We illustrate the usefulness of this property by constructing F_7 out of F_6. Between $\frac{0}{1}$ and $\frac{1}{6}$ in F_7 occurs $\frac{0+1}{1+6} = \frac{1}{7}$. Since $\frac{1+1}{6+5} = \frac{2}{11}$, then $\frac{1}{6}$ and $\frac{1}{5}$ will be adjacent in F_7, F_8, F_9 and F_{10}. In F_{11} the fraction $\frac{2}{11}$ will be between them. Also $\frac{1+1}{5+4} = \frac{2}{9}$ will occur in F_9 for the first time, but $\frac{1+1}{4+3} = \frac{2}{7}$ is between $\frac{1}{4}$ and $\frac{1}{3}$ in F_7, and so on. Finally, we obtain F_7:

$$\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{4}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}.$$

Property 3 is easily confused with the fact that, for each three adjacent fractions $\frac{a}{b} < \frac{e}{f} < \frac{c}{d}$ of some Farey series F_n, the equation $\frac{e}{f} = \frac{a+c}{b+d}$ holds. (For a geometric proof see [1].) This equation does not mean $e = a + c$ and $f = b + d$, as the following example illustrates. In F_3 the fractions $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ are adjacent and $\frac{1+2}{3+3} = \frac{3}{6} = \frac{1}{2}$. Here the fraction $\frac{a+c}{b+d}$ is not reduced; but this is not contradicting Property 3, because $\frac{1}{3}$ and $\frac{2}{3}$ are never adjacent in any F_n.

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105.03 Almost Goldbach theorems

Although it is not included in the list of 'million-dollar millennium problems', there can be no doubt that one of the most difficult unsolved problems in number theory today is the following statement, known as the strong Goldbach conjecture.

Strong: Every even integer greater than or equal to 6 can be written as the sum of two primes.

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It is still not solved as of today, 278 years after it was first posed. This was one of two unsolved problems that the German mathematician Christian Goldbach proposed in a 1741 letter to Leonhard Euler, a Swiss mathematician. The other was a statement known as the weak Goldbach conjecture.

Weak: Every odd integer greater than or equal to 9 can be written as the sum of three odd primes.

Neither Euler nor Goldbach, nor any of the other mathematicians world-wide including Gauss, could prove these.

History of the Weak

In 1923, Hardy and Littlewood [1], by assuming the unproved generalised Riemann hypothesis, proved the weak Goldbach for every odd number larger than a 'sufficiently large' constant C, beyond the computational capacity of any computer. This result, however, was important because it proved that weak is not unassailable, as long as Riemann is true.

Later, in 1937, I. M. Vinogradov [2] proved the weak, for sufficiently large odd numbers but, this time, without the use of Riemann. In Vinogradov's proof the sufficiently large constant C was $3^{3^{15}}$, about 3×10^{6846168} which would probably take a computer the size of a giant Black Hole millions of years to compute. In 2002 Ming-Chit Liu and Tianze Wang [3] lowered the constant to 2×10^{1346}, still too big for a computer to verify all numbers up to that size. The weak could be verified only for odd numbers up to something around 8×10^{30}.

But, in 2013, a major breakthrough occurred when the Peruvian mathematician Harald Helfgott [4] proved that every odd integer $n \geqslant C = 10^{27}$ can be expressed as the sum of three primes, making it easy to show that all odd numbers greater than 9 can be written as the sum of three primes. Thus, 272 years after it was first posed, the weak was finally settled.

History of the Strong

But what about the Strong? Is it so difficult that it may be one of the unsolvable statements that are predicted by Kurt Gödel's incompleteness theorems? Do not think that mathematicians have not been working on this problem. That is what the Almost Goldbach Theorems (AGT) are all about, proving theorems that have slight additional conditions and then peeling away these conditions hoping to arrive at a proof of the Strong. This activity has not only proved these 'lesser' theorems, but has produced other conjectures that have not yet been proven as theorems.

In 1919, Viggo Brun [5] from Norway proved that every (large enough) even number is the sum of two numbers each having, at most, nine prime factors. In 1956, Ming-Chit Liu and Tianze Wang, from China, proved in

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[6] that the numbers in Brun's theorem could be reduced from nine prime factors in each term to no more than three in one and four in the other. In a second article [7] in the same journal he proved the '(1,4) theorem' where every large enough even number could be written as the sum of a prime and a number with, at most, 4 prime factors.

Taking an entirely different approach, in 1930, Lev Schnirelmann, a Russian mathematician, proved in [8] there is some constant, C, such that every positive integer (even or odd) is the sum of at most C primes. He proved it first for $C \leqslant 800\,000$. Fortunately, it was rapidly reduced by other Russians, A. Ya Khinchin [9] in 1932 and, in 1935, N. P. Romanov [10], bringing it down to $C \leqslant 2208$. Other mathematicians contributed to the reduction to smaller values of C. An Italian mathematician, G. Ricci [11], in 1936 reduced it to $C \leqslant 67$. The current record is $C \leqslant 6$ which has stood since 1976, proved by the British mathematician R. C. Vaughan [12]. Helfgott's proof of the ternary makes the new record $C \leqslant 4$ for any even integer $n \geqslant 12$, because then $n - 3$ is odd and, thus, the sum of three odd primes. Whoever can get it down to $C = 2$, when n is even, will be celebrated as the prover of the Strong.

