Math. Proc. Camb. Phil. Soc. (1993), **115**, 253
Printed in Great Britain

Quantum $SU(2)$-invariants dominate Casson's $SU(2)$-invariant

BY HITOSHI MURAKAMI

Department of Mathematics, Osaka City University, Sugimoto, Sumiyoshi-ku, Osaka, 558, Japan

(*Received* 23 *March* 1993)

In 1988, E. Witten proposed an array of invariants (quantum G-invariants) for a 3-manifold associated with a compact simple Lie group G based on the quantum field theory [**22**]. N. Reshetikhin and V. Turaev [**20**] gave a mathematical approach to these invariants for $G = SU(2)$ by using representations of the quantum group $U_q(sl(2, \mathbb{C}))$. See also [**2**, **11**, **12**, **14**, **15**, **16**] for other approaches to these invariants. [**16**] may be the simplest way (for knot theorists) to understand the existence of these invariants.

On the other hand, A. Casson in 1985 introduced a $\mathbb{Z}$-valued invariant for a $\mathbb{Z}$-homology 3-sphere by using representations from its fundamental group to $SU(2)$ [**1**, **21**].

R. Kirby and P. Melvin studied in [**11**] the quantum $SU(2)$-invariants $\tau_r(M)$ $(r > 2)$ more explicitly and found the Symmetry Principle. In particular, they proved by using this principle that $\tau_5(M)$ defines Casson's $SU(2)$-invariant modulo 5 for the case that M is a $\mathbb{Z}$-homology sphere which is obtained by $1/n$-Dehn surgery on a knot in S^3 (and a connected sum of such manifolds) [**11**], Theorem 8·20.

In this paper, we generalize their result in two directions; to every odd prime r and to every $\mathbb{Z}$-homology 3-sphere.

Throughout this paper r is an odd prime. Let $\tau_r(M)$ be the quantum $SU(2)$-invariant corresponding to $q = \exp(2\pi\sqrt{-1}/r)$ and $\lambda(M)$ be Casson's $SU(2)$-invariant. Let $T: \mathbb{Z}[q] \to \mathbb{Z}/r\mathbb{Z}$ be the map which sends $\sum_k f_k q^k \in \mathbb{Z}[q]$ to $\sum_k f_k \bmod r$ [**11**], p. 532, where $\mathbb{Z}[q]$ is the ring of cyclotomic integers. Note that $f \in \mathbb{Z}[q]$ is divisible by $q-1$ if and only if $T(f) \equiv 0 \bmod r$ and that T is a ring homomorphism.

We will show

MAIN THEOREM. *Let r be an odd prime and M a $\mathbb{Z}$-homology 3-sphere. Then $\tau_r(M) \in \mathbb{Z}[q]$ and $T(\tau_r(M)) \equiv 1 \bmod r$. So $\tau_r(M) - 1$ is divisible by $q-1$ in $\mathbb{Z}[q]$. Moreover we have*

$$T\left(\frac{\tau_r(M)-1}{q-1}\right) \equiv 6\lambda(M) \bmod r.$$

Therefore the set of quantum $SU(2)$-invariants $\{\tau_r\}_{r:\text{odd prime}}$ completely determines Casson's $SU(2)$-invariant.

Remark. If we regard q as an indeterminant, T would be an operation 'substituting q with 1'. So we might say that the left hand side of the congruence above is 'the first derivative of $\tau_r(M)$ at $q = 1$'.

Section 1 contains definitions of terminology used in this paper. Especially we describe formulae to calculate the quantum $SU(2)$-invariants and Casson's $SU(2)$-

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press



Fig. 1.

invariant which were obtained by Kirby and Melvin[**11**] and J. Hoste[**4**] respectively. In Section 2 we prove the Main Theorem assuming Proposition 3·1 and Lemmas 2·5 and 2·6. Proposition 3·1 describes a behaviour of the d-th derivative at 1 of the Jones polynomial of an algebraically split link (i.e. the linking number of any pair of components vanishes) in terms of the Conway polynomials of its $(d-1)$-component sublinks and the d-th derivatives of the Jones polynomials of its sublinks with $d-2$ or fewer components. It will be proved in Section 3. Lemmas 2·5 and 2·6 will be proved in Sections 4 and 5 respectively. Section 6 is devoted to various calculations used in this paper.

1. *Preliminaries*

Throughout this paper r is an odd prime, $q = \exp(2\pi\sqrt{-1}/r)$ and $s = \exp(\pi\sqrt{-1}/r)$. Thus q is a primitive r-th root of unity and $s^2 = q$. We also put $[k] = (s^k - s^{-k})/(s - s^{-1})$.

Bold faces (e.g. $\mathbf{k}$) are used for multi-indices. If $\mathbf{k} = (k_1, k_2, \ldots, k_n)$, then $|\mathbf{k}| = k_1 + k_2 + \ldots + k_n$, $\#(\mathbf{k}) = n$, $\mathbf{k}! = k_1! k_2! \ldots k_n!$, and $[\mathbf{k}] = [k_1][k_2]\ldots[k_n]$. For another multi-index $\mathbf{j}$ with $\#(\mathbf{j}) = n$, $\mathbf{k}+\mathbf{j} = (k_1+j_1, k_2+j_2, \ldots, k_n+j_n)$, $\mathbf{kj} = (k_1 j_1, k_2 j_2, \ldots, k_n j_n)$, $\binom{\mathbf{k}}{\mathbf{j}} = \binom{k_1}{j_1}\binom{k_2}{j_2}\cdots\binom{k_n}{j_n}$ and $\mathbf{k} > \mathbf{j}$ means

$$k_1 > j_1,\ k_2 > j_2,\ \ldots,\ k_n > j_n.$$

For a scalar x, $\mathbf{x} = (x, x, \ldots, x)$, $x\mathbf{k} = \mathbf{x}\mathbf{k} = (xk_1, xk_2, \ldots, xk_n)$, $x^{\mathbf{k}} = x^{k_1} x^{k_2} \ldots x^{k_n}$, and $\mathbf{k} > x$ means $\mathbf{k} > \mathbf{x}$, i.e., $k_1 > x$, $k_2 > x, \ldots, k_n > x$.

$R[x_1, x_2, \ldots, x_n]$ is the ring over R generated by $x_1, x_2, \ldots, x_n$. The map $T: \mathbb{Z}[q] \to \mathbb{Z}/r\mathbb{Z}$ sends $\sum_k f_k q^k \in \mathbb{Z}[q]$ to $\sum_k f_k \bmod r$, and may also be defined as a map from $\mathbb{Z}[\frac{1}{2}, q] = (\mathbb{Z}[\frac{1}{2}])[q]$ to $\mathbb{Z}/r\mathbb{Z}$ which sends $\frac{1}{2^n}(\sum_k f_k q^k)$ $(f_k \in \mathbb{Z})$ to $(2^{-1})^n (\sum_k f_k)$ with 2^{-1} the inverse of 2 in $\mathbb{Z}/r\mathbb{Z}$.

Let $L = K_1 \cup K_2 \cup \ldots \cup K_\mu$ be an oriented link in S^3.

The Conway polynomial $\nabla(L; z) \in \mathbb{Z}[z]$ [**3**, **8**] is defined by

$$\nabla(\text{trivial knot}; z) = 1 \quad \text{and} \quad \nabla(L_+; z) - \nabla(L_-; z) = -z\nabla(L_0; z),$$

where L_+, L_-, and L_0 are links identical to L except near a crossing as described in Figure 1. ∇ (empty link; z) is defined to be 0.

Note that the sign in the right hand side of the second axiom differs from that in [**8**]. We follow [**3**, **4**]. (Note also that the sign convention for L_+ and L_- in [**3**] differs from ours.) We denote by $\phi(L)$ the coefficient of $z^{\#(L)+1}$ in $\nabla(L; z)$, where $\#(L)$ is the number of components in L. Thus if $\#(L_+) = \#(L_-) > \#(L_0)$,

$$\phi(L_+) - \phi(L_-) = -\phi(L_0). \tag{1·1}$$

The Jones polynomial $V(L; t) \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ [**6**] is defined by

$$V(\text{trivial knot}; t) = 1 \quad \text{and} \quad tV(L_+; t) - t^{-1}V(L_-; t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\, V(L_0; t).$$

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

V (empty link; t) is defined to be $1/(t^{\frac{1}{2}}+t^{-\frac{1}{2}})$. We note that our normalization differs from the usual one. Ours coincides with $\tilde{V}$ in [**11**]. We also note that V (trivial μ-component link; q) $= [2]^{\mu-1}$ for $\mu \geqslant 0$.

Let M be a three-dimensional $\mathbb{Z}$-homology sphere. Then it is well known that M can be obtained from S^3 by Dehn surgery on a framed link $\mathscr{L} = \mathscr{K}_1 \cup \mathscr{K}_2 \cup \ldots \cup \mathscr{K}_\mu$ with framing $f_\xi = \pm 1$ and $\mathrm{lk}(\mathscr{K}_\xi, \mathscr{K}_\zeta) = 0$ for any ξ and ζ. (First represent M by an integral framed link [**13**]. Then use the Kirby calculus [**10**] to change it into a required one just as we change an integer matrix with determinant ± 1 into a diagonal one.)

A framed link is described by a link diagram with black board framing [**20**], i.e., the writhe (that is the algebraic sum of the crossings) of each component $w(\mathscr{K}_\xi)$ coincides with its framing f_ξ. For a multi-index $\mathbf{m} = (m_1, m_2, \ldots, m_\mu)$, $\mathscr{L}^{\mathbf{m}}$ denotes the $\mathbf{m}$-parallel of $\mathscr{L}$. It has m_ξ copies of $\mathscr{K}_\xi$ which are all parallel in the plane to $\mathscr{K}_\xi$ with the same orientations.

Let $\sigma_+(\mathscr{L})$ and $\sigma_-(\mathscr{L})$ be the numbers of $+1$'s and -1's in $\{f_1, f_2, \ldots, f_\mu\}$ respectively. Put $\sigma(\mathscr{L}) = \sigma_+(\mathscr{L}) - \sigma_-(\mathscr{L})$, $G(q) = \sum_{k=0}^{r-1} q^{k^2}$ (a Gaussian sum), and $\epsilon(r) = \pm 1$ according to whether $r \equiv 1$ or $-1 \bmod 4$.

Now it is a result due to Kirby and Melvin that the quantum $SU(2)$-invariant $\tau_r(M)$ can be calculated as follows. (This holds for any odd integer $r \geqslant 3$.)

THEOREM 1·2 ([**11**], Corollary 8·9).

$$\tau_r(M) = \alpha(\mathscr{L})\,\Sigma(\mathscr{L})$$

with

$$\alpha(\mathscr{L}) = \left(\frac{2}{\sqrt{r}}\sin\frac{\pi}{r}\right)^{\mu} \exp\left(\frac{-\epsilon(r)\,\sigma(\mathscr{L})\,2\pi\sqrt{-1}}{8}\right) \exp\left(\frac{-3(r-2)\,\sigma(\mathscr{L})\,2\pi\sqrt{-1}}{8r}\right)$$

and

$$\Sigma(\mathscr{L}) = \sum_{\mathbf{k}=1}^{(\mathbf{r}-1)/2} \sqrt{-1}^{\,r f \mathbf{E}(\mathbf{k})}\,[\mathbf{k}]\,J_{\mathscr{L},\mathbf{k}}.$$

Here $\mathbf{E}(\mathbf{k}) = (E_1(k_1), E_2(k_2), \ldots, E_\mu(k_\mu))$ *with* $E_\xi(k_\xi) = 1$ *if* k_ξ *is even and* 0 *otherwise and*

$$J_{\mathscr{L},\mathbf{k}} = [2] \sum_{\mathbf{j}=0}^{(\mathbf{k}-1)/2} (-1)^{\mathbf{j}} \binom{\mathbf{k}-\mathbf{j}-1}{\mathbf{j}} \sqrt{s}^{\,3w(\mathscr{L}^{\mathbf{k}-2\mathbf{j}-1})}\, V(|\mathscr{L}^{\mathbf{k}-2\mathbf{j}-1}|; q)$$

with $\sqrt{s} = \exp(\pi\sqrt{-1}/2r)$ *and* $|\mathscr{L}^{\mathbf{k}-2\mathbf{j}-1}|$ *the underlying link of* $\mathscr{L}^{\mathbf{k}-2\mathbf{j}-1}$. *Note that* $\tau_3(M) = 1$ *for any* $\mathbb{Z}$*-homology* 3*-sphere* M.

We put $X(L;t) = V(L;t)/(t^{\frac{1}{2}}+t^{-\frac{1}{2}})^{\#(L)-1}$. (Thus X(trivial link; t) $= 1$ and $V(L;q) = [2]^{\#(L)-1} X(L;q)$.) Note that $X(L;t) \in \mathbb{Z}\left[\frac{1}{t+1}, t, t^{-1}\right]$ and $X(L;q) \in \mathbb{Z}[q]$ since $1/(q+1) = q^{r-1}+q^{r-3}+\ldots+q^2+1$. (This was pointed out by Y. Yokota.)

We use the following formula to prove our Main Theorem. Put $L^{\mathbf{k}-2\mathbf{j}-1} = |\mathscr{L}_0^{\mathbf{k}-2\mathbf{j}-1}|$, where $\mathscr{L}_0$ is the framed link with **0**-framing which is isotopic to $\mathscr{L}$ as a link without framing.

