CONSTRUCTION TECHNIQUES FOR ANTI-PASCH STEINER TRIPLE SYSTEMS

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ABSTRACT

Four methods for constructing anti-Pasch Steiner triple systems are developed. The first generalises a construction of Stinson and Wei to obtain a general singular direct product construction. The second generalises the Bose construction. The third employs a construction due to Lu. The fourth employs Wilson-type inflation techniques using Latin squares having no subsquares of order 2. As a consequence of these constructions we are able to produce anti-Pasch systems of order v for $v \equiv 1$ or 7 (mod 18), for $v \equiv 49$ (mod 72), and for many other values of v.

1. *Introduction*

A *Steiner triple system* $S = (V, \mathscr{B})$ of *order* v, denoted STS(v), is a collection $\mathscr{B}$ of triples (3-element subsets) on a set V, $|V| = v$, such that each unordered pair of elements of V is contained in exactly one triple from $\mathscr{B}$. It is well known that an STS(v) exists if and only if $v \equiv 1, 3 \pmod 6$; such orders are *admissible*.

A (k, ℓ)-*configuration* in a Steiner triple system $(V, \mathscr{B})$ is a subset of ℓ triples of $\mathscr{B}$ whose union is a k-element subset of V. The *Pasch configuration* or *quadrilateral*, P, is the $(6, 4)$-configuration on elements (say) a, b, c, d, e, f with the triples $\{a, b, c\}$, $\{a, d, e\}$, $\{f, b, d\}$ and $\{f, c, e\}$. A Steiner triple system is *anti-Pasch* (or *quadrilateral-free*) if it does not contain P. For instance, the unique Steiner triple system of order 9 is anti-Pasch and of the 80 STS(15), just one (number 80 in [**10**]) is anti-Pasch.

The problem of characterising those v for which there exists an anti-Pasch Steiner triple system of order v appears to be difficult. For every $v \equiv 3 \pmod 6$, an anti-Pasch STS(v) is known to exist [**2**, **6**]. There is no anti-Pasch Steiner triple system of order 7 or 13. It has been conjectured that an anti-Pasch STS(v) exists for all other $v \equiv 1 \pmod 6$. Although this remains far from settled, some progress has been made [**2**, **5**, **6**, **12**]. It is the purpose of this paper to narrow substantially the spectrum of possible exceptions. We shall denote an anti-Pasch (quadrilateral-free) STS(v) by QFSTS(v).

A *group divisible design* (denoted GDD) is a triple $(X, \mathscr{G}, \mathscr{B})$ which satisfies the following properties:

(1) $\mathscr{G}$ is a partition of a set X (of *points*) into subsets called *groups*;

(2) $\mathscr{B}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point;

(3) every pair of points from distinct groups occurs in a unique block.

The *group-type* (*type*) of the group divisible design is the multiset $\{|G| : G \in \mathscr{G}\}$. We usually use an 'exponential' notation to describe group-type: a group-type $1^i 2^j 3^k \ldots$ denotes i occurrences of 1, j occurrences of 2, k occurrences of 3, and so on.

If K is a set of positive integers, each of which is not less than 2, then we say that a group divisible design $(X, \mathscr{G}, \mathscr{B})$ is a K-GDD if $|B| \in K$ for every block B in $\mathscr{B}$. When

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$K = \{k\}$, we simply write k for K. A K-GDD is said to be *uniform* if all groups have the same size, that is, if it is of type g^u. A *transversal design* TD(k, n) is a k-GDD of type n^k.

2. *Stinson and Wei's construction*

In this section, we extend the second recursive construction of Stinson and Wei [**12**]. This is a singular direct product construction. It employs Latin squares with certain properties. A *subsquare* of a Latin square is a square subarray that is itself a Latin square. A Latin square is an N_2-*Latin square* if it contains no subsquare of order 2. An N_2-Latin square of order n exists for all $n \geqslant 3$ and $n \neq 4$ [**7**, **8**, **11**].

We need N_2-Latin squares with additional properties, similar to (but weaker than) the 'special' Latin squares in [**12**]. We shall call such squares Kotzig squares in recognition of Kotzig's work in this area. A *Kotzig square* of order $n = 2w$ is an N_2-Latin square L of order $2w$ with rows, columns and symbols indexed by $\{0, 1, \ldots, n-1\}$, and enjoying three properties:

(1) $\{L(2i, s), L(2i+1, s)\} \neq \{2j, 2j+1\}$ for $0 \leqslant i, j < w$;
(2) $\{L(s, 2i), L(s, 2i+1)\} \neq \{2j, 2j+1\}$ for $0 \leqslant i, j < w$;
(3) $L(2i, 2j) \neq L(2i+1, 2j+1)$, $L(2i, 2j+1) \neq L(2i+1, 2j)$ for $0 \leqslant i, j < w$.

Stinson and Wei used similar N_2-Latin squares to prove the following theorem.

THEOREM 2.1 [**12**]. *If there exists a* QFSTS(u) *and* $u \equiv 1 \pmod 4$, *and* $u-1$ *has an odd divisor exceeding* 3, *then there exists a* QFSTS$(3(u-1)+1)$.

We extend Theorem 2.1 to relax the condition that $u \equiv 1 \pmod 4$, and the condition on divisors.

LEMMA 2.2. *There exists a Kotzig square of order* $2w$ *whenever* $w \geqslant 4$, *except possibly for* $w = 5$, 8, 10 *and* 16.

Proof. The proof relies on three general constructions.

(a) Firstly we prove that a Kotzig square of order $2w$ exists whenever w is odd and $w \geqslant 7$. To do this form a $2w \times 2w$ array L by setting

$$L(i,j) = \begin{cases} 2i-4j \bmod w & \text{if } 0 \leqslant i,j \leqslant w-1 \\ (2i+4j \bmod w)+w & \text{if } 0 \leqslant i \leqslant w-1, w \leqslant j \leqslant 2w-1 \\ (2i+4j \bmod w)+w & \text{if } w \leqslant i \leqslant 2w-1, 0 \leqslant j \leqslant w-1 \\ 2i-4j+3 \bmod w & \text{if } w \leqslant i,j \leqslant 2w-1. \end{cases}$$

It is helpful to regard L as being formed from four $w \times w$ subsquares, two of which are identical:

$$L = \begin{array}{|c|c|} \hline A & B+w \\ \hline B+w & C \\ \hline \end{array}$$

where $A(i,j) = 2i-4j \bmod w$, $B(i,j) = 2i+4j \bmod w$ and $C(i,j) = 2i-4j+3 \bmod w$, all for $0 \leqslant i, j \leqslant w-1$, and $B+w$ denotes the result of adding w to the entries of B. Plainly L is a Latin square of side $2w$.

Within each of the four subsquares, consecutive entries in a given column differ by 2, consecutive entries in a given row differ by 4, and diagonally adjacent entries differ

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by 2 or 6. Since $w \geqslant 7$, properties (1), (2) and (3) are therefore satisfied within each subsquare. Furthermore, each of these subsquares is equivalent to a cyclic Latin square of odd side and so is an N_2-Latin square. To confirm the Kotzig properties for the full $2w \times 2w$ array L, it is necessary to examine entries along the common boundaries of the subsquares.

Property (1) would be violated only if, for some j, we have $\{L(w-1,j), L(w,j)\} = \{w-1, w\}$. If $0 \leqslant j \leqslant w-1$ this requires $L(w-1,j) = w-1$ and $L(w,j) = w$, and hence $2(w-1)-4j \equiv w-1 \pmod{w}$ and $4j \equiv 0 \pmod{w}$; this gives $1 \equiv 0 \pmod{w}$, a contradiction. If $w \leqslant j \leqslant 2w-1$ we similarly obtain $1 \equiv 0$. In the same way, any violation of property (2) requires either $5 \equiv 0$ or $1 \equiv 0 \pmod{w}$. To verify property (3), only the four central entries in L need to be considered. However $L(w-1, w-1) = 2$, $L(w-1, w) = 2w-2$, $L(w, w-1) = 2w-4$ and $L(w, w) = 3$, and so these four entries are distinct.

Finally, to verify that L has no subsquares of order 2, note that any such subsquare would involve entries from all four of the $w \times w$ constituent subsquares forming L. That is, there would be four values i, j, i', j' satisfying $0 \leqslant i, j \leqslant w-1$ and $w \leqslant i', j' \leqslant 2w-1$ for which $L(i,j) = L(i',j')$ and $L(i,j') = L(i',j)$. However these give $2i-4j \equiv 2i'-4j'+3$ and $2i+4j' \equiv 2i'+4j \pmod{w}$, which are inconsistent.

