group of type $\mathcal{T}$ produces a genus 1 non-orientable handlebody, each group of type $\mathcal{U}$ produces a genus 1 orientable handlebody containing a simple loop of conical points, and each group of type $\mathcal{V}$ produces a genus 1 non-orientable handlebody containing a simple loop of conical points. Moreover, each of these isolated conical points, arcs of conical points and loops of conical points are separated from the others by a disc, containing no conical points, whose boundary lies in the conformal boundary of the orbifold.

At the level of closed Riemann surfaces, in several papers, see for instance [**3**, **11**, **12**, **13**], different authors have obtained upper bounds for the number of ovals that a finite set of symmetries on a Riemann surface may have and, in particular, proved that a non-hyperelliptic Riemann surface may have at most one *maximal symmetry* (that is, one with the maximal number of ovals) and that a hyperelliptic surface has at most two (their product being the hyperelliptic involution). For instance, in [**3**] it is proven that the sum of the number of ovals

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of two reflections is bounded by $4g/p + 2$ if p is even and by $2(g-1)/p + 4$ if p is odd, where p is the order of the product of the two reflections and g is the genus of the surface. In [**13**] it was shown that the sum of the number of ovals that may appear with three reflections is bounded by $2g + 4$, where g is the genus of the surface. In this paper, we will be concerned with similar upper bounds for the number of isolated fixed points of pairs and triples of imaginary reflections acting on handlebodies. The study of more than 3 imaginary reflections may be carried out in a similar fashion. Also, as a consequence of Hurwitz's bound on the full group of conformal automorphisms, we note that the number of distinct imaginary reflections is finite.

We are interested in finding upper bounds for the number of fixed points of either two or three different imaginary reflections acting on a handlebody with a Schottky structure. As a consequence of Theorem 1 we are able to obtain the following upper bounds (the proof is provided in Section 6). In what follows, $[x]$ denotes the integral part of x.

THEOREM 4. *Let M be a handlebody with a Schottky structure of genus $g \geqslant 2$ admitting two distinct imaginary reflections, τ_1 with n_1 fixed points, and τ_2 with n_2 fixed points. If the order of the dihedral group generated by τ_1 and τ_2 is $2p$, then $n_1 + n_2 \leqslant 2[(g-1)/p] + 4$. Moreover, if $g = 2$, then $n_1 + n_2 = 2$.*

Theorem 4 should be compared with [**3**, theorem 3] which provides a sharp upper bound on the number of ovals that two reflections may have on a symmetric Riemann surface in terms of the genus of the surface and the order of the product of the reflections.

COROLLARY 5. *Let M be a handlebody of genus $g \geqslant 2$, with a Schottky structure, admitting two distinct imaginary reflections, say τ_1 with n_1 fixed points, and τ_2 with n_2 fixed points.*

(1) $n_1 + n_2 \leqslant g + 3$.
(2) *If $n_1 + n_2 = g + 3$, then τ_1 and τ_2 necessarily commute.*
(3) *If $g > 3N + 10$, and τ_1 and τ_2 do not commute, then $n_1 + n_2 \leqslant g - N - 1$.*
(4) *M admits at most one maximal imaginary reflection.*

Proof. Let $2p$ the order of the dihedral group generated by τ_1 and τ_2. Theorem 4 asserts that $n_1 + n_2 \leqslant 2[(g-1)/p] + 4$. Since $g, p \geqslant 2$, $2[(g-1)/p] + 4 \leqslant g + 3$, from which (1) follows.

We observe that τ_1 and τ_2 commute if and only if they generates the Klein group $\mathbb{Z}_2^2$, that is, $p = 2$. Now, if $n_1 + n_2 = g + 3$, then the inequality $n_1 + n_2 \leqslant 2(g-1)/p + 4$ (by Theorem 4) ensures that $p = 2$, obtaining in this way (2).

Now, let us assume that $n_1 + n_2 \geqslant g - N$, for some $N \geqslant 0$, and that τ_1 and τ_2 do not commute. It follows that $p \geqslant 3$ and $g - N \leqslant n_1 + n_2 < 2[(g-1)/p] + 4 \leqslant (2g + 10)/3$, in particular, $g \leqslant 3N + 10$, obtaining (3).

Since $g > 1$, part (4) is a direct consequence of Part (1).

We can contrast Part (4) of Corollary 5, concerning maximal imaginary reflections acting on hyperbolic handlebodies, with maximal symmetries of Riemann surfaces [**12**]. In that paper Natanzon proved that on a non-hyperelliptic Riemann surface there is at most one maximal reflection and, on a hyperelliptic surface, if there is one then there are two.

THEOREM 6. *Let $g \geqslant 3$ be an odd integer and $n_1, n_2 \in N_g = \{0, 2, ..., g-1, g+1\}$ so that $n_1 + n_2 \leqslant g + 3$. Then there is a handlebody (with Schottky structure) of genus g*

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admitting two commuting imaginary reflections, $\tau_1 \neq \tau_2$, where, for $i = 1, 2$, τ_i has n_i fixed points.

The proof of Theorem 6 is provided in Section 8. We remark that, except for the case that $n_1 = n_2 = 0$, τ_1 and τ_2 commute in the examples given.

Remark 7. We expect that a similar result holds for even genus, where, since each imaginary reflection has an odd number of fixed points, the sum of the fixed points is bounded by $g + 2$. We do give examples in Section 8 of handlebodies of every even genus $g \geqslant 2$ admitting two distinct imaginary reflections.

A result similar to Theorem 6, for the case of reflections on Riemann surfaces, was proved in [**8**]. More precisely, they provided conditions for the existence of pairs of distinct reflections on a Riemann surface for all possible sums of the number of their ovals.

COROLLARY 8. *Let τ_1, τ_2, τ_3 be three different imaginary reflections on a handlebody M with a Schottky structure of genus $g \geqslant 2$. If n_j denotes the number of fixed points of τ_j, then*

$$n_1 + n_2 + n_3 \leqslant \begin{cases} 3, & g = 2 \\ g + 5, & g \geqslant 4 \text{ and } g \text{ even} \\ \dfrac{3g + 8}{2}, & g \geqslant 3 \text{ and } g \text{ odd.} \end{cases}$$

The proof of Corollary 8 is provided in Section 7, where sharper upper bounds are given for the inequality depending on the order of the product of the two imaginary reflections. Examples of handlebodies admitting at least three imaginary reflections are given in Section 8, including an example of a handlebody admitting three imaginary reflections with different numbers of fixed points.

Remark 9. In the proof of Corollary 8 we obtain, for the case that g is odd, a more concrete upper bound. If D_r denotes the dihedral group of order $2r$ and H is the group generated by the three imaginary reflections, then

$$n_1 + n_2 + n_3 \leqslant \begin{cases} g + 5, & g \geqslant 5,\ g \text{ odd and } H \ncong \mathbb{Z}_2 \times D_r \text{ for any } r \geqslant 2 \\ \dfrac{(r+1)g + 5r - 1}{r}, & g \geqslant 5,\ g \text{ odd and } H \cong \mathbb{Z}_2 \times D_r \text{ for some } r \geqslant 2. \end{cases}$$

3. *Structure subgroups of finite-extension-Schottky groups*

We say that an extended Kleinian group K is a *finite-extension-Schottky group* of rank g if it is a finite extension of a Schottky group of rank g and every orientation-reversing element of K is either semi-hyperbolic or an imaginary reflection.

If K is an extended Kleinian group, with region of discontinuity Ω, and K^+ is its index two orientation-preserving half, then there is a natural two-fold (possible branched) cover $\pi : S^+ = \Omega/K^+ \to S = \Omega/K$. The Riemann orbifold S^+ admits an anticonformal involution τ. A simple loop $\delta \subset S$ is called orientation-reversing if its lift to S^+ by π is a single τ-invariant loop; otherwise, the loop is called orientation-preserving.

Let K be a finite-extension-Schottky group of rank $g \geqslant 2$, in particular K is non-elementary. As K contains a Schottky group of rank $g \geqslant 2$ as a finite index subgroup, it follows that K has a connected set of discontinuity Ω. Let $S = \Omega/K$. Let Ω^0 be Ω with the fixed points of elliptic elements of K removed, and let S^0 be the projection of Ω^0 to S.

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Let N denote the normal subgroup defining the covering of S^0 by Ω^0. We use the planarity theorem to construct on S^0 a set of simple disjoint loops, $W_1, \ldots, W_w$, and positive integers, $a_1, \ldots, a_w$, so that each of the loops $W_i^{a_i}$ lifts to a loop in Ω^0, and N is the normal closure of the set of (homotopy classes of the) loops $W_1^{a_1}, \ldots, W_w^{a_w}$.

For our purposes here, we must choose these loops with some care as we have to consider three different types of simple loops on S^0. The first class of loops are those defined by small loops around each puncture of S^0. The second class of loops are those corresponding to orientation-reversing loops and the third class of loops are the orientable ones.