LEMMA 1·3.

$$\alpha(\mathscr{L}) = (-1)^{\tilde{\sigma}(\mathscr{L})}\, q^{\mu(r-1)/2}\, q^{\sigma(\mathscr{L})\{1-\rho(r)\}} \left(\frac{q-1}{G(q)}\right)^{\mu} \tag{1·4}$$

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

and

$$\Sigma(\mathscr{L}) = \sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}[k] \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} [2]^{k-2j-1} X(L^{k-2j-1}; q), \qquad (1\cdot5)$$

where $\tilde{\sigma}(\mathscr{L}) = \sigma_+(\mathscr{L})$ *or* $\sigma(\mathscr{L})$ *according to whether* $r \equiv 1$ *or* $-1 \bmod 4$ *and* $\rho(r) = \{1-\epsilon(r)r\}/4$.

Proof. Noting that

$$G(q) = \begin{cases} \sqrt{r} & (r \equiv 1 \bmod 4) \\ \sqrt{-1}\sqrt{r} & (r \equiv -1 \bmod 4) \end{cases}$$

(see for example [5], chapter 6) and that $\sin(\pi/r) = (s-s^{-1})/(2\sqrt{-1})$, it is not so hard to prove (1·4). Details are omitted.

From [11], Lemma 3·27, $J_{\mathscr{L},k} = \sqrt{s^{f(k^2-1)}} J_{\mathscr{L}_0,k}$. Since $w(\mathscr{L}_0^{k-2j-1}) = 0$, we have

$$\begin{aligned} J_{\mathscr{L},k} &= \sqrt{s^{f(k^2-1)}}[2] \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} V(|\mathscr{L}_0^{k-2j-1}|; q) \\ &= \sqrt{s^{f(k^2-1)}} \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} [2]^{k-2j-1} X(L^{k-2j-1}; q). \end{aligned}$$

Since $\sqrt{-1}^{rf_\xi E_\xi(k)} = \sqrt{-1}^{rf_\xi(k_\xi-1)^2} = \sqrt{s^{r^2 f_\xi(k_\xi-1)^2}}$ and $s = -q^{\{1-\epsilon(r)r\}/2}$, we have

$$\sqrt{-1}^{rfE(k)} \sqrt{s^{f(k^2-1)}} = q^{\rho(r)f(k^2-1)}$$

and (1·5) follows. This completes the proof. ∎

For the Casson invariant, we use J. Hoste's formula.

THEOREM 1·6 [4]. *Let M be a $\mathbb{Z}$-homology 3-sphere which is represented by a framed link $\mathscr{L}$ as above. Then*

$$\lambda(M) = \sum_{l \subset L} \left(\prod_{\xi \in l} f_\xi \right) \phi(l).$$

2. *Outline of the proof*

We will show the following instead of the Main Theorem.

THEOREM 2·1. *Let r be an odd prime and M a $\mathbb{Z}$-homology 3-sphere. Then $\tau_r(M) \in \mathbb{Z}[\frac{1}{2}, q]$ and $T(\tau_r(M)) \equiv 1 \bmod r$. So $\tau_r(M) - 1$ is divisible by $q-1$ in $\mathbb{Z}[\frac{1}{2}, q]$. Moreover we have*

$$T\left(\frac{\tau_r(M)-1}{q-1}\right) \equiv 6\lambda(M) \mod r.$$

Following C. F. Gauss (see, for example, [5], chapter 6 again) we have

$$G(q) = \prod_{h=1}^{(r-1)/2} (q^{2h-1} - q^{1-2h}) = (q-1)^{(r-1)/2} \times u$$

with u invertible in $\mathbb{Z}[q]$. (This is because $\dfrac{q^k-q^{-k}}{q-q^{-1}} \times \dfrac{q-q^{-1}}{q^k-q^{-k}} = 1$ and $\dfrac{q-q^{-1}}{q^k-q^{-k}} \in \mathbb{Z}[q]$ since q is some power of q^k.) So from Lemma 1·3, we have $\alpha(\mathscr{L}) = \dfrac{u(\mathscr{L})}{(q-1)^{\mu(r-3)/2}}$ with

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

$u(\mathscr{L})\in\mathbb{Z}[q]$. Since it is easily seen from Lemma 1·3 that $\Sigma(\mathscr{L})\in\mathbb{Z}[q]$, we have $\tau_r(M)\in\mathbb{Z}\left[\frac{1}{q-1},q\right]$. Therefore if Theorem 2·1 holds, then $\tau_r(M)\in\mathbb{Z}[\frac{1}{2},q]\cap\mathbb{Z}\left[\frac{1}{q-1},q\right]$ and so the Main Theorem follows from Lemma 2·2 below.

LEMMA 2·2. $\mathbb{Z}[\frac{1}{2},q]\cap\mathbb{Z}\left[\frac{1}{q-1},q\right]=\mathbb{Z}[q]$.

Proof. Suppose that there exists $f(q)\in\mathbb{Z}[\frac{1}{2},q]\cap\mathbb{Z}\left[\frac{1}{q-1},q\right]$ which is not in $\mathbb{Z}[q]$. Then $f(q)$ can be written as both $\sum_i f_i q^i$ and $(q-1)^{-n}\sum_j g_j q^j$ with $f_i\in\mathbb{Z}[\frac{1}{2}]$, $g_j\in\mathbb{Z}$, and n a positive integer. We assume that n is minimal over all such expressions. (This means that $T(\sum_j g_j q^j)\neq 0$.) Now we have $(q-1)^n\sum_i f_i q^i=\sum_j g_j q^j$ and so $T((q-1)^n\sum_i f_i q^i)=T(\sum_j g_j q^j)$. The left hand side vanishes since $n>0$ but the right hand side does not. This is a contradiction and the proof is complete. ∎

Using the Taylor expansion, we have

LEMMA 2·3.

$$\begin{aligned}X(L;q)=X^{(0)}(L)+X^{(1)}(L)(q-1)+\tfrac{1}{2}X^{(2)}(L)(q-1)^2\\ +\cdots+\frac{1}{d!}X^{(d)}(L)(q-1)^d\\ +\cdots+\frac{1}{(\mu(r-3)/2+1)!}X^{(\mu(r-3)/2+1)}(L)(q-1)^{\mu(r-3)/2+1}\\ +f(L)(q-1)^{\mu(r-3)/2+2}\end{aligned}$$

for some $f(L)\in\mathbb{Z}[\frac{1}{2},q]$. *Here* $X^{(d)}(L)$ *is the dth derivative at* 1 *of* $X(L;t)$ *with respect to* t. *Note that* $\frac{1}{d!}X^{(d)}(L)\in\mathbb{Z}[\frac{1}{2}]$ *since* $X(L;t)\in\mathbb{Z}\left[\frac{1}{t+1},t,t^{-1}\right]$.

Proof. Put

$$\begin{aligned}\tilde{f}(L;t)=X(L;t)\\ -X^{(0)}(L)-X^{(1)}(L)(t-1)-\tfrac{1}{2}X^{(2)}(L)(t-1)^2\\ -\ldots-\frac{1}{d!}X^{(d)}(L)(t-1)^d\\ -\ldots-\frac{1}{(\mu(r-3)/2+1)!}X^{(\mu(r-3)/2+1)}(L)(t-1)^{\mu(r-3)/2+1}\quad\left(\in\mathbb{Z}\left[\frac{1}{2},\frac{1}{t+1},t,t^{-1}\right]\right).\end{aligned}$$

Then the dth derivative of $\tilde{f}(L;t)$ at 1 vanishes for $0\leqslant d\leqslant\mu(r-3)/2+1$ and so $\tilde{f}(L;t)$ is of the form $f(L;t)(t-1)^{\mu(r-3)/2+2}$ with $f(L;t)\in\mathbb{Z}\left[\frac{1}{2},\frac{1}{t+1},t,t^{-1}\right]$. Since $1/(q+1)=q^{r-1}+q^{r-3}+\ldots+q^2+1$, we have $f(L;q)\in\mathbb{Z}[\frac{1}{2},q]$ and the conclusion follows. ∎

Thus $$\Sigma(\mathscr{L})=\sum_{d=0}^{\mu(r-3)/2+1}\Sigma^{(d)}(\mathscr{L})+F(\mathscr{L})(q-1)^{\mu(r-3)/2+2},$$

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

where

$$\Sigma^{(d)}(\mathscr{L}) = \sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}[k] \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} [2]^{k-2j-1} \times \frac{1}{d!} X^{(d)}(L^{k-2j-1})(q-1)^d$$

and $F(\mathscr{L})$ is some element in $\mathbb{Z}[\frac{1}{2}, q]$.

Now from Proposition 3·1 below we have

$$\Sigma^{(0)}(\mathscr{L}) = \sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}[k] \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} [2]^{k-2j-1}$$

and $\Sigma^{(d)}(\mathscr{L}) = \Sigma_1^{(d)}(\mathscr{L}) + \Sigma_2^{(d)}(\mathscr{L})$ with

$$\Sigma_1^{(d)}(\mathscr{L}) = \sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}[k] \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} [2]^{k-2j-1}$$

$$\times (-1)^{d+1} 12 \times 2^{-d} \sum_{\substack{\#(\ell)=d-1 \\ \ell \subset L^{k-2j-1}}} \phi(\ell)(q-1)^d$$

and

$$\Sigma_2^{(d)}(\mathscr{L}) = \sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}[k] \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} [2]^{k-2j-1}$$

$$\times \frac{1}{d!} \sum_{\nu=1}^{d-2} (-1)^{d+\nu} \binom{|k-2j-1|-\nu-1}{d-\nu-2} \sum_{\substack{\#(\ell)=\nu \\ \ell \subset L^{k-2j-1}}} X^{(d)}(\ell)(q-1)^d$$

for $d > 0$.

The following lemma is nothing but the definition of $\alpha(\mathscr{L})$.

LEMMA 2·4. $\alpha(\mathscr{L})\Sigma^{(0)}(\mathscr{L}) = 1$.

Proof. Since $\tau_r(S^3) = 1$ and the trivial μ-component link $\mathscr{O}$ with framing $\mathbf{f} = (f_1, f_2, \ldots, f_\mu)$ $(f_\xi = \pm 1)$ represents S^3, we have $\alpha(\mathscr{O})\Sigma(\mathscr{O}) = 1$. Now since $X(|\mathscr{O}|; t) = 1$, $\Sigma(\mathscr{O}) = \Sigma^{(0)}(\mathscr{O})$ and so $\alpha(\mathscr{O})\Sigma^{(0)}(\mathscr{O}) = 1$. But $\alpha(\mathscr{L})$ and $\Sigma^{(0)}(\mathscr{L})$ depend only on the number of components and the framing of $\mathscr{L}$. Therefore we have $\alpha(\mathscr{L})\Sigma^{(0)}(\mathscr{L}) = \alpha(\mathscr{O})\Sigma^{(0)}(\mathscr{O}) = 1$, completing the proof. ∎

Now we will prove Theorem 2·1 assuming the following two lemmas.

LEMMA 2·5. *$\alpha(\mathscr{L})\Sigma_1^{(d)}(\mathscr{L}) \in \mathbb{Z}[q]$ and $T(\alpha(\mathscr{L})\Sigma_1^{(d)}(\mathscr{L})) \equiv 0 \bmod r$. Thus $\alpha(\mathscr{L})\Sigma_1^{(d)}(\mathscr{L})$ is divisible by $q-1$ in $\mathbb{Z}[q]$ and*

$$T\left(\frac{\alpha(\mathscr{L})\Sigma_1^{(d)}(\mathscr{L})}{q-1}\right) \equiv 6 \sum_{\substack{\#(\ell)=d-1 \\ \ell \subset L}} \left(\prod_{\xi \in \ell} f_\xi\right) \phi(\ell) \bmod r.$$

LEMMA 2·6. *$\alpha(\mathscr{L})\Sigma_2^{(d)}(\mathscr{L}) \in \mathbb{Z}[\frac{1}{2}, q]$ and $T(\alpha(\mathscr{L})\Sigma_2^{(d)}(\mathscr{L})) \equiv 0 \bmod r$. Thus $\alpha(\mathscr{L})\Sigma_2^{(d)}(\mathscr{L})$ is divisible by $q-1$ in $\mathbb{Z}[\frac{1}{2}, q]$ and*

$$T\left(\frac{\alpha(\mathscr{L})\Sigma_2^{(d)}(\mathscr{L})}{q-1}\right) \equiv 0 \bmod r.$$

Proofs of these lemmas are given in Sections 4 and 5.

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

Proof of Theorem 2·1. From Section 1 we have

$$\begin{aligned}\tau_r(M) &= \alpha(\mathscr{L})\,\Sigma(\mathscr{L}) \\ &= \alpha(\mathscr{L})\,\Sigma^{(0)}(\mathscr{L}) + \sum_{d=1}^{\mu(r-3)/2+1} \alpha(\mathscr{L})\,\Sigma_1^{(d)}(\mathscr{L}) \\ &\quad + \sum_{d=1}^{\mu(r-3)/2+1} \alpha(\mathscr{L})\,\Sigma_2^{(d)}(\mathscr{L}) + \alpha(\mathscr{L})\,F(\mathscr{L})\,(q-1)^{\mu(r-3)/2+2}.\end{aligned}$$

Since $\alpha(\mathscr{L}) = \dfrac{u(\mathscr{L})}{(q-1)^{\mu(r-3)/2}}$ with $u(\mathscr{L}) \in \mathbb{Z}[q]$, we have $\alpha(\mathscr{L})\,F(\mathscr{L})\,(q-1)^{\mu(r-3)/2+2} = (q-1)^2 F(\mathscr{L})\,u(\mathscr{L}) \in \mathbb{Z}[\frac{1}{2}, q]$ and so

$$T\left(\frac{\alpha(\mathscr{L})\,F(\mathscr{L})\,(q-1)^{\mu(r-3)/2+2}}{q-1}\right) = 0.$$

Using Lemmas 2·4–2·6, we finally have $\tau_r(M) \in \mathbb{Z}[\frac{1}{2}, q]$, $T(\tau_r(M)) \equiv 1 \bmod r$, and

$$\begin{aligned}T\left(\frac{\tau_r(M)-1}{q-1}\right) &\equiv 6 \sum_{d=1}^{\mu(r-3)/2+1} \sum_{\substack{\#(\ell)=d-1 \\ \ell \subset L}} \left(\prod_{\xi \in \ell} f_\xi\right) \phi(\ell) \\ &\equiv 6 \sum_{\ell \subset L} \left(\prod_{\xi \in \ell} f_\xi\right) \phi(\ell) \\ &\equiv 6\lambda(M) \bmod r,\end{aligned}$$

where the second congruence is trivial if $r = 3$ and follows from $\mu(r-3)/2+1 > \mu$ if $r \geqslant 5$. ∎

3. *Derivatives of the Jones polynomial*

Let $L = K_1 \cup K_2 \cup \ldots \cup K_\mu$ be a link in S^3. L is called *algebraically split* if $\mathrm{lk}(K_\xi, K_\zeta) = 0$ for any ξ and ζ $(\xi \neq \zeta)$, following P. Melvin. If there are mutually disjoint 3-balls $B_1, B_2, \ldots,$ and B_μ in S^3 with $K_\xi \subset \mathrm{Int}\, B_\xi$, then L is called *geometrically split*.

