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SRB measures and the fiber leaf condition

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Abstract. In this paper, we study (Sinai–Ruelle–Bowen) SRB measures for random dynamical systems and their associated skew-product transformations. Extending Burguet's variational method [SRB measures for C^∞ surface diffeomorphisms. *Invent. Math.* **235**(3) (2024), 1019–1062], we establish the existence of random SRB measures under a fiber leaf condition, which ensures recurrent hyperbolic behavior along unstable leaves within generic fibers. The construction is based on geometric-time techniques and Følner averaging, leading to invariant measures that satisfy the relative Pesin entropy formula. We further identify a spectral-gap condition relating the Lyapunov spectra of the base and fiber dynamics, under which the random SRB measure coincides with an SRB measure for the smooth skew-product system.

Key words: random dynamics, smooth dynamics, classical ergodic theory
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1. *Introduction*

1.1. *Background and motivations.*

Sinai–Ruelle–Bowen (SRB) measure is a fundamental concept in the ergodic theory of smooth dynamical systems. The significance is underscored by a suite of remarkable geometric, variational, and physical properties. From the measure-theoretic viewpoint, the measure-theoretical entropy of an SRB measure of a smooth dynamical system is equal to the sum of positive Lyapunov exponents by Pesin's entropy formula [17]. Geometrically, SRB measures possess smooth conditional densities along unstable manifolds [13]. Orbits that start in close proximity in the unstable direction will diverge over time, and the smooth density of the SRB measure on the unstable manifolds enables a detailed analysis of how these orbits disperse and interact with the overall dynamics of the system. Another crucial aspect of SRB measures is that every hyperbolic ergodic SRB measure is a physical measure [22], capturing the asymptotic statistics of Lebesgue typical orbits. These properties have far-reaching implications in the study of non-uniformly hyperbolic dynamics.

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The concept of SRB measures originated from the classical works of Sinai, Ruelle, and Bowen [6, 23, 26], who established equivalent characterizations and the existence results for SRB measures in the context of Anosov and Axiom A systems. Their contributions opened a fundamental question: *under what general conditions does a smooth system admit SRB measures?* Beyond uniform hyperbolicity, the existence problem becomes far more delicate.

In his talk at ICM in 1998, Viana proposed a celebrated conjecture asserting that any $C^{1+\alpha}$ diffeomorphism that has only non-zero Lyapunov exponents at Lebesgue-almost every (a.e.) point should admit SRB measures. This conjecture has motivated a great deal of research, with partial affirmative results being obtained for different types of systems. Although Viana's conjecture remains open in its full generality, progress has been made under additional conditions. For instance, in the partially hyperbolic setting, [2, 3] prove the existence of SRB measures for partially hyperbolic systems with strongly and weakly almost expanding centers, respectively.

Another condition for the existence of SRB measures in higher-dimensional non-uniform hyperbolicity systems is known as the leaf condition in [4, 11, 12]. This condition, in a related form involving effectively hyperbolic points, was first introduced by Climenhaga *et al* [11]. In the surface setting, Climenhaga *et al* [12] obtained a necessary and sufficient leaf condition for the existence of SRB measures for $C^{1+\alpha}$ diffeomorphisms via the construction of Young towers. Ben Ovadia [4] later established an arbitrary-dimensional analog by proving that the leaf condition introduced in [12] implies the existence of a hyperbolic SRB measure for $C^{1+\alpha}$ diffeomorphisms in any dimension. His proof is based on Sarig's countable Markov coding [25] and shows that, under the leaf condition, the corresponding geometric potential is positive recurrent. The existence of an SRB measure then follows.

Recently, Burguet [7] resolved Viana's conjecture for C^∞ surface diffeomorphisms and also established closely related results for C^r surface diffeomorphisms, while Buzzi *et al* [10] independently obtained the same result by using the method in [9]. More recently, Ben Ovadia and Burguet [5] further developed the methods of [7] and made further progress on Viana's conjecture in arbitrary dimensions. They showed that every C^∞ diffeomorphism for which the set of points with positive kth exponent has positive volume admits a k-dimensional generalized Gibbs u-measure. Moreover, if the system also satisfies suitable assumptions on the negative exponents, then the generalized Gibbs u-measure is an SRB measure.

Burguet's method in [7] provides a new approach to constructing invariant measures by using a set of geometric times and the associated geometric sets. This technique allows one to verify that the resulting measure has positive Lyapunov exponents almost everywhere, and that its entropy equals the sum of its positive Lyapunov exponents. In [7], Burguet strongly proposed that his approach can be used to recover the existence of SRB measures for weakly mostly expanding partially hyperbolic systems and for $C^{1+\alpha}$ diffeomorphisms with leaf condition in any dimension. Later, Burguet and Yang in [8] have made this idea a reality for weakly mostly expanding partially hyperbolic systems. We now apply Burguet's method to show the existence of SRB measures under the leaf condition, not only for diffeomorphisms in arbitrary dimensions, but also for random dynamical systems and their associated skew products.

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Burguet's proof in the C^∞ surface setting contains a substantial Yomdin-theoretic component. More precisely, Yomdin theory is used in two essential ways: first, to obtain geometric times with positive density and, second, to estimate the volume of the corresponding Bowen balls. In our setting, however, Yomdin theory is no longer available due to the limited $C^{1+\alpha}$ regularity. Nevertheless, the positive-density geometric times follow directly from the leaf condition. We then follow Burguet's strategy to construct an invariant measure and establish a Misiurewicz-type entropy argument for the resulting averaged measures. The remaining ingredient, namely the required estimates on the volumes of Bowen balls, is obtained from non-uniform hyperbolic theory (see Proposition 3.2).

Our adaptation is entirely fiberwise and applies to C^2 fiber diffeomorphisms on compact manifolds of arbitrary dimension $\dim M \geq 2$. In particular, neither the surface assumption nor C^∞ regularity is required.

1.2. *Basic settings and main results.* The goal of this paper is to establish the existence of random SRB measures for skew-product transformations satisfying a fiber leaf condition—a natural random analog of the deterministic leaf condition, by extending Burguet's variational framework to the random dynamical setting.

Let Ω be a compact metric space and $\mathcal{M}$ be a compact smooth Riemannian manifold with $\dim \mathcal{M} \geq 2$. Suppose that $\theta : \Omega \to \Omega$ is a homeomorphism and we assume that $f : \Omega \to \mathrm{Diff}^2(\mathcal{M})$ is a continuous mapping. Let $\mathbb{P}$ be a given θ-invariant Borel probability measure on Ω.

We consider the following skew product transformation $F : \Omega \times \mathcal{M} \to \Omega \times \mathcal{M}$ by $F(\omega, x) = (\theta\omega, f_\omega(x))$ for any $(\omega, x) \in \Omega \times \mathcal{M}$. We denote

$$f_\omega^n = \begin{cases} f_{\theta^{n-1}\omega} \circ \cdots \circ f_\omega & \text{if } n > 0, \\ id_{\mathcal{M}} & \text{if } n = 0, \\ (f_{\theta^n\omega})^{-1} \circ \cdots \circ (f_{\theta^{-1}\omega})^{-1} & \text{if } n < 0. \end{cases} \tag{1.1}$$

Under this setting, let μ be an F-invariant probability Borel measure on $\Omega \times \mathcal{M}$, whose marginal on Ω is $\mathbb{P}$. Assume the following integrability condition holds:

$$\int \log^+ |D_x f_\omega^{\pm}| \, d\mu(\omega, x) < +\infty,$$

where $\log^+ a := \max\{\log a, 0\}$. By [15, Theorem 2.2.1], a reformulation of the celebrated Oseledets multiplicative ergodic theorem [21, 24], there exists a measurable set $\Delta_0 \subset \Omega \times \mathcal{M}$ such that $\mu(\Delta_0) = 1$ and for each $(\omega, x) \in \Delta_0$, there are numbers

$$-\infty < \lambda^{(1)}(\omega, x) < \lambda^{(2)}(\omega, x) < \cdots < \lambda^{r(\omega,x)}(\omega, x) < +\infty$$

depending measurably on (ω, x) and a unique splitting of $T_x\mathcal{M}$,

$$T_x\mathcal{M} = E^1(\omega, x) \oplus E^2(\omega, x) \oplus \cdots \oplus E^{r(\omega,x)}(\omega, x)$$

satisfying

$$\lim_{n \to \pm\infty} \frac{1}{n} \log |D_x f_\omega^n \xi| = \lambda^{(i)}(\omega, x) \quad \text{for all } \xi \in E^{(i)}(\omega, x) \backslash \{0\}, \ 1 \leq i \leq r(\omega, x).$$

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Each $\lambda^{(i)}(\omega, x)$ is called a Lyapunov exponent of the random diffeomorphism and $m^{(i)}(\omega, x) := \dim E^{(i)}(\omega, x)$ is called its multiplicity.

In this paper, we characterize the random SRB measure by using Pesin's entropy formula for RDS. This definition turns out to be equivalent to having absolutely continuous conditional measures on W^u-manifolds as stated in [15, Theorem 3.2.4], whose proof for the independent and identically distributed (i.i.d.) RDS can be found in [20].

Definition 1.1. An F-invariant Borel probability measure μ with marginal $\mathbb{P}$ on Ω is called a random SRB measure if

$$h_\mu^{(r)}(F) = \int \sum_i \lambda^{(i)}(\omega, x)^+ m^{(i)}(\omega, x)\, d\mu,$$

where $h_\mu^{(r)}(F)$ is the relative entropy of F defined in §2.4.

We now introduce some notation from non-uniformly hyperbolic theory for random dynamical systems, which can be found in [15, §2]. Assume that there exists an F-invariant subset Λ^χ of $\Omega \times \mathcal{M}$ such that F is non-uniformly hyperbolic on the fiber of this set. To be specific, for $\chi > 0$ and $0 < \epsilon \ll \chi$, denote by $\Lambda^\chi := \Lambda_\epsilon^\chi \subset \Omega \times \mathcal{M}$ the set of all points (ω, x) satisfying:

- for every $(\omega', y) \in \mathrm{Orb}(\omega, x)$, there is a splitting $T_y\mathcal{M} = E^u(\omega', y) \oplus E^s(\omega', y)$ with
$$D_y f_{\omega'} E^\tau(\omega', y) = E^\tau(\theta\omega', f_{\omega'}(y)) \quad \text{for } \tau = s, u;$$
- there exists an ϵ-tempered function $C : \Lambda^\chi \to \mathbb{R}$ such that for any $n \in \mathbb{N}$,
$$m(D_x f_\omega^n|_{E^u(\omega,x)}) \geq C(\omega, x)^{-1} e^{n\chi}, \quad |D_x f_\omega^n|_{E^s(\omega,x)}| \leq C(\omega, x) e^{-n\chi},$$
where $m(A)$ denotes the conorm of a linear map A;
- there exists an ϵ-tempered function $K : \Lambda^\chi \to \mathbb{R}$ such that for any $(\omega, x) \in \Lambda^\chi$,
$$\angle(E^s(\omega, x), E^u(\omega, x)) \geq K(\omega, x)^{-1},$$
where a function $g : \Lambda^\chi \to \mathbb{R}$ is called ϵ-tempered if
$$e^{-\epsilon|n|} \leq \frac{g(F^n(\omega, x))}{g(\omega, x)} \leq e^{\epsilon|n|} \quad \text{for all } n \in \mathbb{Z}, \text{ for all } (\omega, x) \in \Lambda^\chi.$$

For every F-invariant ergodic measure μ on $\Omega \times \mathcal{M}$ with marginal $\mathbb{P}$ on Ω, if all the Lyapunov exponents of μ are outside $[-\chi, \chi]$, then such subset Λ^χ exists and $\mu(\Lambda^\chi) = 1$ by the multiplicative ergodic theorem for RDS [20, Theorem 3.2].

Recall the invariant manifolds theorem for RDS (see e.g. [15, Theorem 2.2.5]). There exists a local unstable manifold $W^u_{\mathrm{loc}}(\omega, x) \subset \mathcal{M}$ with $\dim E^u(\omega, x)$ for any $(\omega, x) \in \Lambda^\chi$. Denote $\lambda^u := \lambda^u_{(\omega,x)}$ to be the inherited Riemannian volume measure on $W^u(\omega, x)$.

For $l > 1$, we define the (l-level) regular set

$$\Lambda_l := \{(\omega, x) \in \Lambda^\chi : \max\{C(\omega, x), K(\omega, x)\} \leq l\}. \tag{1.2}$$

A point $\omega \in \Omega$ is called a *generic point* of $\mathbb{P}$ if $(1/n)\sum_{i=0}^{n-1} \delta_{\theta^i\omega}$ converges to $\mathbb{P}$ in the weak* topology. We emphasize that, in the following theorem, the fiber manifold M may

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have arbitrary dimension with $\dim M \geq 2$ and the fiber maps are assumed only to be C^2 diffeomorphisms.

THEOREM A. *Assume that the skew-product transformation $F : \Omega \times \mathcal{M} \to \Omega \times \mathcal{M}$ is given as above. Let $\mathbb{P}$ be an ergodic θ-invariant measure on Ω. If there exists $l > 0$, a generic point of $\mathbb{P}$, named ω^*, and a Borel measurable set $E_{\omega^*} \subset W^u_{\mathrm{loc}}(\omega^*, x_0) \cap \Lambda_l$ with $\lambda^u_{(\omega^*,x_0)}(E_{\omega^*}) > 0$ such that*

$$\limsup_{n\to\infty} \frac{1}{n}\#\{0 \leq j \leq n : (\theta^j\omega^*, f^j_{\omega^*}(x)) \in \Lambda_l\} > 0 \quad \textit{for any } x \in E_{\omega^*}, \tag{1.3}$$

then there exists a random SRB measure.

Remark 1.1. We refer to the conditions stated in Theorem A as the fiber leaf condition. Theorem A can be applied directly to the random perturbation of Axiom A basic sets, see [19]. When the external forcing of the skew product vanishes, our leaf condition coincides with that in [4]. Consequently, we provide a new proof of the main results of [4].

The key innovation of our approach is that, unlike previous constructions based on Young towers or any Markov structures, we rely purely on Burguet's variational method adapted to the fiberwise recurrence context. The leaf condition guarantees that there exists an unstable leaf with a subset of positive leaf volume of hyperbolic points that are recurrent to a Pesin set with positive frequency. This subset gives the geometric set in Burguet's paper. Using a Borel–Cantelli argument [7, Lemma 2], we can construct a Følner sequence such that the geometric time is uniformly dense in this Følner sequence on a positive leaf volume subset. By averaging the leaf volume measure on the uniformity set along the Følner sequence, Burguet's [7, Lemma 4] can be applied to show that any limit measure has almost everywhere positive Lyapunov exponents. The crucial part is to show the Folner Gibbs property of the leaf volume measure, which guarantees the existence of SRB measure by [7, Proposition 3].

Remark 1.2. Note that the leaf condition does not imply the continuity of the bundles E^u and E^s. Therefore, unlike in [8], we must lift the measure to the Grassmannian bundle to estimate the metric entropy. To estimate the entropy of the random system, we establish a Gibbs–Følner property for random dynamical systems, see §2.4.

Our results also yield the existence of SRB measures for smooth skew-product systems (not random SRB measures). We assume further that Ω is a compact smooth manifold, θ is a C^2 diffeomorphism on Ω, and F is a C^2 diffeomorphism on $\Omega \times \mathcal{M}$. Let $\mathbb{P}$ be a hyperbolic SRB measure of (Ω, θ), and denote $\lambda^+_{\theta,\min}$ and $\lambda^-_{\theta,\max}$ to be the minimum of the positive Lyapunov exponents and the maximum of the negative Lyapunov exponents. The following theorem shows that, under the following spectral gap condition:

$$\lambda^-_{\theta,\max} < \min_{\omega\in\Omega}\min_{x\in\mathcal{M}} \log \|D(f_\omega)^{-1}(x)\|^{-1} \leq \max_{\omega\in\Omega}\max_{x\in\mathcal{M}} \log \|Df_\omega(x)\| < \lambda^+_{\theta,\min}, \tag{1.4}$$

the random SRB measure lifts to an SRB measure for the entire skew product.

THEOREM B. *In the setting of Theorem A, assume further that Ω is a compact smooth manifold, θ is a C^2 diffeomorphism on Ω that admits an ergodic hyperbolic SRB*

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measure $\mathbb{P}$, and F is a C^2 diffeomorphism on $\Omega \times \mathcal{M}$. If (1.4) holds, then there exists an SRB measure for the skew product F.

The rest of the paper is organized as follows. In §2, we collect the necessary preliminary lemmas. Section 3 contains the core of the paper: we adapt Burguet's variational method to the random context. We construct an invariant random hyperbolic measure by averaging leaf-volume measures along suitably chosen Følner sequences and verify that it satisfies the fiber Følner Gibbs property. This property, combined with the subadditive ergodic arguments and the relative entropy estimates, yields the random SRB measure asserted in Theorem A. The proof of Theorem B is addressed in §4 and consists of two steps. In §4.1, we borrow ideas from [**28**, §4.2.5] to prove a dimension reduction lemma. In §4.2, we apply the dimension reduction lemma to prove that under the assumption (1.4), the positive Lyapunov exponents are exactly the positive Lyapunov exponents of $(\Omega, \theta, \mathbb{P})$ combining with the positive Lyapunov exponents generated by the RDS. Finally, we show that the entropy formula holds for a skew product system by using the Abramov–Rohlin formula and the random version of entropy formula.

2. *Preliminary lemma*

2.1. *Invariant manifolds.*

The following lemma is a special version of [**15**, Theorem 2.2.5] for random diffeomorphisms, see also [**20**, Theorem III.3.1].