(b) Next we prove that a Kotzig square of order $4w$ exists whenever w is odd, $w \geqslant 7$, $w \neq 9$. In order to do this, consider first the $2w \times 2w$ array K defined by

$$K = \begin{array}{|c|c|}\hline B+w & A \\ \hline C & B+w \\ \hline \end{array}$$

where A, B, C are as in part (a). It is easily shown, in a similar fashion to the proof for L, that K is a Kotzig square of side $2w$ provided that w is odd, $w \geqslant 7$ and $w \neq 9$. The condition $w \neq 9$ arises from consideration of the four central entries in K. Now define a $4w \times 4w$ array M by

$$M = \begin{array}{|c|c|}\hline L & L+2w \\ \hline K+2w & L \\ \hline \end{array}$$

That M has properties (1), (2) and (3) is immediate since K and L are both Kotzig squares of even order. It is also clear that M is a Latin square. In order to investigate possible subsquares of order 2, write M as

$$M = \begin{array}{|c|c|c|c|}\hline A & B+w & A+2w & B+3w \\ \hline B+w & C & B+3w & C+2w \\ \hline B+3w & A+2w & A & B+w \\ \hline C+2w & B+3w & B+w & C \\ \hline \end{array}$$

and consider the effect of projecting each $w \times w$ subsquare $X+kw$ to the entry k, that is, projecting M to

$$N = \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 3 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}$$

Any subsquare of order 2 in M must project onto a subsquare of order 2 in N. However N only contains four such subsquares and each of these corresponds to one

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of the four $2w \times 2w$ subsquares of M, namely L, $L+2w$, $K+2w$ and L, each of which is free of subsquares of order 2.

(c) The third construction takes a Kotzig square K of order $2w$ and an N_2-Latin square N of order n on the integers $0, 1, \ldots, n-1$, and produces a Kotzig square of order $2wn$. To do this replace each entry k in N by the square $K+2wk$. The resulting array is a Latin square. Properties (1), (2) and (3) and the N_2 property are easily verified.

(d) To complete the proof of the lemma, we next give Kotzig squares of orders 8 and 12.

A Kotzig square of order 8 is as follows.

0	2	1	3	4	6	5	7
5	7	4	6	2	0	3	1
1	3	5	7	0	4	6	2
4	6	0	2	3	7	1	5
2	4	3	5	7	1	0	6
7	1	6	0	5	3	2	4
3	5	7	1	6	2	4	0
6	0	2	4	1	5	7	3

A Kotzig square of order 12 is as follows.

0	6	4	8	2	10	1	7	5	9	3	11
10	2	6	0	8	4	11	3	7	1	9	5
2	8	0	10	4	6	3	9	1	11	5	7
6	4	8	2	10	0	7	5	9	3	11	1
4	10	2	6	0	8	5	11	3	7	1	9
8	0	10	4	6	2	9	1	11	5	7	3
1	7	5	9	3	11	6	0	10	2	8	4
11	3	7	1	9	5	4	8	0	6	2	10
3	9	1	11	5	7	8	2	6	4	10	0
7	5	9	3	11	1	0	10	2	8	4	6
5	11	3	7	1	9	10	4	8	0	6	2
9	1	11	5	7	3	2	6	4	10	0	8

Now suppose that $w \geqslant 4$ and $w \neq 5, 8, 10, 16$. We may write $2w = 2^k d$, where d is odd and $k \geqslant 1$. If $k \geqslant 3$ then $2w = 8(2^{k-3}d)$ and construction (c) above produces a Kotzig square of order $2w$ from the Kotzig square of order 8 and an N_2-Latin square of order $2^{k-3}d$. If $k = 2$ then $2w = 4d$ where $d = 3$ or $d \geqslant 7$ and construction (b) produces the required square except when $d = 3$ or 9. However, $k = 2$, $d = 3$ corresponds to the Kotzig square of order 12 and $k = 2$, $d = 9$ corresponds to $2w = 36$. A Kotzig square of order 36 may be formed from the Kotzig square of order 12 and an N_2-Latin square of order 3. Finally, if $k = 1$ then construction (a) gives a Kotzig square of order $2w$. □

THEOREM 2.3. *If there exists a* QFSTS(u), $u \neq 3, 21, 33$, *then there exists a* QFSTS($3(u-1)+1$).

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Proof. Let $u-1=2w$. Because $u \neq 3, 21, 33$ and there exists a QFSTS(u), we must have $w \geqslant 4$ and $w \neq 5, 8, 10, 16$. Let X, Y and Z be disjoint sets of cardinality $2w$, and let $\infty \notin X \cup Y \cup Z$. Denote the elements of X, Y and Z by $X=\{x_i : 0 \leqslant i < 2w\}$, $Y=\{y_i : 0 \leqslant i < 2w\}$ and $Z=\{z_i : 0 \leqslant i < 2w\}$.

Let $(X \cup \{\infty\}, \mathscr{A})$, $(Y \cup \{\infty\}, \mathscr{B})$ and $(Z \cup \{\infty\}, \mathscr{C})$ be QFSTS(u). Without loss of generality, we can stipulate that the Steiner triple systems contain the blocks $\{\infty, m_{2i}, m_{2i+1}\}$ for $0 \leqslant i < w$, and $m = x$, y, or z as appropriate.

Let L be a Kotzig square of order $2w$. Then define a set of blocks $\mathscr{D} = \{\{x_i, y_j, z_{L(i,j)}\} : 0 \leqslant i < 2w, 0 \leqslant j < 2w\}$.

Now $(\{\infty\} \cup X \cup Y \cup Z, \mathscr{A} \cup \mathscr{B} \cup \mathscr{C} \cup \mathscr{D})$ is an STS($3(u-1)+1$). We prove that it is quadrilateral-free.

Let Q denote the four blocks in a hypothetical quadrilateral. There are the following possible distributions of the four blocks to consider:

(i) $Q \subset \mathscr{A}$, $Q \subset \mathscr{B}$ and $Q \subset \mathscr{C}$: There are no quadrilaterals contained in $\mathscr{A}$, $\mathscr{B}$ or $\mathscr{C}$, since the STS(u) are quadrilateral-free.

(ii) $Q \subset \mathscr{D}$: Such a quadrilateral must look like the following:

$$\{x_i, y_j, z_k\}, \{x_i, y_h, z_g\}, \{x_f, y_j, z_g\}, \{x_f, y_h, z_k\}.$$

Then $L(i,j) = L(f,h) = k$ and $L(f,j) = L(i,h) = g$, so L has a subsquare of order 2, a contradiction.

(iii) $|Q \cap \mathscr{D}| = 3$: If, for example, $|Q \cap \mathscr{A}| = 1$ then the quadrilateral has the form

$$\{x_i, x_j, x_k\}, \{x_i, y_l, z_m\}, \{x_j, y_l, W\}, \{x_k, W, z_m\}.$$

From the third block, $W = z_f$, while from the fourth, $W = y_g$, a contradiction. Thus $|Q \cap \mathscr{D}| = 3$ is not possible.

(iv) $|Q \cap \mathscr{A}| = 1$, $|Q \cap \mathscr{B}| = 1$ and $|Q \cap \mathscr{D}| = 2$: Then Q has the form

$$\{\infty, x_i, x_j\}, \{\infty, y_g, y_h\}, \{x_i, y_g, z_k\} \text{ and } \{x_j, y_h, z_k\}$$

so that $\{i,j\} = \{2a, 2a+1\}$ and $\{g,h\} = \{2b, 2b+1\}$. However $L(i,g) = L(j,h)$, contradicting property (3).

(v) $|Q \cap \mathscr{A}| = 1$, $|Q \cap \mathscr{C}| = 1$ and $|Q \cap \mathscr{D}| = 2$: Then Q has the form

$$\{\infty, x_i, x_j\}, \{\infty, z_g, z_h\}, \{x_i, y_k, z_g\} \text{ and } \{x_j, y_k, z_h\}$$

so that $\{i,j\} = \{2a, 2a+1\}$ and $\{g,h\} = \{2b, 2b+1\}$. Then $\{L(i,k), L(j,k)\} = \{2b, 2b+1\}$, contradicting property (1).

(vi) $|Q \cap \mathscr{B}| = 1$, $|Q \cap \mathscr{C}| = 1$ and $|Q \cap \mathscr{D}| = 2$: Then Q has the form

$$\{\infty, y_i, y_j\}, \{\infty, z_g, z_h\}, \{x_k, y_i, z_g\} \text{ and } \{x_k, y_j, z_h\}$$

so that $\{i,j\} = \{2a, 2a+1\}$ and $\{g,h\} = \{2b, 2b+1\}$. Then $\{L(k,i), L(k,j)\} = \{2b, 2b+1\}$, contradicting property (2).