In this section, we prove a formula for an algebraically split link L which expresses the dth derivative of $X(L;t)$ at $t = 1$ in terms of the dth coefficients of the Conway polynomials of $(d-1)$-component sublinks and the dth derivatives of $X(\ell;t)$ of the sublinks ℓ with $d-2$ or less components. More precisely, we will prove

Proposition 3·1. *Let L be an algebraically split link with μ components. Then for $d > 0$ we have*

$$X^{(d)}(L) = (-1)^{d+1}\, 12 \times d!\, 2^{-d} \sum_{\substack{\#(\ell)=d-1 \\ \ell \subset L}} \phi(\ell) + \sum_{\nu=1}^{d-2} (-1)^{d+\nu} \binom{\mu-\nu-1}{d-\nu-2} \sum_{\substack{\#(\ell)=\nu \\ \ell \subset L}} X^{(d)}(\ell). \tag{3·2}$$

Here we abbreviate $X^{(d)}(L;1)$ to $X^{(d)}(L)$. The binomial coefficient $\binom{m}{n}$ is defined for any integers m and n to be the coefficient of x^n in the formal power expansion of $(1+x)^m$ with non-negative powers. So in particular $\binom{m}{n} = 0$ for a negative n. (H. Morton taught this to the author.) Note that $X^{(0)}(L) = X(L;1) = 1$.

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

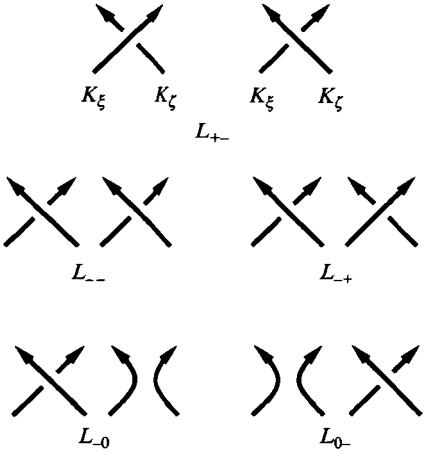


Fig. 2.

Let $K_1, K_2, \ldots, K_\mu$ be the components of L. Consider a pair of crossings of K_ξ and K_ζ which are positive and negative ($\xi \neq \zeta$). Then we denote by $L_{+-}, L_{--}, L_{-+}, L_{0-}$, and L_{-0} the links as indicated in Figure 2. We say that L_{+-} is obtained from L_{-+} by a *double crossing change* at K_ξ and K_ζ, following P. Melvin again. Note that the numbers of components of L_{0-} and L_{-0} are $\mu-1$. We also note that any algebraically split link can be deformed into geometrically split link by applying a finite number of double crossing changes.

Since $tV(L_{+-};t)-t^{-1}V(L_{--};t) = (t^{\frac{1}{2}}-t^{-\frac{1}{2}})V(L_{0-};t)$ and $\#(L_{+-}) > \#(L_{0-})$, we have $(t^2+t)X(L_{+-};t)-(1+t^{-1})X(L_{--};t) = (t-1)X(L_{0-};t)$. Since we also have $(t^2+t)X(L_{-+};t)-(1+t^{-1})X(L_{--};t) = (t-1)X(L_{-0};t)$,

$$(t^2+t)\{X(L_{+-};t)-X(L_{-+};t)\} = (t-1)\{X(L_{0-};t)-X(L_{-0};t)\}.$$

Taking the dth derivatives of both sides at $t=1$, we have

$$X^{(1)}(L_{+-})-X^{(1)}(L_{-+}) = \tfrac{1}{2}\{X^{(0)}(L_{0-})-X^{(0)}(L_{-0})\}-\tfrac{3}{2}\{X^{(0)}(L_{+-})-X^{(0)}(L_{-+})\} = 0 \quad (3\cdot3)$$

and
$$X^{(d)}(L_{+-})-X^{(d)}(L_{-+}) = \frac{d}{2}\{X^{(d-1)}(L_{0-})-X^{(d-1)}(L_{-0})\}$$
$$-\frac{3d}{2}\{X^{(d-1)}(L_{+-})-X^{(d-1)}(L_{-+})\}$$
$$-\frac{d(d-1)}{2}\{X^{(d-2)}(L_{+-})-X^{(d-2)}(L_{-+})\} \quad (3\cdot4)$$

for $d \geqslant 2$. Since $\#(L_{0-}) = \#(L_{-0}) < \#(L_{+-}) = \#(L_{-+}) = \#(L_{--})$, we have from (1·1)

$$\phi(L_{+-})-\phi(L_{--}) = -\phi(L_{0-})$$

and
$$\phi(L_{-+})-\phi(L_{--}) = -\phi(L_{-0}).$$

So we have

$$\phi(L_{+-})-\phi(L_{-+}) = -\{\phi(L_{0-})-\phi(L_{-0})\}. \quad (3\cdot5)$$

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

Note that if L_{+-}(respectively L_{-+}) is algebraically split, then so are L_{-+} (respectively L_{+-}), L_{0-}, and L_{-0}. Therefore the links appearing in (3·3)–(3·5) are all algebraically split.

We divide the proof of Proposition 3·1 into two lemmas. Put

$$Y_d(L) = X^{(d)}(L) - (-1)^{d+1} 12 \times d!\, 2^{-d} \sum_{\substack{\#(\ell)=d-1 \\ \ell \subset L}} \phi(\ell)$$

$$- \sum_{\nu=1}^{d-2} (-1)^{d+\nu} \binom{\mu-\nu-1}{d-\nu-2} \sum_{\substack{\#(\ell)=\nu \\ \ell \subset L}} X^{(d)}(\ell).$$

LEMMA 3·6. *If L is geometrically split, then $Y_d(L) = 0$, i.e.* (3·2) *holds.*

LEMMA 3·7. *If L_{+-} is obtained from L_{-+} by a double crossing change, then $Y_d(L_{+-}) = Y_d(L_{-+})$.*

Proof of Proposition 3·1 *using Lemmas* 3·6 *and* 3·7. We will prove that $Y_d(L) = 0$ for every algebraically split link L. We define the complexity of an algebraically split link to be the minimum number of double crossing changes needed to deform it into a geometrically split link. Then the proof will be done by induction on the complexity $c(L)$ of L. Note that $|c(L_{+-}) - c(L_{-+})| = 1$. If the complexity is greater than zero, we use Lemma 3·7 and if the complexity is zero, i.e., L is geometrically split, we use Lemma 3·6. So the proof is complete if Lemmas 3·6 and 3·7 are proved. ∎

Now we prove Lemmas 3·6 and 3·7.

Proof of Lemma 3·6. First note that for a geometrically split link, $X(L;t) = \prod_{\xi=1}^{\mu} X(K_\xi;t) = \prod_{\xi=1}^{\mu} V(K_\xi;t)$.

For $d = 1$, we have $X^{(1)}(L) = \sum_{\xi=1}^{\mu} X^{(1)}(K_\xi) = \sum_{\xi=1}^{\mu} V^{(1)}(K_\xi)$ since $X(L;1) = 1$ for any link L. Since the first derivative of the Jones polynomial of a knot its zero [**6**], Theorem 16, (3·2) follows.

For $d = 2$, since the first derivative of the Jones polynomial of a knot vanishes, we have $X^{(2)}(L) = \sum_{\xi=1}^{\mu} X^{(2)}(K_\xi) = \sum_{\xi=1}^{\mu} V^{(2)}(K_\xi)$. Now the second derivative of the Jones polynomial of a knot K is equal to $-6\phi(K)$ [**19**], Theorem 2 (see also [**7**], section 12) and (3·2) follows. Note that the normalization for the Jones polynomial in [**19**] differs from that in this paper by the multiplication of $(-1)^{\#(L)+1}$ and so they coincide for knots.

For $d > 2$, since the Conway polynomial of a geometrically split link with two or more components vanishes we only have to prove the following formula.

$$X^{(d)}(L) = \sum_{\nu=1}^{d-2} (-1)^{d+\nu} \binom{\mu-\nu-1}{d-\nu-2} \sum_{\substack{\#(\ell)=\nu \\ \ell \subset L}} X^{(d)}(\ell). \tag{3·8}$$

Since $X(L) = X(K_1)X(K_2)\dots X(K_\mu)$, $X(K_\xi;1) = 1$, and $X^{(1)}(K_\xi) = 0$, we have

$$X^{(d)}(L) = \sum_{s=1}^{d/2} \sum_{\substack{m_1+m_2+\dots+m_s=d \\ m_1>1, m_2>1, \dots, m_s>1}} \sum_{\substack{\#(\tilde{\ell})=s \\ \tilde{\ell} \subset L}} \frac{d!}{m_1!\, m_2! \dots m_s!} X^{(m_1, m_2, \dots, m_s)}(\tilde{\ell})$$

$$= \sum_{\substack{|\mathbf{m}|=d \\ \mathbf{m}>1 \\ 1 \leq \#(\mathbf{m}) \leq d/2}} \sum_{\substack{\#(\tilde{\ell})=\#(\mathbf{m}) \\ \tilde{\ell} \subset L}} \frac{d!}{\mathbf{m}!} X^{(\mathbf{m})}(\tilde{\ell}). \tag{3·9}$$

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

Here the first summation ranges over all partitions of d into positive integers $\mathbf{m} = (m_1, m_2, \ldots, m_s)$ and $X^{(\mathbf{m})}(\tilde{\ell}) = X^{(m_1, m_2, \ldots, m_s)}(\tilde{\ell}) = \prod_{K_\xi \in \tilde{\ell}} X^{(m_\xi)}(K_\xi)$.

For a sublink ℓ in L with ν components we also have

$$X^{(d)}(\ell) = \sum_{\substack{|\mathbf{m}|=d \\ \mathbf{m}>1 \\ 1 \leqslant \#(\mathbf{m}) \leqslant d/2}} \sum_{\substack{\#(\tilde{\ell})=\#(\mathbf{m}) \\ \tilde{\ell} \subset \ell}} \frac{d!}{\mathbf{m}!} X^{(\mathbf{m})}(\tilde{\ell}).$$

Thus the right hand side of (3·8) is

$$\sum_{\nu=1}^{d-2} (-1)^{d+\nu} \binom{\mu-\nu-1}{d-\nu-2} \sum_{\substack{\#(\ell)=\nu \\ \ell \subset L}} \sum_{\substack{|\mathbf{m}|=d \\ \mathbf{m}>1 \\ 1 \leqslant \#(\mathbf{m}) \leqslant d/2}} \sum_{\substack{\#(\tilde{\ell})=\#(\mathbf{m}) \\ \tilde{\ell} \subset \ell}} \frac{d!}{\mathbf{m}!} X^{(\mathbf{m})}(\tilde{\ell})$$

$$= \sum_{\substack{|\mathbf{m}|=d \\ \mathbf{m}>1 \\ 1 \leqslant \#(\mathbf{m}) \leqslant d/2}} \sum_{\substack{\#(\tilde{\ell})=\#(\mathbf{m}) \\ \tilde{\ell} \subset L}} \sum_{\nu=1}^{d-2} (-1)^{d+\nu} \binom{\mu-\nu-1}{d-\nu-2} \binom{\mu-\#(\mathbf{m})}{\nu-\#(\mathbf{m})} \frac{d!}{\mathbf{m}!} X^{(\mathbf{m})}(\tilde{\ell}) \quad (3\cdot10)$$

since there are $\binom{\mu-\#(\mathbf{m})}{\nu-\#(\mathbf{m})}$ ν-component sublinks in L which contains a $\#(\mathbf{m})$-component sublink $\tilde{\ell}$. (Recall that $\binom{\mu-\#(\mathbf{m})}{\nu-\#(\mathbf{m})} = 0$ for $\nu < \#(\mathbf{m})$.)

Now comparing (3·9) and (3·10), we see that it is sufficient to prove

$$\sum_{\nu=1}^{d-2} (-1)^{d+\nu} \binom{\mu-\nu-1}{d-\nu-2} \binom{\mu-s}{\nu-s} = 1$$

for $1 \leqslant s \leqslant d/2$. Since $\binom{\mu-s}{\nu-s} = 0$ for $\nu < s$, the left hand side is

$$\sum_{\nu=s}^{d-2} (-1)^{d+\nu} \binom{\mu-\nu-1}{d-\nu-2} \binom{\mu-s}{\nu-s}$$

$$= \frac{(-1)^{d+s}(\mu-s)!}{(\mu-d+1)!(d-s-2)!} \sum_{a=0}^{d-s-2} \binom{d-s-2}{a} \frac{(-1)^a}{\mu-s-a}$$

$$= \frac{(-1)^{d+s}(\mu-s)!}{(\mu-d+1)!(d-s-2)!} \frac{(-1)^{d-s-2}(d-s-2)!(\mu-d+1)!}{(\mu-s)!}$$

$$= 1,$$

where we put $a = \nu - s$ in the first equality and use Lemma 6·1 in the second equality since $\binom{\mu-\nu-1}{d-\nu-2} = 0$ for $\mu \leqslant d-2$ and $s \leqslant d-2$ for $d > 2$. The proof is complete. ∎

Proof of Lemma 3·7. We prove Lemma 3·7 by induction on d.