LEMMA 2.1. (Local stable and unstable manifolds) *For each $n \in \mathbb{N}$ and $(\omega, x) \in \Lambda_l$, there exists a continuous family of C^1 embedded k-dimensional discs $W^s_{\mathrm{loc}}((\omega, x), n)$ and $(\dim \mathcal{M} - k)$-dimensional discs $W^u_{\mathrm{loc}}((\omega, x), n)$ in $\mathcal{M}$, and there exist positive numbers $\delta(l)$, $L(l) \in (0, 1)$ and $\zeta(l)$ such that the following statements hold.*

(a) *There exists a C^1 map*

$$h^s_{(\omega,x),n} : O^s_n(\omega, x) \to E^u(\theta^n\omega, f^n_\omega(x)),$$
$$h^u_{(\omega,x),n} : O^u_n(\omega, x) \to E^s(\theta^n\omega, f^n_\omega(x)),$$

where $O^\tau_n(\omega, x)$ is an open subset of $E^\tau(\theta^n\omega, f^n_\omega(x))$ containing $\{\xi \in E^\tau(\theta^n\omega, f^n_\omega(x)) : |\xi| < \delta(l)e^{-\epsilon n}\}$ for $\tau = s, u$, such that:

(i) $h^\tau_{(\omega,x),n}(0) = 0$ *for* $\tau = s, u$;

(ii) $\mathrm{Lip}(h^\tau_{(\omega,x),n}) \le L(l)e^{\epsilon n}$, $\mathrm{Lip}(D.h^\tau_{(\omega,x),n}) \le L(l)e^{\epsilon n}$ *for* $\tau = s, u$;

(iii) $W^\tau_{\mathrm{loc}}((\omega, x), n) = \exp_{f^n_\omega(x)} \mathrm{Graph}(h^\tau_{(\omega,x),n})$ *and tangent to* $E^\tau(\theta^n\omega, f^n_\omega(x))$ *at point* $f^n_\omega(x)$ *for* $\tau = s, u$.

(b) *We have* $f_{\theta^n\omega} W^s_{\mathrm{loc}}((\omega, x), n) \subset W^s_{\mathrm{loc}}((\omega, x), n + 1)$ *and* $(f_{\theta^{n+1}\omega})^{-1} W^u_{\mathrm{loc}}((\omega, x), n + 1) \subset W^u_{\mathrm{loc}}((\omega, x), n)$.

(c) *We have* $d^s(f^t_{\theta^n\omega}(y), f^t_{\theta^n\omega}(z)) \le \zeta(l)e^{\epsilon n}e^{(-\chi+\epsilon)t} d^s(y, z)$ *for* $y, z \in W^s_{\mathrm{loc}}((\omega, x), n)$, $t \in \mathbb{N}$, *where d^s denote the distance along $W^s_m(\omega, x)$ for $m \in \mathbb{N}$ induced by the inherited Riemannian metric as submanifolds of $\mathcal{M}$. Here,* $d^u(f^{-t}_{\theta^n\omega}(y), f^{-t}_{\theta^n\omega}(z)) \le \zeta(l)e^{\epsilon n}e^{(-\chi+\epsilon)t} d^u(y, z)$ *for* $y, z \in W^u_{\mathrm{loc}}((\omega, x), n)$, $t \in \mathbb{N}$, *where d^u denotes the distance along $W^u_m(\omega, x)$ for $m \in \mathbb{N}$ induced by the inherited Riemannian metric as submanifolds of $\mathcal{M}$.*

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From now on, we will use the following notation: $W^{\tau}_{\delta(l)}(\omega, x) := W^{\tau}_{\mathrm{loc}}((\omega, x), 0)$, and $h^{\tau}_{(\omega,x)} := h^{\tau}_{(\omega,x),0}$ for $(\omega, x) \in \Lambda_l$ and $\tau = s, u$. By statement (a) of Lemma 2.1, one has $\delta(l)e^{-N\epsilon} \le \delta(le^{N\epsilon})$ for any $N \in \mathbb{N}$. The following is a direct corollary of the local stable and unstable manifolds.

COROLLARY 2.1. *For any given $l > 0$, there exists $C(l) > 0$ such that for any $8\delta < \delta(l)$, as in Lemma 2.1, and any $(\omega, x) \in \Lambda_l$, one has*

$$\frac{1}{C(l)}\delta^{\dim E^u(\omega,x)} \le \lambda^u(W^u_{\delta}(\omega, x)) \le \lambda^u(W^u_{4\delta}(\omega, x)) \le C(l)\delta^{\dim E^u(\omega,x)}.$$

2.2. *Bounded distortion lemmas.* For any $(\omega, x) \in \Lambda_l$, according to the definition of Λ_l, we have that both $(\theta\omega, f_\omega(x))$ and $(\theta^{-1}\omega, f^{-1}_\omega(x))$ belong to Λ_{le^ϵ}. The idea of the following bounded distortion lemma comes from [**18**, Lemma 3.2].

LEMMA 2.2. *There exists positive numbers C_l and $\delta'(l) < \min\{\delta(le^\epsilon), \delta(l)\}$ such that for any $(\theta\omega, f_\omega(x)) \in \Lambda_l$, $f_\omega(y) \in W^u_{\delta(l)}(\theta\omega, f_\omega(x))$ and $y \in W^u_{\delta'(l)}(\omega, x)$, one has*

$$\log \frac{|\det D_y f_\omega|_{T_y W^u_{\delta(le^\epsilon)}(\omega,x)}|}{|\det D_x f_\omega|_{E^u(\omega,x)}|} \le C_l d^u(y, x).$$

Proof of Lemma 2.2. With the help of the normal coordinate chart, we deal with $W^u_{\delta(le^\epsilon)}(\omega, x)$ and $W^u_{\delta(l)}(\theta\omega, f_\omega(x))$ in Euclidean space. We let $\delta'(l) < \delta(le^\epsilon)$ be determined later. By Lemma 2.1, there exist $\xi_y \in E^u(\omega, x)(\delta'(l))$ and $\xi_{f_\omega(y)} \in E^u(\theta\omega, f_\omega(x))(\delta(l))$ such that

$$y = x + \xi_y + h^u_{(\omega,x)}(\xi_y), \tag{2.1}$$

$$f_\omega(y) = f_\omega(x) + \xi_{f_\omega(y)} + h^u_{(\theta\omega,f_\omega(x))}(\xi_{f_\omega(y)}), \tag{2.2}$$

and $T_y W^u_{\delta(le^\epsilon)}(\omega, x) = \mathrm{Graph}((Dh^{uu}_{(x,\omega)})_{\xi_y})$, $T_{f_\omega(y)} W^u_{\delta(l)}(\theta\omega, f_\omega(x)) = \mathrm{Graph}((Dh^{uu}_{(f_\omega(x),\theta\omega)})_{\xi_{f_\omega(y)}})$. By (2.1) and (2.2), we have

$$|x - y| = |\xi_y + h^u_{(\omega,x)}(\xi_y)| \ge |\xi_y| - L(le^\epsilon)|\xi_y|$$

and

$$\begin{aligned}(1 - L(l))|\xi_{f_\omega(y)}| &\le |\xi_{f_\omega(y)} + h^u_{(\theta\omega,f_\omega(x))}(\xi_{f_\omega(y)})| = |f_\omega(y) - f_\omega(x)| \\ &\le R|y - x| \le R(1 + L(le^\epsilon))|\xi_y|,\end{aligned}$$

so $|\xi_{f_\omega(y)}| \le R((1 + L(le^\epsilon))/(1 - L(l)))|\xi_y|$, where $R \ge 1$ is a constant such that

$$\max\left\{\sup_{(x,\omega)\in\mathcal{M}\times\Omega} |D_x f_\omega|, \sup_{(x,\omega)\in\mathcal{M}\times\Omega} |D^2_x f_\omega|\right\} \le R. \tag{2.3}$$

Now, we define the following linear maps $L_{(\omega,x)}, \tilde{L}_{(\omega,y)} : E^u(\omega, x) \to E^u(\theta\omega, f_\omega(x))$ by

$$\begin{aligned}L_{(\omega,x)} &= D_x f_\omega|_{E^u(\omega,x)}, \\ \tilde{L}_{(\omega,y)} &= \pi^u_{(\theta\omega,f_\omega(x))} D_y f_\omega|_{T_y W^u_{\delta(le^\epsilon)}(\omega,x)}(I + (Dh^u_{(\omega,x)})_{\xi_y}),\end{aligned}$$

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where $\pi^u_{(\theta\omega, f_\omega(x))}$ is the projection from $T_{f_\omega(x)}\mathcal{M}$ to $E^u(\theta\omega, f_\omega(x))$ with respect to the splitting $E^u(\theta\omega, f_\omega(x)) \oplus E^s(\theta\omega, f_\omega(x))$. Note that $(\theta\omega, f_\omega(x)) \in \Lambda_l$ and $\angle(E^u(\theta\omega, f_\omega(x)), E^s(\theta\omega, f_\omega(x))) \geq l^{-1}$, there exists $\theta(l) > 0$ such that $\|\pi^u_{(\theta\omega, f_\omega(x))}\| \leq \theta(l)$. Then,

$$\|L_{(\omega,x)}\|, \|\tilde{L}_{(\omega,y)}\| \leq R\theta(l)(1 + L(le^\epsilon)).$$

Now, we have

$$\begin{aligned}
&\sup_{v\in E^u(\omega,x),\ \|v\|=1} |D_x f_\omega v - \pi^u_{(\theta\omega, f_\omega(x))} D_y f_\omega|_{T_y W^u_{\delta(le^\epsilon)}(\omega,x)}(I + (Dh^u_{(\omega,x)})_{\xi_y})v| \\
&\quad \leq \theta(l)(\|D_x f_\omega - D_y f_\omega\| + \|D_y f_\omega (Dh^u_{(\omega,x)})_{\xi_y}\|) \\
&\quad \leq R\theta(l)|y - x| + R\theta(l)L(l)|\xi_y| \\
&\quad \leq (R\theta(l) + R\theta(l)L(l))d^u(x, y).
\end{aligned}$$

Hence, $\|L_{(\omega,x)} - \tilde{L}_{(\omega,y)}\| \leq C_0(R\theta(l) + R\theta(l)L(l))d^u(x, y)$, where C_0 only depends on the normal coordinate chart. Then, by the property of the determinant,

$$|\det(L_{(\omega,x)}) - \det(\tilde{L}_{(\omega,y)})| \leq C_1(l)d^u(x, y), \tag{2.4}$$

where $C_1(l)$ is a polynomial of $R\theta(l) + R\theta(l)L(l)$ and $\dim E^u(\omega, x)$.

Note that

$$\begin{aligned}
&\|(\pi^u_{(\theta\omega, f_\omega(x))}|_{T_{f_\omega(y)} W^u_{\delta(l)}(\theta\omega, f_\omega(x))} - I)(I + (Dh^u_{(\theta\omega, f_\omega(x))})_{\xi_{f_\omega(y)}})\| \\
&\quad \leq \|(Dh^u_{(\theta\omega, f_\omega(x))})_{\xi_{f_\omega(y)}}\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\pi^u_{(\theta\omega, f_\omega(x))}|_{T_{f_\omega(y)} W^u_{\delta(l)}(\theta\omega, f_\omega(x))} - I\| &\leq \frac{\|(Dh^u_{(\theta\omega, f_\omega(x))})_{\xi_{f_\omega(y)}}\|}{1 - \|(Dh^u_{(\theta\omega, f_\omega(x))})_{\xi_{f_\omega(y)}}\|} \\
&\leq \frac{L(l)|\xi_{f_\omega(y)}|}{1 - L(l)|\xi_{f_\omega(y)}|} \\
&\leq \frac{L(l) \cdot R((1 + L(le^\epsilon))/(1 - L(l)))|\xi_y|}{1 - L(l) \cdot R((1 + L(le^\epsilon))/(1 - L(l)))|\xi_y|} \\
&\leq \frac{L(l) \cdot R((1 + L(le^\epsilon))/(1 - L(l)))|\xi_y|}{1 - L(l) \cdot R((1 + L(le^\epsilon))/(1 - L(l)))\delta'(l)} \\
&\leq 2L(l) \cdot R\frac{1 + L(le^\epsilon)}{1 - L(l)}|d^u(x, y),
\end{aligned}$$

where $\delta'(l)$ is chosen so that $L(l) \cdot R((1 + L(le^\epsilon))/(1 - L(l)))\delta'(l) < 1/2$. So, we have

$$|\det(\pi^u_{(\theta\omega, f_\omega(x))}|_{T_{f_\omega(y)} W^u_{\delta(l)}(\theta\omega, f_\omega(x))}) - 1| \leq C_2(l)d^u(x, y) \tag{2.5}$$

where $C_2(l)$ is a polynomial of $L(l) \cdot R((1 + L(le^\epsilon))/(1 - L(l)))$ and $\dim E^u(\omega, x)$. In addition,

$$\|I + (Dh^u_{(\omega,x)})_{\xi_y} - I\| \leq L(le^\epsilon)|\xi_y| \leq L(le^\epsilon)d^u(x, y)$$

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implies that there exists $C_3(l)$ such that

$$|\det(I + (Dh^u_{(\omega,x)})_{\xi_y}) - 1| \le C_3(l) d^u(x, y). \tag{2.6}$$

Combining (2.4), (2.5), and (2.6), we conclude that there exists $C_4(l) > 0$,

$$|\det D_x f_\omega|_{E^u(\omega,x)} - \det D_y f_\omega|_{T_y W^u_{\delta(le^\epsilon)}(\omega,x)}| \le C_4(l) d^u(x, y).$$

Then, the proof of Lemma 2.2 is complete by letting $C_l = C_4(l)/\sup_{(\omega,x,E)} |\det D_x f_\omega|_E|$. □

For any $N \in \mathbb{N}$ and $\Delta \subset \Lambda_l(\omega)$, we say that an interval of integers $[a, b]$ is N-$\{(\omega, x) : x \in \Delta\}$-admissible if for any $x \in \Delta$, there exists a sequence of numbers $a = a_0 < a_1 < \cdots < a_n = b$ (depending on (ω, x)) such that $(\theta^{a_i}\omega, f^{a_i}_\omega(x)) \in \Lambda_l$ for $i = 0, \ldots, n$ and $a_{i+1} - a_i \le N$.

LEMMA 2.3. *For any $N \in \mathbb{N}$, $\gamma > 0$, there exists $\delta_0(l, N, \gamma) > 0$ such that for any $(\omega, x) \in \Lambda_l$, $\delta < \delta_0(l, N, \gamma)$, for any $n > 0$ with $[0, n]$ is N-$\{(\omega, x)\}$-admissible, one has:*

(1) *for any $0 \le i \le n$, the local unstable manifold $W^u_{8\delta}(\theta^i\omega, f^i_\omega(x))$ is well defined and*

$$f^i_\omega(W^u_{4\delta}(\omega, x) \cap B_n(\omega, x, \delta)) \subset W^u_{4\delta}(\theta^i\omega, f^i_\omega(x));$$

(2) *for any Borel measurable set $\Delta \subset W^u_{4\delta}(\omega, x) \cap B_n(\omega, x, \delta)$, one has*

$$e^{-\gamma} \le \frac{\lambda^u(\Delta) \det(D_x f^n_\omega|_{E^u(\omega,x)})}{\lambda^u(f^n_\omega(\Delta))} \le e^{\gamma}.$$

Proof of Lemma 2.3. For any $(\omega, x) \in \Lambda_l$ and $[0, n]$ that is N-$\{(\omega, x)\}$-admissible, then by definition, there exists $0 = a_0 < a_1 < \cdots < a_k = n$ with $a_{i+1} - a_i \le N$ such that $(\theta^{a_i}\omega, f^{a_i}_\omega(x)) \in \Lambda_l$ for $i = 0, \ldots, k$. Therefore, for any $t \in [0, n]$, we have $(\theta^t\omega, f^t_\omega(x)) \in \Lambda_{le^{N\epsilon}}$. As a consequence, for any

$$\delta < \delta_0(l, N, \gamma) := \frac{\min\{\delta'(l)/4, \delta(l)\}e^{-N(\epsilon+R)}}{8} \le \delta(le^{N\epsilon})e^{-NR}/8, \tag{2.7}$$

$W^u_{8\delta}(\theta^t\omega, f^t_\omega(x))$ is well defined, where $\delta'(l)$ is given in Lemma 2.2, and R is a constant as in (2.3).

For $i \in [a_0, a_1]$, we have

$$\operatorname{diam}(f^i_\omega W^u_{4\delta}(\omega, x)) \le 4\delta \cdot e^{iR} \le 4\frac{\delta(l)e^{-N(\epsilon+R)}}{8} \cdot e^{iR} \le \frac{\delta(l)e^{-N\epsilon}}{2} \le \delta(le^{N\epsilon}).$$

As a consequence, $f^i_\omega W^u_{4\delta}(\omega, x)$ belongs to the local stable manifolds. Therefore, for $i \in [a_0, a_1]$, we have

$$f^i_\omega(W^u_{4\delta}(\omega, x) \cap B_n(\omega, x, \delta)) \subset f^i_\omega W^u_{4\delta}(\omega, x) \cap B(\theta^i\omega, f^i_\omega(x), \delta) \subset W^u_{4\delta}(\theta^i\omega, f^i_\omega(x)).$$

Inductively, one can prove statement (1) holds. Next, we prove statement (2).