No other possible distributions of Q need to be considered. Hence the STS($3(u-1)+1$) is quadrilateral-free. □

Our next construction generalises this.

THEOREM 2.4. *If there exist a* QFSTS(v) *and a* QFSTS(u), *and* $u > 3$, $u \neq 21, 33$, *then there exists a* QFSTS($v(u-1)+1$).

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Proof. Take a QFSTS(v) = $(V, \mathscr{B})$ and let $I_{u-1} = \{0, 1, \ldots, u-2\}$. We construct a QFSTS($v(u-1)+1$) on the set $\{\infty\} \cup (V \times I_{u-1})$. For each $a \in V$, put a QFSTS(u) on $V_a = \{\infty\} \cup (\{a\} \times I_{u-1})$. Without loss of generality, we can stipulate that this Steiner triple system contains the blocks $\{\infty, a_{2i}, a_{2i+1}\}$ for $0 \leqslant i < (u-1)/2$. For each block $\{a, b, c\} \in \mathscr{B}$, fix the order of the elements and put the TD($3, u-1$) which arises from a Kotzig square of order $u-1$ on $\{a, b, c\} \times I_{u-1}$. We claim that this produces a QFSTS($v(u-1)+1$). It is easy to see that we obtain an STS($v(u-1)+1$), so we concentrate on proving it is anti-Pasch.

First of all, if there is a Pasch configuration in the Steiner triple system which involves the point ∞, then for some $a, b \in V$ two of the blocks must come from V_a, V_b. If $a = b$ then these and the other two blocks come from the same QFSTS(u), a contradiction. Hence $a \neq b$ and the other two blocks come from the transversal design associated with the block in $\mathscr{B}$ which contains the pair $\{a, b\}$. As in the proof of the previous result this supposition produces a contradiction.

It follows that any possible Pasch configuration contains at most one block from any of the V_a. If it contained one such block, then it would necessarily have the form

$$\{a_i, a_j, a_k\}, \{a_i, b_l, c_m\}, \{a_j, b_l, D\}, \{a_k, D, c_m\}.$$

From the third block we have $D = c_f$ and from the fourth $D = b_g$, a contradiction.

Thus any Pasch configuration must have all four blocks from the transversal designs. It therefore has the form

$$\{a_i, b_j, c_k\}, \{a_i, y_m, z_n\}, \{x_l, b_j, z_n\}, \{x_l, y_m, c_k\},$$

where the blocks obtained from these by deleting the subscripts lie in $\mathscr{B}$. Since there is no quadrilateral in $\mathscr{B}$, at least two (and consequently all four) of these resulting blocks are identical; indeed $x = a$, $y = b$ and $z = c$. Then the supposed quadrilateral comprises

$$\{a_i, b_j, c_k\}, \{a_i, b_m, c_n\}, \{a_l, b_j, c_n\}, \{a_l, b_m, c_k\},$$

and consideration of the suffices establishes that the underlying Latin square has a subsquare of order 2, a final contradiction. □

3. *Bose-type constructions*

THEOREM 3.1. *If there exists a* QFSTS($6n+1$) *and* $3, 7 \nmid (2s+1)$, *then there exists a* QFSTS($6n(2s+1)+1$).

Proof. Take $2s+1$ copies of the QFSTS($6n+1$), one on each of the point sets V_i, where

$$V_i = \{\infty, (0, i_0), (0, i_1), (0, i_2), (1, i_0), (1, i_1), (1, i_2), \ldots, (2n-1, i_0), (2n-1, i_1), (2n-1, i_2)\},$$

$0 \leqslant i \leqslant 2s$. The corresponding sets of blocks $\mathscr{C}_i$ are chosen so that the triples containing ∞ are

$$\{\infty, (0, i_0), (0, i_1)\}, \{\infty, (0, i_2), (1, i_0)\}, \{\infty, (1, i_1), (1, i_2)\}, \{\infty, (2, i_0), (2, i_1)\}, \ldots, \{\infty, (2n-1, i_1), (2n-1, i_2)\}.$$

Next take an Abelian group $(G, \circ)$ of order $2s+1$ represented on $G = \{0, 1, \ldots, 2s\}$ with identity I and define $x * y$ for $x, y \in G$ to be the unique $z \in G$ for which $z \circ z = x \circ y$. Note that $x, y, x * y$ are either all equal or all distinct.

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Let N be an N_2-Latin square of order $2n$ represented on $\{0, 1, \ldots, 2n-1\}$; such squares exist for all $n \geqslant 3$ [**7**, **8**, **11**], a condition which is required for the existence of a QFSTS($6n+1$).

Let $\mathscr{L}$ be the set of triples of the form $\{(a, i_\alpha), (b, j_\alpha), (c, (i*j)_{\alpha+1})\}$ where $a, b, c \in \{0, 1, \ldots, 2n-1\}, 0 \leqslant i < j \leqslant 2s$, subscript arithmetic is modulo 3 and $c = N(a, b)$. Put

$$V = \bigcup_{i=0}^{2s} V_i \quad \text{and} \quad \mathscr{B} = \left(\bigcup_{i=0}^{2s} \mathscr{C}_i\right) \cup \mathscr{L}.$$

We claim that $(V, \mathscr{B})$ is a QFSTS($6n(2s+1)+1$). To verify this, note firstly that

$$|\mathscr{C}_i| = n(6n+1) \quad \text{and} \quad |\mathscr{L}| = 4n^2 \cdot 3 \cdot s(2s+1).$$

Hence $|\mathscr{B}| = n(2s+1)[6n(2s+1)+1]$, the number of blocks required to form an STS($6n(2s+1)+1$). Moreover, any pair of points from V which contains ∞ lies in a block in one of the $\mathscr{C}_i$ as does any pair of the form $\{(a, i_\alpha), (b, i_\beta)\}$. All remaining pairs are of the form $\{(a, i_\alpha), (b, j_\beta)\}$ with $i < j$. We now identify the third element (c, k_γ) of the triple from $\mathscr{L}$ which contains this pair.

(i) If $\beta = \alpha$, then $\gamma = \alpha+1$, $k = i*j$ and $c = N(a, b)$.

(ii) If $\beta = \alpha+1$, then $\gamma = \alpha$ and k is chosen so that $i*k = j$. If $k > i$ then c is chosen so that $b = N(a, c)$; if $k < i$ then c is chosen so that $b = N(c, a)$.

(iii) If $\beta = \alpha+2$, then $\gamma = \alpha+2$ and k is chosen so that $j*k = i$. If $k > j$ then c is chosen so that $a = N(b, c)$; if $k < j$ then c is chosen so that $a = N(c, b)$.

It remains to prove that $\mathscr{B}$ is quadrilateral-free. Since none of the $\mathscr{C}_i$ contain a quadrilateral, any possible quadrilateral in $\mathscr{B}$ must comprise one of the following:

(a) a triple from $\mathscr{C}_i$, and another from $\mathscr{C}_j$, $i \neq j$, plus two from $\mathscr{L}$;

(b) a triple from $\mathscr{C}_i$, plus three from $\mathscr{L}$;

(c) four triples from $\mathscr{L}$.

We examine each of these cases in turn.

Case (a): Suppose there is a quadrilateral comprising

$$\{\infty, (a, i_\alpha), (b, i_\beta)\}, \{\infty, (c, j_\gamma), (d, j_\delta)\},$$
$$\{(a, i_\alpha), (c, j_\gamma), (e, k_\varepsilon)\}, \{(b, i_\beta), (d, j_\delta), (e, k_\varepsilon)\}$$

where $i \neq j$, $\alpha \neq \beta$, $\gamma \neq \delta$.

(i) If $\gamma = \alpha$ then $\varepsilon = \alpha+1$ and $k = i*j$. Then if $\beta = \alpha+1$ we have $\delta = \alpha+2$ and $j = i*k$, which gives $k = i*(i*k)$ or $(k \circ i^{-1})^3 = I$, an impossibility since $3 \nmid (2s+1)$. Similarly, if $\beta = \alpha+2$, we have $\delta = \alpha+1$ and $i = j*k$, giving $k = (j*k)*j$ and hence again a contradiction.

(ii) If $\gamma = \alpha+1$ then $\varepsilon = \alpha$ and $j = i*k$. Then if $\beta = \alpha+1$ we have $\delta = \alpha$ and $i = j*k$. Similarly if $\beta = \alpha+2$ then $\delta = \alpha+2$ and $k = i*j$. Both possibilities again lead to a contradiction.