First, for each orbifold point on S (these are the removed points of S^0) we choose a small loop about this point (i.e., the loop bounds a punctured disc); then this loop, when raised to an appropriate power lifts to a loop. Call this first set of loops, $U_1, \ldots, U_u$. Note that no U_i itself lifts to a loop. Next, if there is an orientation-reversing loop, which when raised to some power lifts to a loop, then there is a simple such loop, disjoint from all the others. Choose the loops $V_1, \ldots, V_v$ so that they are all orientation-reversing, simple, disjoint and disjoint from all the U_i, and so that they, when raised to appropriate powers, necessarily all even, lift to loops. We next choose a set of loops $X_1, \ldots, X_x$, all simple, disjoint and disjoint from every U_i, and from every V_j, so that these loops, when raised to appropriate powers, all lift to loops, and so that N is the normal closure of (the homotopy classes of) all these loops raised to appropriate powers. We further require that the set of loops, $V_1, \ldots, V_v, X_1, \ldots, X_x$ be minimal; that is, there is no other such set of loops with smaller cardinality.

The lifts of the loops, $U_1, \ldots, U_u, V_1, \ldots, V_v, X_1, \ldots, X_x$, each raised to its appropriate minimal power, are all simple loops; they have been chosen so that every loop in Ω^0 is freely homotopic in Ω^0 to a product of these lifted loops. We next fill in the elliptic fixed points, and remove the small loops about their projections. We obtain that, if we consider only the lifts of the loops of the form $V_1, \ldots, V_v, X_1, \ldots, X_x$, then every loop in Ω is freely homotopic in Ω to a product of these loops. These loops, $V_1, \ldots, V_v, X_1, \ldots, X_x$, are called the *base structure loops*, and the simple loops obtained by lifting them to appropriate (minimal) powers, are called the *structure loops*. The base structure loops divide S into *base structure regions*; similarly, the structure loops divide Ω into *structure regions*.

For a Kleinian group, the stabilizer of a structure loop is necessarily elliptic cyclic, perhaps trivial. For our extended Kleinian group K, a structure loop may also be stabilized by an imaginary reflection. However, a structure loop can be stabilized by at most one imaginary reflection, for the product of two distinct imaginary reflections is always hyperbolic, and the entire structure loop must be contained in Ω. Also, a structure loop cannot be stabilized by both an elliptic element and by an imaginary reflection, for the product of these two would be either a semi-elliptic transformation or a reflection, and our extended function group K does not contain elements of either kind. We conclude that the stabilizer of a structure loop can only be elliptic cyclic, a group of order 2 generated by an imaginary reflection (called an *imaginary reflection group*), or the trivial group.

PROPOSITION 10. *Let R be a structure region with stabilizer H, and let $\Delta(H)$ be the component of the set of discontinuity of H containing R. Then $\Delta(H)$ is simply connected. In particular, every structure subgroup of K is finite.*

Proof. As remarked above, each structure loop on the boundary of R is stabilized by a finite group of elliptic transformations, which is necessarily contained in H, or by an imaginary reflection group, which cannot be contained in H, or by the trivial group. In any case, the structure loop on the boundary of R, along with the outside disc bounded by it, is

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stabilized inside H by a finite group, and hence is contained in $\Omega(H)$. Let $\widetilde{\Omega}$ be the union of R, the structure loops on the boundary of R, and the discs outside R bounded by these structure loops. It follows that $\widetilde{\Omega} \subset \Delta(H)$. Since every loop in R is freely homotopic to some number of structure loops, and each of these is contractible in $\widetilde{\Omega}$, we conclude that $\widetilde{\Omega}$ is simply connected. We also observe that $\widetilde{\Omega}$ is H-invariant, and that $\widetilde{\Omega}/H$ is the orbifold R/H, with discs sewn in along the boundary loops. Hence $\widetilde{\Omega}/H$ is a closed orbifold. It follows that $\widetilde{\Omega}$ is a connected component of the set of discontinuity of H and, in particular, equal to $\Delta(H)$.

PROPOSITION 11. *Every structure subgroup of K is finite, and every structure region has only finitely many structure loops on its boundary.*

Proof. Since $\Lambda(H) \subset \Lambda(K)$, which is a Cantor set, $\Delta(H)$, being simply connected, must be either the sphere or the plane; it cannot be hyperbolic. However, since H contains no parabolics, it must be that $\Delta(H)$ is the sphere, obtaining the H is necessarily finite. It then follows that there can be at most finitely many structure loops on the boundary of the structure region stabilized by H.

A typical structure region can be constructed as follows. Fix any finite group H of conformal/anticonformal automorphisms of $\widehat{\mathbb{C}}$ (there are few of them, up to conjugacy in the Möbius group). Construct a connected domain $R \subset \widehat{\mathbb{C}}$ bounded by a finite number of pairwise disjoint simple loops so that R is invariant under H.

4. *Proof of Theorem* 1

In this section we assume that we have two different Klein–Schottky groups, say G_1 and G_2, both of which are $\mathbb{Z}_2$-extensions of the same Schottky group G^+ of rank $g \geqslant 2$. Set $\widetilde{G} = \langle G_1, G_2 \rangle$, which is a dihedral-Klein–Schottky group of rank g. As $g \geqslant 2$, $\widetilde{G}$ is an extended Kleinian group of the second kind, and contains both G_1 and G_2 as subgroups of finite index, in particular, $\widetilde{G}$ is a finite-extension-Schottky group. It follows that G^+, G_1, G_2 and $\widetilde{G}$ all have the same limit set, and hence the same set of discontinuity, Ω, on $\widehat{\mathbb{C}}$. We set $S^+ = \Omega/G^+$; $S_1 = \Omega/G_1$; $S_2 = \Omega/G_2$; and $\widetilde{S} = \Omega/\widetilde{G}$. Since $\widetilde{G}$ is a finite extension of G^+, and S^+ is a closed Riemann surface, $\widetilde{S}$ is a compact non-orientable orbifold, with a finite number of orbifold points. Since G^+ is a subgroup of index 2 in G_i, $i = 1, 2$, there is an orientation-reversing conformal automorphism τ_i of S^+ so that τ_i lifts to an orientation-reversing element of G_i; that is, $S_i = S^+/\langle \tau_i \rangle$. Since no non-trivial element of G_i has a fixed point in Ω, τ_i acts as a fixed point free automorphism of S^+. If we set $J = \langle \tau_1, \tau_2 \rangle$, which is a group of orientation-preserving and orientation-reversing conformal automorphisms of S^+, then we have that $\widetilde{S} = S^+/J$. Since S^+ is closed of genus $g \geqslant 2$, and $\tau_1 \neq \tau_2$, there is some smallest positive power $p > 1$ so that $(\tau_1 \circ \tau_2)^p = 1$, that is, J is a dihedral group of order $2p$. It follows that every orientation-reversing element of J is conjugate in J to either τ_1 or τ_2, from which it follows that every orientation-reversing element of J acts without fixed points on S^+. Lifting to Ω, we see that every orientation-reversing element of $\widetilde{G}$ acts without fixed points on Ω.

Before proceeding, we need to explicate some properties of dihedral groups. The dihedral group J has order $2p$. If p is odd, then every involution in J is orientation-reversing; but if p is even, then there are orientation-preserving involutions in J. In the particular case that $p = 2$, we have that any two involutions in J generate it.

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LEMMA 12. *The dihedral-Klein–Schottky group $\widetilde{G}$ of rank $g \geqslant 2$ contains no parabolic elements. Every orientation-reversing element of $\widetilde{G}$ is either a semi-hyperbolic or an imaginary reflection. In particular, the Klein orbifold $\widetilde{S}$ is closed; that is, it has no boundary components.*

Proof. The first part follows at once from the fact that $[\widetilde{G}:G^+] < \infty$ and G^+ contains no parabolic elements. The second part is a consequence of the fact that every orientation-reversing element of $\widetilde{G}$ is a lift of a conjugate of either τ_1 or τ_2, both of which are involutions and both of which act without fixed points. It follows that if $a \in \widetilde{G}$ is orientation-reversing, then $a^2 \in G^+$, and a acts without fixed points on Ω.

LEMMA 13. *There is a canonical homomorphism $\Phi: \widetilde{G} \to J$, where the kernel of Φ is the Schottky group G. This canonical homomorphism has the following properties.*

(i) *No elliptic element of $\widetilde{G}$ lies in the kernel of Φ.*

(ii) *Let α and β be involutory generators (both orientation-reversing) of the p dihedral group J; that is $J = \langle \alpha, \beta | \alpha^2 = \beta^2 = (\alpha\beta)^p = 1 \rangle$. Then every element of $\Phi^{-1}(\alpha)$, and every element of $\Phi^{-1}(\beta)$, reverses orientation.*

Proof. The existence of Φ follows at once from standard covering space arguments.

The first statement follows at once from the fact that the kernel of Φ is a Schottky group, which is free. The second statement follows from the fact that if $a_1, a_2 \in \Phi^{-1}(\alpha)$, then $a_1 a_2 \in \mathrm{Ker}\,(\Phi) = G^+$, that is, both are orientation-reversing or both orientation-preserving. As we have assumed that α is orientation-reversing, and every element of the kernel of Φ is orientation-preserving, we must have that every element of $\Phi^{-1}(\alpha)$ is orientation-reversing. The argument is similar for β.

We remark that if we restrict Φ to the orientation-preserving half of $\widetilde{G}$, call it $\widetilde{G}^+$, then we obtain a surjective homomorphism from $\widetilde{G}^+$ onto a cyclic group of order p whose kernel is our Schottky group G^+.

Remark 14. We observe that in Lemma 13 both groups, $\Phi^{-1}(\langle\alpha\rangle)$ and $\Phi^{-1}(\langle\beta\rangle)$, are Klein–Schottky groups; for each of these, its index 2 orientation-preserving half is the Schottky group G^+. Moreover, if we choose $\alpha = \tau_1$ and $\beta = \tau_2$, then these groups are exactly G_1 and G_2. This observation will be used in Section 4·2.