For any Borel measurable set $\Delta \subset W^u_{4\delta}(\omega, x) \cap B_n(\omega, x, \delta)$, by statement (1), $f^n_\omega(\Delta) \subset W^u_{4\delta}(\theta^n\omega, f^n_\omega(x))$. Therefore,

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$$\lambda^u(f^n\Delta) = \int_\Delta \det(D_y f_\omega^n|_{T_y W^u_{4\delta}(\omega,x)})\, d\lambda^u(y)$$
$$= \int_\Delta \frac{\det(D_y f_\omega^n|_{T_y W^u_{4\delta}(\omega,x)})}{\det(D_x f_\omega^n|_{T_x W^u_{4\delta}(\omega,x)})}\, d\lambda^u(y) \cdot \det(D_x f_\omega^n|_{T_x W^u_{4\delta}(\omega,x)}).$$

By statement (1) and Lemma 2.2, we have estimate

$$e^{-\sum_{i=0}^{n-1} C_{le^{N\epsilon}}\cdot d^u(f_\omega^i(x),f_\omega^i(y))} \le \left|\frac{\det(D_y f_\omega^n|_{T_y W^u_{4\delta}(\omega,x)})}{\det(D_x f_\omega^n|_{T_x W^u_{4\delta}(\omega,x)})}\right| \le e^{\sum_{i=0}^{n-1} C_{le^{N\epsilon}}\cdot d^u(f_\omega^i(x),f_\omega^i(y))},$$

where $C_{le^{N\epsilon}}$ is given in the statement of Lemma 2.2. This implies

$$\frac{\lambda^u(\Delta)\,\det(D_x f_\omega^n|_{T_x W^u_{4\delta}(\omega,x)})}{\lambda^u(f^n\Delta)} \in (e^{-k\sum_{i=1}^N C_{le^{N\epsilon}}\cdot e^{-\chi i}\cdot 4\delta}, e^{k\sum_{i=1}^N C_{le^{N\epsilon}}\cdot e^{-\chi i}\cdot 4\delta}).$$

The proof is complete by shrinking δ. □

LEMMA 2.4. *For any $N \in \mathbb{N}$ and any $\gamma > 0$, there exists positive number $\delta_1(l, N, \gamma) < \delta_0(l, N, \gamma)$ such that for any $\delta < \delta_1(l, N, \gamma)$, $(\omega, x) \in \Lambda_l$, if $y \in B_n(\omega, x, \delta)$ with $[\![0, n]\!]$ being N-$\{(\omega, x), (\omega, y)\}$-admissible, then we have*

$$\left|\log \frac{\det(D_x f_\omega^n|_{E^u(\omega,x)})}{\det(D_y f_\omega^n|_{E^u(\omega,y)})}\right| < n\gamma.$$

Proof of Lemma 2.4. By the Hölder continuity of the fiber unstable subbundle (see e.g. [**20**, Ch. III, Theorem 4.1]), there exists a constant $C(l)$ such that for any $(\omega, z_1), (\omega, z_2) \in \Lambda_{le^{N\epsilon}}$, one has

$$\left|\log \frac{\det(D_{z_1} f_\omega|_{E^u(\omega,z_1)})}{\det(D_{z_2} f_\omega|_{E^u(\omega,z_2)})}\right| < C(l)e^{N\epsilon} d(z_1, z_2)^\alpha$$

for some exponent $\alpha \in (0, 1)$. Note that $[\![0, n]\!]$ is N-$\{(\omega, x), (\omega, y)\}$-admissible, by the same argument in the proof of Lemma 2.3, for any $i \in [\![0, n]\!]$, $(\theta^i\omega, f_\omega^i(x))$ and $(\theta^i\omega, f_\omega^i(y))$ belong to $\Lambda_{le^{N\epsilon}}$. Then, the conclusion follows from the above inequality by letting

$$\delta_1(l, N, \gamma) = \min\left\{\delta_0(l, N, \gamma), \left(\frac{\gamma}{C(l)e^{N\epsilon}}\right)^{1/\alpha}\right\}.$$

This completes the proof of this lemma. □

2.3. *Følner sequence.* We introduce some notation of integers in this subsection and we use the same notation as in [**7**]. We denote $\mathcal{P}_\mathbb{N}$ and $\mathcal{P}_n$ to be the power sets of $\mathbb{N}$ and $\{0, 1, 2, \ldots, n-1\}$, $n \in \mathbb{N}$. For $a \le b \in \mathbb{N}$, we write $[\![a, b]\!]$ (respectively $[\![a, b[\![$ and $]\!]a, b]\!]$) as the interval of integers k with $a \le k \le b$ (respectively $a \le k < b$ and $a < k \le b$). For any $E \in \mathcal{P}_\mathbb{N}$, an interval of integers $[\![a, b]\!]$ is said to be E-irreducible if $a, b \in E$ and $[\![a, b[\![\cap E = \{a\}$. We denote E_M to be the union of the intervals of integers $[\![a, b[\![$ with $a, b \in E$ and $0 < b - a \le M$. The boundary ∂E of E is the symmetric difference of E and $E + 1$ with $E + 1 := \{k + 1,\ k \in E\}$. In particular, $\partial[\![a, b[\![= \{a, b\}$. We denote $\mathfrak{N}$ to be the set of increasing sequences of natural integers. For $\mathfrak{n} \in \mathfrak{N}$, the generalized power set of $\mathfrak{n}$ is defined by $\mathcal{Q}_\mathfrak{n} := \prod_{n\in\mathfrak{n}} \mathcal{P}_n$.

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For $n \in \mathbb{Z}^+$ and $E \in \mathcal{P}_{\mathbb{N}}$, we denote

$$d_n(E) = \frac{\#(E \cap [\![0, n[\![)}{n}.$$

The upper and lower asymptotic densities of $E \in \mathcal{P}_{\mathbb{N}}$ along a sequence $\mathfrak{n} \in \mathfrak{N}$ are defined by

$$\bar{d}^{\mathfrak{n}}(E) := \limsup_{n \in \mathfrak{n}} d_n(E) \quad \text{and} \quad \underline{d}^{\mathfrak{n}}(E) := \liminf_{n \in \mathfrak{n}} d_n(E).$$

We write $d^{\mathfrak{n}}(E)$ when $\bar{d}^{\mathfrak{n}}(E) = \underline{d}^{\mathfrak{n}}(E)$. For $\mathfrak{n} \in \mathfrak{N}$ and $\mathcal{F} = (F_n)_{n \in \mathfrak{n}} \in \mathcal{Q}_{\mathfrak{n}}$, we define

$$\bar{d}^{\mathfrak{n}}(\mathcal{F}) := \limsup_{n \in \mathfrak{n}} d_n(F_n) \quad \text{and} \quad \underline{d}^{\mathfrak{n}}(\mathcal{F}) := \liminf_{n \in \mathfrak{n}} d_n(F_n).$$

We write $d^{\mathfrak{n}}(\mathcal{F})$ when $\bar{d}^{\mathfrak{n}}(\mathcal{F}) = \underline{d}^{\mathfrak{n}}(\mathcal{F})$. The classical definition of Følner sequence only requires that $\lim_{n \in \mathfrak{n}}((\#\partial F_n)/\#F_n) = 0$. In this paper, we consider a stronger version of the Følner property.

Definition 2.1. For $\mathfrak{n} \in \mathfrak{N}$ and $\mathcal{F} = (F_n)_{n \in \mathfrak{n}} \in \mathcal{Q}_{\mathfrak{n}}$, we let $\partial\mathcal{F} := (\partial F_n)_{n \in \mathfrak{n}} \in \mathcal{Q}_{\mathfrak{n}}$. We call $\mathcal{F}$ Følner if $d^{\mathfrak{n}}(\partial\mathcal{F}) = 0$.

2.4. *Relative entropy.* We recall the definition of relative entropy in a general setting. In this section, we consider an RDS F over a metric dynamical system $(\Omega, \mathfrak{F}, \mathbb{P}, \theta)$ on a compact metric space $(X, \mathcal{B})$ generated by $g_\omega : X \to X$ so that the map $(\omega, x) \mapsto g_\omega(x)$ is measurable. We use the notation $G(\omega, x) = (\theta\omega, g_\omega(x))$ to denote the skew product transformation. For any Borel probability measure μ, on $\Omega \times X$, a finite measurable partition α, β of $\Omega \times X$, and a σ-sub-algebra $\mathcal{B}_1$ on $\Omega \times X$, the conditional entropy function has the following property (see e.g. [20, Ch. 0]):

- $H_\mu(\alpha|\mathcal{B}_1) \le \log \#\alpha$, $H_\mu(\alpha \vee \beta|\mathcal{B}_1) \le H_\mu(\alpha|\mathcal{B}_1) + H_\mu(\beta|\mathcal{B}_1)$;
- $H_{\sum_i p_i\mu_i}(\alpha|\mathcal{B}_1) \ge \sum_i p_i H_{\mu_i}(\alpha|\mathcal{B}_1)$ for any probability measures $\{\mu_i\}_{i=1}^n$ and probability vector $\{p_i\}_{i=1}^n$, see [14];
- $H_\mu(G^{-1}\alpha|G^{-1}\mathcal{B}_1) = H_{G_*\mu}(\alpha|\mathcal{B}_1)$. If $\mathcal{B}_1$ is G-invariant, then $H_\mu(G^{-1}\alpha|\mathcal{B}_1) = H_{G_*\mu}(\alpha|\mathcal{B}_1)$.

For any G-invariant measure μ with marginal $\mathbb{P}$ on Ω, let $\{\mu_\omega\}_{\omega\in\Omega}$ be its disintegration. We recall that the relative entropy of G, $h_\mu^{(r)}(G)$, is defined by the following:

$$\begin{aligned} h_\mu^{(r)}(G) &:= \sup_{\mathcal{Q}} h_\mu^{(r)}(G, \mathcal{Q}), \\ h_\mu^{(r)}(G, \mathcal{Q}) &= \lim_{n\to\infty} \frac{1}{n} H_\mu\Big(\bigvee_{i=0}^{n-1} (G^i)^{-1}\mathcal{Q}|\mathfrak{F}_{\Omega\times X}\Big) \\ &= \lim_{n\to\infty} \frac{1}{n} \int H_{\mu_\omega}\Big(\bigvee_{i=0}^{n-1} (g_\omega^i)^{-1}\mathcal{P}\Big)\, d\mathbb{P}(\omega), \end{aligned}$$

where the supremum is taken over all finite measurable partitions $\mathcal{Q}$ of $\Omega \times X$ in the form $\mathcal{Q} = \{\Omega \times P_i\}$ with a finite measurable partition $\mathcal{P} = \{P_i\}$ of X, and $\mathfrak{F}_{\Omega\times X}$ is the σ-algebra formed by all sets $A \times X$ with $A \in \mathfrak{F}$. Denote $h_\mu(G)$ to be the classical measure-theoretic entropy of skew product transformation G. There is the following

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Abramov–Rohlin formula [1]:

$$h_\mu(G) = h_\mu^{(r)}(G) + h_{\mathbb{P}}(\theta). \tag{2.8}$$

LEMMA 2.5. *Let $\mathcal{F} = (F_n)_{n\in\mathfrak{n}}$ be a Følner sequence with $\underline{d}^{\mathfrak{n}}(\mathcal{F}) > 0$. Then, for any finite measurable partition $\mathcal{Q}$ of $\Omega \times X$ and a fixed natural number q, there exists a sequence $\epsilon_n = \epsilon_n(q, \#\mathcal{Q}, \#F_n) \to 0$ as $n \in \mathfrak{n} \to \infty$ such that for any $n \in \mathfrak{n}$,*

$$\frac{1}{q} H_{\mu_n^{F_n}}\Big(\bigvee_{i=0}^{q-1} (G^i)^{-1}\mathcal{Q}|\mathfrak{F}_\Omega \times X\Big) \geq \frac{1}{\#F_n} H_{\mu_n}\Big(\bigvee_{i\in F_n} (G^i)^{-1}\mathcal{Q}|\mathfrak{F}_\Omega \times X\Big) - \epsilon_n,$$

where $\{\mu_n\}_{n\in\mathfrak{n}}$ is a sequence of Borel probability measures on $\Omega \times X$.

Proof of Lemma 2.5. We first consider the case where $F_n = [\![n_0, n_1]\!]$ is an interval of integers. Fix a natural number $q \in \mathbb{N}$. We consider the case where $1 < q \leq n_1 - n_0$. For $1 \leq j \leq q-1$, we define $a(j) = \lceil (n_1 - n_0 - j/q) \rceil$. For any fixed $j \in [\![0, q-1]\!]$, we have decomposition

$$\bigvee_{i=0}^{n_1-n_0} G^{-i} G^{-n_0}\mathcal{Q} = \bigvee_{r=0}^{a(j)-1} G^{-(rq+j+n_0)}\Big(\bigvee_{i=0}^{q-1} G^{-i}\mathcal{Q}\Big) \vee \bigvee_{l\in S} G^{-l}\mathcal{Q},$$

where S has cardinality at most $2q$. Therefore, we have

$$\begin{aligned} H_{\mu_n}\Big(\bigvee_{i=n_0}^{n_1-1} G^{-i}\mathcal{Q}|\mathfrak{F}_\Omega \times X\Big) &\leq \sum_{r=0}^{a(j)-1} H_{\mu_n}(G^{-(rq+j+n_0)} \bigvee_{i=0}^{q-1} G^{-i}\mathcal{Q}|\mathfrak{F}_\Omega \times X) + 2q \log \#\mathcal{Q} \\ &= \sum_{r=0}^{a(j)-1} H_{\mu_n \circ G^{-(rq+j+n_0)}}\Big(\bigvee_{i=0}^{q-1} G^{-i}\mathcal{Q}|\mathfrak{F}_\Omega \times X\Big) + 2q \log \#\mathcal{Q}. \end{aligned}$$

Summing the above over j from 0 to $q-1$ and dividing it by $\#F_n$, we obtain

$$\begin{aligned} &\frac{q}{\#F_n} H_{\mu_n}\Big(\bigvee_{i=n_0}^{n_1} G^{-i}\mathcal{Q}|\mathfrak{F}_\Omega \times X\Big) \\ &\quad\leq \frac{1}{\#F_n} \sum_{k=0}^{n_1-n_0} H_{\mu_n \circ G^{-(k+n_0)}}\Big(\bigvee_{i=0}^{q-1} G^{-i}\mathcal{Q}|\mathfrak{F}_\Omega \times X\Big) + \frac{2q^2 \log \#\mathcal{Q}}{\#F_n} \\ &\quad\leq H_{\mu_n^{F_n}}\Big(\bigvee_{i=0}^{q-1} G^{-i}\mathcal{Q}|\mathfrak{F}_\Omega \times X\Big) + \frac{2q^2 \log \#\mathcal{Q}}{\#F_n}, \end{aligned}$$

where the second inequality is due to the concavity of the conditional entropy. Then, dividing it by q, we obtain the desired inequality. Now, we consider the case where $q > n_1 - n_0$. By direct computation, we have

$$\begin{aligned} \frac{1}{\#F_n} H_{\mu_n}\Big(\bigvee_{i\in F_n} (G^i)^{-1}\mathcal{Q}|\mathfrak{F}_\Omega \times X\Big) \leq \log \#\mathcal{Q} &\leq \frac{q}{\#F_n} \log \#\mathcal{Q} \\ &\leq \frac{1}{q} H_{\mu_n^{F_n}}\Big(\bigvee_{i=0}^{q-1} (G^i)^{-1}\mathcal{Q}|\mathfrak{F}_\Omega \times X\Big) + \frac{q}{\#F_n} \log \#\mathcal{Q}. \end{aligned}$$

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Now, we consider a general set $F_n \in \mathcal{P}_n$, which can be decomposed into connected components $F_n = \bigsqcup_{j\in J_n} F_n^j$. Noticing that $\#J_n \le \#\partial F_n$, then we obtain

$$\begin{aligned}\frac{1}{q}H_{\mu_n^{F_n}}\Big(\bigvee_{i=0}^{q-1} G^{-i}\mathcal{Q}|\mathfrak{F}_\Omega\times X\Big) &\ge \sum_{j\in J_n}\frac{\#F_n^j}{q\#F_n}H_{\mu_n^{F_n^j}}\Big(\bigvee_{i=0}^{q-1} G^{-i}\mathcal{Q}|\mathfrak{F}_\Omega\times X\Big)\\ &\ge \sum_{j\in J_n}\Big(H_{\mu_n}\Big(\bigvee_{i\in F_n^j} G^{-i}\mathcal{Q}|\mathfrak{F}_\Omega\times X\Big)-\frac{2q\log\#\mathcal{Q}}{\#F_n}\Big)\\ &\ge H_{\mu_n}\Big(\bigvee_{i\in F_n} G^{-i}\mathcal{Q}|\mathfrak{F}_\Omega\times X\Big)-\frac{2q\#J_n\log\#\mathcal{Q}}{\#F_n},\end{aligned}$$

where the first inequality is due to the concavity of the conditional entropy function and the second inequality is due to the first case for intervals. Note that $(F_n)_{n\in\mathfrak{n}}$ is a Følner sequence with $\underline{d}^{\mathfrak{n}}(\mathcal{F})>0$. Therefore, we have

$$\limsup_{n\in\mathfrak{n}\to\infty}\frac{2q\#J_n\log\#\mathcal{Q}}{\#F_n}\le\limsup_{n\in\mathfrak{n}\to\infty}\frac{2q\#\partial F_n\log\#\mathcal{Q}}{\#F_n}=0. \qquad \square$$

2.5. *Principle extension.* For $k\in\mathbb{N}$, we denote $\mathcal{G}^k(T\mathcal{M}):=\bigsqcup_{x\in\mathcal{M}}\mathcal{G}^k(T_x\mathcal{M})$, where $\mathcal{G}^k(T_x\mathcal{M})$ is the Grassmannian that consists of all k-dim subspaces of $T_x\mathcal{M}$. Let $\mathcal{G}^k(T\Omega)$ be denoted in the same way. Let

$$\begin{aligned}\mathbb{F}:\Omega\times\mathcal{G}^k(T\mathcal{M})&\to\Omega\times\mathcal{G}^k(T\mathcal{M})\\ (\omega,x,E)&\mapsto(\theta\omega,f_\omega(x),D_xf_\omega E).\end{aligned}$$

We denote

$$\mathbb{F}_\omega(x,E):=(f_\omega(x),D_xf_\omega(E)) \tag{2.9}$$

for any $(x,E)\in\mathcal{G}^k(T\mathcal{M})$ and let $\mathbb{F}_\omega^i$ be generated as (1.1) by $\mathbb{F}_\omega$. We note that $\mathbb{F}$ is isomorphic to the following map:

$$\begin{aligned}\tilde{F}:\mathcal{G}^0(T\Omega)\times\mathcal{G}^k(T\mathcal{M})&\to\mathcal{G}^0(T\Omega)\times\mathcal{G}^k(T\mathcal{M})\\ ((\omega,x),(0,E)^T)&\mapsto(F(\omega,x),D_{(\omega,x)}F(0,E)^T)=((\theta\omega,f_\omega(x)),(0,D_xf_\omega E)^T).\end{aligned}$$

Both $\tilde{F}$ and $\mathbb{F}$ are extensions of F. Denote $\pi:\Omega\times\mathcal{G}^k(T\mathcal{M})\to\Omega\times\mathcal{M}$ to be the natural projection. The following lemma says that both $\tilde{F}$ and $\mathbb{F}$ are principle extensions of F, whose proof basically follows the proof of [27, Theorem 1.1].