For $d = 1$, the formula is clear from (3·3).

We assume $Y_{d'}(L_{+-}) = Y_{d'}(L_{-+})$ for $d' < d$.

First of all we note that Proposition 3·1 is true for d' from Lemma 3·6. Therefore $Y_{d'}(L) = 0$ for any algebraically split link L. (See proof of Proposition 3·1.)

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

We may assume L_{-+} is obtained from L_{+-} by a double crossing change at K_1 and K_2. We denote by k_{+-} and k_{-+} the two-component sublinks consisting of K_1 and K_2 in L_{+-} and L_{-+} respectively. By k we denote either k_{+-} or k_{-+} and L denotes either L_{+-} or L_{-+}. We use these notations only for $L\backslash k$ and so no confusion can occur. The knot appearing in Figure 2 of L_{0-} (L_{-0}, respectively) is denoted by k_{0-} (k_{-0}, respectively) k_0 is either by k_{0-} or k_{-0} and L_0 either L_{0-} or L_{-0}.

From (3·4) and (3·5) we have

$$
\begin{aligned}
&Y_d(L_{+-})-Y_d(L_{-+})\\
&\quad= X^{(d)}(L_{+-})-X^{(d)}(L_{-+})-(-1)^{d+1}12\times d!\,2^{-d}\sum_{\substack{\#(\ell)=d-3\\ \ell\subset L\backslash k}}\{\phi(k_{+-}\cup\ell)-\phi(k_{-+}\cup\ell)\}\\
&\qquad-\sum_{\nu=1}^{d-2}(-1)^{d+\nu}\binom{\mu-\nu-1}{d-\nu-2}\sum_{\substack{\#(\ell)=\nu-2\\ \ell\subset L\backslash k}}\{X^{(d)}(k_{+-}\cup\ell)-X^{(d)}(k_{-+}\cup\ell)\}\\
&\quad=\frac{d}{2}\{X^{(d-1)}(L_{0-})-X^{(d-1)}(L_{-0})\}-\frac{3d}{2}\{X^{(d-1)}(L_{+-})-X^{(d-1)}(L_{-+})\}\\
&\qquad-\frac{d(d-1)}{2}\{X^{(d-2)}(L_{+-})-X^{(d-2)}(L_{-+})\}\\
&\qquad+(-1)^{d+1}12\times d!\,2^{-d}\sum_{\substack{\#(\ell)=d-3\\ \ell\subset L_0\backslash k_0}}\{\phi(k_{0-}\cup\ell)-\phi(k_{-0}\cup\ell)\}\\
&\qquad-\frac{d}{2}\sum_{\nu=2}^{d-2}(-1)^{d+\nu}\binom{\mu-\nu-1}{d-\nu-2}\sum_{\substack{\#(\ell)=\nu-2\\ \ell\subset L_0\backslash k_0}}\{X^{(d-1)}(k_{0-}\cup\ell)-X^{(d-1)}(k_{-0}\cup\ell)\}\\
&\qquad+\frac{3d}{2}\sum_{\nu=2}^{d-2}(-1)^{d+\nu}\binom{\mu-\nu-1}{d-\nu-2}\sum_{\substack{\#(\ell)=\nu-2\\ \ell\subset L\backslash k}}\{X^{(d-1)}(k_{+-}\cup\ell)-X^{(d-1)}(k_{-+}\cup\ell)\}\\
&\qquad+\frac{d(d-1)}{2}\sum_{\nu=2}^{d-2}(-1)^{d+\nu}\binom{\mu-\nu-1}{d-\nu-2}\sum_{\substack{\#(\ell)=\nu-2\\ \ell\subset L\backslash k}}\{X^{(d-2)}(k_{+-}\cup\ell)-X^{(d-2)}(k_{-+}\cup\ell)\}.
\end{aligned}
$$

From the inductive hypothesis (i.e. $Y_{d'}=0$ for $d'<d$), we have

$$
\begin{aligned}
&X^{(d-1)}(L_{0-})-X^{(d-1)}(L_{-0})\\
&\quad=(-1)^d12\times(d-1)!\,2^{-d+1}\sum_{\substack{\#(\ell)=d-3\\ \ell\subset L_0\backslash k_0}}\{\phi(k_{0-}\cup\ell)-\phi(k_{-0}\cup\ell)\}\\
&\qquad+\sum_{\nu=1}^{d-3}(-1)^{d+\nu+1}\binom{\mu-\nu-2}{d-\nu-3}\sum_{\substack{\#(\ell)=\nu-1\\ \ell\subset L_0\backslash k_0}}\{X^{(d-1)}(k_{0-}\cup\ell)-X^{(d-1)}(k_{-0}\cup\ell)\},
\end{aligned}
$$

$$
\begin{aligned}
&X^{(d-1)}(L_{+-})-X^{(d-1)}(L_{-+})\\
&\quad=(-1)^d12\times(d-1)!\,2^{-d+1}\sum_{\substack{\#(\ell)=d-4\\ \ell\subset L\backslash k}}\{\phi(k_{+-}\cup\ell)-\phi(k_{-+}\cup\ell)\}
\end{aligned}
$$

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

$$+\sum_{\nu=2}^{d-3}(-1)^{d+\nu+1}\binom{\mu-\nu-1}{d-\nu-3}\sum_{\substack{\#(\ell)=\nu-2\\ \ell\subset L\setminus k}}\{X^{(d-1)}(k_{+-}\cup\ell)-X^{(d-1)}(k_{-+}\cup\ell)\},$$

and

$$X^{(d-2)}(L_{+-})-X^{(d-2)}(L_{-+})$$

$$=(-1)^{d+1}12\times(d-2)!\,2^{-d+2}\sum_{\substack{\#(\ell)=d-5\\ \ell\subset L\setminus k}}\{\phi(k_{+-}\cup\ell)-\phi(k_{-+}\cup\ell)\}$$

$$+\sum_{\nu=2}^{d-4}(-1)^{d+\nu}\binom{\mu-\nu-1}{d-\nu-4}\sum_{\substack{\#(\ell)=\nu-2\\ \ell\subset L\setminus k}}\{X^{(d-2)}(k_{+-}\cup\ell)-X^{(d-2)}(k_{-+}\cup\ell)\}.$$

Thus

$$Y_d(L_{+-})-Y_d(L_{-+})$$

$$=(-1)^{d+1}12\times d!\,2^{-d}\sum_{\substack{\#(\ell)=d-3\\ \ell\subset L_0\setminus k_0}}\{\phi(k_{0-}\cup\ell)-\phi(k_{-0}\cup\ell)\}$$

$$+\frac{d}{2}(-1)^{d}12\times(d-1)!\,2^{-d+1}\sum_{\substack{\#(\ell)=d-3\\ \ell\subset L_0\setminus k_0}}\{\phi(k_{0-}\cup\ell)-\phi(k_{-0}\cup\ell)\}$$

$$-\frac{3d}{2}(-1)^{d}12\times(d-1)!\,2^{-d+1}\sum_{\substack{\#(\ell)=d-4\\ \ell\subset L\setminus k}}\{\phi(k_{+-}\cup\ell)-\phi(k_{-+}\cup\ell)\}$$

$$-\frac{d(d-1)}{2}(-1)^{d+1}12\times(d-2)!\,2^{-d+2}\sum_{\substack{\#(\ell)=d-5\\ \ell\subset L\setminus k}}\{\phi(k_{+-}\cup\ell)-\phi(k_{-+}\cup\ell)\}$$

$$+\frac{d}{2}\Bigg\{\sum_{\nu=1}^{d-3}(-1)^{d+\nu+1}\binom{\mu-\nu-2}{d-\nu-3}\sum_{\substack{\#(\ell)=\nu-1\\ \ell\subset L_0\setminus k_0}}\{X^{(d-1)}(k_{0-}\cup\ell)-X^{(d-1)}(k_{-0}\cup\ell)\}$$

$$-\sum_{\nu=2}^{d-2}(-1)^{d+\nu}\binom{\mu-\nu-1}{d-\nu-2}\sum_{\substack{\#(\ell)=\nu-2\\ \ell\subset L_0\setminus k_0}}\{X^{(d-1)}(k_{0-}\cup\ell)-X^{(d-1)}(k_{-0}\cup\ell)\}\Bigg\}$$

$$+\frac{3d}{2}\Bigg\{-\sum_{\nu=2}^{d-3}(-1)^{d+\nu+1}\binom{\mu-\nu-1}{d-\nu-3}\sum_{\substack{\#(\ell)=\nu-2\\ \ell\subset L\setminus k}}\{X^{(d-1)}(k_{+-}\cup\ell)-X^{(d-1)}(k_{-+}\cup\ell)\}$$

$$+\sum_{\nu=2}^{d-2}(-1)^{d+\nu}\binom{\mu-\nu-1}{d-\nu-2}\sum_{\substack{\#(\ell)=\nu-2\\ \ell\subset L\setminus k}}\{X^{(d-1)}(k_{+-}\cup\ell)-X^{(d-1)}(k_{-+}\cup\ell)\}\Bigg\}$$

$$+\frac{d(d-1)}{2}\Bigg\{-\sum_{\nu=2}^{d-4}(-1)^{d+\nu}\binom{\mu-\nu-1}{d-\nu-4}\sum_{\substack{\#(\ell)=\nu-2\\ \ell\subset L\setminus k}}\{X^{(d-2)}(k_{+-}\cup\ell)-X^{(d-2)}(k_{-+}\cup\ell)\}$$

$$+\sum_{\nu=2}^{d-2}(-1)^{d+\nu}\binom{\mu-\nu-1}{d-\nu-2}\sum_{\substack{\#(\ell)=\nu-2\\ \ell\subset L\setminus k}}\{X^{(d-2)}(k_{+-}\cup\ell)-X^{(d-2)}(k_{-+}\cup\ell)\}\Bigg\}.$$

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

Since the first and the second terms cancel and the fifth term (beginning with $d/2$) vanishes, we have

$$
\begin{aligned}
&Y_d(L_{+-})-Y_d(L_{-+})\\
&\quad=-\frac{3d}{2}(-1)^d\,12\times(d-1)!\,2^{-d+1}\sum_{\#(\ell)=d-4,\ \ell\subset L\setminus k}\{\phi(k_{+-}\cup\ell)-\phi(k_{-+}\cup\ell)\}\\
&\qquad-\frac{d(d-1)}{2}(-1)^{d+1}\,12\times(d-2)!\,2^{-d+2}\sum_{\#(\ell)=d-5,\ \ell\subset L\setminus k}\{\phi(k_{+-}\cup\ell)-\phi(k_{-+}\cup\ell)\}\\
&\qquad+\frac{3d}{2}\sum_{\nu=2}^{d-3}(-1)^{d+\nu}\left\{\binom{\mu-\nu-1}{d-\nu-2}+\binom{\mu-\nu-1}{d-\nu-3}\right\}\\
&\qquad\quad\times\sum_{\#(\ell)=\nu-2,\ \ell\subset L\setminus k}\{X^{(d-1)}(k_{+-}\cup\ell)-X^{(d-1)}(k_{-+}\cup\ell)\}\\
&\qquad+\frac{3d}{2}(-1)^{d+(d-2)}\binom{\mu-(d-2)-1}{d-(d-2)-2}\\
&\qquad\quad\times\sum_{\#(\ell)+(d-2)-2,\ \ell\subset L\setminus k}\{X^{(d-1)}(k_{+-}\cup\ell)-X^{(d-1)}(k_{-+}\cup\ell)\}\\
&\qquad+\frac{d(d-1)}{2}\sum_{\nu=2}^{d-4}(-1)^{d+\nu}\left\{\binom{\mu-\nu-1}{d-\nu-2}-\binom{\mu-\nu-1}{d-\nu-4}\right\}\\
&\qquad\quad\times\sum_{\#(\ell)=\nu-2,\ \ell\subset L\setminus k}\{X^{(d-2)}(k_{+-}\cup\ell)-X^{(d-2)}(k_{-+}\cup\ell)\}\\
&\qquad+\frac{d(d-1)}{2}(-1)^{d+(d-3)}\binom{\mu-(d-3)-1}{d-(d-3)-2}\\
&\qquad\quad\times\sum_{\#(\ell)=(d-3)-2,\ \ell\subset L\setminus k}\{X^{(d-2)}(k_{+-}\cup\ell)-X^{(d-2)}(k_{-+}\cup\ell)\}\\
&\qquad+\frac{d(d-1)}{2}(-1)^{d+(d-2)}\binom{\mu-(d-2)-1}{d-(d-2)-2}\\
&\qquad\quad\times\sum_{\#(\ell)=(d-2)-2,\ \ell\subset L\setminus k}\{X^{(d-2)}(k_{+-}\cup\ell)-X^{(d-2)}(k_{-+}\cup\ell)\}\\
&\quad=\frac{3d}{2}\Bigg\{\sum_{\#(\ell)=d-4,\ \ell\subset L\setminus k}\{X^{(d-1)}(k_{+-}\cup\ell)-X^{(d-1)}(k_{-+}\cup\ell)\}\\
&\qquad-(-1)^d\,12\times(d-1)!\,2^{-d+1}\sum_{\#(\ell)=d-4,\ \ell\subset L\setminus k}\{\phi(k_{+-}\cup\ell)-\phi(k_{-+}\cup\ell)\}\\
&\qquad+\sum_{\nu=2}^{d-3}(-1)^{d+\nu}\binom{\mu-\nu}{d-\nu-2}\sum_{\#(\ell)=\nu-2,\ \ell\subset L\setminus k}\{X^{(d-1)}(k_{+-}\cup\ell)-X^{(d-1)}(k_{-+}\cup\ell)\}\Bigg\}
\end{aligned}
$$