(iii) If $\gamma = \alpha+2$ then $\varepsilon = \alpha+2$ and $i = j*k$. Then if $\beta = \alpha+1$ we have $\delta = \alpha+1$ and $k = i*j$. Similarly if $\beta = \alpha+2$ then $\delta = \alpha$ and $j = i*k$. Again, both possibilities give a contradiction.

It follows that there can be no quadrilateral of the form described in (a).

In cases (b) and (c) we may take three blocks from $\mathscr{L}$ as $\{(a, i_\alpha), (b, j_\beta), (c, k_\gamma)\}$,

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$\{(a, i_\alpha), (b', j'_{\beta'}), (c', k'_{\gamma'})\}$ and $\{(a', i'_{\alpha'}), (b, j_\beta), (b', j'_{\beta'})\}$, with the fourth block forming a possible quadrilateral as $\{(a', i'_{\alpha'}), (c, k_\gamma), (c', k'_{\gamma'})\}$.

Case (b): Here $i' = k = k'$ so that the fourth block lies in $\mathscr{C}_k$. If $\gamma' = \gamma$ then from the first two blocks we can deduce that $j_\beta = j'_{\beta'}$, which contradicts the fact that the third block lies in $\mathscr{L}$. Hence we may assume that $\gamma' \neq \gamma$ and, consequently, $\beta' \neq \beta$.

(i) If $\beta = \alpha$ then $\gamma = \alpha + 1$. Then if $\beta' = \alpha + 1$ we have $\gamma' = \alpha$ and $j' = i * k$; also $\alpha' = \alpha$ and $j' = j * k$, so $j * k = i * k$, a contradiction. On the other hand if $\beta' = \alpha + 2$ we have $\gamma' = \alpha + 2$ and $i = j' * k$; also $\alpha' = \alpha + 2$ and $j = j' * k$, so $i = j$, a contradiction.

(ii) If $\beta = \alpha + 1$ then $\gamma = \alpha$ and $j = i * k$. Then if $\beta' = \alpha$ we have $\alpha' = \alpha$ and $j = j' * k$, so $i * k = j' * k$, a contradiction. On the other hand if $\beta' = \alpha + 2$ we have $\gamma' = \alpha + 2$ and $i = j' * k$; also $\alpha' = \alpha + 1$ and $j' = j * k$. It then follows that $j = ((j * k) * k) * k$, giving $(j \circ k^{-1})^7 = I$, a contradiction since $7 \nmid (2s + 1)$.

(iii) If $\beta = \alpha + 2$ then $\gamma = \alpha + 2$ and $i = j * k$. Then if $\beta' = \alpha$ we have $\alpha' = \alpha + 2$ and $j' = j * k$, so $i = j'$, a contradiction. On the other hand if $\beta' = \alpha + 1$ we have $\gamma' = \alpha$ and $j' = i * k$; also $\alpha' = \alpha + 1$ and $j = j' * k$. It then follows that $i = ((i * k) * k) * k$, giving $(i \circ k^{-1})^7 = I$, again a contradiction.

It follows that there can be no quadrilateral of the form described in (b).

Case (c): Here i', k and k' are distinct. Consider the triples formed from the second components, namely $\{i_\alpha, j_\beta, k_\gamma\}$, $\{i_\alpha, j'_{\beta'}, k'_{\gamma'}\}$, $\{i'_{\alpha'}, j_\beta, j'_{\beta'}\}$, $\{i'_{\alpha'}, k_\gamma, k'_{\gamma'}\}$. Suppose that two of these triples are identical. Without loss of generality we may assume that the first and second are equal. However $j \neq j'$ and $k \neq k'$ from the third and fourth triples and so we must have $j_\beta = k'_{\gamma'}$ and $j'_{\beta'} = k_\gamma$. Then from the first and fourth triples $i'_{\alpha'} = i_\alpha$ and so all four triples are identical. Again, without loss of generality we may then assume that $\beta = \alpha$, $\gamma = \alpha + 1$, $i < j$ and $k = i * j$. The supposed quadrilateral then has the form

$$\{(a, i_\alpha), (b, j_\alpha), (c, k_{\alpha+1})\}, \{(a, i_\alpha), (c' j_\alpha), (b', k_{\alpha+1})\},$$
$$\{(a', i_\alpha), (b, j_\alpha), (b', k_{\alpha+1})\}, \{(a', i_\alpha), (c' j_\alpha), (c, k_{\alpha+1})\},$$

where $c = N(a, b) = N(a', c')$ and $b' = N(a, c') = N(a', b)$. This is impossible because N is an N_2-Latin square.

It follows that the triples formed from the second components must be distinct. Without loss of generality we may assume that $\alpha = \beta$ and $\gamma = \alpha + 1$ so that $k = i * j$. If $\beta' = \alpha$ then $\gamma' = \alpha + 1$ and $\alpha' = \alpha + 1$ so that the fourth block gives $\gamma = \alpha + 2$, a contradiction. Similarly if $\beta' = \alpha + 2$ then $\gamma' = \alpha + 2$ and $\alpha' = \alpha + 2$ so that the fourth block gives $\gamma = \alpha$, a contradiction. Thus we must have $\beta' = \alpha + 1$ which gives $\gamma' = \alpha = \alpha'$. Examination of the four triples reveals that $k = i * j$, $j' = i * k'$, $j' = i' * j$, $k = i' * k'$. Thus $i \circ j = i' \circ k'$ and $i \circ k' = i' \circ j$, from which we can deduce that $(k' \circ j^{-1})^2 = I$, and so $k' = j$. Then $j' = k$ and $i' = i$, but then the four triples are not distinct, a contradiction. Thus there can be no quadrilateral of the form described in (c). □

THEOREM 3.2. *If there exist a* QFSTS($6n + 1$), $n \geqslant 4$, $n \neq 5, 8, 10, 16$, *and a* QFSTS($6s + 3$), *the latter having a parallel class, then there exists a* QFSTS($6n(2s + 1) + 1$).

Proof. Take $2s + 1$ copies of the QFSTS($6n + 1$), one on each of the point sets V_i, where

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$$V_i = \{\infty, (0, i_0), (0, i_1), (0, i_2), (1, i_0), (1, i_1), (1, i_2), \ldots, (2n-1, i_0), (2n-1, i_1), (2n-1, i_2)\},$$

$0 \leqslant i \leqslant 2s$. The corresponding sets of blocks $\mathscr{C}_i$ are chosen so that the triples containing ∞ are

$$\{\infty, (0, i_0), (1, i_0)\}, \{\infty, (0, i_1), (1, i_1)\}, \{\infty, (0, i_2), (1, i_2)\},$$
$$\{\infty, (2, i_0), (3, i_0)\}, \{\infty, (2, i_1), (3, i_1)\}, \{\infty, (2, i_2), (3, i_2)\},$$
$$\vdots$$
$$\{\infty, (2n-2, i_0), (2n-1, i_0)\}, \{\infty, (2n-2, i_1), (2n-1, i_1)\}, \{\infty, (2n-2, i_2), (2n-1, i_2)\}.$$

Take the QFSTS($6s+3$) on the points $\{i_\alpha : 0 \leqslant i \leqslant 2s, \alpha = 0, 1 \text{ or } 2\}$ with the parallel class $\mathscr{P} = \{\{i_0, i_1, i_2\} : 0 \leqslant i \leqslant 2s\}$. Let $\mathscr{B}_s$ denote the set of blocks forming this system.

Let L be a Kotzig square of side $2n$ represented on $\{0, 1, \ldots, 2n-1\}$, and take $\mathscr{L}$ to be the set of triples of the form $\{(a, i_\alpha), (b, j_\beta), (c, k_\gamma)\}$ where $\{i_\alpha, j_\beta, k_\gamma\} \in \mathscr{B}_s \setminus \mathscr{P}$, $i < j < k$, $a, b, c \in \{0, 1, \ldots, 2n-1\}$ and $c = L(a, b)$. Put

$$V = \bigcup_{i=0}^{2s} V_i \quad \text{and} \quad \mathscr{D} = \left(\bigcup_{i=0}^{2s} \mathscr{C}_i\right) \cup \mathscr{L}.$$

We claim that $(V, \mathscr{D})$ is a QFSTS($6n(2s+1)+1$). To verify this, note firstly that

$$|\mathscr{C}_i| = n(6n+1) \quad \text{and} \quad |\mathscr{L}| = 4n^2 \cdot 3s(2s+1).$$

Hence $|\mathscr{D}| = n(2s+1)[6n(2s+1)+1]$, the number of blocks required to form an STS($6n(2s+1)+1$). Moreover, any pair of points from V which contains ∞ lies in a block in one of the C_i as does any pair of the form $\{(a, i_\alpha), (b, i_\beta)\}$. All remaining pairs are of the form $\{(a, i_\alpha), (b, j_\beta)\}$ with $i < j$. We now identify the third element (c, k_γ) of the triple from $\mathscr{L}$ which contains this pair. Firstly, take k_γ such that $\{i_\alpha, j_\beta, k_\gamma\} \in \mathscr{B}_s \setminus \mathscr{P}$. Note that $k \neq i, j$.