LEMMA 2.6. *For any $\mathbb{F}$-invariant Borel probability measure μ on $\Omega\times\mathcal{G}^k(T\mathcal{M})$, one has $h_\mu(\mathbb{F})=h_{\pi_*\mu}(F)$. As a consequence of the Abramov–Rohlin formula,*

$$h_\mu^{(r)}(\mathbb{F})=h_{\pi_*\mu}^{(r)}(F). \tag{2.10}$$

Proof of Lemma 2.6. We denote $E_k(x):=\Lambda^k(T_x\mathcal{M})$ and $E_k=\bigsqcup_{x\in\mathcal{M}}E_k(x)$, where Λ^k is the kth exterior power. Let $A_k:\mathcal{G}^0(T\Omega)\times E_k\to\mathcal{G}^0(T\Omega)\times E_k$ be a vector bundle endomorphism over $F:\Omega\times\mathcal{M}\to\Omega\times\mathcal{M}$ and be defined by $(A_k)_{(\omega,x)}(0,e_1\wedge\cdots\wedge e_k)=(0,D_xf_\omega e_1\wedge\cdots\wedge D_xf_\omega e_k)$. Note that for any two bases

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$\{e_1, \dots, e_k\}$ and $\{w_1, \dots, w_k\}$ spanning the same k-dimensional subspaces of $T_x\mathcal{M}$ if and only if $w_1 \wedge \cdots \wedge w_k = ae_1 \wedge \cdots \wedge e_k$ for some $a \in \mathbb{R}$, it gives an equivalence relation $\sim$. Let $D_k(x) = \{e_1 \wedge \cdots \wedge e_k |\ e_1, \dots, e_k$ be linearly independent in $T_x\mathcal{M}\}$ and set $G_k(x) = D_k(x)/\sim$, $G_k = \bigsqcup_{x\in\mathcal{M}} G_k(x)$. We note that

$$G_k(x) \subset P(E_k(x)) := \{[u] |\ v \in [u] \text{ iff } v = au,\ a \neq 0,\ u, v \in E_k(x)\}$$

is a subbundle whose fiber over x is the projectivization of the set of decomposable k-vectors in $E_k(x)$. We note that $\mathcal{G}^k(T_x\mathcal{M})$ is isomorphic to $G_k(x)$ by

$$E \subset \mathcal{G}^k(T_x\mathcal{M}) \mapsto \{[e_1 \wedge \cdots \wedge e_k] |\ e_1, \dots, e_k \text{ forms a basis of } E\}.$$

Moreover, this isomorphism conjugates $\tilde{F}$ to $P(A_k) : P(\mathcal{G}^0(T\Omega)) \times G_k \subset P(\mathcal{G}^0(T\Omega) \times E_k) \to P(\mathcal{G}^0(T\Omega)) \times G_k$ over F, which is defined by $P(A_k)_{(\omega,x)}([0] \times [e_1 \wedge \cdots \wedge e_k]) = [0] \times [D_x f_\omega e_1 \wedge \cdots \wedge D_x f_\omega e_k]$. Now, for any $\mathbb{F}$-invariant Borel probability measure μ on $\Omega \times \mathcal{G}^k(T\mathcal{M})$, let $\hat{\mu}$ be the corresponding $\tilde{F}$-invariant Borel probability measure on $\mathcal{G}^0(T\Omega) \times \mathcal{G}^k(T\mathcal{M})$, and $\hat{\mu}^*$ be the $P(A_k)$-invariant Borel probability measure on $P(\mathcal{G}^0(T\Omega)) \times G_k$ by conjugation. Then, the projection of $\hat{\mu}^*$ on $\Omega \times \mathcal{M}$ is $\pi_*\mu$. By [27, Theorem 3.7], we have $h_\mu(\mathbb{F}) = h_{\hat{\mu}^*}(P(A_k)) = h_{\pi_*\mu}(F)$. □

3. *The existence of the random SRB measures*

3.1. *Construction of the invariant measure.* By assumption (1.3), we can assume that $E_{\omega^*} \subset W^u_{\text{loc}}(\omega^*, x_0)$ with $\lambda^u(E_{\omega^*}) > 0$ and there exists $\beta_0 > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\#\{0 \le j \le n :\ (\theta^j\omega^*, f^j_{\omega^*}(x)) \in \Lambda_l\} > \beta_0 > 0 \quad \text{for any } x \in E_{\omega^*}.$$

Let $T \in \mathbb{N}$ to be determined later. Furthermore, we can assume that $E_{\omega^*} \subset W^u_{\text{loc}}(\omega^*, x_0)$ with $\lambda^u(E_{\omega^*}) > 0$, and there exist $\beta := \beta_0/T > 0$ and $t \in \{0, \dots, T-1\}$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\#\{j \in \{0, \dots, n\} \cap \{t + T\mathbb{N}\} : (\theta^j\omega^*, f^j_{\omega^*}(x)) \in \Lambda_l\} > \beta > 0 \quad \text{for any } x \in E_{\omega^*}.$$

Denote $I(\omega^*, x) := I_T(\omega^*, x) = \{j \in \{t + T\mathbb{N}\} : (\theta^j\omega^*, f^j_{\omega^*}(x)) \in \Lambda_l\}$ for any $x \in E_{\omega^*}$. The following lemma is a direct corollary of [7, Lemma 2].

LEMMA 3.1. *There exists $\mathfrak{n} \in \mathfrak{N}$, Borel measurable subsets $(A^{\omega^*}_n)_{n\in\mathfrak{n}}$ of E_{ω^*} and $\mathcal{F} = \{F_n\}_{n\in\mathfrak{n}} \in \mathfrak{Q}_\mathfrak{n}$ with $\partial F_n \subset I(x)$ for all $x \in A^{\omega^*}_n$, $n \in \mathfrak{n}$ such that:*

- *$\lambda^u(A^{\omega^*}_n) \ge e^{-n\delta_n}/n^2$ for all $n \in \mathfrak{n}$ with $\delta_n \to 0$ as $n \in \mathfrak{n} \to \infty$;*
- *$\mathcal{F}$ is a Følner sequence;*
- *I is dense in $\mathcal{F}$ uniformly on $A^{\omega^*}_n$, that is,*

$$\limsup_{n\in\mathfrak{n}} \sup_{x\in A^{\omega^*}_n} d_n(F_n \backslash I_M(\omega^*, x)) \to 0 \quad \text{as } M \to \infty,$$

where $I_M(\omega^*, x)$ is the union of the interval of integers $[\![a, b[\![$ with $a, b \in E$ and $0 < b - a \le M$; and

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$$\underline{d}^{\mathfrak{n}}(\mathcal{F}) \geq \liminf_{n\in\mathfrak{n}} \inf_{x\in A_n^{\omega^*}} d_n(I(\omega^*, x) \cap F_n) \geq \beta.$$

The following is the main result in this section, which basically follows the idea of [7, Lemma 4].

PROPOSITION 3.1. *Let* $\mu_n = \lambda^u|_{A_n^{\omega^*}}/\lambda^u(A_n^{\omega^*})$. *There exists a limit measure of* $\{(1/\#F_n)\sum_{k\in F_n} F_*^k(\delta_{\omega^*} \times \mu_n)\}_{n\in\mathfrak{n}}$, *named* μ, *which is F-invariant with marginal* $\mathbb{P}$ *on* Ω. *Moreover, for* μ*-a.e.* $(\omega, x) \in \Omega \times \mathcal{M}$, *there are k positive Lyapunov exponents greater than* $\chi/2$, *and* $\dim \mathcal{M} - k$ *negative Lyapunov exponents less than* $-\chi/2$.

Proof of Proposition 3.1. Denote $\mathbb{A}_n^{\omega^*} = \{(\omega^*, x, E^u(\omega^*, x)) : x \in A_n^{\omega^*}\}$, $\mathbb{E}_{\omega^*} = \{(\omega^*, x, E^u(\omega^*, x)) : x \in E_{\omega^*}\}$, and let $\hat{\mu}_n$ be the measure on $\mathbb{A}_n^{\omega^*}$ corresponding to μ_n on $A_n^{\omega^*}$.

Let $\mathfrak{n}'$ be a subsequence of $\mathfrak{n}$ such that $\{\hat{\mu}_n^{F_n} := (1/\#F_n)\sum_{k\in F_n} \mathbb{F}_*^k\hat{\mu}_n\}_{n\in\mathfrak{n}'}$ is converging to $\hat{\mu}$ under the weak* topology. By the property of the Følner sequence and a standard argument, one can see that $\hat{\mu}$ is $\mathbb{F}$-invariant. Since $F : \Omega \times \mathcal{M} \to \Omega \times \mathcal{M}$ is a factor system of $\mathbb{F}$, the projection of $\hat{\mu}$ is F-invariant. The projection of $\hat{\mu}$ has marginal $\mathbb{P}$ on Ω since ω^* is a generic point of $\mathbb{P}$.

For $(\omega, x, E) \in \mathcal{G}^0(T\Omega) \times \mathcal{G}^k(T\mathcal{M})$, we define $\phi_n(\omega, x, E) = \log m(D_x f_\omega^n|_E)$ for any $n \in \mathbb{N}$, where m denotes the conorm. Then, ϕ_n is a continuous function on $\mathcal{G}^0(T\Omega) \times \mathcal{G}^k(T\mathcal{M})$. Moreover, one has

$$\phi_{n+m}(\omega, x, E) \geq \phi_n(\omega, x, E) + \phi_m(\mathbb{F}^n(\omega, x, E)).$$

Recall the definition of $I(\omega^*, x) := I_T(\omega^*, x)$. We can pick T sufficiently large such that for any $q > p \in I(\omega^*, x)$ and $(\omega^*, x, E^u(\omega^*, x)) \in \mathbb{E}_{\omega^*}$,

$$\phi_{q-p}(\mathbb{F}^p(\omega^*, x, E^u(\omega^*, x))) \geq (q-p)\frac{\chi}{2}. \tag{3.1}$$

In fact, for any $q > p \in I(\omega^*, x)$, then $q - p \geq T$ and $F^p(\omega^*, x) \in \Lambda_l$; therefore, we have

$$\begin{aligned}\phi_{q-p}(\mathbb{F}^p(\omega^*, x, E^u(\omega^*, x))) &= \log m(D_{f_{\omega^*}^p(x)} f_{\theta^p\omega^*}^{q-p}|_{E^u(\theta^p\omega^*, f_{\omega^*}^p(x))}) \\ &\geq \log(C(F^p(\omega^*, x))^{-1} e^{(q-p)\chi}) \geq \log(e^{-l} e^{(q-p)\chi}) \\ &\geq (q-p)\cdot\frac{\chi}{2}\end{aligned}$$

by picking T sufficiently large.

For convenience, define $\phi_n'(\omega^*, x, E) = \phi_n(\omega^*, x, E) - n\cdot(\chi/2)$ for $n \in \mathbb{N}$. For a finite subset of integers J, we denote $\phi_J(\omega^*, x, E)$ (respectively $\phi_J'(\omega^*, x, E)$) to be the sum of $\phi_{q-p}(\mathbb{F}^p(\omega^*, x, E))$ (respectively $\phi_{q-p}'(\mathbb{F}^p(\omega^*, x, E))$), where $[\![p, q[\![$ runs over all maximal intervals of integers contained in J. Finally, define $\|\phi_1'\|_{\sup} := \sup_{(\omega,x,E)\in\mathcal{G}^0(T\Omega)\times\mathcal{G}^k(T\mathcal{M})} |\phi_1'(\omega, x, E)|$.

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Fix $\alpha < 0$, $M < N \in \mathbb{N}$. For $x \in A_n^{\omega^*}$, $n \in \mathfrak{n}'$, we define

$$F_n^\alpha(\omega^*, x) = \left\{ k \in F_n,\ \frac{\phi'_N(\mathbb{F}^k(\omega^*, x, E^u(\omega^*, x)))}{N} \le \alpha \right\}.$$

For $l \in \mathbb{N}$ and $k \in \{0, \dots, N-1\}$ with $k + lN \in F_n$, define $J_{k,l} = [k + lN, k + (l+1)N) \cap \mathbb{N}$. The interval of integers $J_{k,l}$ can be written as $J_{k,l} = I_{k,l}^1(x) \sqcup I_{k,l}^2(x) \sqcup I_{k,l}^3(x) \sqcup I_{k,l}^4(x)$, where we have the following:

- $I_{k,l}^1(x)$ is the union of disjoint $I(\omega^*, x)$ irreducible interval of length less than M contained in $J_{k,l}$; by (3.1), we have $\phi'_{I_{k,l}^1}(\omega^*, x, E^u(\omega^*, x)) \ge 0$;
- $I_{k,l}^2(x)$ consists of subintervals of an $I(\omega^*, x)$ irreducible interval of length less than M, which does not lie in $J_{k,l}$, but may contain the right extreme of $J_{k,l}$; in particular, $\#I_{k,l}^2 \le 2M$ and, therefore, $\phi'_{I_{k,l}^2}(\omega^*, x, E^u(\omega^*, x)) \ge -2M\|\phi_1'\|_{\sup}$;
- $I_{k,l}^3(x) \subset F_n \backslash I_M(\omega^*, x)$;
- $I_{k,l}^4(x) \subset \mathbb{N} \backslash F_n$.

Fix $0 \le k < N$. Since $k + lN \in F_n$, if $I_{k,l}^4(x) \ne \emptyset$, then $J_{k,l}$ must contain a point of ∂F_n. By the intervals $\{J_{k,l}\}_l$ are pairwise disjoint, we have $\#(\bigcup_{l:\ I_{k,l}^4(x) \ne \emptyset} J_{k,l}) \le N \cdot \#\partial F_n$. Therefore, by the superadditivity of ϕ', for the fixed k, we obtain

$$\begin{aligned}\sum_{\substack{l:\ k+lN \in F_n^\alpha(\omega^*, x) \\ I_{k,l}^4(x) \ne \emptyset}} \phi'_{J_{k,l}}(\omega^*, x, E^u(\omega^*, x)) &\ge -\|\phi_1'\|_{\sup} \cdot \sum_{\substack{l:\ k+lN \in F_n^\alpha(\omega^*, x) \\ I_{k,l}^4(x) \ne \emptyset}} \#J_{k,l} \\ &\ge -N \cdot \#\partial F_n \cdot \|\phi_1'\|_{\sup}.\end{aligned}$$

If $I_{k,l}^4(x) = \emptyset$, by the superadditivity of ϕ', we obtain

$$\begin{aligned}\phi'_{J_{k,l}}(\omega^*, x, E^u(\omega^*, x)) &\ge \phi'_{I_{k,l}^1}(\omega^*, x, E^u(\omega^*, x)) + \sum_{i=2,3} \phi'_{I_{k,l}^i}(\omega^*, x, E^u(\omega^*, x)) \\ &\ge -(\#I_{k,l}^3 + 2M) \cdot \|\phi_1'\|_{\sup}.\end{aligned}$$

As $F_n \subset \{0, \dots, n-1\}$, the cardinality of $\{l :\ k + lN \in F_n\}$ is less than or equal to $\lceil n/N \rceil$. Also by $\{J_{k,l}\}_l$ are pairwise disjoint, $\sum_\ell \#I_{k,l}^3 \le \#(F_n \backslash I_M(\omega^*, x))$. Therefore, we have

$$\sum_{\substack{l:\ k+lN \in F_n^\alpha(\omega^*, x) \\ I_{k,l}^4(x) = \emptyset}} \phi'_{J_{k,l}}(\omega^*, x, E^u(\omega^*, x)) \ge -\left(\#(F_n \backslash I_M(\omega^*, x)) + 2M \left\lceil \frac{n}{N} \right\rceil \right) \|\phi_1'\|_{\sup}.$$

Combining these two cases, for each fixed $0 \le k < N$, we have

$$\begin{aligned}&\sum_{l:\ k+lN \in F_n^\alpha(\omega^*, x)} \phi'_{J_{k,l}}(\omega^*, x, E^u(\omega^*, x)) \\ &\quad \ge -\left(N \cdot \partial F_n + \#(F_n \backslash I_M(\omega^*, x)) + 2M \left\lceil \frac{n}{N} \right\rceil \right) \cdot \|\phi_1'\|_{\sup}. \qquad (3.2)\end{aligned}$$

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For each $(\omega^*, x, E^u(\omega^*, x)) \in \mathbb{A}_n^{\omega^*}$, define $\delta^{F_n}_{(\omega^*,x,E^u(\omega^*,x))} = (1/\#F_n)\sum_{j\in F_n} \mathbb{F}^k_* \delta_{(\omega^*,x,E^u(\omega^*,x))}$. Note that (recall that $\phi'_{J_{k,l}} = \phi'_N$)

$$\int_{\{\phi'_N/N\le\alpha\}} \frac{\phi'_N}{N}\, ad\delta^{F_n}_{(\omega^*,x,E^u(\omega^*,x))} = \frac{1}{N\#F_n} \sum_{(k,l):\ k+lN\in F_n^\alpha(\omega^*,x)} \phi'_{J_{k,l}}(\omega^*, x, E^u(\omega^*, x)).$$

By (3.2), after summing over $k \in \{0, \dots, N-1\}$,

$$\begin{aligned}&\sum_{(k,l):\ k+lN\in F_n^\alpha(\omega^*,x)} \phi'_{J_{k,l}}(\omega^*, x, E^u(\omega^*, x))\\ &\quad\ge -N\left(N\cdot\partial F_n + \#(F_n\backslash I_M(\omega^*, x)) + 2M\left\lceil\frac{n}{N}\right\rceil\right)\cdot\|\phi'_1\|_{\sup}.\end{aligned}$$

Hence, dividing by $N\cdot\#F_n$, we have

$$\begin{aligned}&\int_{\{\phi'_N/N\le\alpha\}} \frac{\phi'_N}{N}\, d\delta^{F_n}_{(\omega^*,x,E^u(\omega^*,x))}\\ &\quad\ge -\frac{1}{\#F_n}\left(N\cdot\partial F_n + \#(F_n\backslash I_M(\omega^*, x)) + 2M\left\lceil\frac{n}{N}\right\rceil\right)\cdot\|\phi'_1\|_{\sup}.\end{aligned}$$

Recall that $\hat\mu_n$ is supported on $\mathbb{A}_n^{\omega^*}$. We integrate with $\hat\mu_n^{F_N}$, then

$$\begin{aligned}\int_{\{\phi'_N/N\le\alpha\}} \frac{\phi'_N}{N}\, d\hat\mu_n^{F_n} &\ge -\frac{1}{\#F_n}\left(N\cdot\partial F_n + \#(F_n\backslash I_M(\omega^*, x)) + 2M\left\lceil\frac{n}{N}\right\rceil\right)\cdot\|\phi'_1\|_{\sup}\\ &\ge -\frac{1}{d_n(F_n)}\left(Nd_n(\partial F_n) + \sup_{x\in A_n^{\omega^*}} d_n(F_n\backslash I_M(\omega^*, x))\right.\\ &\qquad\left. + \frac{M}{n}\left\lceil\frac{n}{N}\right\rceil\right)\|\phi'_1\|_{\sup}.\end{aligned}$$