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

$$+\frac{d(d-1)}{2}\Bigg\{-(\mu-d+2)\sum_{\#_{\ell\subset L\setminus k}^{(\ell)=d-5}}\{X^{(d-2)}(k_{+-}\cup\ell)-X^{(d-2)}(k_{-+}\cup\ell)\}$$

$$+\sum_{\#_{\ell\subset L\setminus k}^{(\ell)=d-4}}\{X^{(d-2)}(k_{+-}\cup\ell)-X^{(d-2)}(k_{-+}\cup\ell)\}$$

$$+\sum_{\nu=2}^{d-4}(-1)^{d+\nu}\left\{\binom{\mu-\nu-1}{d-\nu-2}-\binom{\mu-\nu-1}{d-\nu-4}\right\}$$

$$\times\sum_{\#_{\ell\subset L\setminus k}^{(\ell)=\nu-2}}\{X^{(d-2)}(k_{+-}\cup\ell)-X^{(d-2)}(k_{-+}\cup\ell)\}$$

$$-(-1)^{d+1}12\times(d-2)!\,2^{-d+2}\sum_{\#_{\ell\subset L\setminus k}^{(\ell)=d-5}}\{\phi(k_{+-}\cup\ell)-\phi(k_{-+}\cup\ell)\}\Bigg\}. \tag{3·11}$$

Now for a link ℓ with $d-4$ components we have from the inductive hypothesis

$$X^{(d-1)}(k_{+-}\cup\ell)-X^{(d-1)}(k_{-+}\cup\ell)$$
$$=(-1)^{d}12\times(d-1)!\,2^{-d+1}\{\phi(k_{+-}\cup\ell)-\phi(k_{-+}\cup\ell)\}$$
$$+\sum_{\nu=2}^{d-3}(-1)^{d+\nu-1}\sum_{\#_{\tilde{\ell}\subset\ell}^{(\tilde{\ell})=\nu-2}}\{X^{(d-1)}(k_{+-}\cup\ell)-X^{(d-1)}(k_{-+}\cup\ell)\}.$$

So

$$\sum_{\#_{\ell\subset L\setminus k}^{(\ell)=d-4}}\{X^{(d-1)}(k_{+-}\cup\ell)-X^{(d-1)}(k_{-+}\cup\ell)\}$$

$$=(-1)^{d}12\times(d-1)!\,2^{-d+1}\sum_{\#_{\ell\subset L\setminus k}^{(\ell)=d-4}}\{\phi(k_{+-}\cup\ell)-\phi(k_{-+}\cup\ell)\}$$

$$+\sum_{\nu=2}^{d-3}(-1)^{d+\nu-1}\sum_{\#_{\ell\subset L\setminus k}^{(\ell)=d-4}}\sum_{\#_{\tilde{\ell}\subset\ell}^{(\tilde{\ell})=\nu-2}}\{X^{(d-1)}(k_{+-}\cup\ell)-X^{(d-1)}(k_{-+}\cup\ell)\}$$

$$=(-1)^{d}12\times(d-1)!\,2^{-d+1}\sum_{\#_{\ell\subset L\setminus k}^{(\ell)=d-4}}\{\phi(k_{+-}\cup\ell)-\phi(k_{-+}\cup\ell)\}$$

$$+\sum_{\nu=2}^{d-3}(-1)^{d+\nu-1}\binom{\mu-\nu}{d-\nu-2}$$

$$\times\sum_{\#_{\tilde{\ell}\subset L\setminus k}^{(\tilde{\ell})=\nu-2}}\{X^{(d-1)}(k_{+-}\cup\ell)-X^{(d-1)}(k_{-+}\cup\ell)\},$$

since there are $\binom{\mu-2-(\nu-2)}{d-4-(\nu-2)}=\binom{\mu-\nu}{d-\nu-2}$ $(d-4)$-component sublinks in $L\setminus k$ which contain $(\nu-2)$-component link $\tilde{\ell}$. Thus the first term in (3·11) vanishes.

In a similar way, we have

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

$$\sum_{\substack{\#(\ell)=d-5\\ \ell\subset L\setminus k}}\{X^{(d-2)}(k_{+-}\cup\ell)-X^{(d-2)}(k_{-+}\cup\ell)\}$$

$$=(-1)^{d+1}12\times(d-2)!\,2^{-d+2}\sum_{\substack{\#(\ell)=d-5\\ \ell\subset L\setminus k}}\{\phi(k_{+-}\cup\ell)-\phi(k_{-+}\cup\ell)\}$$

$$+\sum_{\nu=2}^{d-4}(-1)^{d+\nu}\binom{\mu-\nu}{d-\nu-3}\sum_{\substack{\#(\ell)=\nu-2\\ \ell\subset L\setminus k}}\{X^{(d-2)}(k_{+-}\cup\ell)-X^{(d-2)}(k_{-+}\cup\ell)\}$$

and

$$\sum_{\substack{\#(\ell)=d-4\\ \ell\subset L\setminus k}}\{X^{(d-2)}(k_{+-}\cup\ell)-X^{(d-2)}(k_{-+}\cup\ell)\}$$

$$=(-1)^{d+1}12\times(d-2)!\,2^{-d+2}\sum_{\substack{\#(\ell)=d-5\\ \ell\subset L\setminus k}}(\mu-d+3)\{\phi(k_{+-}\cup\ell)-\phi(k_{-+}\cup\ell)\}$$

$$+\sum_{\nu=2}^{d-4}(-1)^{d+\nu}(d-\nu-3)\binom{\mu-\nu}{d-\nu-2}\sum_{\substack{\#(\ell)=\nu-2\\ \ell\subset L\setminus k}}\{X^{(d-2)}(k_{+-}\cup\ell)-X^{(d-2)}(k_{-+}\cup\ell)\}.$$

Thus the second term in (3·11) also vanishes since a direct computation shows that

$$(d-\nu-3)\binom{\mu-\nu}{d-\nu-2}-(\mu-d+2)\binom{\mu-\nu}{d-\nu-3}+\binom{\mu-\nu-1}{d-\nu-2}-\binom{\mu-\nu-1}{d-\nu-4}=0.$$

This completes the proof of Lemma 3·7. ∎

Putting $d=\mu+1$, we have

COROLLARY 3·12.

$$\phi(L)=\frac{2^{\mu}}{6(\mu+1)!}\sum_{\ell\subseteq L}(-1)^{\#(\ell)}X^{(\mu+1)}(\ell).$$

Since L is algebraically split, $X(\ell)$ does not depend on the orientation of any component [**7**, **9**, **17**, **18**]. Thus we have the following corollary which was suggested by H. Morton.

COROLLARY 3·13. *If L is algebraically split, $\phi(L)$ does not depend on the orientation of any component.*

Remark 3·14. As noted in [**4**], Lemma 1·5, $\phi(L)$ vanishes for an algebraically split link with more than 3 components. So this corollary is non-trivial only for 2- and 3-component links.

4. *Calculation of* $\Sigma_1^{(d)}(\mathscr{L})$

In this section we will prove Lemma 2·5.
We first note that

LEMMA 4·1. $\phi(L^c)=0$ *if* $c_\xi>1$ *for some* ξ.

Proof. First recall that L^c is algebraically split since L^c is the **c**-parallel of L with

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

0-framing. From Corollary 3·6, we may assume that a pair of components in the parallels of K_ξ are oppositely oriented if $c_\xi > 1$. So these two components can bound an annulus disjoint from other components of L^c since it is algebraically split. Thus L^c is a boundary link and its Conway polynomial vanishes, completing the proof. ∎

Therefore we have

$$\sum_{\substack{\#(\ell)=d-1 \\ \ell \subset L^{k-2j-1}}} \phi(\ell) = \sum_{\substack{\#(\ell)=d-1 \\ \ell \subset L}} \left\{ \prod_{K_\xi \in \ell} (k_\xi - 2j_\xi - 1) \right\} \phi(\ell).$$

Thus

$$\Sigma_1^{(d)}(\mathscr{L}) = \sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}[k] \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} [2]^{k-2j-1}$$

$$\times (-1)^{d+1} 12 \times 2^{-d} \sum_{\substack{\#(\ell)=d-1 \\ \ell \subset L}} \left\{ \prod_{K_\xi \in \ell} (k_\xi - 2j_\xi - 1) \right\} \phi(\ell) (q-1)^d$$

$$= (q-1)^d (-1)^{d+1} 12 \times 2^{-d} \sum_{\substack{\#(\ell)=d-1 \\ \ell \subset L}} \phi(\ell)$$

$$\times \prod_{K_\xi \in \ell} \left\{ \sum_{k_\xi=1}^{(r-1)/2} q^{\rho(r) f_\xi (k_\xi^2-1)} [k_\xi] \sum_{j_\xi=0}^{(k_\xi-1)/2} (-1)^{j_\xi} \binom{k_\xi - j_\xi - 1}{j_\xi} [2]^{k_\xi - 2j_\xi - 1} (k_\xi - 2j_\xi - 1) \right\}$$

$$\times \prod_{K_\zeta \notin \ell} \left\{ \sum_{k_\zeta=1}^{(r-1)/2} q^{\rho(r) f_\zeta (k_\zeta^2-1)} [k_\zeta] \sum_{j_\zeta=0}^{(k_\zeta-1)/2} (-1)^{j_\zeta} \binom{k_\zeta - j_\zeta - 1}{j_\zeta} [2]^{k_\zeta - 2j_\zeta - 1} \right\}.$$

We want to calculate

$$\sum_{k=1}^{(r-1)/2} q^{\rho(r) f(k^2-1)} [k] \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} [2]^{k-2j-1} \tag{4·2}$$

and

$$\sum_{k=1}^{(r-1)/2} q^{\rho(r) f(k^2-1)} [k] \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} [2]^{k-2j-1} (k-2j-1) \tag{4·3}$$

for scalars k, j, and f.

LEMMA 4·4. (4·2) *is of the form*

$$-f\left(\frac{f}{r}\right) \frac{G(q)}{q-1} + a_f (q-1)^{(r-1)/2}$$

with $a_f \in \mathbb{Z}[q]$. *Note that* $\dfrac{G(q)}{q-1}$ *is of the form* $w(q-1)^{(r-3)/2}$ *with* $w \in \mathbb{Z}[q]$.

Proof. The following identity is well known.

$$[k] = \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} [2]^{k-2j-1}. \tag{4·5}$$

(For a proof, see for example [15] or it is not so hard to prove it by induction.)

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

Therefore (4·2) becomes

$$\sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}[k]^2 = \frac{1}{(s-s^{-1})^2}\sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}(s^k-s^{-k})^2$$

$$= \frac{1}{2(s-s^{-1})^2}\sum_{k=0}^{r-1} q^{\rho(r)f(k^2-1)}(q^k-2+q^{-k}) \tag{4·6}$$

since $q^{\rho(r)f(k^2-1)}(q^k-2+q^{-k})$ does not change if we replace k with $r-k$ and vanishes if $k=0$.

From Lemma 6·2, we have

$$\sum_{k=0}^{r-1} q^{\rho(r)f(k^2-1)}q^k = \sum_{k=0}^{r-1} q^{\rho(r)f(k^2-1)}q^{-k} = q^{-\{\rho(r)+1\}f}\left(\frac{\rho(r)f}{r}\right)G(q)$$

and

$$\sum_{k=0}^{r-1} q^{\rho(r)f(k^2-1)} = q^{-\rho(r)f}\left(\frac{\rho(r)f}{r}\right)G(q).$$

Thus (4·6) becomes

$$\frac{q}{(q-1)^2}q^{-\rho(r)f}(q^{-f}-1)\left(\frac{\rho(r)f}{r}\right)G(q) = \frac{q}{(q-1)^2}q^{-\rho(r)f}(q^{-f}-1)\left(\frac{f}{r}\right)G(q) \tag{4·7}$$

since $\left(\frac{\rho(r)}{r}\right) = \left(\frac{\{1-\epsilon(r)r\}/4}{r}\right) = 1$. Since $q\times q^{-\rho(r)f}(q^{-f}-1) = -f(q-1)+(q-1)^2a'_f$ and $G(q) = (q-1)^{(r-1)/2}\times u$ for some a'_f, $u\in\mathbb{Z}[q]$, we have the required formula, completing the proof. ∎

To calculate (4·3), we need to calculate

$$F_k = \sum_{j=0}^{(k-1)/2}(-1)^j\binom{k-j-1}{j}[2]^{k-2j-1}(k-2j-1).$$

LEMMA 4·8.

$$F_k = \frac{k(s^k+s^{-k})(s+s^{-1})}{(s-s^{-1})^2} - \frac{(s+s^{-1})^2[k]}{(s-s^{-1})^2}.$$

Proof. Since $[k] = (s^k-s^{-k})/(s-s^{-1})$, we have from (4·5)

$$s^k-s^{-k} = (s-s^{-1})\sum_{j=0}^{(k-1)/2}(-1)^j\binom{k-j-1}{j}(s+s^{-1})^{k-2j-1}.$$

Differentiating both sides, we have

$$ks^{k-1}+ks^{-k-1}$$

$$= (1+s^{-2})\sum_{j=0}^{(k-1)/2}(-1)^j\binom{k-j-1}{j}(s+s^{-1})^{k-2j-1}$$

$$+(s-s^{-1})\sum_{j=0}^{(k-1)/2}(-1)^j\binom{k-j-1}{j}(k-2j-1)(s+s^{-1})^{k-2j-2}(1-s^{-2})$$

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

$$= (1+s^{-2}) \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} [2]^{k-2j-1}$$

$$+ \frac{(s-s^{-1})(1-s^{-2})}{(s+s^{-1})} \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} (k-2j-1)[2]^{k-2j-1}$$

$$= (1+s^{-2})[k] + \frac{(s-s^{-1})(1-s^{-2})}{(s+s^{-1})} F_k.$$

Thus we have the required formula. ∎

Now we can calculate (4·3).