(i) If $i < j < k$, put $c = L(a, b)$.
(ii) If $i < k < j$, take c so that $b = L(a, c)$.
(iii) If $k < i < j$, take c so that $b = L(c, a)$.

It remains to prove that $\mathscr{D}$ is quadrilateral-free. Since none of the $\mathscr{C}_i$ contains a quadrilateral, any possible quadrilateral in $\mathscr{D}$ must comprise one of the following:

(a) a triple from $\mathscr{C}_i$, and another from $\mathscr{C}_j$, $i < j$, plus two from $\mathscr{L}$;
(b) a triple from $\mathscr{C}_i$, plus three from $\mathscr{L}$;
(c) four triples from $\mathscr{L}$.

We examine each of these cases in turn.

Case (a): The quadrilateral must comprise

$$\{\infty, (2a, i_\alpha), (2a+1, i_\alpha)\}, \{\infty, (2b, j_\beta), (2b+1, j_\beta)\}$$

together with either

$$\{(2a, i_\alpha), (2b, j_\beta), (c, k_\gamma)\}, \{(2a+1, i_\alpha), (2b+1, j_\beta), (c, k_\gamma)\}$$

or

$$\{(2a, i_\alpha), (2b+1, j_\beta), (c, k_\gamma)\}, \{(2a+1, i_\alpha), (2b, j_\beta), (c, k_\gamma)\}.$$

In the former case one of the following must apply: $c = L(2a, 2b) = L(2a+1, 2b+1)$, or $2b = L(2a, c)$ and $2b+1 = L(2a+1, c)$, or $2b = L(c, 2a)$ and $2b+1 = L(c, 2a+1)$. In the latter case we have one of the following: $c = L(2a, 2b+1) = L(2a+1, 2b)$, or $2b+1 = L(2a, c)$ and $2b = L(2a+1, c)$, or $2b+1 = L(c, 2a)$ and $2b = L(c, 2a+1)$.

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All of these possibilities are in contradiction with the fact that L is a Kotzig square.

Case (b): The quadrilateral has the form

$$\{(a,i_\alpha),(b,i_\beta),(c,i_\gamma)\},\ \{(a,i_\alpha),(x,j_\delta),(y,k_\varepsilon)\},$$
$$\{(b,i_\beta),(x,j_\delta),(z,l_\phi)\},\ \{(c,i_\gamma),(y,k_\varepsilon),(z,l_\phi)\},$$

where $i_\alpha, j_\delta, k_\varepsilon$ are distinct, $i_\beta, j_\delta, l_\phi$ are distinct and $i_\gamma, k_\varepsilon, l_\phi$ are distinct. Now if $i_\alpha = i_\beta$ then from the second and third blocks we obtain $k_\varepsilon = l_\phi$, a contradiction. Thus $i_\alpha \neq i_\beta$ and similarly $i_\alpha \neq i_\gamma, i_\beta \neq i_\gamma$. Hence $\{i_\alpha, i_\beta, i_\gamma\} = \{i_0, i_1, i_2\}$ and the blocks formed from the second components of the four blocks in the quadrilateral would therefore form a quadrilateral in $\mathscr{B}_s$, a contradiction.

Case (c): The quadrilateral has the form

$$\{(a,i_\alpha),(b,j_\beta),(c,k_\gamma)\},\ \{(a,i_\gamma),(x,l_\delta),(y,m_\varepsilon)\},$$
$$\{(b,j_\beta),(x,l_\delta),(z,n_\phi)\},\ \{(c,k_\gamma),(y,m_\varepsilon),(z,n_\phi)\}.$$

Without loss of generality we may assume that $i<j<k$ and from the third and fourth blocks we see that $l \neq j$, $m \neq k$.

If $m_\varepsilon = j_\beta$ then the second and fourth blocks give $l_\delta = k_\gamma$ and $n_\phi = i_\alpha$ and we then have $c = L(a,b) = L(z,y)$, $x = L(a,y) = L(z,b)$, but this contradicts the fact that L is an N_2-square.

If $m_\varepsilon \neq j_\beta$ then $l_\delta \neq k_\gamma$ and $n_\phi \neq i_\alpha$. Consequently $i_\alpha, j_\beta, k_\gamma, l_\delta, m_\varepsilon, n_\phi$ are all distinct. Thus the blocks formed from the second components of the four blocks in the quadrilateral would form a quadrilateral in $\mathscr{B}_s$, a contradiction. □

In connection with Theorem 3.2 we observe that there exists a QFSTS($6s+3$) with a parallel class for all $s \geqslant 0$. This may be established from the constructions in [**6**] as follows. Firstly, the generalisation of the Bose construction given in [**6**] always produces a parallel class (the 'type A' blocks with $\{a,b,c\} \in P$). Secondly, if we construct an STS(uv) from an STS(u) and an STS(v) ($=(V,\mathscr{B})$) by the standard product construction and if the STS(u) has a parallel class, say $\mathscr{P}$, then the resulting STS(uv) also has a parallel class comprising all blocks of the form $\{(a,x),(b,x),(c,x)\}$, where $\{a,b,c\} \in \mathscr{P}$ and $x \in V$. Following through the proof of the final theorem in [**6**], it is then easy to see that the systems constructed always contain a parallel class. Consequently we may state the following.

COROLLARY 3.3. *If there exists a* QFSTS($6n+1$), $n \geqslant 4$, $n \neq 5, 8, 10, 16$, *then for any odd integer* m *there exists a* QFSTS($6nm+1$).

4. *Lu's construction*

We employ a construction of Lu [**9**] to obtain the following result which has previously, and independently, been established by Chen Demeng [**4**].

THEOREM 4.1. *If there exists a* QFSTS($m+2$) *and a* QFSTS($n+2$), *then there exists a* QFSTS($mn+2$).

Proof. Let $(Z_m \cup \{a,b\}, \mathscr{A})$ be a QFSTS($m+2$) with $\{a,b,0\} \in \mathscr{A}$, and let $(Z_n \cup \{a,b\}, \mathscr{B})$ be a QFSTS($n+2$). To avoid trivialities we may assume that $m,n>1$. Let $N_{ab} = \{\{x_i,x_j\} : x_i, x_j \in Z_m \backslash \{0\}, \ell \in \{a,b\} \text{ and } \{\ell, x_i, x_j\} \in \mathscr{A}\}$. N_{ab} is a set of pairs on $Z_m \backslash \{0\}$

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with every element appearing in two pairs. Each pair can then be ordered so that each element is the first element of one pair, and the second element of another; call this set of ordered pairs Q_{ab}. Define a permutation π on $Z_m\backslash\{0\}$ by setting $\pi(i)=j$ whenever $(i,j)\in Q_{ab}$. Subsequently, it is crucial that since $\{a,b,0\}$, $\{a,i,\pi(i)\}$ and $\{b,\pi(i),\pi^2(i)\}$ appear in $\mathscr{A}$ (or the three blocks obtained by interchanging a and b appear in $\mathscr{A}$), no block of the form $\{0,i,\pi^2(i)\}$ can appear in $\mathscr{A}$ since it is anti-Pasch.

We construct an STS($mn+2$) on the point set $(Z_m\times Z_n)\cup\{a,b\}$ with triples of the following forms where $x_1,x_2,x_3\in Z_m$ and $y_1,y_2,y_3\in Z_n$:

(i) $\{(0,y_1),(0,y_2),(0,y_3)\}$ whenever $\{y_1,y_2,y_3\}\in\mathscr{B}$, and $\{\ell,(0,y_2),(0,y_3)\}$ whenever $\{\ell,y_2,y_3\}\in\mathscr{B}$ and $\ell\in\{a,b\}$, and $\{a,b,(0,y_3)\}$ when $\{a,b,y_3\}\in\mathscr{B}$;

(ii) $\{(x_1,y_1),(x_1,y_2),(x_2,y_3)\}$ where $(x_1,x_2)\in Q_{ab}$ and $y_1+y_2\equiv 2y_3 \pmod n$;

(iii) $\{\ell,(x_1,y_1),(x_2,y_1)\}$ where $\ell\in\{a,b\}$ and $\{\ell,x_1,x_2\}\in\mathscr{A}$;

(iv) $\{(x_1,y_1),(x_2,y_2),(x_3,y_3)\}$ where $\{x_1,x_2,x_3\}\in\mathscr{A}$, $x_1<x_2<x_3$ and $y_1+y_2+y_3\equiv 0 \pmod n$.