Note that $\{\phi'_N/N \le \alpha\}$ is closed, so by weak* convergence, we have

$$\begin{aligned}\alpha\hat\mu\left\{\frac{\phi'_N}{N}\le\alpha\right\} &\ge \int_{\{\phi'_N/N\le\alpha\}} \frac{\phi'_N}{N}\, d\hat\mu\\ &\ge -\frac{1}{\beta}\limsup_{n\in\mathfrak{n}'}\left(\sup_{x\in A_n^{\omega^*}} d_n(F_n\backslash I_M(\omega^*, x)) + \frac{M}{N}\right)\|\phi'_1\|_{\sup}.\end{aligned}$$

Note that $\alpha < 0$; therefore, we get for all M:

$$\lim_{N\to\infty}\hat\mu\left\{\frac{\phi'_N}{N}\le\alpha\right\} \le -\frac{1}{\alpha\beta}\limsup_{n\in\mathfrak{n}'}\left(\sup_{x\in A_n^{\omega^*}} d_n(F_n\backslash I_M(\omega^*, x))\right)\|\phi'_1\|_{\sup}.$$

By taking M to infinity, we arrive at $\lim_{N\to\infty}\hat\mu\{\phi'_N/N \le \alpha\} = 0$ by Lemma 3.1. Therefore, $\lim_{N\to\infty}(\phi'_N/N) = \lim_{N\to\infty}(\phi_N/N) - \chi/2 \ge \alpha$ for $\hat\mu$-a.e $(\omega, x, E) \in \mathcal{G}^0(T\Omega) \times \mathcal{G}^k(T\mathcal{M})$, where the limit exists due to the Kingman's subadditive ergodic theorem. Since $\alpha < 0$ is arbitrary, we conclude that

$$\phi_* = \lim_N \frac{\phi_N}{N} \ge \frac{\chi}{2} \quad \text{for } \hat\mu\text{-a.e. } (\omega, x, E) \in \mathcal{G}^0(T\Omega) \times \mathcal{G}^k(T\mathcal{M}). \tag{3.3}$$

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For $(\omega, x, E) \in \mathcal{G}^0(T\Omega) \times \mathcal{G}^{(\dim \mathcal{M}-k)}(T\mathcal{M})$, we define $\psi_n(\omega, x, E) = \log \|D_x f_\omega^n|_E\|^{-1}$. Then, ψ_n is a continuous function on $\mathcal{G}^0(T\Omega) \times \mathcal{G}^{(\dim \mathcal{M}-k)}(T\mathcal{M})$. It is clear that $\psi_{n+m}(\omega, x, E) \geq \psi_n(\omega, x, E) + \psi_m(\mathbb{F}^n(\omega, x, E))$. Denote $\bar{E}_{\omega^*} = \{(\omega^*, x, E^s(\omega^*, x)) : x \in E_{\omega^*}\}$ and $\bar{A}_n^{\omega^*} = \{(\omega^*, x, E^s(\omega^*, x)) : x \in A_n^{\omega^*}\}$. We let $\bar{\mu}_n$ be the measure on $\bar{A}_n^{\omega^*}$ corresponding to μ_n on $A_n^{\omega^*}$. Note that for any $q > p \in I(\omega^*, x)$, then $q - p \geq T$ and $F^p(\omega^*, x) \in \Lambda_l$; therefore, we have

$$\begin{aligned}\psi_{q-p}(\mathbb{F}^p(\omega^*, x, E^s(\omega^*, x))) &= \log \|D_{f_{\omega^*}^p(x)} f_{\theta^p\omega^*}^{q-p}|_{E^s(\theta^p\omega^*, f_{\omega^*}^p(x))}\|^{-1} \\ &\geq \log(C(F^p(\omega^*, x))^{-1} e^{(q-p)\chi}) \geq \log(e^{-l} e^{(q-p)\chi}) \\ &\geq (q-p)\cdot\frac{\chi}{2}.\end{aligned}$$

By the same proof as (3.3), for any $\mathfrak{n}''$ being a subsequence of $\mathfrak{n}'$ such that $\{(1/\#F_n)\sum_{k\in F_n} \mathbb{F}_*^k \bar{\mu}_n\}_{n\in\mathfrak{n}''}$ converges to $\bar{\mu}$, one has

$$\psi_* = \lim_N \frac{\psi_N}{N} \geq \frac{\chi}{2} \quad \text{for } \bar{\mu}\text{-a.e. } (\omega, x, E) \in \mathcal{G}^0(T\Omega) \times \mathcal{G}^{\dim \mathcal{M}-k}(T\mathcal{M}). \tag{3.4}$$

By the construction, we have

$$Proj_{\Omega\times\mathcal{M}}(\hat{\mu}) = Proj_{\Omega\times\mathcal{M}}(\bar{\mu}) = \lim_{n\in\mathfrak{n}''\to\infty} \frac{1}{\#F_n} \sum_{k\in F_n} F_*^k(\delta_{\omega^*} \times \mu_n).$$

We name $Proj_{\Omega\times\mathcal{M}}(\hat{\mu})$ by μ, and this measure has marginal $\mathbb{P}$ on Ω since ω^* is a generic point of $\mathbb{P}$. We conclude from (3.3) and (3.4) that for μ-a.e. $(\omega, x) \in \Omega \times \mathcal{M}$, there are k positive Lyapunov exponents greater than $\chi/2$ and $\dim \mathcal{M} - k$ negative Lyapunov exponents less than $-\chi/2$. The proof of Proposition 3.1 is complete. □

3.2. *Fiber Følner Gibbs property.* In this subsection, we prove the fiber Følner Gibbs property. Recall that $\{A_n^{\omega^*}\}_{n\in\mathfrak{n}}$ and $\mathcal{F} = \{F_n\}_{n\in\mathfrak{n}}$ are given in Lemma 3.1. We first denote

$$\tilde{C} := \sup_{(\omega,x,E)\in\Omega\times\mathcal{G}^{\dim E^u}(T\mathcal{M})} \log|\det(D_x f_\omega|_E)|.$$

Fix $\gamma > 0$. For each $M > 0$, let $\delta_1 = \delta_1(l, M, \gamma)$ be defined as in Lemma 2.4 and let $C(l)$ be given in Corollary 2.1. We denote

$$C_M^* = C(l)\delta_1(l, M, \gamma)^{-\dim E^u}. \tag{3.5}$$

By direct computation, one can see that for fixed l and γ,

$$\limsup_{M\to\infty} \frac{C_M^*}{M} < \infty. \tag{3.6}$$

Then, we have

$$(C_M^*)^{-1} \leq \lambda^u(W_{\delta_1}^u(\omega, x)) \leq \lambda^u(W_{8\delta_1}^u(\omega, x)) \leq 1. \tag{3.7}$$

PROPOSITION 3.2. *For any $n \gg M$, any finite measurable partition $\mathcal{P}$ of M with*

$$\operatorname{diam}(\mathcal{P}) < \delta < \delta_1(l, M, \gamma),$$

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and any $x \in A_n^{\omega^*}$, *one has*

$$\lambda^u(A_n^{\omega^*} \cap P_{\omega^*}^{F_n}(x)) \le (C_M^* \cdot e^{\tilde{C}} \cdot e^{\gamma})^{\#\partial F_n + \sup_{y \in A_n^{\omega^*}} \lceil (\#(F_n \setminus I_M(\omega^*, y))/M) \rceil} \cdot e^{3n\gamma} \times e^{-\sum_{i \in F_n} J^u(f_{\omega^*}^i(x))} \cdot e^{3\tilde{C} \sup_{y \in A_n^{\omega^*}} F_n \setminus I_M(\omega^*, y)} \cdot e^{3n/M},$$

where $\mathcal{P}_{\omega^*}^{F_n} = \bigvee_{i \in F_n} (f_{\omega^*}^i)^{-1} \mathcal{P}$ *and* $J^u(f_{\omega^*}^i(x)) := \log |\det(D_{f_{\omega^*}^i(x)} f_{\theta^i(\omega^*)}|_{E^u(\theta^i \omega^*, f_{\omega^*}^i(x))})|$ *for* $i \in F_n$.

Proof of Proposition 3.2. For any $x \in E_{\omega^*}$, $M > 0$, and any $n \gg M$, let $L_n(\omega^*, x) \subset I(\omega^*, x) \cap [\![0, n[\![$ be the disjoint union of $\{a, b\}$ satisfying

$$a, b \in I(\omega^*, x), \quad b - a > M \quad \text{and} \quad]\!]a, b[\![\cap I(\omega^*, x) = \emptyset. \tag{3.8}$$

Denote $n_{(\omega^*, x)}$ to be the number of pairs in $L_n(\omega^*, x)$. By (3.8), we have $n_{(\omega^*, x)} \le \lceil n/M \rceil$ and, therefore,

$$\#L_n(\omega^*, x) \le \left\lceil \frac{n}{M} \right\rceil.$$

We consider $S_n = \{\bigsqcup_{i=1}^k \{a_i, b_i\} |\ b_i - a_i > M,\ k \le \lceil n/m \rceil\}$, that is, all the possibilities of $L_n(\omega^*, x)$. When n is sufficiently large, one has

$$\#S_n \le \sum_{k=1}^{2\lceil n/M \rceil} \binom{n}{l} \le e^{3n/M}. \tag{3.9}$$

For any $s \in S_n$, we denote $A_{n,s}^{\omega^*} = \{z \in A_n^{\omega^*} |\ L_n(x) = s \in S_n\}$. Then, for any $x \in A_n^{\omega^*}$,

$$\lambda^u(A_n^{\omega^*} \cap \mathcal{P}_{\omega^*}^{F_n}(x)) \le \sum_{s \in S_n} \lambda^u(A_{n,s}^{\omega^*} \cap \mathcal{P}_{\omega^*}^{F_n}(x)). \tag{3.10}$$

Next, we are going to estimate the right side of the above inequality for fixed $x \in A_n^{\omega^*}$.

Recall that $I_M(\omega^*, x)$ is the union of the intervals $[\![c, d[\![$ with $c, d \in I(\omega^*, x)$ and $0 < d - c \le M$. Fix $s = \bigsqcup_{i=1}^k \{a_i, b_i\}$ and for any $x \in A_{n,s}^{\omega^*}$, we let $I_M(s) = I_M(\omega^*, x) \cap [\![0, n[\![$. By the definition of $A_{n,s}^{\omega^*}$, we have

$$I_M(s) = [\![0, a_1[\![\cup [\![b_1, a_2[\![\cup \cdots \cup [\![b_k, n[\![.$$

We decompose F_n into the following two parts:

- $F_{n,s}^2 := F_n \setminus I_M(s)$;
- $F_{n,s}^1 := F_n \setminus F_{n,s}^1 = \bigcup_{i=0}^{m_s - 1} [\![c_i, d_i]\!]$.

Without loss of generality, we can assume $0 \in F_n$ for all $n \in \mathfrak{n}$ and, therefore, $C_0 = 0$. Note that $F_{n,s}^1 = F_n \cap I_M(s)$; therefore, for any $0 \le i \le m_s - 1$, $[\![c_i, d_i]\!]$ is M-$\{(\omega^*, x) : x \in A_{n,s}^{\omega^*}\}$-admissible and

$$m_s \le \#\partial F_n + \left\lceil \frac{\#(F_n \setminus I_M(s))}{M} \right\rceil. \tag{3.11}$$

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CLAIM 3.1. *There exists a constant $\tilde{C} > 0$ such that for any $x_s \in A^{\omega^*}_{n,s} \cap P^{F_n}_{\omega^*}(x)$, one has*

$$\begin{aligned}\lambda^u(A^{\omega^*}_{n,s} \cap P^{F_n}_{\omega^*}(x)) &\le \lambda^u(A^{\omega^*}_{n,s} \cap P^{F^1_{n,s}}_{\omega^*}(x)) \\ &\le (C^*_M)^{m_s} \cdot e^{m_s\gamma + 2(d_{m_s-1}-d_0)\gamma} \cdot e^{m_s\tilde{C}} \cdot e^{-\sum_{i\in F^1_{n,s}} J^u(f^i_{\omega^*}(x_s))}, \quad (3.12)\end{aligned}$$

where $\mathcal{P}^{F^1_{n,s}}_{\omega^*} = \bigvee_{i\in F^1_{n,s}} (f^i_{\omega^*})^{-1}\mathcal{P}$.

Proof of Claim 3.1. The first inequality is clear; therefore, we only need to prove the second inequality. For $k \in [\![0, m_s - 1]\!]$, we denote $F^1_{n,s}(k) := \bigcup_{i=0}^{k} [\![c_i, d_i]\!]$.

Step 1. When $k = 0$, we are going to estimate $\lambda^u(A^{\omega^*}_{n,s} \cap P^{F^1_{n,s}(0)}_{\omega^*}(x))$. Note that $[\![c_0, d_0]\!] = [\![0, d_0]\!]$ is M-$\{(\omega^*, x) : x \in A^{\omega^*}_{n,s}\}$-admissible. Denote $\Delta_0 = A^{\omega^*}_{n,s} \cap \mathcal{P}^{F^1_{n,s}(0)}_{\omega^*}(x)$. By Lemma 2.3 and $\operatorname{diam}(\mathcal{P}) < \delta$, and note that $x_s \in A^{\omega^*}_{n,s} \cap P^{F_n}_{\omega^*}(x)$, we have

$$f^{d_0}_{\omega^*}(\Delta_0) \subset f^{d_0}_{\omega^*}(W^u_{4\delta}(\omega^*, x_s) \cap B_{d_0}(\omega^*, x_s, \delta)) \subset W^u_{4\delta}(\theta^{d_0}\omega^*, f^{d_0}_{\omega^*}(x_s)).$$

Then, we have

$$\begin{aligned}\lambda^u(\Delta_0) &\le e^{\gamma}\lambda^u(f^{d_0}_{\omega^*}\Delta_0)|\det(D_x f^{d_0}_{\omega^*}|_{E^u(\omega^*,x_s)})|^{-1} \quad \text{by Lemma 2.3(2),} \\ &\le e^{\gamma}\lambda^u(W^u_{4\delta}(\theta^{d_0}\omega^*, f^{d_0}_{\omega^*}(x_s))) \cdot |\det(D_x f^{d_0}_{\omega^*}|_{E^u(\omega^*,x_s)})|^{-1} \\ &\le e^{\gamma} \cdot e^{-\sum_{i\in[0,d_0]} J^u(f^i_{\omega^*}(x_s))} \cdot e^{\tilde{C}} \quad \text{by Corollary 2.1.}\end{aligned}$$

Step 2. We consider the case of $k = 1$. Denote $\Delta_1 = A^{\omega^*}_{n,s} \cap \mathcal{P}^{F^1_{n,s}(1)}_{\omega^*}(x)$, then

$$f^{d_0}_{\omega^*}(\Delta_1) = f^{d_0}_{\omega^*}(\Delta_0) \cap f^{-(c_1-d_0)}_{\theta^{c_1}\omega^*}\mathcal{P}^{[\![0,d_1-c_1]\!]}_{\theta^{c_1}\omega^*}(f^{c_1}_{\omega^*}(x)).$$

In addition, note that $[\![c_1, d_1]\!]$ is M-$\{(\omega^*, x) : x \in A^{\omega^*}_{n,s}\}$-admissible; therefore, we have

$$\begin{aligned}f^{d_0}_{\omega^*}(\Delta_1) \subset\ & W^u_{4\delta}(\theta^{d_0}\omega^*, f^{d_0}_{\omega^*}(x_s)) \cap \Lambda_l(\theta^{d_0}\omega^*) \cap f^{-(c_1-d_0)}_{\theta^{d_0}\omega^*}\Lambda_l(\theta^{c_1}\omega^*) \cap f^{-(c_1-d_0)}_{\theta^{c_1}\omega^*} \\ &\times B(f^{c_1}_{\omega^*}(x_s), \delta).\end{aligned}$$

Then, there exists finite points $\{y^1_i\}_{i\in I_1} \subset f^{d_0}_{\omega^*}(\Delta_1)$ such that:

(1) $W^u_{4\delta}(\theta^{c_1}\omega^*, f^{c_1-d_0}_{\theta^{d_0}\omega^*}y^1_i)$ are mutually disjoint; and

(2) $f^{d_0}_{\omega^*}(\Delta_1) \subset \bigcup_{i\in I_1} f^{-(c_1-d_0)}_{\theta^{c_1}\omega^*}(W^u_{4\delta}(\theta^{c_1}\omega^*, f^{c_1-d_0}_{\theta^{d_0}\omega^*}(y^1_i)) \cap B(f^{c_1}_{\omega^*}x_s, \delta)) \subset W^u_{8\delta}(\theta^{d_0}\omega^*, f^{d_0}_{\omega^*}(x_s))$.