LEMMA 4·9. (4·3) *is of the form*

$$2\left(\frac{f}{r}\right)\frac{G(q)}{(q-1)^2} + b_f(q-1)^{(r-3)/2}$$

with $b_f \in \mathbb{Z}[q]$. *Note that* $\dfrac{G(q)}{(q-1)^2}$ *is of the form* $w'(q-1)^{(r-5)/2}$ *with* $w' \in \mathbb{Z}[q]$.

Proof. Since (4·3) vanishes if $r = 3$, we assume $r \geqslant 5$. From Lemma 4·8, (4·3) becomes

$$\sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}[k]F_k$$

$$= \frac{s+s^{-1}}{(s-s^{-1})^2} \sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}[k]\,k(s^k+s^{-k}) - \frac{(s+s^{-1})^2}{(s-s^{-1})^2} \sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}[k]^2$$

$$= \frac{q(q+1)}{(q-1)^3} \sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}\,k(q^k-q^{-k}) - \frac{(q+1)^2}{(q-1)^2} \sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}[k]^2.$$

Now since

$$\sum_{k=0}^{r-1} q^{\rho(r)f(k^2-1)}\,kq^k$$

$$= \sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}\,kq^k + \sum_{k=(r+1)/2}^{r-1} q^{\rho(r)(k^2-1)}\,kq^k$$

$$= \sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}\,kq^k + \sum_{k'=1}^{(r-1)/2} q^{\rho(r)f(k'^2-1)}(r-k')\,q^{-k'} \quad (k' = r-k)$$

$$= \sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}\,k(q^k-q^{-k}) + r\sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}q^{-k}$$

and $r = \pm G(q)^2$ is divisible by $(q-1)^{r-1}$, we have

$$\sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}\,k(q^k-q^{-k}) = -2fq^{-\{\rho(r)+1\}f}\left(\frac{f}{r}\right)G(q) + v_f(q-1)^{r-1}$$

with $v_f \in \mathbb{Z}[q]$ from Lemma 6·3.

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

Thus from (4·7), we have

$$\sum_{k=1}^{(r-1)/2} q^{\rho(r)f(k^2-1)}[k]F_k$$

$$= \frac{q(q+1)}{(q-1)^3}\left\{-2fq^{-\{\rho(r)+1\}f}\left(\frac{f}{r}\right)G(q)+v_f(q-1)^{r-1}\right\}-\frac{q(q+1)^2}{(q-1)^4}q^{-\rho(r)f}(q^{-f}-1)\left(\frac{f}{r}\right)G(q)$$

$$= \frac{q(q+1)}{(q-1)^4}q^{-\rho(r)f}\left(\frac{f}{r}\right)G(q)\{-2fq^{-f}(q-1)-(q+1)(q^{-f}-1)\}+v'_f(q-1)^{r-4}$$

with $v'_f \in \mathbb{Z}[q]$. Since

$$-2fq^{-f}(q-1)-(q+1)(q^{-f}-1) = \begin{cases} q^{-1}(q-1)^2 & (f=1) \\ (q-1)^2 & (f=-1), \end{cases}$$

we have the required formula since $r-4 \geqslant (r-3)/2$ if $r \geqslant 5$. The proof is complete. ∎

Therefore

$$\Sigma_1^{(d)}(\mathscr{L}) = (q-1)^d(-1)^{d+1}12\times 2^{-d}\sum_{\substack{\#(\ell)=d-1 \\ \ell\subset L}}\phi(\ell)$$

$$\times\prod_{K_\xi\in\ell}\left\{2\left(\frac{f_\xi}{r}\right)G(q)(q-1)^{-2}+b_f(q-1)^{(r-3)/2}\right\}$$

$$\times\prod_{K_\zeta\notin\ell}\left\{-f_\zeta\left(\frac{f_\zeta}{r}\right)G(q)(q-1)^{-1}+a_f(q-1)^{(r-1)/2}\right\}$$

$$= (q-1)^d(-1)^{d+1}12\times 2^{-d}\sum_{\substack{\#(\ell)=d-1 \\ \ell\subset L}}\phi(\ell)$$

$$\times 2^{d-1}G(q)^\mu(q-1)^{-\mu-d+1}\prod_{K_\xi\in\ell}\left(\frac{f_\xi}{r}\right)\prod_{K_\zeta\notin\ell}\left\{-f_\zeta\left(\frac{f_\zeta}{r}\right)\right\}+v(\mathscr{L})(q-1)^{\mu(r-3)/2+2}$$

for some $v(\mathscr{L}) \in \mathbb{Z}[q]$. Since

$$\alpha(\mathscr{L}) = \frac{(-1)^{\tilde{\sigma}(\mathscr{L})}(q-1)^\mu + w(\mathscr{L})(q-1)^{\mu+1}}{G(q)^\mu}$$

with $w(\mathscr{L}) \in \mathbb{Z}[q]$, we have

$\alpha(\mathscr{L})\Sigma_1^{(d)}(\mathscr{L})$

$$= 6(q-1)(-1)^{\tilde{\sigma}(\mathscr{L})+d+1}\sum_{*\substack{(\ell)=d-1 \\ \ell\subset L}}\phi(\ell)\prod_{K_\xi\in\ell}\left(\frac{f_\xi}{r}\right)\prod_{K_\zeta\notin\ell}\left\{-f_\zeta\left(\frac{f_\zeta}{r}\right)\right\}+V(\mathscr{L})(q-1)^2$$

for $V(\mathscr{L}) \in \mathbb{Z}[q]$.

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

Thus $\alpha(\mathscr{L})\Sigma_1^{(d)}(\mathscr{L})\in\mathbb{Z}[q]$ and $T(\alpha(\mathscr{L})\Sigma_1^{(d)}(\mathscr{L}))\equiv 0 \bmod r$.

We finally have

$$T\left(\frac{\alpha(\mathscr{L})\Sigma_1^{(d)}}{q-1}\right)$$

$$\equiv 6(-1)^{\tilde{\sigma}(\mathscr{L})+d+1}\sum_{\substack{\#(\ell)=d-1\\ \ell\subset L}}\phi(\ell)\prod_{K_\xi\in\ell}\left(\frac{f_\xi}{r}\right)\prod_{K_\zeta\notin\ell}\left\{-f_\zeta\left(\frac{f_\zeta}{r}\right)\right\}$$

$$\equiv\begin{cases}6(-1)^{\sigma_+(\mathscr{L})+d+1}\displaystyle\sum_{\substack{\#(\ell)=d-1\\ \ell\subset L}}\left\{\prod_{K_\zeta\notin\ell}(-f_\zeta)\right\}\phi(\ell) & (r\equiv 1 \bmod 4)\\ 6(-1)^{\sigma(\mathscr{L})+d+1}\displaystyle\sum_{\substack{\#(\ell)=d-1\\ \ell\subset L}}\left\{\prod_{K_\xi\in\ell}f_\xi\right\}\left\{\prod_{K_\zeta\notin\ell}(-1)\right\}\phi(\ell) & (r\equiv 3 \bmod 4)\end{cases}$$

$$\equiv\begin{cases}6(-1)^{\sigma_+(\mathscr{L})+d+1}\displaystyle\sum_{\substack{\#(\ell)=d-1\\ \ell\subset L}}\{(-1)^{\sigma_+(\mathscr{L})-\sigma_+(\ell)}\}\phi(\ell) & (r\equiv 1 \bmod 4)\\ 6(-1)^{\sigma(\mathscr{L})+d+1}\displaystyle\sum_{\substack{\#(\ell)=d-1\\ \ell\subset L}}\{(-1)^{\sigma_-(\ell)+\mu-d+1}\}\phi(\ell) & (r\equiv 3 \bmod 4)\end{cases}$$

$$\equiv 6\sum_{\substack{\#(\ell)=d-1\\ \ell\subset L}}\{(-1)^{\sigma_-(\ell)}\}\phi(\ell)$$

$$\equiv 6\sum_{\substack{\#(\ell)=d-1\\ \ell\subset L}}\left(\prod_{K_\xi\in\ell}f_\xi\right)\phi(\ell) \mod r,$$

where $\sigma_+(\ell)$ ($\sigma_-(\ell)$, respectively) is the number of $+1$'s (-1's, respectively) in $f_\xi(K_\xi\in\ell)$.

The proof of Lemma 2·5 is now complete.

5. *Calculation of* $\Sigma_2^{(d)}(\mathscr{L})$.

In this section, we will prove Lemma 2·6.

Since

$$\sum_{\substack{\#(\ell)=\nu\\ \ell\subset L^{k-2j-1}}}{}^{\cdot}X^{(d)}(\ell)=\sum_{\mathbf{c}=(c_1,c_2,\dots,c_\mu)_{|\mathbf{c}|=\nu}}\binom{\mathbf{k}-2\mathbf{j}-1}{\mathbf{c}}X^{(d)}(L^{\mathbf{c}}),$$

we have

$$\Sigma_2^{(d)}(\mathscr{L})=\sum_{\mathbf{k}=1}^{(r-1)/2}q^{\rho(r)\,f(\mathbf{k}^2-1)}[\mathbf{k}]\sum_{\mathbf{j}=0}^{(\mathbf{k}-1)/2}(-1)^{\mathbf{j}}\binom{\mathbf{k}-\mathbf{j}-1}{\mathbf{j}}[2]^{\mathbf{k}-2\mathbf{j}-1}$$

$$\times\frac{1}{d!}\sum_{\nu=1}^{d-2}(-1)^{d+\nu}\binom{|\mathbf{k}-2\mathbf{j}-1|-\nu-1}{d-\nu-2}$$

$$\times\sum_{|\mathbf{c}|=\nu}\binom{\mathbf{k}-2\mathbf{j}-1}{\mathbf{c}}X^{(d)}(L^{\mathbf{c}})\,(q-1)^d$$

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

$$= (q-1)^d \sum_{\nu=1}^{d-2} (-1)^{d+\nu} \frac{1}{d!} \sum_{|\mathbf{c}|=\nu} X^{(d)}(L^{\mathbf{c}})$$
$$\times \left\{ \sum_{\mathbf{k}=1}^{(\mathbf{r}-1)/2} q^{\rho(r)\mathbf{f}(\mathbf{k}^2-1)} [\mathbf{k}] \sum_{\mathbf{j}=0}^{(\mathbf{k}-1)/2} (-1)^{\mathbf{j}} \binom{\mathbf{k}-\mathbf{j}-1}{\mathbf{j}} [2]^{\mathbf{k}-2\mathbf{j}-1} \right.$$
$$\left. \times \binom{|\mathbf{k}-2\mathbf{j}-1|-\nu-1}{d-\nu-2} \binom{\mathbf{k}-2\mathbf{j}-1}{\mathbf{c}} \right\}.$$

Since $\frac{1}{d!} X^{(d)}(L^{\mathbf{c}}) \in \mathbb{Z}[\frac{1}{2}]$ from the recursive formula (3·4) and $\alpha(\mathscr{L})$ is of the form $\frac{u(\mathscr{L})}{(q-1)^{\mu(r-3)/2}}$ with $u(\mathscr{L}) \in \mathbb{Z}[q]$, it suffices to prove

PROPOSITION 5·1.

$$\sum_{\mathbf{k}=1}^{(\mathbf{r}-1)/2} q^{\rho(r)\mathbf{f}(\mathbf{k}^2-1)} [\mathbf{k}] \sum_{\mathbf{j}=0}^{(\mathbf{k}-1)/2} (-1)^{\mathbf{j}} \binom{\mathbf{k}-\mathbf{j}-1}{\mathbf{j}} [2]^{\mathbf{k}-2\mathbf{j}-1}$$
$$\times \binom{|\mathbf{k}-2\mathbf{j}-1|-\nu-1}{d-\nu-2} \binom{\mathbf{k}-2\mathbf{j}-1}{\mathbf{c}} \quad (5\cdot2)$$

is divisible by $(q-1)^{\mu(r-3)/2-d+2}$ *in* $\mathbb{Z}[q]$ *for* $d \geqslant 3$ *and* $|\mathbf{c}| = \nu \geqslant 1$.