As previously observed, our extended Kleinian group $\widetilde{G}$ is a finite-extension-Schottky group. We assume that we have constructed a minimal set of structure loops and structure regions for $\widetilde{G}$ as in Section 3. We start by deriving some basic properties of these structure loops and regions for our special case.

LEMMA 15. *If H is a structure subgroup of $\widetilde{G}$, then H is either elliptic cyclic, an imaginary reflection group or trivial.*

Proof. Assume that H is non-trivial; we saw in Proposition 11 that it is finite. We also know that there is a homomorphism $\Phi: \widetilde{G} \to D_p = \langle \alpha, \beta: \alpha^2 = \beta^2 = (\alpha\beta)^p = 1 \rangle$, as in Lemma 13. Hence the same is true for H, although of course, $\Phi | H$ need not be surjective. Since H is finite, no non-trivial element of H can lie in the kernel of Φ; i.e., $\Phi: H \to D_p$ is a monomorphism. If H does not contain orientation-reversing elements, then every element of H is mapped into some power of $\alpha\beta$; hence H is cyclic, necessarily elliptic cyclic. In the case that H contains orientation-reversing elements, as H is finite, we have that the only possible orientation-reversing element it can contain is an imaginary reflection. We

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only need to eliminate the possibility that H contains both an imaginary reflection a and an elliptic element b. If there were such elements, then since H is finite, the fixed point of a must lie on the axis of b. If b has order greater than 2, then the product ab is semi-elliptic, which cannot be. If b has order 2, then the product ab is a reflection, which also cannot be.

We now have a first picture of $\Omega(\widetilde{G})$. There are a finite number of conjugacy classes of structure loops; the stabilizer of each structure loop is either an elliptic cyclic group, an imaginary reflection group, or trivial. The structure loops divide $\Omega(\widetilde{G})$ into structure regions; as with the structure loops, the stabilizer of every structure region is either elliptic cyclic, an imaginary reflection group, or trivial. Our next goal is to use the minimality of the set of basic structure loops so as to gain a clearer picture of this structure. We first examine the structure regions with elliptic cyclic stabilizers.

LEMMA 16. *If the stabilizer H of the structure region R is elliptic cyclic, and the structure loop W on the boundary of R is stabilized by the maximal elliptic cyclic group $H' \neq \{I\}$, then $H' = H$.*

Proof. Since H' is elliptic cyclic, it stabilizes both discs bounded by W; hence it stabilizes R, that is, $H' \subset H$; hence H' and H commute. Since they are both maximal, they are equal.

LEMMA 17. *Suppose the structure loop W is stabilized by the elliptic cyclic group H. Then W separates the fixed points of H.*

Proof. Both discs bounded by W are also necessarily stabilized by H; so they each contain a fixed point.

LEMMA 18. *Suppose there is a structure region R stabilized by the elliptic cyclic group H, where there is a fixed point of H in R. Then both fixed points of H lie in R.*

Proof. Assume there is exactly one fixed point of H in R; then there is exactly one structure loop W stabilized by H on the boundary of R. If there were no other structure loops on the boundary of R, then it would be clear that we could just eliminate the projection of W from the set of basic structure loops. If there are other structure loops on the boundary of R, then, since the stabilizer of R is necessarily cyclic, the stabilizer of each of these is either an imaginary reflection group or trivial. We can choose a fundamental set for H of these other structure loops and surround them by a simple loop V, where V projects to a simple loop on the orbifold below. Next connect V to W by a simple arc U, which is also disjoint from all its H translates. Now replace W by W', which is the boundary loop of a regular neighborhood of the union of the H translates of $W \cup V \cup U$, and correspondingly replace the $\widetilde{G}$-translates of W. After this replacement, we have replaced our original set of basic structure loops by another set of basic structure loops of the same cardinality. Now, there is a new structure region containing one fixed point of H, and having exactly one boundary structure loop W', whose projection is clearly redundant. We have shown that our set of basic structure loops could be replaced by another such set of smaller cardinality; this contradicts the minimality of the set of basic structure loops.

LEMMA 19. *If there is a structure region R stabilized by an elliptic cyclic group H, and there are two structure loops, W_1 and W_2, on the boundary of R, each stabilized by H, then W_1 and W_2 are conjugate in $\widetilde{G}$.*

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Proof. Assume that W_1 and W_2 project to different basic structure loops. If there were no other structure loops on the boundary of R, then it would be clear that we could just eliminate the projection of W_2 from the set of basic structure loops, which would contradict the minimality of the set of basic structure loops. If there are other structure loops on the boundary of R, then, since the stabilizer of R is necessarily cyclic, the stabilizer of each of these is either an imaginary reflection group or trivial. As in the previous proof, we choose an appropriate fundamental set for H of these other structure loops and surround them by a simple loop V, which projects to a simple loop. We then connect V to W_2 by a simple arc U, which also projects to a simple arc, disjoint from the projection of V. Now replace W_2 by W_2', which is the boundary loop of a regular neighborhood of the union of the H translates of $W_2 \bigcup V \bigcup U$, and correspondingly replace all the $\widetilde{G}$-translates of W_2. After this replacement, we have replaced our original set of basic structure loops with another such set of the same cardinality. Now, there is a new structure region bounded by just the two H-invariant loops W_1 and W_2', which project to distinct basic structure loops; as above, this contradicts the minimality of the set of basic structure loops.

We next take up the case of structure regions with trivial stabilizers.

LEMMA 20. *If R is a structure region with stabilizer $H = \{I\}$, then the projection map from Ω to S is a homeomorphism on R.*

Proof. If $a \in \widetilde{G}$, then, since a preserves the set of structure loops, $a(R) \bigcap R \neq \varnothing$ if and only if $a(R) = R$, in which case $a \in H$.

LEMMA 21. *If there is a structure loop W, on the common boundary of the structure regions R_1 and R_2, where the stabilizers of W and R_1 are both trivial, then R_1 and R_2 are $\widetilde{G}$-equivalent.*

Proof. Assume that R_1 and R_2 are not $\widetilde{G}$-equivalent. We first observe that, since the stabilizer of R_1 is trivial, no structure loop on the boundary of R_1 can be stabilized by an elliptic cyclic group. We proceed as in the previous proofs by first replacing our minimal set of basic structure loops by a new set of basic structure loops of the same cardinality, and then showing that one of these new basic structure loops can be eliminated, contradicting the minimality of the original set. Suppose the structure loop W_1 on the boundary of R_1 is stabilized by an imaginary reflection group. Draw a loop V in R_1, with base point on W, where V separates W_1 from the other structure loops on the boundary of R_1. Now replace W by the obvious simple loop W' which is homotopic to $W \cdot V$, and correspondingly replace all $\widetilde{G}$-translates of W. After this replacement, we obtain a new set of basic structure loops of the same cardinality as the original set. With this new set of structure loops, we have two new structure regions, R_1', which also has trivial stabilizer, still has a structure loop on its boundary with trivial stabilizer, namely W', and which has one fewer boundary structure loop with non-trivial stabilizer than did R_1. Of course, R_2' has one more such boundary structure loop than did R_2. After a finite number of steps as above, we will achieve that we have a structure region, where every structure loop on its boundary has trivial stabilizer, and there is a structure loop W' on its boundary, where the structure region R_2' on the other side of W' is not $\widetilde{G}$-equivalent to R_1'. It follows that no structure loop on the boundary of R_1' is $\widetilde{G}$-equivalent to W'. Hence the projection of W' is freely homotopic to a product of the projections of the other structure loops on the boundary of R_1', from which it follows that our set of structure loops is not minimal, which is a contradiction.

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LEMMA 22. *If there are structure regions R and R' with trivial stabilizers, then R and R' are $\widetilde{G}$-equivalent.*

Proof. Assume not. There is an obvious distance between structure regions given by the minimal number of structure loops a path between the two structure regions must cross. We assume that V is such a path, and that we have replaced R by a $\widetilde{G}$ equivalent structure region so that the distance between R and R' is minimal. Label the structure regions V passes through as $R_1 = R, R_2, \ldots, R_n = R'$. Also, label the structure loops V crosses in order as $W_1, \ldots, W_{n-1}$. Since the distance between R and R' is minimal, R_1 and R_2 are not $\widetilde{G}$ equivalent; it follows that the stabilizer of W_1 does not contain an imaginary reflection; since the stabilizer of R_1 is trivial, the stabilizer of W_1 cannot be elliptic cyclic. Hence the stabilizer of W_1 is trivial. The desired result now follows from Lemma 21.

As a consequence of all the previous lemmas, the following two results give us a new and clearer picture of the structure of $\widetilde{G}$.

PROPOSITION 23. *If there is a structure region R whose stabilizer is trivial, then every structure region is $\widetilde{G}$-equivalent to R.*

Proof. It suffices to show that if W is a structure loop on the boundary of R separating it from the structure region R', then R and R' are $\widetilde{G}$-equivalent. The stabilizer of W cannot be elliptic, so must either be either trivial or an imaginary reflection group. If the stabilizer of W is trivial, the desired result follows from Lemma 21; if the stabilizer of W contains an imaginary reflection a, then $a(R) = R'$.