We denote $\Delta^i_1(\theta^{c_1}\omega^*) = f^{c_1}_{\omega^*}(\Delta_1) \cap W^u_{4\delta}(\theta^{c_1}\omega^*, f^{c_1-d_0}_{\theta^{d_0}\omega^*}(y^1_i)) \cap B(f^{c_1}_{\omega^*}(x_s), \delta)$ for $i = 1 \in I_1$, and by the choice of $\{y^1_i\}_{i\in I_1}$,

$$f^{c_1}_{\omega^*}(\Delta_1) = \bigcup_{i=1}^{k} \Delta^i_1(\theta^{c_1}\omega) \quad \text{and} \quad \Delta^i_1(\theta^{c_1}\omega^*) \quad \text{are mutually disjoint.}$$

We note that $[\![0, d_1 - c_1]\!]$ is M-$\{(\theta^{c_1}\omega^*, x) : x \in \Delta^i_1(\theta^{c_1}\omega^*)\}$-admissible. Applying Lemma 2.3 on $\Delta^i_1(\theta^{c_1}\omega^*)$, we have

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$$f^{d_1-c_1}_{\theta^{c_1}\omega^*}(\Delta^i_1(\theta^{c_1}\omega^*)) \subset f^{d_1-c_1}_{\theta^{c_1}\omega^*}(W^u_{4\delta}(\theta^{c_1}\omega^*, f^{c_1-d_0}_{\theta^{d_0}\omega^*}(y^1_i)) \cap B_{d_1-c_1}(f^{c_1-d_0}_{\theta^{d_0}\omega^*}(y^1_i), \theta^{c_1}\omega^*, 2\delta))$$
$$\subset W^u_{4\delta}(\theta^{d_1}\omega^*, f^{d_1-c_1}_{\theta^{c_1}\omega^*} f^{c_1-d_0}_{\theta^{d_0}\omega^*}(y^1_i))$$

and

$$\begin{aligned}
&\lambda^u(\Delta^i_1(\theta^{c_1}\omega^*)) \\
&\quad\le e^{\gamma}\lambda^u(f^{d_1-c_1}_{\theta^{c_1}\omega^*}\Delta^i_1(\theta^{c_1}\omega^*)) \cdot |\det D_{f^{c_1-d_0}_{\theta^{d_0}\omega^*}(y^1_i)} f^{d_1-c_1}_{\theta^{c_1}\omega^*}|_{E^u(\theta^{c_1}\omega^*, f^{c_1-d_0}_{\theta^{d_0}\omega^*}(y^1_i))}|^{-1} \\
&\quad\le e^{\gamma} \cdot \frac{|\det D_{f^{c_1-d_0}_{\theta^{d_0}\omega^*}(y^1_i)} f^{d_1-c_1}_{\theta^{c_1}\omega^*}|_{E^u(\theta^{c_1}\omega^*, f^{c_1-d_0}_{\theta^{d_0}\omega^*}(y^1_i))}|^{-1}}{|\det D_{f^{c_1}_{\omega^*}(x_s)} f^{d_1-c_1}_{\theta^{c_1}\omega^*}|_{E^u(\theta^{c_1}\omega^*, f^{c_1}_{\omega^*}(x_s))}|^{-1}} \\
&\qquad\cdot |\det D_{f^{c_1}_{\omega^*}(x_s)} f^{d_1-c_1}_{\theta^{c_1}\omega^*}|_{E^u(\theta^{c_1}\omega^*, f^{c_1}_{\omega^*}(x_s))}|^{-1} \\
&\quad\le e^{\gamma} \cdot e^{(d_1-c_1)\gamma} \cdot |\det D_{f^{c_1}_{\omega^*}(x_s)} f^{d_1-c_1}_{\theta^{c_1}\omega^*}|_{E^u(\theta^{c_1}\omega^*, f^{c_1}_{\omega^*}(x_s))}|^{-1},
\end{aligned} \tag{3.13}$$

where the second inequality is due to Corollary 2.1 and the third inequality is due to Lemma 2.4. Note that $\sum_i \lambda^u(f^{-(c_1-d_0)}_{\theta^{c_1}\omega^*} W^u_{4\delta_1}(\theta^{c_1}\omega^*, f^{c_1-d_0}_{\theta^{d_0}\omega^*}(y^1_i))) \le \lambda^u(W^u_{8\delta_1}(\theta^{d_0}\omega^*, f^{d_0}_{\omega^*}(x_s))) \le 1$ and

$$\begin{aligned}
&\lambda^u(f^{-(c_1-d_0)}_{\theta^{c_1}\omega^*} W^u_{4\delta_1}(\theta^{c_1}\omega^*, f^{c_1-d_0}_{\theta^{d_0}\omega^*}(y^1_i))) \\
&\quad= \int_{W^u_{4\delta_1}(\theta^{c_1}\omega^*, f^{c_1-d_0}_{\theta^{d_0}\omega^*}(y^1_i))} \det(D_y f^{-(c_1-d_0)}_{\theta^{c_1}\omega^*}|_{E^u(\theta^{c_1}\omega^*, y)})\, d\lambda^u(y) \\
&\quad\ge e^{-(c_1-d_0)\gamma} \cdot (\det(D_{y^1_i} f^{c_1-d_0}_{\theta^{f_0}\omega^*}|_{E^u(\theta^{d_0}\omega^*, y^1_i)}))^{-1} \cdot \lambda^u(W^u_{4\delta_1}(\theta^{c_1}\omega^*, f^{c_1-d_0}_{\theta^{d_0}\omega^*}(y^1_i))) \\
&\quad\ge e^{-(c_1-d_0)\gamma} \cdot (\det(D_{y^1_i} f^{c_1-d_0}_{\theta^{f_0}\omega^*}|_{E^u(\theta^{d_0}\omega^*, y^1_i)}))^{-1} \cdot (C^*_M)^{-1},
\end{aligned} \tag{3.14}$$

where the first inequality is due to Lemma 2.4 and the second inequality is due to (3.7). The inequality (3.14) enables us to estimate the sum of determinants in the gaps of $F^1_{n,s}(1)$, that is,

$$\begin{aligned}
&\sum_i \left(\det(D_{y^1_i} f^{c_1-d_0}_{\theta^{f_0}\omega^*}|_{E^u(\theta^{d_0}\omega^*, y^1_i)}) \right)^{-1} \\
&\quad\le C^*_M e^{(c_1-d_0)\gamma} \sum_i \lambda^u(f^{-(c_1-d_0)}_{\theta^{c_1}\omega^*} W^u_{4\delta}(\theta^{c_1}\omega^*, f^{c_1-d_0}_{\theta^{d_0}\omega^*}(y^1_i))) \\
&\quad\le C^*_M e^{(c_1-d_0)\gamma}
\end{aligned} \tag{3.15}$$

Then, we have estimates

$$\begin{aligned}
\lambda^u(f^{d_0}_{\omega^*}(\Delta_1)) &= \sum_i \lambda^u(f^{-(c_1-d_0)}_{\theta^{c_1}\omega^*}\Delta^i_1(\theta^{c_1}\omega^*)) \\
&\le \sum_i e^{(c_1-d_0)\gamma} \cdot (\det(D_{y^1_i} f^{c_1-d_0}_{\theta^{d_0}\omega^*}|_{E^u(\theta^{d_0}\omega^*, y^1_i)}))^{-1} \cdot \lambda^u(\Delta^i_1(\theta^{c_1}\omega^*)) \\
\text{by (3.13)}, &\le e^{\gamma+(d_1-d_0)\gamma} |\det D_{f^{c_1}_{\omega^*}(x_s)} f^{d_1-c_1}_{\theta^{c_1}\omega^*}|_{E^u(\theta^{c_1}\omega^*, f^{c_1}_{\omega^*}(x_s))}|^{-1}.
\end{aligned}$$

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$$\times \sum_i (\det(D_{y_i^1} f^{c_1-d_0}_{\theta^{f_0}\omega^*}|_{E^u(\theta^{d_0}\omega^*, y_i^1)}))^{-1}$$

$$\text{by (3.15)}, \le e^{\gamma+(d_1-d_0)\gamma} |\det D_{f^{c_1}_{\omega^*}(x_s)} f^{d_1-c_1}_{\theta^{c_1}\omega^*}|_{E^u(\theta^{c_1}\omega^*, f^{c_1}_{\omega^*}(x_s))}|^{-1} C^*_M \cdot e^{(c_1-d_0)\gamma}.$$

Then, finally, we have

$$\begin{aligned}\lambda^u(\Delta_1) &\le e^{\gamma}\lambda^u(f^{d_0}_{\omega^*}\Delta_1)|\det(D_x f^{d_0}_{\omega^*}|_{E^u(\omega^*,x_s)})|^{-1} \quad \text{by Lemma 2.3(2)}\\ &\le e^{2\gamma+2(d_1-d_0)\gamma}\cdot C^*_M \cdot e^{-\sum_{i\in F^1_{n,s}(1)} J^u(f^i_{\omega^*}(x_s))}\cdot e^{2\tilde{C}}. \end{aligned} \tag{3.16}$$

Step 3. We consider the case $k = 2$. Denote $\Delta_2 = A^{\omega^*}_{n,s} \cap \mathcal{P}^{F^1_{n,s}(2)}_{\omega^*}(x) = \Delta_1 \cap \mathcal{P}^{[\![c_2,d_2]\!]}_{\omega^*}(x)$. Notice that $[\![c_2, d_2]\!]$ is M-$\{(\omega^*, x) : x \in A^{\omega^*}_{n,s}\}$-admissible, and

$$f^{d_1}_{\omega^*}(\Delta_2) \subset \bigcup_{i\in I_1} (f^{d_1-c_1}_{\theta^{c_1}\omega^*}(\Delta^j_1(\theta^{c_1}\omega^*)) \cap f^{-(c_2-d_1)\mathcal{P}^{[\![0,d_2-c_2]\!]}_{\theta^{c_2}\omega^*}}_{\theta^{c_2}\omega}(f^{c_2}_{\omega^*}(x))),$$

then for all $j \in I_1$, there exist finite points $\{z^{j,2}_i\}_{i\in I_{j,2}} \subset f^{d_0}_{\omega^*}\Delta_2$ such that $\{W^u_{4\delta}(\theta^{c_2}\omega^*, f^{c_2-d_0}_{\theta^{d_0}\omega^*}(z^{j,2}_i))\}_{i\in I_{j,2}}$ are mutually disjoint, and

$$\begin{aligned} f^{d_1-c_1}_{\theta^{c_1}\omega^*}(\Delta^j_1(\theta^{c_1}\omega^*)) &\subset \bigcup_{i\in I_{j,2}} f^{-(c_2-d_1)}_{\theta^{c_2}\omega^*}(W^u_{4\delta}(\theta^{c_2}\omega^*, f^{c_2-d_0}_{\theta^{d_0}\omega^*}(z^{j,2}_i)) \cap B(f^{c_2}_{\omega^*}(x_s), \delta))\\ &\subset W^u_{8\delta}(\theta^{d_1}\omega^*, f^{d_1-d_0}_{\theta^{d_0}\omega^*}(y^1_j)).\end{aligned}$$

For $j \in I_1$ and $i \in I_{j,2}$, we denote

$$\Delta^{j,i}_1(\theta^{c_1}\omega^*) = f^{c_1}_{\omega^*}(\Delta_2) \cap f^{-(c_2-d_1)}_{\theta^{c_2}\omega^*}(W^u_{4\delta}(\theta^{c_2}\omega^*, f^{c_2-d_0}_{\theta^{d_0}\omega^*}(z^{j,2}_i)) \cap B(f^{c_2}_{\omega^*}(x_s), \delta)).$$

Now, $\bigcup_{i\in I_{j,2}} \Delta^{j,i}_1(\theta^{c_1}\omega^*)$ is in the position of Δ_1 in Step 2. By the similar estimates, we obtain

$$\lambda^u\Big(\bigcup_{i\in I_{j,2}} \Delta^{j,i}_1(\theta^{c_1}\omega^*)\Big) \le e^{2\gamma+2(d_2-c_1)\gamma}\cdot C^*_M \cdot e^{-\sum_{i\in[\![c_1,d_1]\!]\cup[\![c_2,d_2]\!]} J^u(f^i_{\omega^*}(x_s))}\cdot e^{2\tilde{C}}. \tag{3.17}$$

Then,

$$\begin{aligned}&\lambda^u(f^{d_0}_{\omega^*}(\Delta_2))\\ &= \sum_{j\in I_1} \lambda^u\Big(f^{-(c_1-d_0)}_{\theta^{c_1}\omega^*}\Big(\bigcup_{i\in I_{j,2}} \Delta^{j,i}_1(\theta^{c_1}\omega^*)\Big)\Big)\\ &\le \sum_{j\in I_1} e^{(c_1-d_0)\gamma}\cdot(\det(D_{y^1_j} f^{c_1-d_0}_{\theta^{d_0}\omega^*}|_{E^u(\theta^{d_0}\omega^*, y^1_j)}))^{-1}\cdot \lambda^u\Big(\bigcup_{i\in I_{j,2}} \Delta^{j,i}_1(\theta^{c_1}\omega^*)\Big)\\ &\le (C^*_M)^2\cdot e^{2\gamma+2(d_2-d_0)\gamma}\cdot e^{2\tilde{C}}\cdot e^{-\sum_{i\in[\![c_1,d_1]\!]\cup[\![c_2,d_2]\!]} J^u(f^i_{\omega^*}(x_s))} \quad \text{by (3.15) and (3.17)}.\end{aligned}$$

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Similar as (3.16), we obtain

$$\begin{aligned}\lambda^u(\Delta_2) &\le e^{\gamma}\lambda^u(f_{\omega^*}^{d_0}\Delta_2)|\det(D_x f_{\omega^*}^{d_0}|_{E^u(\omega^*,x_s)})|^{-1} \quad \text{by Lemma 2.3(2)}\\ &\le (C_M^*)^2\cdot e^{3\gamma+2(d_2-d_0)\gamma}\cdot e^{3\tilde{C}}\cdot e^{-\sum_{i\in F_{n,s}^1(2)} J^u(f_{\omega^*}^i(x_s))}.\end{aligned}$$

Repeating the above procedure, we can obtain

$$\lambda^u(A_{n,s}^{\omega^*}\cap P_{\omega^*}^{F_{n,s}^1}(x)) \le (C_M^*)^{m_s}\cdot e^{m_s\gamma+2(d_{m_s}-d_0)\gamma}\cdot e^{m_s\tilde{C}}\cdot e^{-\sum_{i\in F_{n,s}^1} J^u(f_{\omega^*}^i(x_s))}$$

The proof of Claim 3.1 is complete. □

Note that for $x\in A_n^{\omega^*}$, $I_M(\omega^*,x)$ may not coincide with $I_M(s)$. For the x_s in the statement of Claim 3.1 and note that $x_s\in A_{n,s}^{\omega^*}\cap P_{\omega^*}^{F_n}(x)$, moreover for those $i\in I_M(\omega^*,x)\cap I_M(\omega^*,x_s)$, one has $(\theta^i\omega^*, f_{\omega^*}^i(x))$ and $(\theta^i\omega^*, f_{\omega^*}^i(x_s))$ both belong to $\Lambda_{le^{M\epsilon}}$. Then, by Lemma 2.4, for those $i\in I_M(\omega^*,x)\cap I_M(\omega^*,x_s)$, one has

$$\left|\log\frac{\det D_{f_{\omega^*}^i(x)}f_{\theta^i\omega^*}|_{E^u(\theta^i\omega^*,f_{\omega^*}^i(x))}}{\det D_{f_{\omega^*}^i(x_s)}f_{\theta^i\omega^*}|_{E^u(\theta^i\omega^*,f_{\omega^*}^i(x_s))}}\right|\le\gamma.$$

For those i such that it may not satisfy the above estimate, we also have estimates

$$\#F_n\backslash(I_M(\omega^*,x)\cap I_M(\omega^*,x_s)) \le 2\sup_{y\in A_n^{\omega^*}} F_n\backslash I_M(\omega^*,y). \tag{3.18}$$

Combining the above two cases, we obtain

$$\left|\log\prod_{i\in F_n}\frac{\det D_{f_{\omega^*}^i(x)}f_{\theta^i\omega^*}|_{E^u(\theta^i\omega^*,f_{\omega^*}^i(x))}}{\det D_{f_{\omega^*}^i(x_s)}f_{\theta^i\omega^*}|_{E^u(\theta^i\omega^*,f_{\omega^*}^i(x_s))}}\right|\le n\gamma+2\tilde{C}\sup_{y\in A_n^{\omega^*}} F_n\backslash I_M(\omega^*,y). \tag{3.19}$$

Therefore, for $x\in A_n^{\omega^*}$ and $x_s\in A_{n,s}^{\omega^*}\cap P_{\omega^*}^{F_n}(x)$, we have

$$\begin{aligned}&\lambda^u(A_{n,s}^{\omega^*}\cap P_{\omega^*}^{F_n}(x))\\ &\le (C_M^*\cdot e^{\tilde{C}}\cdot e^{\gamma})^{m_s}\cdot e^{2n\gamma}\cdot e^{-\sum_{i\in F_{n,s}^1} J^u(f_{\omega^*}^i(x_s))} \quad \text{by Claim 3.1}\\ &\le (C_M^*\cdot e^{\tilde{C}}\cdot e^{\gamma})^{m_s}\cdot e^{2n\gamma}\cdot e^{\tilde{C}\cdot\#(F_n\backslash I_M(s))}\cdot e^{-\sum_{i\in F_n} J^u(f_{\omega^*}^i(x_s))}\\ &\le (C_M^*\cdot e^{\tilde{C}}\cdot e^{\gamma})^{\#\partial F_n+\lceil(\#(F_n\backslash I_M(s))/M)\rceil}\cdot e^{2n\gamma}\cdot e^{\tilde{C}\cdot\#(F_n\backslash I_M(s))}\\ &\quad\times e^{-\sum_{i\in F_n} J^u(f_{\omega^*}^i(x_s))} \quad \text{by (3.11)}\\ &\le (C_M^*\cdot e^{\tilde{C}}\cdot e^{\gamma})^{\#\partial F_n+\lceil(\#(F_n\backslash I_M(s))/M)\rceil}\cdot e^{2n\gamma}\cdot e^{\tilde{C}\cdot\#(F_n\backslash I_M(s))}\cdot e^{-\sum_{i\in F_n} J^u(f_{\omega^*}^i(x))}\\ &\quad\times\cdot e^{n\gamma+2\tilde{C}\sup_{y\in A_n^{\omega^*}} F_n\backslash I_M(\omega^*,y)} \quad \text{by (3.19)}\\ &\le (C_M^*\cdot e^{\tilde{C}}\cdot e^{\gamma})^{\#\partial F_n+\sup_{y\in A_n^{\omega^*}}\lceil(\#(F_n\backslash I_M(\omega^*,y))/M)\rceil}\cdot e^{2n\gamma}\cdot e^{\tilde{C}\cdot\sup_{y\in A_n^{\omega^*}}\#(F_n\backslash I_M(\omega^*,y))}\\ &\quad\times e^{-\sum_{i\in F_n} J^u(f_{\omega^*}^i(x))}\cdot e^{n\gamma+2\tilde{C}\sup_{y\in A_n^{\omega^*}} F_n\backslash I_M(\omega^*,y)}.\end{aligned}$$

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By (3.10) and (3.9), when $n \gg M$, we have

$$\begin{aligned}\lambda^u(A_n^{\omega^*} \cap P_{\omega^*}^{F_n}(x)) &\le (C_M^* \cdot e^{\tilde{C}} \cdot e^{\gamma})^{\#\partial F_n + \sup_{y\in A_n^{\omega^*}} \lceil (\#(F_n \setminus I_M(\omega^*, y))/M)\rceil} \cdot e^{2n\gamma} \\ &\times e^{\tilde{C}\cdot \sup_{y\in A_n^{\omega^*}} \#(F_n \setminus I_M(\omega^*, y))} \\ &\times e^{-\sum_{i\in F_n} J^u(f^i_{\omega^*}(x))} \cdot e^{n\gamma + 2\tilde{C}\sup_{y\in A_n^{\omega^*}} F_n \setminus I_M(\omega^*, y)} \cdot e^{3n/M}.\end{aligned}$$

The proof of Proposition 3.2 is complete. □

3.3. *Entropy formula.* In this subsection, we estimate the relative entropy of μ constructed in Proposition 3.1. Let us first recall some notation from Proposition 3.1. We recall that $\mathbb{A}_n^{\omega^*} = \{(\omega^*, x, E^u(\omega^*, x)) : x \in A_n^{\omega^*}\}$ and let $\hat{\lambda}^u$ be the measure concentrated on $\mathbb{A}_n^{\omega^*}$ corresponding to λ^u on $A_n^{\omega^*}$, that is,

$$\hat{\lambda}^u\{(\omega^*, x, E^u(\omega^*, x)) : x \in B\} = \lambda^u(B) \quad \text{for any Borel set } B \subset A_n^{\omega^*}.$$

We recall that $\mu_n = \lambda^u|_{A_n^{\omega^*}}/\lambda^u(A_n^{\omega^*})$ and $\hat{\mu}_n = \hat{\lambda}^u/\hat{\lambda}^u(\mathbb{A}_n^{\omega^*})$. Note that $\hat{\mu}_n$ is concentrated on the fiber $\{\omega^*\}$ and then $\hat{\mu}_n$ has disintegration

$$\hat{\mu}_n = \delta_{\omega^*} \times (\hat{\mu}_n)_{\omega^*}. \tag{3.20}$$

We recall that in the proof of Proposition 3.1, $\hat{\mu}$ is the weak* limit of $\{(1/\#F_n)\sum_{k\in F_n} \mathbb{F}_*^k \hat{\mu}_n\}_{n\in\mathfrak{n}''}$, $\mu = Proj_{\Omega\times\mathcal{M}}(\hat{\mu})$ and here, $\mathfrak{n}''$ is a subsequence of $\mathfrak{n}$.