Proof. From Lemma 6·4, (5·2) becomes

$$\sum_{\mathbf{k}=1}^{(\mathbf{r}-1)/2} q^{\rho(r)\mathbf{f}(\mathbf{k}^2-1)} [\mathbf{k}] \sum_{\mathbf{j}=0}^{(\mathbf{k}-1)/2} (-1)^{\mathbf{j}} \binom{\mathbf{k}-\mathbf{j}-1}{\mathbf{j}} [2]^{\mathbf{k}-2\mathbf{j}-1}$$
$$\times \sum_{\mathbf{e}=0}^{(\mathbf{r}-3)/2} \binom{\mathbf{k}-2\mathbf{j}-1}{\mathbf{e}} \binom{-\nu-1}{d-\nu-2-|\mathbf{e}|} \binom{\mathbf{k}-2\mathbf{j}-1}{\mathbf{c}}$$
$$= \sum_{\mathbf{e}=0}^{(\mathbf{r}-3)/2} \binom{-\nu-1}{d-\nu-2-|\mathbf{e}|} \sum_{\mathbf{k}=1}^{(\mathbf{r}-1)/2} q^{\rho(r)\mathbf{f}(\mathbf{k}^2-1)} [\mathbf{k}]$$
$$\times \sum_{\mathbf{j}=0}^{(\mathbf{k}-1)/2} (-1)^{\mathbf{j}} \binom{\mathbf{k}-\mathbf{j}-1}{\mathbf{j}} \binom{\mathbf{k}-2\mathbf{j}-1}{\mathbf{e}} \binom{\mathbf{k}-2\mathbf{j}-1}{\mathbf{c}} [2]^{\mathbf{k}-2\mathbf{j}-1}$$
$$= \sum_{\mathbf{e}=0}^{(\mathbf{r}-3)/2} \binom{-\nu-1}{d-\nu-2-|\mathbf{e}|} \prod_{\xi=1}^{\mu} \left\{ \sum_{k_\xi=1}^{(r-1)/2} q^{\rho(r) f_\xi(k_\xi^2-1)} [k_\xi] \right.$$
$$\left. \times \sum_{j_\xi=0}^{(k_\xi-1)/2} (-1)^{j_\xi} \binom{k_\xi-j_\xi-1}{j_\xi} \binom{k_\xi-2j_\xi-1}{j_\xi} \binom{k_\xi-2j_\xi-1}{c_\xi} [2]^{k_\xi-2j_\xi-1} \right\}. \quad (5\cdot3)$$

Proposition 5·4 below shows that

$$\prod_{\xi=1}^{\mu} \left\{ \sum_{k_\xi=1}^{(r-1)/2} q^{\rho(r) f_\xi(k_\xi^2-1)} [k_\xi] \sum_{j_\xi=0}^{(k_\xi-1)/2} (-1)^{j_\xi} \binom{k_\xi-j_\xi-1}{j_\xi} \binom{k_\xi-2j_\xi-1}{e_\xi} \binom{k_\xi-2j_\xi-1}{c_\xi} [2]^{k_\xi-2j_\xi-1} \right\}$$

is divisible by $q-1$ raised to the power of $\sum_{\xi=1}^{\mu} \{(r-3)/2 - e_\xi - c_\xi\}$. Since $\binom{-\nu-1}{d-\nu-2-|\mathbf{e}|} = 0$

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

if $d-\nu-2-|\mathbf{e}|<0$, we may assume $|\mathbf{e}|\leqslant d-\nu-2$ in the summation of (5·3). Thus we have

$$\begin{aligned}\sum_{\xi=1}^{\mu}\{(r-3)/2-e_\xi-c_\xi\} &= \mu(r-3)/2-|\mathbf{e}|-|\mathbf{c}|\\ &\geqslant \mu(r-3)/2-d+\nu+2-\nu\\ &= \mu(r-3)/2-d+2\end{aligned}$$

since $|\mathbf{c}|=\nu$. Therefore (5·2) is divisible by $(q-1)^{\mu(r-3)/2-d+2}$ and the proof is complete. ∎

The following proposition is due to T. Ohtsuki.

PROPOSITION 5·4 (T. Ohtsuki). *Let r be an odd prime. Then for any $a\in\mathbb{Z}$,*

$$\sum_{k=1}^{(r-1)/2} q^{ak^2}[k]\sum_{j=0}^{(k-1)/2}(-1)^j\binom{k-j-1}{j}\binom{k-2j-1}{e}\binom{k-2j-1}{c}[2]^{k-2j-1}$$

is divisible by $(q-1)^{(r-3)/2-e-c}$.

To prove Proposition 5·4, we prepare some lemmas.

$D:(\mathbb{Z}/r\mathbb{Z})[q]\to(\mathbb{Z}/r\mathbb{Z})[q]$ is defined by $D(\sum_h f_h q^h)=\sum_h h f_h q^{h-1}$. D is well defined since $q^r=1$. We use the same notation D for the map $\mathbb{Z}[q]\to(\mathbb{Z}/r\mathbb{Z})[q]$ defined similarly, which is also well defined. The following lemma may be well known.

LEMMA 5·5. *If $n\leqslant r-1$, $f\in\mathbb{Z}[q]$ is divisible by $(q-1)^n$ if and only if $T(D^d(f))\equiv 0 \bmod r$ for $d=0,1,\dots,(n-1)$. Here $D^d=\underbrace{D\circ\dots\circ D}_{d\text{times}}$ and D^0 is the identity.*

Proof. Since $r=\pm G(q)^2$ is divisible by $(q-1)^{r-1}$, we can investigate the divisibility of f as if it is in $(\mathbb{Z}/r\mathbb{Z})[q]$.

Noting that every element $f\in\mathbb{Z}[q]$ can be expressed as $f_0+f_1(q-1)+f_2(q-1)^2+\dots+f_{r-2}(q-1)^{r-2}$ and that $q^r=1$, the proof is straightforward. ∎

LEMMA 5·6. *If $1\leqslant n\leqslant(r-3)/2$, then* $\sum_{k=1}^{(r-1)/2}k^{2n}\equiv 0 \bmod r$.

Proof. It suffices to show that

$$\sum_{k=1}^{(r-1)/2}k^2(k^2-1^2)(k^2-2^2)\dots(k^2-(m-2)^2)(k^2-(m-1)^2)=\sum_{k=m}^{(r-1)/2}k\frac{(k+m-1)!}{(k-m)!}\qquad(5\cdot7)$$

is divisible by r for every $1\leqslant m\leqslant n$ since (5·7) is a polynomial in k^2 of the form $(k^2)^m+$lower terms.

Since

$$k\frac{(k+m-1)!}{(k-m)!}=\frac{(k+m-1)!}{(k-m-1)!}+m\frac{(k+m-1)!}{(k-m)!}$$

for $k\geqslant m+1$, (5·7) becomes

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

$$\sum_{k=m+1}^{(r-1)/2}\frac{(k+m-1)!}{(k-m-1)!}+m\sum_{k=m}^{(r-1)/2}\frac{(k+m-1)!}{(k-m)!}$$

$$=\sum_{i=1}^{(r-1)/2-m}\frac{(2m+i-1)!}{(i-1)!}+m\sum_{j=1}^{(r-1)/2-m+1}\frac{(2m+j-2)!}{(j-1)!}\quad(i=k-m, j=k-m+1)$$

$$=\frac{1}{2m+1}\frac{\{(r-1)/2+m\}!}{\{(r-1)/2-m-1\}!}+m\frac{1}{2m}\frac{\{(r-1)/2+m\}!}{\{(r-1)/2-m\}!}$$

$$=\frac{1}{2(2m+1)}\frac{\{(r-1)/2+m\}!}{\{(r-1)/2-m\}!}r.$$

Here we use the identity

$$\sum_{h=1}^{N}\frac{(h+M-1)!}{(h-1)!}=\frac{1}{M+1}\frac{(N+M)!}{(N-1)!}$$

in the second equality. Since $\dfrac{\{(r-1)/2+m\}!}{\{(r-1)/2-m\}!}$ is even for $m\geqslant 1$, $2m+1<r$, and r is prime, (5·7) is divisible by r. ∎

Let $\mathrm{Poly}_d(x)$ denote the set of polynomials of x of degree at most d with coefficients in $\mathbb{Z}/r\mathbb{Z}$. $\mathrm{Poly}_d^{+}(x)$ and $\mathrm{Poly}_d^{\mathrm{Odd}}(x)$ are the subsets of $\mathrm{Poly}_d(x)$ consisting of polynomials with only positive and odd powers respectively.

As a corollary to Lemma 5·6 we have

COROLLARY 5·8.

$$\sum_{k=1}^{(r-1)/2}P^{+}_{(r-3)/2}(k^2)\equiv 0 \bmod r,$$

for any $P^{+}_{(r-3)/2}(k^2)\in\mathrm{Poly}^{+}_{(r-3)/2}(k^2)$.

LEMMA 5·9. *For a fixed* d, $T(D^d([2]^{2n}))\equiv Q_d(n)\,2^{2n} \bmod r$ *for some* $Q_d(n)\in\mathrm{Poly}_d(n)$.

Proof. First note that $[2]^{2n}=q^{-n}(q+1)^{2n}\in\mathbb{Z}[q]$. Now we have

$$T(D^d([2]^{2n}))$$

$$\equiv T\left(\sum_{h=0}^{d}\binom{d}{h}D^{d-h}(q^{-n})D^h((q+1)^{2n})\right)$$

$$\equiv\sum_{h=0}^{d}\binom{d}{h}\underbrace{(-n)(-n-1)\dots(-n-d+h+1)}_{d-h}\underbrace{2n(2n-1)\dots(2n-h+1)}_{h}2^{2n-h}.$$

Since 2 is invertible in $\mathbb{Z}/r\mathbb{Z}$, the conclusion follows. ∎

LEMMA 5·10. *For a fixed* n $(0\leqslant n\leqslant(r-3)/2)$, $\sum_{j=0}^{(k-1)/2}(-1)^j\binom{k-j-1}{j}R_n(k-2j)\,2^{k-2j-1}\in\mathrm{Poly}^{\mathrm{Odd}}_{2n+1}(k)$, *for any* $R_n(k-2j)\in\mathrm{Poly}_n(k-2j)$. *Note that the summation above is defined only for* $k\geqslant 1$. *So strictly speaking, our claim is that it coincides with an element in* $\mathrm{Poly}^{\mathrm{Odd}}_{2n+1}(k)$ *for* $k\geqslant 1$.

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

Proof. It suffices to show that

$$F_{k,m} = \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} \binom{k-2j-1}{m} 2^{k-2j-1} \in \mathrm{Poly}_{2m+1}^{\mathrm{Odd}}(k)$$

for every $m \leqslant n$. This is because

$$\binom{k-2j-1}{m} = \frac{1}{m!}(k-2j-1)(k-2j-2)\dots(k-2j-m) \in \mathrm{Poly}_m(k-2j)$$

since $m!$ is invertible in $\mathbb{Z}/r\mathbb{Z}$.

Now we will calculate $F_{k,m}$. Note that $F_{k,0} = k \equiv T([k])$ and $F_{k,1} \equiv T(F_k) \bmod r$. It is not so hard to see that $F_{k,m}$ satisfies the following recursive formula.

$$F_{k,m} = 2(F_{k-1,m} + F_{k-1,m-1}) - F_{k-2,m}$$

with $F_{k,0} = k$. Since the quantity $2^m \binom{k+m}{2m+1}$ also satisfies the same recursive formula and the initial condition, we see that $F_{k,m} = 2^m \binom{k+m}{2m+1}$. Since

$$\binom{k+m}{2m+1} = \frac{1}{(2m+1)!}(k+m)(k+m-1)\dots(k+1)k(k-1)\dots(k-m+1)(k-m)$$

$$= \frac{1}{(2m+1)!}(k^2-m^2)(k^2-(m-1)^2)\dots(k^2-1^2)k \in \mathrm{Poly}_{2m+1}^{\mathrm{Odd}}(k),$$

the proof is complete. ∎

Proof of Proposition 5·4. We will show more generally that

$$S = \sum_{k=1}^{(r-1)/2} q^{ak^2}[k] \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} R_n(k-2j)\,[2]^{k-2j-1}$$

is divisible by $(q-1)^{(r-3)/2-n}$ for any $R_n(k-2j) \in \mathrm{Poly}_n(k-2j)$ since

$$\binom{k-2j-1}{e}\binom{k-2j-1}{c} \in \mathrm{Poly}_{e+c}(k-2j).$$

From Lemma 5·5, it suffices to show that $T(D^{d'}(S)) \equiv 0 \bmod r$ for $d' = 0, 1, \dots, (r-3)/2-n-1$.

Putting

$$G_{k,n} = \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} R_n(k-2j)\,[2]^{k-2j-1},$$

$T(D^{d'}(S))$ can be written as

$$\sum_{k=1}^{(r-1)/2} \sum_{d=0}^{d'} \binom{d'}{d} T(D^{d'-d}(q^{ak^2}))\, T(D^d([k]\,G_{k,n})).$$

Since $D^{d'-d}(q^{ak^2}) \in \mathrm{Poly}_{d'-d}(k^2)$, it is sufficient to show $T(D^d([k]\,G_{k,n})) \in \mathrm{Poly}_{n+d+1}^{+}(k^2)$ for $d = 0, 1, \dots, (r-3)/2-n-1$ from Corollary 5·8.

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

Now since

$$[k] = \sum_{i=0}^{(k-1)/2} (-1)^i \binom{k-i-1}{i} [2]^{k-2i-1},$$

we have

$$[k]\, G_{k,n} = \sum_{i,j=0}^{(k-1)/2} (-1)^{i+j} \binom{k-i-1}{i} \binom{k-j-1}{j} R_n(k-2j)\, [2]^{2(k-i-j-1)}.$$

From Lemma 5·9, we see

$$\begin{aligned} &T(D^d([k]\, G_{k,n})) \\ &\quad \equiv \sum_{i,j=0}^{(k-1)/2} (-1)^{i+j} \binom{k-i-1}{i} \binom{k-j-1}{j} R_n(k-2j)\, T(D^d([2]^{2(k-i-j-1)})) \\ &\quad \equiv \sum_{i,j=0}^{(k-1)/2} (-1)^{i+j} \binom{k-i-1}{i} \binom{k-j-1}{j} R_n(k-2j)\, Q_d(k-i-j)\, 2^{2(k-i-j-1)} \bmod r, \end{aligned}$$

with $Q_d(k-i-j) \in \mathrm{Poly}_d(k-i-j)$. Since 2 is invertible in $\mathbb{Z}/r\mathbb{Z}$, we can regard $Q_d(k-i-j)$ as a polynomial of $2(k-i-j) = (k-2i)+(k-2j)$ and so $R_n(k-2j)\, Q_d(k-i-j)$ is a polynomial of $k-2i$ and $k-2j$ with total degree at most $n+d$. Therefore we can write $R_n(k-2j)\, Q_d(k-i-j) = \sum_{h=0}^{n+d} S_h(k-2i)\, T_{n+d-h}(k-2j)$ with $S_h(k-2i) \in \mathrm{Poly}_h(k-2i)$ and $T_{n+d-h}(k-2j) \in \mathrm{Poly}_{n+d-h}(k-2j)$.