PROPOSITION 24. *Suppose R is a structure region with non-trivial stabilizer H. Then either*:

(i) *(first type) H is elliptic cyclic and there are no structure loops on the boundary of R stabilized by H; in this case, both fixed points of H lie in R, and no structure loop on the boundary of R is stabilized by any elliptic element of $\widetilde{G}$;*
(ii) *(second type) H is elliptic cyclic, and there are exactly two $\widetilde{G}$-equivalent structure loops on the boundary of R stabilized by H; in this case, no other structure loop on the boundary of R is stabilized by any elliptic element of $\widetilde{G}$, and no fixed point of any elliptic element lies in R; or*
(iii) *(third type) H is an imaginary reflection group; in this case, no structure loop on the boundary of R is stabilized by any elliptic element of $\widetilde{G}$.*

Proof. We know that H is non-trivial, so it must be either elliptic cyclic or an imaginary reflection group. If H is elliptic cyclic, then by Lemma 18 the number of fixed points of H in R is either 0 or 2. If both fixed points of H lie in R, then we have the first type; if no fixed point of H lies in R, then, by Lemma 19, R is of the second type. For the third type, we have already remarked that if H contains an imaginary reflection, then it cannot also contain an elliptic transformation.

4·1. *Proof of the first part of Theorem* 1

The first part of Theorem 1 is proved by using standard combination theorem techniques, as in [**9**].

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4·1·1. *Case* 1

We first take up the case that every structure subgroup of $\widetilde{G}$ is stabilized only by the identity. Choose a structure region R, and list the structure loops on its boundary as $W_1, W_1', \ldots, W_a, W_a', \widehat{W}_1, \ldots, \widehat{W}_b$, where, for $i = 1, \ldots, a$, there is an element $g_i \in \widetilde{G}$, with $g_i(W_i) = W_i'$, and for $i = 1, \ldots, b$, there is an imaginary reflection h_i mapping $\widehat{W}_i$ onto itself. We first use the second combination theorem (HNN-extension across a trivial subgroup) a times to construct $\widetilde{G}_1 = \langle g_1, \ldots, g_a \rangle$, which is a free group on the a generators, $g_1, \ldots, g_a$. That is, G_1 is a free product of infinite cyclic groups of types S and/or T. We next use the third combination theorem [**10**] b times to construct our final group $\widetilde{G}$, which is the free product of G_1 and the b groups of order 2 generated by $h_1, \ldots, h_b$. We have shown that, in this case, $\widetilde{G}$ is a free product of cyclic groups of types $\mathcal{Q}$, $\mathcal{S}$ and $\mathcal{T}$.

4·1·2. *Case* 2

We next take up the case that there is no structure region stabilized only by the identity. Then every structure region R, with stabilizer H, is of one of the three types listed in Proposition 23. It is clear that we can choose a finite set of structure regions $R_1, \ldots, R_k$, with corresponding structure subgroups, $H_1, \ldots, H_k$, so that the following hold.

(i) Every structure region is $\widetilde{G}$ equivalent to exactly one of $R_1, \ldots, R_k$.
(ii) For every $1 < j \leqslant k$, there is an $i < j$ so that there is a structure loop W_j lying on the boundary of both R_i and R_j.

We need the following lemma in order to continue the proof in this case.

LEMMA 25. *Each of the structure loops, $W_2, \ldots, W_k$ is stabilized only by the identity.*

Proof. Assume that say W_2 is stabilized by a non-trivial element $a \in \widetilde{G}$. Clearly, a cannot be an imaginary reflection, for then we would have $a(R_1) = R_2$. The only other possibility is that a is elliptic, in which case a stabilizes both R_1 and R_2. But if a stabilizes R_1, then by Lemma 18, there is a distinct structure loop W_1', also on the boundary of R_1, and there is an element $b \in \widetilde{G}$, where $b \notin H$, with $b(W_1') = W_1$. Then $b(R_1) = R_2$, which is a contradiction.

Before proceeding with the proof of Theorem 1, we make one more adjustment. For each structure region R_i of the second type, there are two distinct structure loops, V_i and V_i', on the boundary of R_i, and there is an element $a_i \in \widetilde{G}$ with $a_i(V_i) = V_i'$. Let $H_i' = \langle H_i, a_i \rangle$. Using the HNN-extension combination theorem it is easy to see that, in the case that a_i is orientation-preserving, H_i' is of type $\mathcal{U}$, while if a_i is orientation-reversing, H_i' is of type $\mathcal{V}$. In any case, for the next step, we replace H_i with H_i', and replace R_i with the union of all H_i'-translates of R_i, together with the structure loops on its boundary, which we denote by R_i'; note that R_i' is a connected H_i'-invariant set. For the sake of simplicity of notation; if H_i is not of the second type, then we set $H_i' = H_i$ and we set $R_i' = R_i$. It is now an easy exercise in the use of combination theorems to show that $\widehat{G} = \langle H_1', \ldots, H_k' \rangle$ is the free product in the sense of combination theorems, of these k groups. Further if we let $\widehat{R}$ be the union of the $\widehat{G}$ translates of $(R_i' \cup \cdots \cup R_k' \cup W_2 \cup \cdots \cup W_k)$, then $\widehat{R}$ is $\widehat{G}$-invariant, and, modulo $\widehat{G}$, has only a finite number of structure loops on its boundary. We observe that every structure loop stabilized by an elliptic cyclic group is already in the interior of $\widehat{R}$, and so does not lie on its boundary. Hence every structure loop on the boundary of $\widehat{R}$ is stabilized either by an imaginary reflection group or only by the identity.

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If the structure loop W on the boundary of $\widehat{R}$ is stabilized by an imaginary reflection, then we can use the third combination theorem [**10**] to adjoin this imaginary reflection to $\widehat{G}$; we observe that this yields a bigger group, which is a combination theorem free product of $\widehat{G}$ and the imaginary reflection group.

If the structure loop W on the boundary of $\widehat{R}$ is stabilized only by the identity, then let S denote the structure region on the other side of W. Since S is $\widetilde{G}$-equivalent to some structure region in $\widehat{R}$, there is an element $a \in \widetilde{G}$ mapping S onto some structure region in $\widehat{R}$; of necessity, $a(W)$ also lies on the boundary of $\widehat{R}$. Hence, in this case, we can enlarge $\widehat{G}$ to the group generated by $\widehat{G}$ and a; this is the combination theorem free product of $\widehat{G}$ and $\langle a \rangle$. Hence, after a finite number of steps, we arrive at the group $\widetilde{G}$, which we now have shown is a combination theorem free product of groups of the requisite types.

4·2. *Proof of second part of Theorem* 1

A group constructed from the basic groups 1–6 of the first part of Theorem 1, using the Klein–Maskit free product combination theorem, will be called a *general group*. Let us consider a general group G, which is, in particular, a finite-extension group. We need to establish that conditions (i) and (ii) in the second part of Theorem 1 are necessary and sufficient for a general group to be a dihedral-Klein–Schottky group. We start with the following fact (see also Remark 14).

LEMMA 26. *A necessary and sufficient condition for a finite-extension Schottky group K to be a dihedral-Klein–Schottky group is the existence, for some $p \geqslant 2$, of an epimorphism $\Phi\colon K \to D_p = \langle \alpha, \beta : \alpha^2 = \beta^2 = (\alpha\beta)^p = 1 \rangle$, where the kernel of Φ is a Schottky group, and where $\Phi^{-1}(\alpha)$ and $\Phi^{-1}(\beta)$ both contain orientation-reversing elements.*

Proof. The necessity is given in Lemma 13. For the sufficiency, choose $G_1 = \langle \Phi^{-1}(1), \Phi^{-1}(\alpha) \rangle$ and choose $G_2 = \langle \Phi^{-1}(1), \Phi^{-1}(\beta) \rangle$. Note that these are distinct Klein–Schottky groups that are index 2 extensions of the kernel of Φ, which is a Schottky group.

Let us assume that K is the combination theorem free product of q groups of type $\mathcal{Q}$; r groups of type $\mathcal{R}$; s groups of type $\mathcal{S}$; t groups of type $\mathcal{T}$; u groups of type $\mathcal{U}$; and v groups of type $\mathcal{V}$.

4·2·1. *Necessary conditions*

We first prove that the conditions of the second part of Theorem 1 are necessary. The first statement follows from the fact that K must contain orientation-reversing elements; the second follows from the fact that K must contain two orientation-reversing elements lying in distinct cyclic subgroups.

4·2·2. *Sufficient conditions*

To prove the sufficiency of the second part of Theorem 1, we choose p to be the least common multiple of the elliptic elements appearing in the groups of type $\mathcal{R}, \mathcal{U}$ and $\mathcal{V}$. Then, since no elliptic element can lie in the kernel, the image of Φ on any elliptic generator is essentially determined.

If $q+t+v > 1$, divide the groups of types $\mathcal{Q}, \mathcal{T}$ and $\mathcal{V}$ into two non-empty subsets; define Φ on these generators by sending the orientation-reversing generators of the first subset to α, sending the orientation-reversing generators of the second subset to β, and sending every loxodromic generator to the identity.