First of all, we prove that the construction gives an STS($mn+2$). The number of type (i) blocks is $(n+2)(n+1)/6$. The number of type (ii) blocks is $(m-1)n(n-1)/2$. The number of type (iii) blocks is $(m-1)n$. The number of type (iv) blocks is $[((m+2)(m+1)/6)-m]n^2$. Thus the total number of blocks is $(mn+2)(mn+1)/6$ as expected. Therefore, it suffices to show that every pair of points is contained in a triple. All possibilities are exhausted as follows:

(1) Pairs $\{a,b\}$, $\{a,(0,y_1)\}$, $\{b,(0,y_2)\}$ and $\{(0,y_1),(0,y_2)\}$ are contained in some type (i) triple.

(2) When $x\in Z_m\backslash\{0\}$, $\{(x,y_1),(x,y_2)\}$ is contained in some type (ii) triple, since x is the first element of some pair in Q_{ab} and n is odd, the equation $y_1+y_2\equiv\simeq 2y_3 \pmod n$ has a unique solution for y_3.

(3) Pairs $\{a,(x,y)\}$ and $\{b,(x,y)\}$ for $x\in Z_m\backslash\{0\}$ are contained in some type (iii) triple.

(4) If $\{x,x'\}\in N_{ab}$, then $\{(x,y_1),(x',y_2)\}$ for $x\neq x'$ is contained in a type (ii) triple when $y_1\neq y_2$, or a type (iii) triple when $y_1=y_2$. If $\{x,x'\}\notin N_{ab}$, then $\{(x,y_1),(x',y_2)\}$ for $x\neq x'$ is contained in some type (iv) triple.

Next, we show that the STS($mn+2$) is anti-Pasch. Assume to the contrary that there exists a Pasch configuration, P say, in the STS($mn+2$). We treat all of the cases.

(a) Suppose that P contains the block $\{a,b,(0,y)\}$. Firstly let us assume that the other block of P containing a is $\{a,(0,y_1),(0,y_2)\}$. Then, without loss of generality, we may also assume that P contains $\{b,(0,y_1),(0,y_3)\}$. The remaining block must then be $\{(0,y),(0,y_2),(0,y_3)\}$. This gives a contradiction because it implies that $\mathscr{B}$ contains the blocks $\{a,b,y\}$, $\{a,y_1,y_2\}$, $\{b,y_1,y_3\}$ and $\{y,y_2,y_3\}$ which form a Pasch configuration, so, alternatively, let us assume that the other blocks of P containing a and b are $\{a,(i,r),(\pi(i),r)\}$ and $\{b,(\pi(i),r),(\pi^2(i),r)\}$. Then the fourth block must be $\{(0,y),(i,r),(\pi^2(i),r)\}$ and once again we obtain a contradiction since $\mathscr{A}$ would then contain blocks forming a Pasch configuration. It follows that P cannot contain the block $\{a,b,(0,y)\}$.

(b) Suppose that P contains a block of the form $\{a,(0,y_1),(0,y_2)\}$. Firstly let us assume that it also contains a block $\{a,(0,y_3),(0,y_4)\}$. Then, without loss of generality, the remaining blocks are $\{(0,y_1),(0,y_3),X\}$ and $\{(0,y_2),(0,y_4),X\}$ where either $X=b$ or $X=(0,y_5)$. In either case $\mathscr{B}$ would have to contain a Pasch configuration,

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giving a contradiction so, alternatively, let us assume that P contains a block $\{a,(i,r),(\pi(i),r)\}$. Then, without loss of generality, the remaining blocks are $\{(0,y_1),\ (i,r),\ (x,y)\}$ and $\{(0,y_2),(\pi(i),r),(x,y)\}$. This implies that both $\{0,i,x\}$ and $\{0,\pi(i),x\}$ are blocks in $\mathscr{A}$, a contradiction. Hence P contains no block of the form $\{a,(0,y_1),(0,y_2)\}$ and, similarly, no block of the form $\{b,(0,y_1),(0,y_2)\}$.

(c) Suppose that P contains a block of the form $\{(0,y_1),(0,y_2),(0,y_3)\}$. Then the other blocks must be of the form $\{(0,y_1),(r_1,s_1),(r_2,s_2)\}$, $\{(0,y_2),(r_1,s_1),(r_3,s_3)\}$ and $\{(0,y_3),(r_2,s_2),(r_3,s_3)\}$. We obtain a contradiction by restricting to the first coordinates.

(Together (a), (b) and (c) establish that P contains no type (i) blocks.)

(d) Suppose that P contains a block of the form $\{a,(i,r),(\pi(i),r)\}$. Then P must also contain a block $\{a,(j,s),(\pi(j),s)\}$. Because $\mathscr{A}$ contains no Pasch configurations, it is not possible to have both $\pi^2(i)=j$ and $\pi^2(j)=i$. By relabelling i and j (if necessary) we can assume that $\pi^2(j)\neq i$.

Suppose firstly that the two remaining blocks are $\{(i,r),(j,s),X\}$ and $\{(\pi(i),r),(\pi(j),s),X\}$. If $i=j$ then $X=(\pi(i),y)=(\pi^2(i),y)$, a contradiction. If $i\neq j$ then $X=(x,y)$ where $\{i,j,x\}$, $\{\pi(i),\pi(j),x\}\in\mathscr{A}$ and consequently $\mathscr{A}$ has a Pasch configuration formed from these latter two blocks and from $\{a,i,\pi(i)\}$ and $\{a,j,\pi(j)\}$.

Suppose, on the other hand, that the two remaining blocks are $\{(i,r),(\pi(j),s),X\}$ and $\{(\pi(i),r),(j,s),X\}$. If $i=j$ then $r\neq s$, so $X=(i,y)$ and we have $y+r\equiv 2s$ and $y+s\equiv 2r \pmod n$. Since $n\equiv\pm 1 \pmod 6$ these give $r=s$, a contradiction. If $j=\pi^2(i)$ and $r=s$ then the fourth block gives $X=b$ and the third block gives $\pi^4(i)=i$, which is impossible because $\mathscr{A}$ contains no Pasch configurations. If $j=\pi^2(i)$ and $r\neq s$ then the fourth block gives $X=(\pi(i),y)$ and the third block gives $\{i,\pi^3(i),\pi(i)\}\in\mathscr{A}$, contradicting $\{a,i,\pi(i)\}\in\mathscr{A}$. Finally, if $i\neq j\neq\pi^2(i)$ then $X=(x,y)$ where $\{i,\pi(j),x\},\{\pi(i),j,x\}\in\mathscr{A}$ which would form a Pasch configuration in $\mathscr{A}$ with $\{a,i,\pi(i)\}$ and $\{a,j,\pi(j)\}$.

It follows that P cannot contain a block $\{a,(i,r),(\pi(i),r)\}$ or, similarly, a block $\{b,(i,r),(\pi(i),r)\}$. Hence P contains no type (iii) blocks.

(e) Suppose that P contains a block of the form $\{(i,r),(i,2s-r),(\pi(i),s)\}$.

Assume firstly that P also contains a block $\{(i,r),(i,2t-r),(\pi(i),t)\}$. We consider the possible coordinates of the sixth point in the Pasch configuration. If $(i,2s-r)$ and $(i,2t-r)$ are joined then the first coordinate of this sixth point must be $\pi(i)$. However no block has all three points with first coordinate $\pi(i)$. If $(i,2s-r)$ is joined to $(\pi(i),t)$, then the sixth point must be $(i,2t-2s+r)$. Similarly, the remaining pair of points forces the sixth point to be of the form $(i,2s-2t+r)$. We must therefore have $2t-2s+r\equiv 2s-2t+r \pmod n$ and, again because $n\equiv\pm 1 \pmod 6$, then $t=s$ which is a contradiction.

Alternatively, assume that P also contains a block of the form $\{(i,r),(j,u),(k,-r-u)\}$. A block is needed containing $(i,2s-r)$ and (j,u) and hence the first coordinate of the sixth point must be k. The first coordinates in the last block must then be k,k and $\pi(i)$, but $k\neq i$ as we have a block $\{(i,r),(j,u),(k,-r-u)\}$, a contradiction.

Hence P contains no type (ii) blocks.