For any $\gamma > 0$, by (3.6) and the third item of Lemma 3.1, we can pick M sufficiently large such that

$$\frac{(1/M)\log(C_M^*) + 4\tilde{C}}{\underline{d}^{\mathfrak{n}}(\mathcal{F})} \limsup_{n\in\mathfrak{n}''\subset\mathfrak{n}} \sup_{y\in A_n^{\omega^*}} d_n(F_n \setminus I_M(\omega^*, x)) \le \gamma,$$

and

$$\frac{3}{M\underline{d}^{\mathfrak{n}}(\mathcal{F})} < \gamma.$$

Pick any finite measurable partition $\mathcal{P} = \{P_i\}$ of $\mathcal{M}$ with

$$\operatorname{diam}(\mathcal{P}) < \delta < \min\{\delta_0(l, M, \gamma), \delta_1(l, M, \gamma)\}$$

as in Proposition 3.2. Denote $\hat{\mathcal{P}} = \{\pi^{-1}(\Omega \times P_i)\}$ to be a corresponding measurable partition of $\Omega \times \mathcal{G}^k(T\mathcal{M})$, where $\pi : \Omega \times \mathcal{G}^k(T\mathcal{M}) \to \Omega \times \mathcal{M}$ is the natural projection. We can pick $\mathcal{P}$ such that

$$\hat{\mu}(\partial\hat{\mathcal{P}}) = \int_\Omega \mu_\omega(\partial\mathcal{P})\, d\mathbb{P}(\omega) = 0. \tag{3.21}$$

By Proposition 3.2, for any $x \in A_n^{\omega^*}$ and $E \in \mathcal{G}^k(T_x\mathcal{M})$, one has

$$\begin{aligned}&\hat{\lambda}^u(\mathbb{A}_n^{\omega^*} \cap \hat{\mathcal{P}}^{F_n}(\omega^*, x, E)) = \lambda^u(A_n^{\omega^*} \cap P_{\omega^*}^{F_n}(x)) \\ &\le (C_M^* \cdot e^{\tilde{C}} \cdot e^{\gamma})^{\#\partial F_n + \sup_{y\in A_n^{\omega^*}} \lceil (\#(F_n \setminus I_M(\omega^*, y))/M)\rceil} \cdot e^{2n\gamma} \cdot e^{\tilde{C}\cdot \sup_{y\in A_n^{\omega^*}} \#(F_n \setminus I_M(\omega^*, y))} \\ &\quad \cdot e^{-\sum_{i\in F_n} J^u(f^i_{\omega^*}(x))} \cdot e^{n\gamma + 2\tilde{C}\sup_{y\in A_n^{\omega^*}} F_n \setminus I_M(\omega^*, y)} \cdot e^{3n/M}.\end{aligned} \tag{3.22}$$

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Let $n \gg M \gg m$. By [16, Lemma 3.2 (ii)], we have

$$\frac{1}{m} H_{\hat{\mu}}\Big(\bigvee_{i=0}^{m-1} \mathbb{F}^{-i}\hat{\mathcal{P}} | \mathcal{F}_{\Omega \times \mathcal{G}^k(T\mathcal{M})}\Big) \geq \frac{1}{m} \limsup_{n\in\mathfrak{n}''\to\infty} H_{(\hat{\mu}_n)^{F_n}}\Big(\bigvee_{i=0}^{m-1} \mathbb{F}^{-i}\hat{\mathcal{P}} | \mathcal{F}_{\Omega \times \mathcal{G}^k(T\mathcal{M})}\Big).$$

To apply [16, Lemma 3.2 (ii)], we use the fact that in (3.21), Ω is a compact metric space and $\mathcal{F}_{\Omega \times \mathcal{G}^k(T\mathcal{M})}$ is the σ-algebra formed by all sets $A \times \mathcal{G}^k(T\mathcal{M})$ with $A \in \mathcal{B}(\Omega)$. Using Lemma 2.5, we obtain

$$\frac{1}{m} H_{\hat{\mu}}\Big(\bigvee_{i=0}^{m-1} \mathbb{F}^{-i}\hat{\mathcal{P}} | \mathcal{F}_{\Omega \times \mathcal{G}^k(T\mathcal{M})}\Big) \geq \limsup_{n\in\mathfrak{n}''\to\infty} \frac{1}{\#F_n} H_{\hat{\mu}_n}\Big(\bigvee_{i\in F_n} (\mathbb{F})^{-i}\hat{\mathcal{P}} | \mathcal{F}_{\Omega \times \mathcal{G}^k(T\mathcal{M})}\Big).$$

Note that $\hat{\mu}_n$ has disintegration by (3.20), so by the definition of conditional entropy,

$$\frac{1}{m} H_{\hat{\mu}}\Big(\bigvee_{i=0}^{m-1} \mathbb{F}^{-i}\hat{\mathcal{P}} | \mathcal{F}_{\Omega \times \mathcal{G}^k(T\mathcal{M})}\Big) \geq \limsup_{n\in\mathfrak{n}''\to\infty} \frac{1}{\#F_n} H_{(\hat{\mu}_n)_{\omega^*}}\Big(\bigvee_{i\in F_n} (\mathbb{F}^i_{\omega^*})^{-1}\hat{\mathcal{P}}(\theta^i\omega^*)\Big),$$

where we recall $\mathbb{F}^i_\omega$ is generated by (2.9), and here, partition $\hat{\mathcal{P}}(\theta^i\omega^*)$ equal to the projection of $\hat{\mathcal{P}}$ on $\mathcal{G}^k(T\mathcal{M})$ for all $i \in \mathbb{Z}$. Then, by the definition of entropy, we have

$$\begin{aligned}
&\frac{1}{m} H_{\hat{\mu}}\Big(\bigvee_{i=0}^{m-1} \mathbb{F}^{-i}\hat{\mathcal{P}} | \mathcal{F}_{\Omega \times \mathcal{G}^k(T\mathcal{M})}\Big) \\
&\quad \geq \limsup_{n\in\mathfrak{n}''\to\infty} \frac{1}{\#F_n} \int_{\mathcal{G}^k(T\mathcal{M})} -\log \hat{\lambda}^u(\mathbb{A}_n^{\omega^*} \cap \hat{\mathcal{P}}^{F_n}(\omega^*, x, E)) + \log \hat{\lambda}^u(\mathbb{A}_n^{\omega^*}) d(\hat{\mu}_n)_{\omega^*}.
\end{aligned}$$

By the first item of Lemma 3.1 and the fourth item $\underline{d}^{\mathfrak{n}}(\mathcal{F}) \geq \beta > 0$, we have

$$\limsup_{n\in\mathfrak{n}''\to\infty} \frac{\log \hat{\lambda}^u(\mathbb{A}_n^{\omega^*})}{\#F_n} = \limsup_{n\in\mathfrak{n}''\to\infty} \frac{\log \lambda^u(A_n^{\omega^*})}{\#F_n} \geq \limsup_{n\in\mathfrak{n}''\to\infty} \frac{-n\delta_n - 2\log n}{d_n(F_n)\cdot n} = 0.$$

By (3.22), the third item of Lemma 3.1, and direct computation, one can deduce that

$$\begin{aligned}
&\limsup_{n\in\mathfrak{n}''\to\infty} \frac{1}{\#F_n} \int_{\mathcal{G}^k(T\mathcal{M})} -\log \hat{\lambda}^u(\mathbb{A}_n^{\omega^*} \cap \hat{\mathcal{P}}^{F_n}(\omega^*, x, E))\, d(\hat{\mu}_n)_{\omega^*} \\
&\quad \geq \limsup_{n\in\mathfrak{n}''\to\infty} \frac{1}{\#F_n} \int \sum_{i\in F_n} J^u(f^i_{\omega^*}(x))\, d(\hat{\mu}_n)_{\omega^*} - \Big(2 + \frac{1}{\underline{d}^{\mathfrak{n}}(\mathcal{F})}\Big)\gamma \\
&\quad = \limsup_{n\in\mathfrak{n}''\to\infty} \frac{1}{\#F_n} \int_{\Omega\times\mathcal{G}^k(T\mathcal{M})} \log|\det(D_x f_\omega|_E)| d(\hat{\mu}_n)^{F_n} - \Big(2 + \frac{1}{\underline{d}^{\mathfrak{n}}(\mathcal{F})}\Big)\gamma \\
&\quad \geq \int_{\Omega\times\mathcal{G}^k(T\mathcal{M})} \log|\det(D_x f_\omega|_E)|\, d\hat{\mu} - \Big(2 + \frac{1}{\underline{d}^{\mathfrak{n}}(\mathcal{F})}\Big)\gamma \\
&\quad = \int_{\Omega\times\mathcal{M}} \log|\det(D_x f_\omega|_{E^u(\omega,x)})|\, d\mu - \Big(2 + \frac{1}{\underline{d}^{\mathfrak{n}}(\mathcal{F})}\Big)\gamma,
\end{aligned}$$

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where in the last inequality, we use the continuity of $\log|\det(D_x f_\omega|_E)|$ on $\Omega \times \mathcal{G}^k(T\mathcal{M})$, and in the last equality, we use (3.3). Since γ is arbitrary, we have

$$\frac{1}{m} H_{\hat{\mu}}\Big(\bigvee_{i=0}^{m-1} \mathbb{F}^{-i}\hat{\mathcal{P}}|\mathcal{F}_{\Omega\times\mathcal{G}^k(T\mathcal{M})}\Big) \geq \int_{\Omega\times\mathcal{M}} \log|\det(D_x f_\omega|_{E^u(\omega,x)})|\, d\mu.$$

Letting $m \to \infty$, we obtain

$$\begin{aligned}\int_{\Omega\times\mathcal{M}} \log|\det(D_x f_\omega|_{E^u(\omega,x)})|\, d\mu &\leq \lim_{m\to\infty} \frac{1}{m} H_{\hat{\mu}}\Big(\bigvee_{i=0}^{m-1} \mathbb{F}^{-i}\hat{\mathcal{P}}|\mathcal{F}_{\Omega\times\mathcal{G}^k(T\mathcal{M})}\Big)\\ &\leq h^{(r)}_{\hat{\mu}}(\mathbb{F}) \overset{(2.10)}{=} h^{(r)}_{\mu}(F).\end{aligned}$$

By Ruelle's inequality for RDS (see [15, Theorem 3.1.1]), we obtain

$$\int_{\Omega\times\mathcal{M}} \log|\det(D_x f_\omega|_{E^u(\omega,x)})|\, d\mu = \int \sum_i \lambda^{(i)}(\omega,x)^+ m^{(i)}(\omega,x)\, d\mu = h^{(r)}_{\mu}(F),$$

that is, μ is a random SRB measure.

4. *The existence of SRB measure*

In this section, we assume that Ω is a closed smooth manifold and θ is a C^2 diffeomorphism on Ω that admits an ergodic hyperbolic SRB measure $\mathbb{P}$, and F is a C^2 diffeomorphism on $\Omega \times \mathcal{M}$. We assume (1.4) holds, that is,

$$\begin{aligned}\lambda^-_{\theta,\max} < m_f &:= \min_{\omega\in\Omega}\min_{x\in\mathcal{M}} \log\|D(f_\omega)^{-1}(x)\|^{-1} < M_f\\ &:= \max_{\omega\in\Omega}\max_{x\in\mathcal{M}} \log\|Df_\omega(x)\| < \lambda^+_{\theta,\min}.\end{aligned} \tag{4.1}$$

Let μ be any random SRB measure with marginal $\mathbb{P}$ on Ω obtained in Theorem A. We are going to show that μ is an SRB measure for F. Since $\mathbb{P}$ is an SRB measure, we denote $T_\omega\Omega = H^u(\omega) \oplus H^s(\omega)$ to be the Oseledets splitting corresponding to positive and negative Lyapunov exponents, and we denote $P(H^u(\omega))$ and $P(H^s(\omega))$ to be the projection operator with respect to this splitting.

Let $\|\cdot\|$ be the norm on $T_{(\omega,x)}(\Omega\times\mathcal{M}) = T_\omega\Omega \times T_x\mathcal{M}$ defined by $\|(u,v)\| := \sqrt{\|u\|^2_{T_\omega\Omega} + \|v\|^2_{T_x\mathcal{M}}}$ for any $u \in T_\omega\Omega$ and $v \in T_x\mathcal{M}$. We denote

$$D_{(\omega,x)}F = \begin{pmatrix} D_\omega\theta & 0\\ C(\omega,x) & D_x f_\omega\end{pmatrix} \quad\text{and}\quad D_{(\omega,x)}F^{-1} = \begin{pmatrix} D_\omega\theta^{-1} & 0\\ \tilde{C}(\omega,x) & D_x f_\omega^{-1}\end{pmatrix} \tag{4.2}$$

relative to the direct sum decomposition $T_{(\omega,x)}(\Omega\times\mathcal{M}) = T_\omega\Omega \times \{0\} \oplus \{0\} \times T_x\mathcal{M}$.

4.1. *Dimension reduction.*

The idea of the following lemma comes from [28, §4.2.5].

LEMMA 4.1. *For μ-a.e. $(\omega,x) \in \Omega\times\mathcal{M}$, $u \in T_\omega\Omega\backslash\{0\}$, $v \in T_x\mathcal{M}$:*

(1) *if $P(H^u(\omega))u \neq 0$, then* $\limsup_n (1/n)\log\|D_\omega\theta^n u\| = \limsup_n (1/n)\log\|D_{(\omega,x)}F^n(u,v)^T\|$;

(2) *if $P(H^s(\omega))u \neq 0$, then* $\limsup_n (1/n)\log\|D_\omega\theta^{-n} u\| = \limsup_n (1/n)\log\|D_{(\omega,x)}F^{-n}(u,v)^T\|$.