If $q + t + v = 1$, and $v = 0$, then either $q = 1$ or $t = 1$, and $r + s + u > 0$. As a consequence of Lemma 27, below, we may find an orientation-reversing generator a of a

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group of type $\mathcal{Q}$ or $\mathcal{T}$; define $\Phi(a) = \alpha$. If $r + u > 0$, then there is a product b of elliptic elements so that $\Phi(b) = \alpha\beta$; then $\Phi(ab) = \beta$. If $r + u = 0$, then let b be some generator of a group of type $\mathcal{S}$ and send b to $\alpha\beta$, so that $\Phi(ab) = \beta$. If there are other loxodromic generators, let Φ send them to the identity.

Finally, if $v = 1$, and $q = t = 0$, let a be a semi-hyperbolic generator of the group of type $\mathcal{V}$, and, as above, let b be a product of elliptic elements so that $\Phi(b) = \alpha\beta$. As above, send a to α, so that $\Phi(ab) = \beta$ and send any other loxodromic generators to the identity.

LEMMA 27. *If $q = 0$, then every orientation-reversing element of a general group K is of infinite order; in particular, it is a semi-hyperbolic transformation.*

Proof. If $a \in K$ is of finite order, then a is conjugate in K to some element of one of the generating subgroups. However, no subgroup of type other than Q contains any orientation-reversing element of finite order.

5. *Some consequences of Theorem* 1

In this section we provide some simple consequences of the topological classification of dihedral-Klein–Schottky groups that we will need later.

THEOREM 28. *Let $\widetilde{G}$ be a dihedral-Klein–Schottky group of rank g and $\widetilde{G}^+$ its orientation-preserving half. Assume there exists a surjective homomorphism $\Psi\colon \widetilde{G}^+ \to \mathbb{Z}_p$, with kernel a Schottky group G of rank g. Then $\widetilde{G}^+$ is a combination theorem free product of Abelian groups of the following types:*

(i) *elliptic cyclic groups whose order divides p (groups of type $\mathcal{R}$);*
(ii) *loxodromic (including hyperbolic) cyclic groups (groups of type $\mathcal{S}$);*
(iii) *abelian groups generated by a loxodromic element and by an elliptic element whose order divides p (groups of type $\mathcal{U}$).*

Proof. This can be obtained directly from Theorem 1, but we provide another argument for completeness. We decompose $\widetilde{G}^+$ into structure subgroups, as in the classification of function groups [**9**]. The general theory says that $\widetilde{G}^+$ is obtained as a free product of a Schottky group, which is itself a free product of loxodromic cyclic groups — these are groups of type $\mathcal{S}$, and groups obtained from structure subgroups by amalgamated free product and HNN-extensions across finite subgroups. Since $\widetilde{G}^+$ is a finite extension of a Schottky group, every such structure subgroup must be finite. It follows from the existence of the surjective homomorphism Ψ that every non-trivial structure subgroup must be finite cyclic, with order dividing p; these are groups of type $\mathcal{R}$. Since the structure subgroups are all finite cyclic, we cannot amalgamate any two of them, and we can use an HNN-extension only to conjugate one of these cyclic groups into itself, obtaining a subgroup of type $\mathcal{U}$.

COROLLARY 29. *Let $\widetilde{G}$ be a dihedral-Klein–Schottky group of rank g, and let $\widetilde{G}^+$ be its orientation-preserving half. Assume $\widetilde{G}$ is constructed by using q groups of type $\mathcal{Q}$, r groups of type $\mathcal{R}$, s groups of type $\mathcal{S}$, t groups of type $\mathcal{T}$ and u groups of type $\mathcal{U}$, as in Theorem* 1, *then $\Omega(\widetilde{G}^+)/\widetilde{G}^+$ is an orientable orbifold of genus $\tilde{g} = q + 2(s + t + u + v) - 1$ with $4r$ orbifold points.*

Proof. By Lemma 26, there is a surjective homomorphism $\Phi\colon \widetilde{G} \to D_p$, with kernel a Schottky group G of rank g, where $D_p = \langle \alpha, \beta : \alpha^2 = \beta^2 = (\alpha\beta)^p = 1 \rangle$, so that $G_1 = \langle \Phi^{-1}(1), \Phi^{-1}(\alpha) \rangle$ and $G_2 = \langle \Phi^{-1}(1), \Phi^{-1}(\beta) \rangle$ are distinct Klein–Schottky groups of the same rank, g. We start by constructing a quasiconformally equivalent model of $\widetilde{G}$, which,

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without loss of generality, we again call $\widetilde{G}$. For this model, we have a fundamental domain D for $\widetilde{G}$ bounded by a set of disjoint circles and circular polygons as follows: q disjoint circles, each kept invariant by a group of type $\mathcal{Q}$; r circular bigons, each bounding a fundamental domain containing D for a group of type $\mathcal{R}$; $s+t$ pairs of circles, each pair bounding a fundamental domain containing D for a group of type $\mathcal{S}$ or $\mathcal{T}$; $u+v$ circular quadrilaterals, each bounding a fundamental domain containing D for a group of type $\mathcal{U}$ or $\mathcal{V}$. If $q>0$, we choose one of the circles invariant under a group of type $\mathcal{Q}$; use the generator to imaginary reflect D in that circle; and regroup the identifications so as to obtain a fundamental domain for $\widetilde{G}^+$. We obtain that $\Omega(\widetilde{G}^+)/\widetilde{G}^+$ is an orientable orbifold of genus $q-1+2(s+t+u+v)$ with $4r$ orbifold points. If $q=0$ and $t>0$, we choose one of the pairs of circles for a group of type $\mathcal{T}$, use the generator of this group to map D into the other side of this circle so as to obtain a fundamental domain for $\widetilde{G}^+$.

As above, we see that $\Omega(\widetilde{G}^+)/\widetilde{G}^+$ is an orientable orbifold of genus $2s-1+2(t+u+v)$ with $4r$ orbifold points. If $q=t=0$, and $v>0$, then we choose one of the groups of type $\mathcal{V}$, choose a semi-hyperbolic generator of this group, and map D into the other side of the corresponding fundamental domain. As above, we see that $\Omega(\widetilde{G}^+)/\widetilde{G}^+$ is an orientable orbifold of genus $2v-1+2(s+u)$ with $4r$ orbifold points.

The next example, which is a consequence of Corollary 29, is that of two different imaginary reflections acting on a symmetric Riemann surface S, where there is no Schottky covering of S to which these two imaginary reflections both lift.

Example 1. Consider the orbifold $\widetilde{S}$ with one handle, one crosscap, and three orbifold points of order 3. This has an orbifold fundamental group $\pi_1^{orb}(\widetilde{S})$, which we regard as a group of isometries of the hyperbolic plane $\mathbb{H}^2$, with the following presentation

$$\pi_1^{orb}(\widetilde{S})=\langle A, B, E_1, E_2, E_3, C : ABA^{-1}B^{-1}E_1E_2E_3C^2=E_1^3=E_2^3=E_3^3=1\rangle.$$

Let $D_3=\langle \alpha,\beta:\alpha^2=\beta^2=(\alpha\beta)^p=1\rangle$, the dihedral group of order 6. Define the homomorphism $\Phi:\pi_1^{orb}(\widetilde{S})\to D_3$ by $\Phi(A)=\Phi(B)=1$; $\Phi(E_1)=\Phi(E_2)=\Phi(E_3)=\alpha\beta$; $\Phi(C)=\alpha$. The kernel F has index 6, and is purely hyperbolic, so, by the Riemann–Hurwitz theorem, $S=\mathbb{H}^2/F$ is a closed Riemann surface of genus 10. We now have that D_3 acts on S, where α and β act as imaginary reflections. Of course α and β are conjugate since their product has order 3. However, since $\tilde{S}=S/D_3$ has three orbifold points of order 3, it follows from Corollary 29 that there is no Schottky covering of S to which they both lift.

COROLLARY 30. *Let $\widetilde{G}$ be a dihedral-Klein–Schottky group of rank $g\geqslant 2$ and $\Phi:\widetilde{G}\to D_p$ be a surjective homomorphism with kernel a Schottky group G of rank g. Assume $\widetilde{G}$ is constructed by using q groups of type $\mathcal{Q}$, r groups of type $\mathcal{R}$, s groups of type $\mathcal{S}$, t groups of type $\mathcal{T}$ and u groups of type $\mathcal{U}$, as in Theorem* 1. *Then $S^+=\Omega(G^+)/G^+$ is an orientable surface of genus*

$$g=p(\tilde{g}-1)+1+2\sum\frac{p(r_i-1)}{r_i},$$

where $\tilde{g}$ is the genus of $\widetilde{S}^+=\Omega(\widetilde{G}^+)/\widetilde{G}^+$, and $r_1,\ldots,r_r$ are the orders of the r groups of type R. In particular, g is even if and only if p is odd and $\tilde{g}$ is even.

Proof. This follows from the Riemann–Hurwitz formula, and the fact that $S^+=\Omega(G)/G$ is a (possibly branched) covering of $\widetilde{S}^+$, of order p, with $4r$ branch points. The last part is just a consequence of the fact that $p(r_i-1)/r_i$ is an integer, as r_i is a factor of p.

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A consequence of the previous fact is the following.