(f) It is now clear that any Pasch configuration P can only contain blocks of type (iv), so suppose that P contains $\{(i,r),(j,s),(k,-s-r)\}$. If it also contains $\{(i,r),(j_1,s_1),(k_1,-s_1-r)\}$ where $\{j_1,k_1\}\neq\{j,k\}$ then the remaining two blocks will project to a Pasch configuration in the STS$(m+2)$. If, however, P also contains $\{(i,r),(j,t),(k,-r-t)\}$ with $t\neq s$ then the remaining blocks must be

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$\{(j,s),(k,-r-t),(i,y)\}$ and $\{(j,t),(k,-s-r),(i,y)\}$ where $y+s-r-t \equiv y+t-s-r \equiv 0 \pmod{n}$, giving $s=t$, a contradiction. □

5. *Group divisible design constructions*

A TD$(3,n)$ without any sub-TD$(3,2)$ is equivalent to an N_2-Latin square of order n. We call such a TD$(3,n)$ an N_2-TD$(3,n)$.

THEOREM 5.1. *If there exists a* QFSTS$(2v+1)$ *and a* QFSTS$(2n+1)$, *and* $n>4$, *then there exists a* QFSTS$(2vn+1)$.

Proof. The construction employed is originally due to Wilson [**13**]. Delete a point from the QFSTS$(2v+1)$ to form a 3-GDD of type 2^v. Replace each remaining point x by n points $x_1, x_2, \ldots, x_n$ and then use an N_2-TD$(3,n)$ to produce a 3-GDD of type $(2n)^v$. Add one new point ∞ and, on each group together with ∞, place a copy of the QFSTS$(2n+1)$ so that when $\{\infty, a_i, b_j\}$ is a triple, then $a \neq b$. Call the triples of the 3-GDD of type $(2n)^v$ *vertical*, and the triples of each STS$(2n+1)$ *horizontal*. Points and blocks from the same copy of the STS$(2n+1)$ will be said to be on the same *level*. The resulting set of triples forms an STS$(2vn+1)$, which we prove is anti-Pasch. Suppose to the contrary that a Pasch configuration P is present.

Suppose firstly that P contains ∞. Then P contains at least two horizontal blocks. Since the STS$(2n+1)$ used is anti-Pasch these two horizontal blocks cannot be from the same level. Hence P must contain two horizontal and two vertical blocks. Then the STS$(2v+1)$ would have to contain a Pasch configuration. It follows that P cannot contain ∞.

Suppose next that P contains two horizontal blocks which do not contain ∞. These must intersect in a common point and consequently they and the remaining two blocks must be from the same level. Then the STS$(2n+1)$ would have to contain a Pasch configuration. Hence P contains at most one horizontal block.

Suppose now that P contains precisely one horizontal block. Without loss of generality this block has the form $\{a_i, a_j, b_k\}$ (where b may or may not equal a) and there are vertical blocks $\{a_i, c_l, d_m\}$ and $\{a_j, c_l, e_p\}$. Then d_m and e_p come from the same level and so the fourth block must be horizontal, a contradiction.

Finally suppose that P contains four vertical triples. If all six points come from different levels then the STS$(2v+1)$ would contain a Pasch configuration. This would also be the case if just two of the six points came from the same level (with the remaining four from different levels). If two points come from one level and a further two from another level then the third pair must also share a common level, which is impossible since the STS$(2v+1)$ used was anti-Pasch and the transversal design used was an N_2-TD. □

We have one more recursive construction using group divisible designs.

THEOREM 5.2. *If there exist a* QFSTS$(2v+1)$, *a* QFSTS$(4n+1)$ *and a* QFSTS$(2n(v-1)+1)$, *and if* $n, v>4$ *then there exists a* QFSTS$(2n(3v-1)+1)$.

Proof. We shall require a particular 3-GDD of type $(2n)^v$ which is constructed as in the proof of Theorem 5.1. Note from the preceding proof that the blocks of this group divisible design cannot form a Pasch configuration.

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For the main construction take an N_2-TD$(3, v)$, and delete a point to obtain a $\{3, v\}$-GDD (say $\mathcal{G}$) of type $2^v(v-1)^1$. Replace each point x of $\mathcal{G}$ by $2n$ points $x_1, x_2, \ldots, x_{2n}$ (we shall refer to x as the *projection* of x_i). Then using an N_2-TD$(3, 2n)$ for the original $\mathcal{G}$-blocks of size 3 and using our 3-GDD of type $(2n)^v$ for each of the two original $\mathcal{G}$-blocks of size v, we may construct a 3-GDD of type $(4n)^v(2n(v-1))^1$. Add one new point ∞ and, on each group together with ∞, place a copy of the QFSTS$(4n+1)$ or a copy of the QFSTS$(2n(v-1)+1)$, as appropriate, so that when $\{\infty, a_i, b_j\}$ is a triple then $a \neq b$. Call the triples of the 3-GDD of type $(4n)^v(2n(v-1))^1$ *vertical*, and triples of each QFSTS$(4n+1)$ and the QFSTS$(2n(v-1)+1)$ *horizontal*. Points and blocks from the same copy of the QFSTS$(4n+1)$ will be said to be on the same *level*; those from the copy of the QFSTS$(2n(v-1)+1)$ will be said to be on the *bottom* level. The resulting set of triples forms an STS$(2n(3v-1)+1)$ which we prove is anti-Pasch. Suppose that this is not the case and that a Pasch configuration P is present.

Suppose firstly that P contains ∞. Then P contains at least two horizontal blocks. Since the STS$(4n+1)$ and the STS$(2n(v-1)+1)$ used are anti-Pasch, these two horizontal blocks cannot be from the same level. Hence P must contain two horizontal and two vertical blocks. If one vertical block comes from the 3-GDD of type $(2n)^v$ then so does the other and they project back into different $\mathcal{G}$-blocks of size v; consequently they are disjoint and cannot contribute to a Pasch configuration. Thus both vertical blocks must come from the N_2-TD$(3, 2n)$. If neither horizontal block comes from the bottom level then, by projecting back to the TD$(3, v)$, we find a Pasch configuration amongst the blocks of this design, contradicting the fact that it is an N_2-TD. If one of the horizontal blocks comes from the bottom level then again by projecting back to the N_2-TD$(3, v)$ we find that the projection of the sixth point (that is, the point which does not lie in a horizontal block) must lie in two different groups of the TD$(3, v)$, which is impossible. It follows that P cannot contain ∞.

Suppose next that P contains two horizontal blocks which do not contain ∞. These must intersect in a common point and consequently they and the remaining two blocks must be from the same level. Then the STS$(4n+1)$ or the STS$(2n(v-1)+1)$ would have to contain a Pasch configuration. Hence P contains at most one horizontal block.

Suppose now that P contains precisely one horizontal block. Assume firstly that this block is from a copy of the QFSTS$(4n+1)$ so that, without loss of generality, it has the form $\{a_i, a_j, b_k\}$ (where b may or may not equal a) and there are vertical blocks $\{a_i, c_l, d_m\}$ and $\{a_j, c_l, e_p\}$. Then d_m and e_p come from the same level and so the fourth block must be horizontal, a contradiction. Next assume that the single horizontal block is from the bottom level. If this block has the form $\{a_i, a_j, b_k\}$ (where b may or may not equal a) then, as above, we have a contradiction. If the horizontal block has the form $\{a_i, b_j, c_k\}$ (a, b, c all distinct) then the projections of two of the remaining three points of P must lie in the same group of v points of $\mathcal{G}$ and be distinct; this is not compatible with these two points of P lying in a block containing one of a_i, b_j or c_k.

Finally suppose that P contains four vertical triples. If none of the six points of P come from the bottom level then the four triples of P come from the 3-GDD of type $(2n)^v$ and they must come from the *same* copy of this GDD because the two copies of this design are pointwise disjoint. Thus the blocks of this GDD would contain a Pasch configuration. However, as noted above, this is not the case. If precisely one of the six points comes from the bottom level then consideration of the two vertical

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blocks through this point, and the fact that there are no horizontal blocks, gives no location for the remaining sixth point. If precisely two of the six points come from the bottom level then either the remaining four points come from just two levels (with two of the four points at each level) and the two points from each level have the same projection, or the remaining four points come from four different levels and the two points from the bottom level have different projections. In the former case we would have a Pasch configuration in the N_2-TD$(3, 2n)$ and in the latter case one in the N_2-TD$(3, v)$. Lastly, we note that we cannot have three or more points of P from the bottom level as this would imply the existence of a horizontal block. □

6. *Conclusions*

Using the constructions above we can produce QFSTS(v) for a wide range of admissible v. Such systems are already known to exist for all $v \equiv 3 \pmod 6$ and so interest naturally focuses on $v \equiv 1 \pmod 6$. We show firstly below how one may cover the residue classes $v \equiv 1$ or $7 \pmod{18}$, $v \neq 7$. By further subdivision of the remaining residue class, $v \equiv 13 \pmod{18}$, it is possible to make further progress. However, the existing constructions for QFSTS(v) do not appear to be capable of establishing the conjectured result that such systems exist for all $v \equiv 1$ or $3 \pmod 6$ apart from $v = 7$ or 13. We go on to show that the existence of QFSTS(v) in the special cases $v = 6p+1$, $p \equiv 5 \pmod 6$, a prime, and $v = 12p+1$, $p \equiv 1 \pmod 6$, a prime, would suffice to complete a proof of the conjecture. In the course of this section we shall make use of some specific QFSTS(v) for $v < 100$; these may be found in [**3**] and the references cited therein.