So we have from Lemma 5·10

$$\begin{aligned} &T(D^d([k]\, G_{k,n})) \\ &\quad \equiv \sum_{i,j=0}^{(k-1)/2} (-1)^{i+j} \binom{k-i-1}{i} \binom{k-j-1}{j} \\ &\qquad \times \sum_{h=0}^{n+d} S_h(k-2i)\, T_{n+d-h}(k-2j)\, 2^{(k-2i)+(k-2j)-2} \\ &\quad \equiv \sum_{h=0}^{n+d} \sum_{i=0}^{(k-1)/2} (-1)^i \binom{k-i-1}{i} S_h(k-2i)\, 2^{k-2i-1} \\ &\qquad \times \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-j-1}{j} T_{n+d-h}(k-2j)\, 2^{k-2j-1} \\ &\quad \equiv \sum_{h=0}^{n+d} S^{\mathrm{Odd}}_{2h+1}(k)\, T^{\mathrm{Odd}}_{2(n+d-h)+1}(k) \bmod r \in \mathrm{Poly}^+_{n+d+1}(k^2), \end{aligned}$$

where $S^{\mathrm{Odd}}_{2h+1}(k) \in \mathrm{Poly}^{\mathrm{Odd}}_{2h+1}(k)$ and $T^{\mathrm{Odd}}_{2(n+d-h)+1}(k) \in \mathrm{Poly}^{\mathrm{Odd}}_{2(n+d-h)+1}(k)$.

The proof of Proposition 5·4 is now complete. ∎

6. *Technical lemmas*

In this section we prove some technical lemmas which are used in previous sections.

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

LEMMA 6·1. *For non-negative integers M and N with $M > N$,*

$$\sum_{a=0}^{N} \binom{N}{a} \frac{(-1)^a}{M-a} = \frac{(-1)^N N!(M-N-1)!}{M!}$$

Proof. Put $S(M,N) = \int_0^1 x^{M-1}(1-x^{-1})^N dx$. Integrating by parts, we have

$$\begin{aligned} S(M,N) &= \left[\frac{x^M}{M}(1-x^{-1})^N\right]_0^1 - \int_0^1 \frac{x^M}{M} N(1-x^{-1})^{N-1}x^{-2}\,dx \\ &= -\frac{N}{M}S(M-1,N-1) \\ &= \cdots \\ &= (-1)^N \frac{N!(M-N)!}{M!} S(M-N,0) \\ &= (-1)^N \frac{N!(M-N)!}{M!} \int_0^1 x^{M-N-1}\,dx \\ &= (-1)^N \frac{N!(M-N-1)!}{M!}. \end{aligned}$$

But since

$$\begin{aligned} S(M,N) &= \int_0^1 \sum_{a=0}^{N} \binom{N}{a} (-x^{-1})^a x^{M-1}\,dx \\ &= \sum_{a=0}^{N} \binom{N}{a} (-1)^a \int_0^1 x^{M-a-1}\,dx \\ &= \sum_{a=0}^{N} \binom{N}{a} (-1)^a \frac{1}{M-a}, \end{aligned}$$

the formula follows. ∎

LEMMA 6·2. *For $(a,r) = 1$ we have*

$$\sum_{k=0}^{r-1} q^{ak^2+bk+c} = q^{-(b^2-4ac)(4a)^{-1}} \left(\frac{a}{r}\right) G(q),$$

where $x^{-1} \in \mathbb{Z}/r\mathbb{Z}$ is the inverse of x and $\left(\frac{a}{r}\right)$ is the Legendre symbol; $\left(\frac{a}{r}\right) = 1$ if $a \equiv y^2 \bmod r$ for some integer y and -1 otherwise.

Proof.

$$\begin{aligned} \sum_{k=0}^{r-1} q^{ak^2+bk+c} &= \sum_{k=0}^{r-1} q^{a(k+b(2a)^{-1})^2-(b^2-4ac)(4a)^{-1}} \\ &= q^{-(b^2-4ac)(4a)^{-1}} \sum_{k \in \mathbb{Z}/r\mathbb{Z}} q^{a(k+b(2a)^{-1})^2} \\ &= q^{-(b^2-4ac)(4a)^{-1}} G(q^a). \end{aligned}$$

Since $G(q^a) = \left(\frac{a}{r}\right) G(q)$ (see for example [5], chapter 6), we have the required formula, completing the proof. ∎

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

LEMMA 6·3. *For* $(a,r)=1$, *we have*

$$\sum_{k=0}^{r-1} kq^{ak^2+bk+c} \equiv -b(2a)^{-1}q^{-(b^2-4ac)(4a)^{-1}}\left(\frac{a}{r}\right)G(q) \bmod r$$

for some $v \in \mathbb{Z}[q]$.

Proof.

$$\begin{aligned}
&\sum_{k=0}^{r-1} kq^{ak^2+bk+c} \\
&\quad = q^{-(b^2-4ac)(4a)^{-1}} \sum_{k=0}^{r-1} kq^{a(k+b(2a)^{-1})^2} \\
&\quad \equiv q^{-(b^2-4ac)(4a)^{-1}} \left\{ \sum_{k\in\mathbb{Z}/r\mathbb{Z}} (k+b(2a)^{-1}) q^{a(k+b(2a)^{-1})^2} - b(2a)^{-1} \sum_{k\in\mathbb{Z}/r\mathbb{Z}} q^{a(k+b(2a)^{-1})^2} \right\} \\
&\quad \equiv q^{-(b^2-4ac)(4a)^{-1}} \left\{ \sum_{k=0}^{r-1} kq^{ak^2} - b(2a)^{-1} \sum_{k=0}^{r-1} q^{ak^2} \right\} \bmod r.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{k=0}^{r-1} kq^{ak^2} &= \sum_{k=1}^{(r-1)/2} kq^{ak^2} + \sum_{k=(r+1)/2}^{r-1} kq^{ak^2} \\
&= \sum_{k=1}^{(r-1)/2} kq^{ak^2} + \sum_{k'=1}^{(r-1)/2} (r-k')q^{ak'^2} \quad (k'=r-k) \\
&\equiv 0 \bmod r,
\end{aligned}$$

the formula follows. ∎

LEMMA 6·4. *For* $\mathbf{0} \leqslant \mathbf{g} = (g_1, g_2, \ldots, g_n) \leqslant \mathbf{M} = (M, M, \ldots, M)$, *we have*

$$\binom{|\mathbf{g}|+x}{y} = \sum_{\mathbf{e}=\mathbf{0}}^{\mathbf{M}} \binom{\mathbf{g}}{\mathbf{e}} \binom{x}{y-|\mathbf{e}|},$$

where $\#(\mathbf{e}) = n$.

Proof. If $\#(\mathbf{g}) = 1$, we have to prove

$$\binom{g+x}{y} = \sum_{e=0}^{M} \binom{g}{e}\binom{x}{y-e}. \tag{6·5}$$

This is proved by induction on g. For $g=0$, (6·5) is trivial. Now assume that (6·5) is true for $g < M$. Then we have

$$\begin{aligned}
\binom{g+1+x}{y} &= \sum_{e=0}^{M} \binom{g}{e}\binom{x+1}{y-e} \\
&= \sum_{e=0}^{M} \binom{g}{e}\left\{\binom{x}{y-e} + \binom{x}{y-e-1}\right\} \\
&= \sum_{e=0}^{M} \binom{g}{e}\binom{x}{y-e} + \sum_{e=1}^{M+1} \binom{g}{e-1}\binom{x}{y-e} \\
&= \sum_{e=0}^{M} \left\{\binom{g}{e} + \binom{g}{e-1}\right\}\binom{x}{y-e}
\end{aligned}$$

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

$$= \sum_{e=0}^{M} \binom{g+1}{e} \binom{x}{y-e},$$

where the fourth identity follows from the fact that $\binom{g}{-1} = \binom{g}{M} = 0$ since $g < M$. Thus (6·5) is true for every $g \leqslant M$.

Now we will prove the lemma by induction on $\#(\mathbf{g})$. We assume that the lemma is true for $\mathbf{g}'$ with $\#(\mathbf{g}') < n$. Then for $\mathbf{g}$ with $\#(\mathbf{g}) = n$, we have

$$\begin{aligned}
\binom{|\mathbf{g}|+x}{y} &= \binom{|\mathbf{g}'|+g_n+x}{y} \\
&= \sum_{\mathbf{e}'=0}^{\mathbf{M}} \binom{\mathbf{g}'}{\mathbf{e}'} \binom{g_n+x}{y-|\mathbf{e}'|} \\
&= \sum_{\mathbf{e}'=0}^{\mathbf{M}} \binom{\mathbf{g}'}{\mathbf{e}'} \sum_{e_n=0}^{M} \binom{g_n}{e_n} \binom{x}{y-|\mathbf{e}'|-e_n} \\
&= \sum_{\mathbf{e}=0}^{\mathbf{M}} \binom{\mathbf{g}}{\mathbf{e}} \binom{x}{y-|\mathbf{e}|},
\end{aligned}$$

where $\mathbf{g}' = (g_1, g_2, \ldots, g_{n-1})$ and $\mathbf{e}' = (e_1, e_2, \ldots, e_{n-1})$.

The proof is complete. ∎

Acknowledgements. This work was mainly done when the author was visiting the Department of Pure Mathematics, Cambridge. He thanks Dr W. B. R. Lickorish for his kindness and the Department for its hospitality. He also thanks the Isaac Newton Institute for Mathematical Sciences which gave him opportunities to discuss with R. Kirby, P. Melvin, H. Morton, V. Turaev, and His Royal Highness the Duke of Edinburgh.

The author is grateful to M. Kurihara, who taught him about Gaussian sums, to P. Melvin for helpful conversations and encouragement (he also checked an earlier version of Proposition 5·4 for $r \leqslant 31$ using MATHEMATICA), to T. Ohtsuki, who proved Proposition 5·4 completely, and to Y. Yokota, who pointed out an error in an earlier version.

REFERENCES

[1] S. AKBULUT and J. D. MCCARTHY. *Casson's invariant for oriented homology 3-spheres. An exposition.* Mathematical Notes 36. (Princeton University Press, 1990).

[2] C. BLANCHET, N. HABEGGER, G. MASBAUM, and P. VOGEL. Three-manifold invariants derived from the Kauffman bracket. *Topology* **31** (1992), 685–699.

[3] J. H. CONWAY. An enumeration of knots and links, and some of their algebraic properties. In *Computational Problems in Abstract Algebra* (Pergamon Press, 1969).

[4] J. HOSTE, A formula for Casson's invariant. *Trans. Amer. Math. Soc.* **297** (1986), 547–562.

[5] K. IRELAND and M. ROSEN. *A Classical Introduction to Modern Number Theory, Second Edition* (Springer-Verlag, 1990).

[6] V. F. R. JONES. A polynomial invariant for knots via von Neumann algebras. *Bull. Amer. Math. Soc.* **12** (1985), 103–111.

[7] V. F. R. JONES. Hecke algebra representations of braid groups and link polynomials. *Ann. Math.* **126** (1987), 335–388.

[8] L. H. KAUFFMAN. The Conway polynomial. *Topology* **20** (1981), 101–108.

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press

[9] L. H. KAUFFMAN. State models and the Jones polynomial. *Topology* **26** (1987), 395–407.
[10] R. C. KIRBY. A calculus for framed links in S^3. *Invent. math.* **45** (1978), 35–56.
[11] R. KIRBY and P. MELVIN. The 3-manifold invariants of Witten and Reshetikhin–Turaev for sl(2, C). *Invent. math.* **105** (1991), 473–545.
[12] T. KOHNO. Topological invariants for 3-manifolds using representations of mapping class groups. *Topology* **31** (1992), 203–230.
[13] W. B. R. LICKORISH. A representation of orientable combinatorial 3-manifolds. *Ann. Math.* **76** (1962), 531–540.
[14] W. B. R. LICKORISH. Three-manifolds and the Temperley–Lieb algebra. *Math. Ann.* **290** (1991), 657–670.
[15] W. B. R. LICKORISH. Calculations with the Temperley–Lieb algebra. *Comm. Math. Helv.*, **67**, (1992), 571–591.
[16] W. B. R. LICKORISH. The skein method for three-manifold invariants. *J. Knot Theory and Its Ramification*, **2**, (1993), 171–194.
[17] W. B. R. LICKORISH and K. C. MILLETT. The reversing result for the Jones polynomial. *Pacific J. Math.* **124** (1986), 173–176.
[18] H. R. MORTON. The Jones polynomial for unoriented links. *Quart. J. Math.* **37** (1986), 55–60.
[19] H. MURAKAMI. On derivatives of the Jones polynomial. *Kobe J. Math.* **3** (1986), 61–64.
[20] N. YU. RESHETIKHIN and V. G. TURAEV. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. math.* **103** (1991), 547–597.
[21] K. WALKER. *An extension of Casson's invariant*, Annals of Mathematics Studies. 126 (Princeton University Press, 1992).
[22] E. WITTEN. Quantum field theory and the Jones polynomial. *Commun. Math. Phys.* **121** (1989), 351–399.

https://doi.org/10.1017/S0305004100072078 Published online by Cambridge University Press