COROLLARY 31. *If G_1 and G_2 are both Klein–Schottky groups of even rank $g \geqslant 2$, with a common orientation-preserving half, then G_1 and G_2 are conjugate by some Möbius transformation. In particular, if M is a handlebody with a Schottky structure of even genus $g \geqslant 2$ admitting two distinct imaginary reflections, say τ_1 and τ_2, then the dihedral group generated by τ_1 and τ_2 has order $2p$ with p odd. In this case, if n_j denotes the number of fixed points of τ_j, then $n_1 = n_2$.*

Proof. Let $\widetilde{G}$ be the dihedral-Klein–Schottky group generated by G_1 and G_2, and let G be the Schottky group that is their common orientation-preserving half. Let η_j be the imaginary reflection induced by G_j on the Riemann surface S uniformized by G, $j = 1, 2$. These two imaginary reflections generate a dihedral group of automorphisms of S of order $2p$. As g is even, Corollary 30 asserts that p is odd and, in particular, η_1 and η_2 are conjugate, inside this dihedral group, by some power $(\eta_1\eta_2)^a$. Since η_1 and η_2 both lift to actions on $\Omega(G)$, so does their product. The lifting of the power $(\eta_1\eta_2)^a$ conjugates G_1 into G_2.

Remark 32. In Corollary 31 the condition that the genus be even is necessary as we gave examples of handlebodies of odd genus admitting at least two imaginary reflections with different numbers of fixed points in [**7**]; these are necessarily not conjugate in $Aut(M)$.

6. *Proof of Theorem* 4

We are given a handlebody M of genus $g \geqslant 2$, with a Schottky structure provided by the Schottky group G; $S = \Omega(G)/G$ is the conformal boundary of M; τ_1, with n_1 fixed points, and τ_2 with n_2 fixed points, are distinct imaginary reflections acting on M.

By lifting both imaginary reflections to the universal covering space $\mathbb{H}^3$, we obtain a dihedral-Klein–Schottky group $\widetilde{G}$, which contains G as a normal subgroup. Then $\widetilde{G}/G \cong D_p$, the dihedral group of order $2p$, where p is the order of $\tau_1\tau_2$. Clearly, $\widetilde{G}$ is the (orbifold) universal covering group of $\widetilde{M} = \mathbb{H}^3/\widetilde{G}$. Let $\widetilde{G}^+$ be its orientation-preserving half, which is the (orbifold) universal cover of $\widetilde{M}^+$, and let G_j denote the extension of G by a lift of τ_j, $j = 1, 2$. It is clear from the construction that there is a surjective homomorphism $\Phi\colon \widetilde{G}^+ \to \mathbb{Z}_p$, with kernel G. It follows from Theorem 28 that $\widetilde{G}^+$ is a combination theorem free product of r groups of type $\mathcal{R}$, s groups of type $\mathcal{S}$, and u groups of $\mathcal{U}$. These numbers, r, s and u, are uniquely determined by the group $\widetilde{G}^+$.

We note that τ_1 and τ_2 both define a single imaginary reflection τ acting on $\widetilde{M}^+$; let n denote the number of fixed points of τ. We note that n is the number of G conjugacy classes of fixed points of its imaginary reflections.

LEMMA 33. *The topological structure of the 3-orbifold $\widetilde{M}^+$ is that of a handlebody of genus $\widetilde{g} = s + u$. Likewise, the topological structure of the orbifold $\widetilde{S}^+$ is that of a closed surface of genus $\widetilde{g}$.*

Proof. It follows from the above that $\widetilde{M}^+$ is a boundary connected sum of the three types of orbifolds defined by the Kleinian groups of types $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{U}$.

It is well known, and one easily sees that the 3-orbifold determined by a Kleinian group of type $\mathcal{R}$ is topologically a 3-ball, with a simple, unknotted, properly embedded arc of orbifold points. Similarly, a Kleinian group of type $\mathcal{S}$ determines a 3-manifold that is topologically a

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handlebody of genus 1. A Kleinian group of type $\mathcal{U}$ determines a 3-orbifold that is topologically a handlebody of genus 1, and has an unknotted, homotopically non-trivial circle of orbifold points.

It is clear that one can read off from this information, the topological and orbifold structure of a boundary connected sum of these, from which the result follows.

LEMMA 34. $\widetilde{g} \leqslant [(g-1)/p] + 1$.

Proof. It follows at once from the Riemann–Hurwitz formula that $2p\widetilde{g} = 2g-2+2p-B$, where $B \geqslant 0$ is the total branching. Hence $\widetilde{g} \leqslant (g-1)/p+1$. The desired conclusion follows from the fact that $\widetilde{g}$ is an integer.

LEMMA 35. *If $x \in M$ is a fixed point of τ_1, then x is not a fixed point of $\tau_2 \neq \tau_1$.*

Proof. The lift of τ_1 that fixes a point $\widehat{x}$ lying over x is necessarily an imaginary reflection, which is completely determined by its fixed point. Since G_1 is a $\mathbb{Z}_2$-extension of G, it is completely determined by this lift of τ_1. Since $\tau_1 \neq \tau_2$, no non-trivial element of G_2 can have $\widehat{x}$ as a fixed point.

LEMMA 36. $2n = n_1 + n_2$.

Proof. The group $\Gamma = \langle \tau_1, \tau_2 \rangle$, being a dihedral group of order $2p$, contains exactly p distinct orientation-reversing elements. Lemma 35 asserts that distinct imaginary reflections have distinct fixed points. It follows from this that an element of G can map a fixed point of τ_1 to another fixed point of τ_1 if and only if it conjugates τ_1 into itself. One of the basic facts about dihedral groups is that there is such an element if and only if p is even, and then this element is unique. If p is odd, then $n_1 = n_2$, as every orientation-reversing element of Γ is conjugate to τ_1, and no element of Γ conjugates τ_1 onto itself. Hence, in this case, $n = n_1 = (n_1 + n_2)/2$. If p is even, then there is a unique oientation-preserving non-trivial element of Γ that conjugates τ_1 into itself and conjugates τ_2 into itself. This element permutes the fixed points of both τ_1 and τ_2, while fixing none of them. Hence, in this case half the fixed points of each of them project to different fixed points of τ; i.e., $n = (n_1 + n_2)/2$.

LEMMA 37. $\widetilde{g} \geqslant n - 1$.

Proof. Since τ has n distinct fixed points on $\widetilde{M}^+$, the orbifold $\widetilde{M}$ has exactly n conical points; in particular, there must be n non-conjugate elements of $\widetilde{G}$ having isolated fixed points in $\mathbb{H}^3$. Of the groups listed in Theorem 1, only groups of type $\mathcal{Q}$ have isolated fixed points in the interior of $\mathbb{H}^3$. It follows that $\widetilde{G}$ must contain n non-conjugate groups of type $\mathcal{Q}$; from which it follows that the free product decomposition of $\widetilde{G}$ must contain at least n distinct imaginary reflection groups. It follows from this that the free product decomposition of $\widetilde{G}^+$ contains at least $n-1$ groups of type $\mathcal{S}$, and then the desired result follows from Lemma 33.

Now we are ready to finish the proof of Theorem 4. Combining Lemmas 34 and 37, we obtain $n \leqslant \widetilde{g} + 1 \leqslant [(g-1)/p] + 2$, and, in particular, by Lemma 36,

$$n_1 + n_2 = 2n \leqslant 2[(g-1)/p] + 4.$$

Moreover, if $g = 2$, it follows from Corollory 30 that the product, $\tau_1\tau_2$, has odd order, from which it follows that τ_1 and τ_2 are conjugate; in particular, $n_1 = n_2$. Also, we know that n_1 can only be one of the numbers $\{1, 3\}$. As $n_1 + n_2 \leqslant 4$, a consequence of the above inequality, $n_1 = n_2 = 1$.

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7. *Proof of Corollary* 8

We are given a handlebody M, with a Schottky structure provided by a Schottky group G of genus $g \geqslant 2$, and we are given three distinct imaginary reflections, τ_1, τ_2 and τ_3 on M. Set $H = \langle \tau_1, \tau_2, \tau_3 \rangle$ and let H^+ be its index two subgroup of orientation-preserving isometries.

The group H^+ is generated by $a = \tau_1\tau_2$ and $b = \tau_2\tau_3$. Let p be the order of a, q be the order of b and r be the order of $c = ab$. Up to permutation of indices, there is no problem in assuming the inequality $2 \leqslant p \leqslant q \leqslant r$. Let n_j be the number of fixed points of τ_j. We already noted in Section 2 that $n_j \in N_g$. It follows from Theorem 4 that

$$n_1 + n_2 \leqslant \frac{2(g-1)}{p} + 4,\ n_2 + n_3 \leqslant \frac{2(g-1)}{q} + 4,\ n_1 + n_3 \leqslant \frac{2(g-1)}{r} + 4 \quad (*)$$

and, for g even, $n_1 = n_2 = n_3$ and $p, q, r \geqslant 3$ are odd integers. We will also need the fact that no handlebody can admit more than one maximal imaginary reflection; that is, an imaginary reflection with the maximal number, $g + 1$, of fixed points.

7·1. *Case* $\mathbf{g \geqslant 2}$ *even*

In this case, $p, q, r \geqslant 3$ are odd integers and $n_1 = n_2 = n_3 = n$. It follows from $(*)$ that $2n \leqslant 2(g-1)/3 + 4$, that is, $n_1 + n_2 + n_3 = 3n \leqslant g + 5$. This takes care of the general case of even g.

In the special case that $g = 2$, we have already seen in Theorem 4 that if a handlebody of genus 2 admits two distinct imaginary reflections, they must each have exactly one fixed point. Hence, if it admits three distinct imaginary reflections, they must each have exactly one fixed point. The (essentially unique) example for genus 2, where the sum of the fixed points of the three imaginary reflections is equal to 3, appears in Section 8, Family 8, with $q = 1$.