THEOREM 6.1. *If* $v \equiv 1$ *or* $7 \pmod{18}$ *and* $v \neq 7$ *then there exist a* QFSTS(v).

Proof. (a) $v \equiv 7 \pmod{18}$: Suppose that $v = 18s+7$ and $s \geqslant 1$. Since there exists a QFSTS($6s+3$), an application of Theorem 2.3 gives a QFSTS($18s+7$) except in the cases $s = 3$ and $s = 5$, but these correspond to a QFSTS(61) and a QFSTS(97) which are known.

(b) $v \equiv 1 \pmod{18}$: Reapplying Theorem 2.3 gives a QFSTS($54s+19$) for $s \geqslant 1$. Since there exists a QFSTS(19), we may assert the existence of QFSTS($54s+19$) for $s \geqslant 0$. Next, taking a QFSTS(37) and using Theorem 5.1 with $n = 18$ and $v = 3s+1$ we obtain a QFSTS($108s+37$) for $s \geqslant 0$. Taking a QFSTS(55), we may apply Theorem 3.2 with $n = 9$ to obtain a QFSTS($108s+55$) for $s \geqslant 0$. Again taking a QFSTS(19) and using Theorem 3.1 with $n = 3$ and $s = 3t+2$ ($t \geqslant 0$), we obtain a QFSTS($108s+91$) *except* in the case when $7|(6t+5)$, that is, when $t = 7u+5$ (corresponding to a QFSTS($756u+631$)). To deal with this exception we may use Theorem 3.2 with $n = 21$; this requires a QFSTS(127) which may be obtained from Theorem 2.3 using a QFSTS(43). We obtain from Theorem 3.2 a QFSTS($252s+127$) ($s \geqslant 0$) which covers the case of QFSTS($756u+631$). At this point it is opportune to summarise the residue classes modulo 108 which are covered by the previous paragraph. These are 19, 37, 55, 73 and 91. To deal with the remaining case, $v \equiv 1 \pmod{108}$, we use induction. Firstly take a QFSTS(37) and apply Theorem 2.3 to obtain a QFSTS(109). Next assume that QFSTS(v) exist for all $v \equiv 1 \pmod{18}$ satisfying $19 \leqslant v \leqslant v_0$. (We have established this for $v_0 = 109$.) Now choose $v' \equiv 1 \pmod{18}$ satisfying $19 \leqslant v' \leqslant 3v_0$. If $v' \equiv 19$, 37, 55, 73 or $91 \pmod{108}$, we already have a QFSTS(v'), so suppose that $v' = 108s+1$. Put $u = (v'+2)/3 = 36s+1$. Then

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$u \leqslant v_0$ and $u \equiv 1 \pmod{18}$ and so, by assumption, there exists a QFSTS(u). Theorem 2.3 then gives a QFSTS($3u-2$), that is, a QFSTS(v'). It now follows by induction that a QFSTS(v) exists for all $v \equiv 1 \pmod{18}$, $v \geqslant 19$. □

THEOREM 6.2. *If* $v \equiv 49 \pmod{72}$ *then there exists a* QFSTS(v).

Proof. Firstly suppose that $v \equiv 49 \pmod{144}$. Take a QFSTS(49) and a QFSTS($6s+3$) and apply Theorem 5.1. This gives a QFSTS($144s+49$) for $s \geqslant 0$.

Secondly suppose that $v \equiv 121 \pmod{144}$. Take a QFSTS(25) and apply Corollary 3.3 with $n = 4$ and $m = 6s+5$. This gives a QFSTS($144s+121$) for $s \geqslant 0$. □

THEOREM 6.3. *If* $v \equiv 31 \pmod{36}$ *and* v *is not of the form* $v = 6p+1$, *where* p *is a prime, then there exists a* QFSTS(v).

Proof. If $v \equiv 31 \pmod{36}$ then we may write $v-1 = 6p_1p_2\ldots p_k$, where each p_i is an odd prime, at least 5, and an odd number of these primes satisfy $p_i \equiv 5 \pmod{6}$. From the assumption in the statement of the theorem, $k \geqslant 2$, and we may also assume that $p_k \equiv 5 \pmod{6}$. Then $3p_k \equiv 3 \pmod{6}$ and $2p_1p_2\ldots p_{k-1}+1 \equiv 3 \pmod{6}$. Applying Theorem 2.4, we obtain a QFSTS(v). □

THEOREM 6.4. *If* $v \equiv 13 \pmod{72}$ *and* $v-1$ *has a factor of the form* $6t+5$ *then there exists a* QFSTS(v).

Proof. If $v \equiv 13 \pmod{72}$ then we may write $v-1 = 12p_1p_2\ldots p_k$, where, since $v \neq 13$, each p_i is an odd prime, at least 5, and an even number of these primes satisfy $p_i \equiv 5 \pmod{6}$. From the assumption in the statement of the theorem, $k \geqslant 2$ and one of the primes is congruent to 5 (mod 6), say $p_k \equiv 5 \pmod{6}$. Then $3p_k \equiv 3 \pmod{6}$ and $4p_1p_2\ldots p_{k-1}+1 \equiv 3 \pmod{6}$. Provided that $4p_1p_2\ldots p_{k-1}+1 \neq 21$, Theorem 2.4 may be applied to obtain a QFSTS(v). If $4p_1p_2\ldots p_{k-1}+1 = 21$, then $k = 2$ and $p_1 = 5$; if $p_2 \neq 5$, then we simply exchange p_1 and p_2 and apply Theorem 2.4. The remaining case, $k = 2$, $p_1 = p_2 = 5$, corresponds to a QFSTS(301). This latter design may be obtained from a QFSTS(21) and a QFSTS(31) using Theorem 5.1. □

Our next observation is something between a conjecture and a theorem. We state it as a proposition.

PROPOSITION 6.5. *If it could be shown that there existed a* QFSTS(v) *for all* $v = 6p+1$ *where* $p \equiv 5 \pmod{6}$ *is a prime, and for all* $v = 12p+1$ *where* $p \equiv 1 \pmod{6}$ *is a prime, then it would follow that there existed a* QFSTS(v) *if and only if* $v \equiv 1$ *or* $3 \pmod{6}$ *and* $v \neq 7$ *or* 13.

Proof. After the preceding theorems, it remains only to deal with the cases $v \equiv 13$, 31 and 67 (mod 72). In the latter two cases $v \equiv 31 \pmod{36}$ and a QFSTS(v) exists either as a consequence of the assumption in the statement of the proposition or as a consequence of Theorem 6.3.

To deal with the case $v \equiv 13 \pmod{72}$, $v \neq 13$, we note that Theorem 6.4 produces a QFSTS(v) unless $v-1$ has no factor of the form $6t+5$. Then $v-1 = 12p_1p_2\ldots p_k$ where each p_i is an odd prime, at least 7, and all of these primes are congruent to 1 (mod 6). If $k = 1$ then the assumption in the statement of the proposition gives a QFSTS(v). If $k \geqslant 2$ then take a QFSTS($12p_1+1$) and apply Corollary 3.3 with

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$n = 2p_1$ and $m = p_2 p_3 \dots p_k$. This produces a QFSTS($12p_1 p_2 \dots p_k + 1$), that is, a QFSTS(v). □

In Proposition 6.5 we have, essentially, investigated the factorisation of $v-1$. It is possible to obtain similar (but more complicated) criteria which would ensure the existence of QFSTS(v) by investigating either the factorisation of v itself or of $v-2$. In the case of factorising v one uses the product construction [**6**] and in the case of factorising $v-2$ one uses Theorem 4.1. It is also possible to combine these criteria with the ones given in our proposition and with other constructions for QFSTS(v) given in [**2**] and [**5**]. We may thereby narrow the range of values of v, needed as the basis of an inductive argument, from the $6p+1$ and $12p+1$ of the proposition. However, this range still (apparently) remains infinite.

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