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Proof of Lemma 4.1. Denote $\Omega_0 \subset \Omega$ to be the Oseledets full measure set relative to the system $(\Omega, \theta, \mathbb{P})$. The following proof is processing in the set $\Omega_0 \times \mathcal{M}$. We first prove condition (1). We observe that

$$D_{(\omega,x)}F^n = \begin{pmatrix} D_\omega\theta^n & 0 \\ C_n(\omega, x) & D_x f_\omega^n \end{pmatrix},$$

where

$$C_n(\omega, x) = \sum_{j=0}^{n-1} D_{f_\omega^{j+1}(x)} f_{\theta^{j+1}\omega}^{n-1-j} C(F^j(\omega, x)) D_\omega\theta^j.$$

By the right side of assumption (1.4), we have

$$\begin{aligned} \limsup_n \frac{1}{n} \log \|D_{(\omega,x)}F^n(u, 0)^T\| &\geq \limsup_n \frac{1}{n} \log \|D_\omega\theta^n u\| \geq \lambda_{\theta,\min}^+ \\ &\geq \limsup_n \frac{1}{n} \log \|D_x f_\omega^n v\| \quad \text{by (1.4)} \\ &= \limsup_n \frac{1}{n} \log \|D_{(\omega,x)}F^n(0, v)^T\|, \end{aligned} \tag{4.3}$$

and, similarly,

$$\begin{aligned} &\limsup_n \frac{1}{n} \log \|D_{(\omega,x)}F^n(u, v)^T\| \\ &\geq \limsup_n \frac{1}{n} \log \|D_\omega\theta^n u\| \geq \limsup_n \frac{1}{n} \log \|D_{(\omega,x)}F^n(0, v)^T\|. \end{aligned} \tag{4.4}$$

Therefore, we have

$$\begin{aligned} &\limsup_n \frac{1}{n} \log \|D_{(\omega,x)}F^n(u, v)^T\| \\ &\leq \limsup_n \frac{1}{n} \log(\|D_{(\omega,x)}F^n(u, 0)^T\| + \|D_{(\omega,x)}F^n(0, v)^T\|) \\ &= \max\left\{\limsup_n \frac{1}{n} \log \|D_{(\omega,x)}F^n(u, 0)^T\|, \limsup_n \frac{1}{n} \log \|D_{(\omega,x)}F^n(0, v)^T\|\right\} \\ &= \limsup_n \frac{1}{n} \log \|D_{(\omega,x)}F^n(u, 0)^T\| \quad \text{by (4.3)}, \end{aligned}$$

and

$$\begin{aligned} &\limsup_n \frac{1}{n} \log \|D_{(\omega,x)}F^n(u, 0)^T\| \\ &\leq \limsup_n \frac{1}{n} \log(\|D_{(\omega,x)}F^n(u, v)^T\| + \|D_{(\omega,x)}F^n(0, v)^T\|) \\ &= \max\left\{\limsup_n \frac{1}{n} \log \|D_{(\omega,x)}F^n(u, v)^T\|, \limsup_n \frac{1}{n} \log(\|D_{(\omega,x)}F^n(0, v)^T\|)\right\} \\ &= \limsup_n \frac{1}{n} \log \|D_{(\omega,x)}F^n(u, v)^T\| \quad \text{by (4.4)}. \end{aligned}$$

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Consequently, we obtain

$$\limsup_n \frac{1}{n} \log \|D_{(\omega,x)} F^n(u,0)^T\| = \limsup_n \frac{1}{n} \log \|D_{(\omega,x)} F^n(u,v)^T\|. \tag{4.5}$$

Now, we need to estimate $\limsup_n (1/n) \log \|C_n(\omega,x)u\|$. Note that for $P(H^u(\omega))u \neq 0$, the following limit exists $\lambda_\theta(u) := \limsup_n (1/n) \log \|D_\omega \theta^n u\|$. Therefore, for any $\epsilon > 0$, there exists $b_\epsilon > 0$ such that

$$\|D_\omega \theta^j u\| \leq b_\epsilon e^{j(\lambda_\theta(u)+\epsilon)} \quad \text{for all } j \in \mathbb{Z}^+. \tag{4.6}$$

Note that F is a C^2 diffeomorphism; therefore, $\|C(F^j(\omega,x))\| \leq C_*$ for some constant $C_* > 0$ only depending on the system. By (4.1), one has $\|D_{f_\omega^{j+1}(x)} f_{\theta^{j+1}\omega}^{n-1-j}\| \leq e^{(n-1-j)M_f}$. These estimates give

$$\begin{aligned}\|C_n(\omega,x)\| &\leq \sum_{j=0}^{n-1} e^{(n-1-j)M_f} \cdot C_* \cdot b_\epsilon e^{j(\lambda_\theta(u)+\epsilon)} \leq \sum_{j=0}^{n-1} e^{(n-1-j)\lambda_\theta(u)} \cdot C_* \cdot b_\epsilon e^{j(\lambda_\theta(u)+\epsilon)} \\ &\leq \frac{C_* b_\epsilon}{e^\epsilon - 1} e^{(\lambda_\theta(u)+\epsilon)n}.\end{aligned}$$

Therefore, we have estimate $\limsup_n (1/n) \log \|C_n(\omega,x)u\| \leq \lambda_\theta(u) + \epsilon$. Note that

$$\|D_{(\omega,x)} F^n(u,0)^T\|^2 = \|D_\omega \theta^n u\|^2 + \|C_n(\omega,x)u\|^2,$$

and we obtain

$$\lambda_\theta(u) := \limsup_n \frac{1}{n} \log \|D_\omega \theta^n u\| \leq \limsup_n \frac{1}{n} \log \|D_{(\omega,x)} F^n(u,0)^T\| \leq \lambda_\theta(u) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we arrive at $\limsup_n (1/n) \log \|D_\omega \theta^n u\| = \limsup_n (1/n) \log \|D_{(\omega,x)} F^n(u,0)^T\|$. Combining with (4.5), we complete the proof of condition (1).

The proof of condition (2) is similar to the proof of condition (1). Now, we assume $P(H^s(\omega))u \neq 0$. We observe that

$$D_{(\omega,x)} F^{-n} = \begin{pmatrix} D_\omega \theta^{-n} & 0 \\ C_{-n}(\omega,x) & D_x f_\omega^{-n} \end{pmatrix},$$

where

$$C_{-n}(\omega,x) = \sum_{j=0}^{n-1} D_{f_\omega^{-(j+1)}(x)} f_{\theta^{-(j+1)}\omega}^{-(n-1-j)} \tilde{C}(F^{-j}(\omega,x)) D_\omega \theta^{-j}.$$

By the left side of assumption (1.4), we have

$$\begin{aligned}\limsup_n \frac{1}{n} \log \|D_{(\omega,x)} F^{-n}(u,v)^T\| &\geq \limsup_n \frac{1}{n} \log \|D_\omega \theta^{-n} u\| \geq -\lambda_{\theta,\max}^- \\ &\geq \limsup_n \frac{1}{n} \log \|D_x f_\omega^{-n} v\| \quad \text{by (1.4)} \\ &= \limsup_n \frac{1}{n} \log \|D_{(\omega,x)} F^{-n}(0,v)^T\|, \end{aligned} \tag{4.7}$$

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and, similarly,

$$\limsup_n \frac{1}{n} \log \|D_{(\omega,x)} F^{-n}(u, 0)^T\| \geq \limsup_n \frac{1}{n} \log \|D_\omega \theta^{-n} u\| \geq \limsup_n \frac{1}{n} \log \|D_\omega \theta^{-n} u\|. \tag{4.8}$$

Therefore,

$$\begin{aligned}
&\limsup_n \frac{1}{n} \log \|D_{(\omega,x)} F^{-n}(u, v)^T\| \\
&\leq \limsup_n \frac{1}{n} \log(\|D_{(\omega,x)} F^{-n}(u, 0)^T\| + \|D_{(\omega,x)} F^{-n}(0, v)^T\|) \\
&= \max\left\{\limsup_n \frac{1}{n} \log \|D_{(\omega,x)} F^{-n}(u, 0)^T\|, \limsup_n \frac{1}{n} \log(\|D_{(\omega,x)} F^{-n}(0, v)^T\|)\right\} \\
&= \limsup_n \frac{1}{n} \log \|D_{(\omega,x)} F^{-n}(u, 0)^T\| \quad \text{by (4.8)}
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_n \frac{1}{n} \log \|D_{(\omega,x)} F^{-n}(u, 0)^T\| \\
&\leq \limsup_n \frac{1}{n} \log(\|D_{(\omega,x)} F^{-n}(u, v)^T\| + \|D_{(\omega,x)} F^{-n}(0, v)^T\|) \\
&= \max\left\{\limsup_n \frac{1}{n} \log \|D_{(\omega,x)} F^{-n}(u, v)^T\|, \limsup_n \frac{1}{n} \log \|D_{(\omega,x)} F^{-n}(0, v)^T\|\right\} \\
&= \limsup_n \frac{1}{n} \log \|D_{(\omega,x)} F^{-n}(u, v)^T\| \quad \text{by (4.7)}.
\end{aligned}$$

Consequently, we have

$$\limsup_n \frac{1}{n} \log \|D_{(\omega,x)} F^{-n}(u, 0)^T\| = \limsup_n \frac{1}{n} \log \|D_{(\omega,x)} F^{-n}(u, v)^T\|. \tag{4.9}$$

Now, we are going to estimate $\limsup_n (1/n) \log \|\tilde{C}_{-n}(\omega, x)u\|$. Since $P(H^s(\omega))u \neq 0$, the following limit exists:

$$\lambda_{\theta,-}(u) := \limsup_n \frac{1}{n} \log \|D_\omega \theta^{-n} u\|.$$

For any $\epsilon > 0$, there exists $b'_\epsilon > 0$ such that $\|D_\omega \theta^{-j} u\| \leq b'_\epsilon e^{(\lambda_{\theta,-}(u)+\epsilon)j}$ for any $j \geq 0$. Note that since F is a C^2 diffeomorphism, there exists $C'_* > 0$ only depending on the system such that $\|\tilde{C}(F^{-j}(\omega, x))\| \leq C'_*$ for any $j \geq 0$. By (4.1), for any $0 \leq j \leq n-1$, we have

$$\|D_{f_\omega^{-(j+1)}(x)} f_{\theta^{-(j+1)}\omega}^{-(n-1-j)}\| \leq e^{-m_f(n-1-j)} \leq e^{\lambda_{\theta,-}(u)(n-1-j)}.$$

These estimates give

$$\|C_{-n}(\omega, x)u\| \leq \sum_{j=0}^{n-1} e^{\lambda_{\theta,-}(u)(n-1-j)} \cdot C'_* \cdot b'_\epsilon e^{(\lambda_{\theta,-}(u)+\epsilon)j} \leq \frac{C'_* b'_\epsilon}{e^\epsilon - 1} e^{n(\lambda_{\theta,-}(u)+\epsilon)}.$$

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Therefore, we have estimate $\limsup_n (1/n) \log \|C_{-n}(\omega, x)u\| \le \lambda_{\theta,-}(u) + \epsilon$. Note that

$$\|D_{(\omega,x)}F^{-n}(u,0)^T\|^2 = \|D_\omega \theta^{-n}u\|^2 + \|C_{-n}(\omega,x)u\|^2,$$

so we obtain

$$\begin{aligned}\lambda_{\theta,-}(u) &:= \limsup_n \frac{1}{n} \log \|D_\omega \theta^{-n}u\| \\ &\le \limsup_n \frac{1}{n} \log \|D_{(\omega,x)}F^{-n}(u,0)^T\| \le \lambda_{\theta,-}(u) + \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we arrive at $\limsup_n (1/n) \log \|D_\omega \theta^{-n}u\| = \limsup_n (1/n) \log \|D_{(\omega,x)}F^{-n}(u,0)^T\|$. Combining with (4.9), we complete the proof of condition (2). □

4.2. *Oseledets splitting and Lyapunov exponents.* Now, we first apply a reformulation of the celebrated Oseledets multiplicative ergodic theorem. For random dynamical system $(f_\omega^n)_{\omega\in\Omega, n\in\mathbb{Z}}$ with random SRB measure μ [21, 24], there exists a measurable set $\Delta_0 \subset \Omega \times \mathcal{M}$ such that $\mu(\Delta_0) = 1$ and for each $(\omega, x) \in \Delta_0$, there are numbers

$$-\infty < \lambda_r^1(\omega,x) < \lambda_r^2(\omega,x) < \cdots < \lambda_r^{r(\omega,x)}(\omega,x) < +\infty$$

measurable in (ω, x) and an associated sequence of subspaces of $T_x\mathcal{M}$,

$$E_r^1(\omega,x) \oplus \cdots \oplus E_r^{r(\omega,x)}(\omega,x) = T_x\mathcal{M}$$

satisfying

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f_\omega^n \xi\| = \lambda_r^i(\omega,x) \quad \text{for all } \xi \in E_r^i(\omega,x),\ 1 \le i \le r(\omega,x).$$

Here, the subscript 'r' represents RDS. By noticing (4.2), subspaces $\{0\} \times E_r^i$ are invariant for DF and moreover, for all $\xi \in E_r^i(\omega,x)$, $1 \le i \le r(\omega,x)$,

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_{(\omega,x)}F^n(0,\xi)^T\| = \lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f_\omega^n \xi\| = \lambda_r^i(\omega,x).$$

We apply the classical multiplicative ergodic theorem. For the driven system (Ω, θ) with ergodic SRB measure $\mathbb{P}$, there exists a measurable set $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ and for each $\omega \in \Omega_0$, there are non-zero numbers

$$-\infty < \lambda_\theta^1 < \cdots \lambda_\theta^{l_1} = \lambda_{\theta,\max}^- < 0 < l_\theta^{l_1+1} = \lambda_{\theta,\min}^+ < \cdots < \lambda_\theta^l < +\infty$$

and an associated sequence of subspaces of $T_\omega\Omega$,

$$H_\theta^1(\omega) \oplus \cdots \oplus H_\theta^l(\omega) = T_\omega\Omega$$

satisfying

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_\omega \theta^n \xi\| = \lambda_\theta^i \quad \text{for all } \xi \in H_\theta^i(\omega),\ 1 \le i \le l.$$

We apply the classical multiplicative ergodic theorem. For the skew product system $(\Omega \times \mathcal{M}, F)$ with invariant measure μ, there exists a measurable set $\Delta_1 \subset \Omega \times \mathcal{M}$

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with $\mu(\Delta_1) = 1$ and for each $(\omega, x) \in \Delta_1$, there are numbers

$$-\infty < \lambda_F^1(\omega, x) < \lambda_F^2(\omega, x) < \cdots < \lambda_F^{k(\omega,x)}(\omega, x) < +\infty$$

measurable in (ω, x) and an associated sequence of subspaces of $T_{(\omega,x)}\Omega \times \mathcal{M}$,

$$V_F^1(\omega, x) \oplus \cdots \oplus V_F^{k(\omega,x)}(\omega, x) = T_{(\omega,x)}\Omega \times \mathcal{M}$$

satisfying

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_{(\omega,x)}F^n\eta\| = \lambda_F^i(\omega, x) \quad \text{for all } \eta \in V_F^i(\omega, x),\ 1 \le i \le k(\omega, x).$$

By the uniqueness of the Oseledets splitting, for $(\omega, x) \in \Delta_0 \cap \Delta_1$, all subspaces $\{0\} \times E_r^i(\omega)$ for $i = 1, \ldots, r(\omega, x)$ must appear in $V_F^j(\omega, x)$ for some $j = 1, \ldots, k(\omega, x)$.

Now, we consider $(\omega, x) \in \Delta_0 \cap \Delta_1 \cap (\Omega_0 \times \mathcal{M})$. For any $\eta \in V_F^{k(\omega,x)}(\omega, x)$, then $\eta = (u_\eta, v_\eta)$ with respect to the decomposition $T_{(\omega,x)}(\Omega \times \mathcal{M}) = T_\omega\Omega \times \{0\} \oplus \{0\} \times T_x\mathcal{M}$. We claim that $P(H_\theta^l(\omega))u_\eta \neq 0$. Otherwise, by Lemma 4.1(1), for any $\eta' = (u_{\eta'}, v_{\eta'})$ with $P(H_\theta^l(\omega))u_{\eta'} \neq 0$,

$$\begin{aligned}\lim_{n\to+\infty} \frac{1}{n} \log \|D_{(\omega,x)}F^n\eta\| &= \limsup_{n\to+\infty} \frac{1}{n} \log \|D_\omega\theta^n u_\eta\| < \limsup_{n\to+\infty} \frac{1}{n} \log \|D_{(\omega,x)}F^n\eta'\| \\ &= \limsup_{n\to+\infty} \frac{1}{n} \log \|D_\omega\theta^n u_{\eta'}\| = \lambda_\theta^l,\end{aligned}$$

which contradicts the largest Lyapunov exponents. This argument also implies that $\lambda_F^{k(\omega,x)} = \lambda_\theta^l$ and $\dim V_F^{k(\omega,x)} = \dim H_\theta^l$. Similarly, one can show that

$$\lambda_F^{k(\omega,x)-i} = \lambda_\theta^{l-i} \quad \text{and} \quad \dim V_F^{k(\omega,x)-i}(\omega, x) = \dim H_\theta^{l-i} \quad \text{for all } 0 \le i \le l - (l_1 + 1).$$

For any $\xi \in V_F^1(\omega, x)$, then $\xi = (u_\xi, v_\xi)$ with respect to the decomposition $T_{(\omega,x)}(\Omega \times \mathcal{M}) = T_\omega\Omega \times \{0\} \oplus \{0\} \times T_x\mathcal{M}$. We claim that $P(H_\theta^1(\omega))u_\xi \neq 0$. Otherwise, by Lemma 4.1(2), for any $\xi' = (u_{\xi'}, v_{\xi'})$ with $P(H_\theta^1(\omega))u_{\xi'} \neq 0$,

$$\begin{aligned}\lim_{n\to+\infty} \frac{1}{-n} \log \|D_{(\omega,x)}F^{-n}\xi\| &= -\limsup_{n\to+\infty} \frac{1}{n} \log \|D_\omega\theta^{-n}u_\xi\| \\ &> -\limsup_{n\to+\infty} \frac{1}{n} \log \|D_\omega\theta^n u_{\xi'}\| = \lambda_\theta^1 \\ &= -\limsup_{n\to+\infty} \frac{1}{n} \log \|D_{(\omega,x)}F^n\xi'\|,\end{aligned}$$

which contradicts the smallest Lyapunov exponents. This argument also implies that $\lambda_F^1 = \lambda_\theta^1$ and $\dim V_F^1 = \dim H_\theta^1$. Similar, one can show that

$$\lambda_F^i = \lambda_\theta^i \quad \text{and} \quad \dim V_F^i(\omega, x) = \dim H_\theta^i \quad \text{for all } 1 \le i \le l_1.$$

By (1.4), $\lambda_r^i(\omega, x)$ for $i = 1, \ldots, r(\omega)$ are bounded between $\lambda_\theta^{l_1} = \lambda_{\theta,\max}^-$ and $\lambda_\theta^{l_1+1} = \lambda_{\theta,\min}^+$. Therefore, the Lyapunov spectrum of F at $(\omega, x) \in \Delta_0 \cap \Delta_1 \cap (\Omega_0 \times \mathcal{M})$ is $\{(\lambda_\theta^1, \dim H_\theta^1), \ldots, (\lambda_\theta^{l_1}, \dim H_\theta^{l_1}), \quad (\lambda_r^1(\omega, x), \dim E_r^1(\omega, x)), \ldots, (\lambda_r^{r(\omega,x)}, \dim E_r^{r(\omega)}(\omega, x)), (\lambda_\theta^{l_1+1}, \dim H_\theta^{l_1+1}(\omega)), \ldots, (\lambda_\theta^l, \dim H_\theta^l(\omega))\}$.

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Using the Abramov–Rohlin formula, we finally prove that

$$h_\mu(F) = h_\mu^{(r)}(F) + h_{\mathbb{P}}(\theta) = \int \sum_i \lambda_i^{(i)}(\omega, x)^+ \dim E_r^i(\omega, x)\, d\mu + \sum_{i=l_1}^{l} \lambda_\theta^i \dim H_\theta^l(\omega),$$

which implies that μ is an SRB measure for $(\Omega \times \mathcal{M}, F)$.

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