Remark 38. We see from the above that in order to attain the upper bound $n_1 + n_2 + n_3 = g + 5$, for $g \geqslant 4$ even, we must have $p = q = r = 3$ and $n = (g+5)/3$. In Section 8, Family 7, we construct examples for this extremal situation.

7·2. *Case* $\mathbf{g \geqslant 3}$ *odd*

In this case $(*)$ asserts that

$$n_1 + n_2 + n_3 \leqslant \left(p^{-1} + q^{-1} + r^{-1}\right)(g-1) + 6. \quad (**)$$

It follows that if $(p^{-1} + q^{-1} + r^{-1}) \leqslant 1$, then we get $n_1 + n_2 + n_3 \leqslant g + 5$ as desired. We now only need to consider the following cases:

(i) $p = 2, q = 3, r \in \{3, 4, 5\}$;

(ii) $p = q = 2$ and $r \geqslant 2$.

7·2·1. *Case* $p = 2$, $q = 3$ *and* $r \in \{3, 4, 5\}$

The condition $(\tau_2\tau_3)^3 = 1$ asserts that τ_2 and τ_3 are conjugated, that is, $n_2 = n_3$. In this way, we need to find an upper bound for $n_1 + 2n_2$. As $2n_2 = n_2 + n_3 \leqslant 2(g-1)/3 + 4$, we have that $n_2 \leqslant (g-1)/3 + 2$. Adding this last inequality with $n_1 + n_3 \leqslant 2(g-1)/r + 4$ gives us $n_1 + n_2 + n_3 \leqslant g + 5$. Note that in this case, no involution in H belongs to the center of H; in particular, H cannot be isomorphic to the direct product of $\mathbb{Z}_2$ with any other subgroup.

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7·2·2. *Case* $p = q = 2$

In this case $(*)$ asserts that

$$n_1 + n_2 + n_3 \leqslant \frac{(1+r)g + 5r - 1}{r}$$

so we are done. In this case, $H \cong \mathbb{Z}_2 \times D_r$.

8. *Examples*

In this section we give examples of handlebodies admitting two, and sometimes three distinct imaginary reflections. We give several examples of handlebodies admitting two imaginary reflections with different numbers of fixed points; in particular, we prove Theorem 6. We also give examples showing that some of our inequalities are sharp. We first exhibit several families of examples, and a couple of exceptional examples; we summarize our results at the end of the section.

In each of the following, except for the last exceptional example, we will construct a dihedral-Klein–Schottky group $\widetilde{G}$, and describe it using the notation of Theorem 1. That is, $\widetilde{G}$ will be a combination theorem free product of q groups of order 2, generated by the imaginary reflections, $Q_1, \ldots, Q_q$, $q \geqslant 0$; q' groups of order 2, generated by the imaginary reflections, $Q'_1, \ldots, Q'_{q'}$, $0 \leqslant q' \leqslant q$; s infinite cyclic groups generated by the loxodromic transformations, $S_1, \ldots, S_s$, $s \geqslant 0$; and t infinite cyclic groups generated by the semi-hyperbolic transformations, $T_1, \ldots, T_t$, $t \geqslant 0$; and perhaps a finite cyclic group of order p, generated by the elliptic transformation R, $p \geqslant 2$. We will not need quite this much generality, so in each set of examples, we will have either $s = 0$ or $t = 0$. In each case, it is easy to draw a fundamental domain bounded by disjoint circles; if there is an elliptic element R of order p, then its boundary component consists of two arcs of circles meeting at an angle of $2\pi/p$.

In each case, again except for the last exceptional example, we construct an epimorphism $\Phi\colon \widetilde{G} \to D_p$, the dihedral group of order $2p$, generated by the involutions α and β. It is also easy to see that Φ maps every non-loxodromic element to a non-trivial element of D_p in such a way that the kernel of Φ consists only of loxodromic elements, from which it follows that the kernel is a Schottky group. We set $G_1 = \Phi^{-1}(\langle\alpha\rangle)$ and $G_2 = \Phi^{-1}(\langle\beta\rangle)$; these are distinct Klein–Schottky groups. Using the fundamental domain described above, and the translates of its boundary curves under appropriate elements of $\widetilde{G}$, we write down explicit generators for G_1 and G_2.

In each case, G_1 and G_2 have the same orientation-preserving half, G^+, which is the kernel of Φ. For $i = 1, 2$, G_i induces an imaginary reflection τ_i on the handlebody of genus g defined by G^+. We denote the number of fixed points of τ_i by n_i.

We recall that if $p > 2$, then, for example, $G_3 = \Phi^{-1}(\langle\alpha\beta\alpha\rangle)$ induces another imaginary reflection on the handlebody defined by G^+. If p is odd, there are then p such imaginary reflections, all distinct, and all having the same number of fixed points.

We note that the generarators we write down are Klein–Schottky generators; that is, for example, G_1 is the free product of the groups generated by these generators written down. It follows that the number of fixed points of say τ_1 is equal to the number of imaginary reflections among these generators.

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8·1. *Family* 1: $p = 2$, *R not present,* $q = q' = 0$, $s = 0$ *and* $t > 1$

Here we set

$$\Phi(T_1) = \alpha,\ \Phi(T_2) = \cdots = \Phi(T_t) = \beta.$$

Then

$$G_1 = \langle T_1, T_2T_1T_2^{-1}, T_2^2, T_3T_2, T_3^{-1}T_2, \ldots, T_tT_2, T_t^{-1}T_2\rangle,$$

$$G_2 = \langle T_2, \ldots, T_t, T_1^2, T_1T_2T_1, \ldots, T_1T_tT_1\rangle.$$

In this example, $n_1 = n_2 = 0$, and $g = 4t - 3$. Note that, since $p = 2$, τ_1 and τ_2 commute. Also note that $g \geqslant 5$.

8·2. *Family* 2: $p = 2$, *R has order* 2, $q = q' = 0$, $s = 0$, $t > 0$

Here we set

$$\Phi(T_1) = \cdots = \Phi(T_t) = \alpha,\ \Phi(R) = \alpha\beta.$$

Then

$$G_1 = \langle T_1, \ldots, T_t, RT_1R, \ldots, RT_tR\rangle,$$

$$G_2 = \langle RT_1, RT_1^{-1}, \ldots, RT_t, RT_t^{-1}\rangle.$$

Here, $n_1 = n_2 = 0$, and $g = 4t - 1$. Note that τ_1 and τ_2 commute, and that $g \geqslant 3$.

8·3. *Family* 3: $p = 2$, *R not present,* $q' = 0$, $q > 0$, $s = 0$, $t > 0$

We set

$$\Phi(Q_1) = \cdots \Phi(Q_q) = \alpha,\ \Phi(T_1) = \cdots = \Phi(T_t) = \beta.$$

Then

$$G_1 = \langle Q_1, \ldots, Q_q, T_1Q_1T_1^{-1}, \ldots, T_1Q_qT_1^{-1}, T_1^2, T_1T_2, T_1T_2^{-1}, \ldots, T_1T_t, T_1T_t^{-1}\rangle$$

$$G_2 = \langle Q_1Q_2, \ldots, Q_1Q_q, T_1, \ldots, T_t, Q_1T_1Q_1, \ldots, Q_1T_tQ_1\rangle.$$

Here $n_1 = 2q$, $n_2 = 0$, and $g = 2q + 4t - 3$. Here again, $p = 2$, so τ_1 and τ_2 commute. Note that $g + 3 - n_1 - n_2 = 4t \geqslant 4$.

8·4. *Family* 4: $p = 2$, *R has order* 2, $q' = 0$, $q > 0$, $s \geqslant 0$, $t = 0$

Set

$$\Phi(Q_1) = \cdots = \Phi(Q_q) = \alpha,\ \Phi(R) = \alpha\beta,\ \Phi(S_1) = \Phi(S_s) = 1.$$

Then

$$G_1 = \langle Q_1, \ldots, Q_q, RQ_1R, \ldots, RQ_qR, S_1, \ldots, S_s, RS_1R, \ldots, RS_sR\rangle,$$

$$G_2 = \langle RQ_1, \ldots, RQ_q, S_1, \ldots, S_s, RS_1R, \ldots, RS_sR\rangle.$$

Here, $n_1 = 2q$, $n_2 = 0$, and $g = 2q + 4s - 1$. Note that τ_1 and τ_2 commute, and that $g + 3 - n_1 - n_2 = 4s + 2 \geqslant 2$.

8·5. *Family* 5: $p = 2$, *R not present,* $0 < q' \leqslant q$, $s \geqslant 0$, $t = 0$

Set

$$\Phi(Q_1) = \cdots = \Phi(Q_q) = \alpha,\ \Phi(Q_1') = \cdots = \Phi(Q_{q'}') = \beta\ \Phi(S_1) = \cdots = \Phi(S_s) = 1.$$

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Then

$$G_1 = \langle Q_1, \dots, Q_q, Q_1' Q_1 Q_1', \dots, Q_1' Q_q Q_1', Q_1' Q_2', \dots, Q_1' Q_{q'}', S_1, \dots, S_s, Q_1' S_1 Q_1', \dots, Q_1' S_s Q_1' \rangle,$$

$$G_2 = \langle Q_1', \dots, Q_{q'}', Q_1 Q_1' Q_1, \dots, Q_1 Q_{q'}' Q_1, S_1, \dots, S_s, Q_1 S_1 Q_1, \dots, Q_1 S_s Q_1 \rangle.$$

Here $n_1 = 2q$, $n_2 = 2q'$, and $g = 2q + 2q' + 4s - 3$. Note that τ_1 and τ_2 commute, and that $g + 3 - n_1 - n_2 = 4s \geqslant 0$.

8·6. *Family* 6: $p = 2$, *R has order* 2, $0 < q' \leqslant q$, $s \geqslant 0$, $t = 0$

Set

$$\Phi(Q_1) = \cdots = \Phi(Q_q) = \alpha,\ \Phi(Q_1') = \cdots = \Phi(Q_{q'}') = \beta,\ \Phi(R) = \alpha\beta,\ \Phi(S_1) = \cdots = \Phi(S_s) = 1.$$

Then

$$G_1 = \langle Q_1, \dots, Q_q, RQ_1R, \dots, RQ_qR, RQ_1' \dots, RQ_{q'}', S_1, \dots, S_s, RS_1R, \dots, RS_sR \rangle,$$

$$G_2 = \langle Q_1', \dots, Q_{q'}', RQ_1'R, \dots, RQ_{q'}'R, RQ_1, \dots, RQ_q, S_1, \dots, S_s, RS_1R, \dots, RS_sR \rangle.$$

Here $n_1 = 2q$, $n_2 = 2q'$, and $g = 2q + 2q' + 4s - 1$. Note that τ_1 and τ_2 commute and that $g + 3 - n_1 - n_2 = 2s + 2 \geqslant 2$.

8·7. *Family* 7: $p = 3$, *R not present*, $q' = 1$, $q > 0$, $s \geqslant 0$, $t = 0$

Since $q' = 1$, we set $Q' = Q_1'$. Set

$$\Phi(Q_1) = \cdots = \Phi(Q_q) = \alpha,\ \Phi(Q') = \beta,\ \Phi(S_1) = \cdots = \Phi(S_s) = 1.$$

Then

$$G_1 = \langle Q_1, \dots, Q_q, (Q'Q_1)Q'(Q'Q_1)^{-1}, Q'Q_1Q_iQ', i = 2, \dots, q, S_j, Q'S_jQ', j = 1, \dots, s \rangle,$$

$$G_2 = \langle Q', Q_2Q_1, \dots, Q_qQ_1, (Q_1Q')Q_i(Q_1Q')^{-1}, i = 1, \dots, q, S_j, Q'S_jQ', j = 1 \dots, s \rangle.$$

Here $n_1 = n_2 = q + 1$, and $g = 3q - 2 + 4s$. We remark that we do not gain any new examples by having $q' > 1$. We also note that g is even if and only if q is even. Here $p = 3$, so $\Phi^{-1}(\langle \alpha\beta\alpha \rangle)$ yields a third imaginary reflection acting on $\mathbb{H}^3/G^+$, with the same number, $n = n_1 = n_2$, of fixed points. If q is even, so g is even, and $s = 0$, then $3n = 3q + 3 = g + 5$, which is the maximal number possible.

8·8. *Family* 8: $p = 3$, *R has order* 3, $q' = 0$, $q > 0$, $s \geqslant 0$, $t = 0$

Set

$$\Phi(Q_1) = \cdots = \Phi(Q_q) = \alpha,\ ,\ \Phi(R) = \alpha\beta,\ \Phi(S_1) = \cdots = \Phi(S_s) = 1.$$

Then

$$G_1 = \langle Q_1, \dots, Q_q, RQ_1R, \dots, RQ_qR, S_1, \dots, S_s, RS_1R^2, \dots, RS_sR^2, R^2S_1R, \dots, R^2S_sR \rangle,$$

$$G_2 = \langle RQ_1R^2, \dots, RQ_qR^2, R^2Q_1, \dots, R^2Q_q, S_1, \dots, S_s, RS_1R^2, \dots, RS_sR^2, R^2S_1R, \dots, R^2S_sR \rangle.$$

Here $n_1 = n_2 = q$, and $g = 3q - 1 + 6s$.

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8·9. *Family* 9: $p = 2m + 1$, *R has order* p, $q' = 0$, $q > 0$, $s = t = 0$

Set

$$\Phi(R) = \alpha\beta,\ \Phi(Q_1) = \cdots = \Phi(Q_q) = \alpha.$$

Then

$$G_1 = \langle Q_i, R^j Q_i R^j, i = 1, \ldots, q, j = 1, \ldots, m\rangle,$$

$$G_2 = \langle R^j Q_i R^{j+1}, i = 1, \ldots, q, j = 0, \ldots, m\rangle.$$

Here $n_1 = n_2 = q$, and $g = (2m + 1)q - 1$. Note that $R^m Q_i R^{m+1} = R^m Q_i R^{-m}$.

8·10. *Exceptional example* 1: $p = 3$, *R is not present,* $q' = 0$ $q = 1$, $s = 1$, $t = 0$

Since $q = 1$ and $s = 1$, we set $Q_1 = Q$ and $S_1 = S$. Set

$$\Phi(Q) = \alpha,\ \Phi(S) = \alpha\beta.$$

Then

$$G_1 = \langle Q, SQS, S^3\rangle,$$

$$G_2 = \langle SQS^{-1}, SQ, S^3\rangle.$$

Here $n_1 = n_2 = 1$ and $g = 4$.

8·11. *Exceptional example* 2: *three imaginary reflections with* $n_1 < n_2 < n_3$

Let Σ_0 be the unit circle $\{|z| = 1\}$. Choose 3 pairwise disjoint circles, Σ_1, Σ_2, Σ_3, each orthogonal to Σ_0 and all three bounding a domain of connectivity 3. Let a denote the reflection in Σ_0 and let b_j denote the imaginary reflection that preserves Σ_j, interchanges both discs bounded by Σ_j and commutes with a. Set $K = \langle a, b_1, b_2, b_3\rangle$. The extended Kleinian group K uniformizes the closed disc with exactly 3 conical points of order 2 in its interior.

Consider the surjective homomorphism

$$\Phi{:}K \to \mathbb{Z}_2^3 = \langle \alpha, \beta, \gamma\rangle$$

defined by $\Phi(a) = \alpha\beta\gamma$, $\Phi(b_1) = \beta$, $\Phi(b_2) = \Phi(b_3) = \gamma$. Set

$$K_1 = \Phi^{-1}(\langle\gamma\rangle) = \langle b_2, b_3, b_1b_2b_1, b_1b_3b_1\rangle$$

$$K_2 = \Phi^{-1}(\langle\beta\rangle) = \langle b_1, b_2b_1b_2, b_2b_3\rangle$$

and

$$K_3 = \Phi^{-1}(\langle\alpha\rangle) = \langle ab_1b_2, ab_1b_3\rangle.$$

The groups K_1, K_2, K_3 are Klein–Schottky groups of genus 3 and types (4, 0), (2, 1) and (0, 2), respectively. Each of them has index 4 in K. It is not difficult to see that $K_1^+ = K_2^+ = K_3^+$; this is the kernel of Φ, which is a Schottky group of genus 3. Then K_1 induces an imaginary reflection on the underlying handlebody with four fixed points; K_2 induces an imaginary reflection with 2 fixed points, and K_3 induces an imaginary reflection with no fixed points.

8·12. *Proof of Theorem* 6

We first take up the case that $g \geqslant 3$ is odd. We observe that Families 1, 3 and 5 cover the cases that $g + 3 - n_1 - n_2 \equiv 0 \bmod 4$, while Families 2, 4 and 6 cover the cases that

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$g + 3 - n_1 - n_2 \equiv 2 \bmod 4$. Families 1 and 2 give examples where neither imaginary reflection has any fixed points. Families 3 and 4 give examples where exactly one of the imaginary reflections has no fixed points; in this case, the imaginary reflection with fixed points can have at most $g + 1$ fixed points, so all possible cases are covered. Families 5 and 6 give examples where both imaginary reflections have fixed points; here again, we easily see that all possible cases are covered. This completes the proof of Theorem 6.

8·13. *Even genus*

Family 7 provides some examples, for even and odd genus, of handlebodies admitting two non-commuting imaginary reflections; in these examples, both imaginary reflections have the same number of fixed points. Family 9, with $q = 1$, shows that there is at least one handlebody admitting two different imaginary reflections for every even genus $\geqslant 2$.

We expect that, as with the case of odd genus, every possible example exists. That is, we expect that for every even $g \geqslant 2$, and for every odd positive integer $q \leqslant (g + 2)/2$, there is a handlebody of genus g admitting two distinct imaginary reflections, each with exactly q fixed points.

There is only one possible example for genus 2; it is provided by Family 8 with $q = 1$ and $s = 0$. There are 3 possible examples for genus 4; one of these is provided by Family 7 with $q = 2$ and $s = 0$. The second example is provided by Family 9 with $p = 5$ and $q = 1$. Exceptional Example 1 provides the third example.

8·14. *Handlebodies admitting three distinct imaginary reflections*

We note that Families 7, 8 and 9 yield examples of handlebodies admitting three distinct imaginary reflections for every genus $g \geqslant 2$. In these examples, the three imaginary reflections all have the same number of fixed points. Exceptional Example 3 is a handlebody of genus 3 admitting imaginary reflections with 0, 2 and 4 fixed points.

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