A third attempt, unlike Brun's or Schnirelmann's, to prove the Strong was one devised by Y. V. Linnik [13], who wanted to prove that every even number could be written as the sum of *two* primes plus powers of the number 2. In 1953 he proved that every even number can be written as the sum of two primes plus K powers of 2, where $K = 54000$. Needless to say, this number was also subject to reduction in subsequent papers. Between 1953 and 1976 several mathematicians brought K down to 2250. Finally in 2002, two British mathematicians D. R. Heath-Brown and J. C. Puchta [14] proved it for $K \leqslant 13$. What we want is a proof that K can be reduced to 0.

For the Strong, the best result, so far, is one called the '(1, 2) theorem,' proved by the Chinese mathematician Chen Jing Run [15]. This says that there is a constant C such that for every integer, $n \geqslant C$, the even number $2n$ can be written either as the sum of two primes or the sum of a prime and a product of not more than two primes. Although Chen stated his theorem in 1966, he did not publish his proof until 1973 [16]. We state it here as the prime semi-prime theorem.

Theorem 1: Prime Semi-prime Theorem (PSP)

Every even number $2n$, with n sufficiently large, can be written either as the sum of two primes or the sum of a prime and a semi-prime.

In this case 'sufficiently large' is pinned down by Yamada [17] to be $e^{36^{36}}$, or about $C = 10^{10^{16}}$, which will take ten raised to ten quadrillion (American definition) or ten raised to ten thousand billion (British) Black-Hole-sized computers. This theorem is an improvement on the (1,4) theorem in Wang's 1956 paper mentioned above. We will not try to write anything about this difficult proof here, but suffice it to say that this is a spectacular theorem. It says that if n is sufficiently large, there exists a pair of primes, or a prime and a semi-prime, whose sum is $2n$. Obviously, what we want for the Strong

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Goldbach conjecture is to have someone prove the '(1,1) theorem'. It would even be nice if someone could prove (1, 2) for a computationally reachable C.

A New Theorem

It is a matter of opinion as to whether or not other theorems that are not Goldbach might qualify as AGT. Here, we present a new AGT, for all $n \geqslant 5$, almost as neat as Chen's but with an easy proof.

Theorem 2: Prime Relatively Prime Theorem (PRP)

If $n \geqslant 5$, then every even number $2n$ can be written either as the sum of two primes or the sum of a prime and an odd number relatively prime to n.

Proof: If n is prime then $2n = n + n$. Otherwise, if n is composite, let p be the largest prime factor of n. Then by the Bertrand-Chebyshev theorem there is a prime between p and $2p$. Let q be the smallest such prime. Thus,

$$p < q < 2p < 2n.$$

Let $h = 2n - q$, So, h is a positive odd integer greater than 1, and $h + q = 2n$, and if h is not prime, it is, at least, relatively prime to n. This is because if a is a common factor of n and h then a divides q, so $a = 1$ or $a = q$. But q does not divide n (by the definition of p), so $a = 1$ and h is relatively prime to n.

To simplify our discourse at this point, we shall introduce a definition of what is meant by a 'Goldbach pair for an even number'.

Definition: The unordered number pair (p, q) is called a *Goldbach Pair* for the even number $2n$ if, and only if, each of p and q is a prime and $p + q = 2n$.

Note that, if $n \geqslant 6$, and (p, q) is a Goldbach pair for $2n$, then either $p = q = n$, or one of p, q is less than n and the other is greater than n, and, of course, less than $2n$. Let us say that $p \geqslant q$, Then from $p = 2n - q$, we get

$$p - n = n - q.$$

Both sides are equal to some common non-negative number r, less than n. Thus $p = n + r$, and $q = n - r$. This means we can rewrite the Strong as follows:

> For any integer $n \geqslant 6$, there exists a non-negative integer r such that both $n - r$ and $n + r$ are prime.

Practical Application

This gives us a useful way to find Goldbach pairs, when they exist, for even numbers $2n$. If n itself is even, just look at all of the odd numbers r less than n, for which $n - r$ is prime; if any such r makes $n + r$ prime, then $(n - r, n + r)$ is a Goldbach pair.

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Example:

Let $2n = 40$, then the odd primes less than 20 are

$$\{20 - 1,\ 20 - 3,\ 20 - 7,\ 20 - 9,\ 20 - 13,\ 20 - 15,\ 20 - 17\}$$

and the primes between 20 and 40 are

$$\{20 + 3,\ 20 + 9,\ 20 + 11,\ 20 + 17\}.$$

Thus, re-using the odd numbers 3, 9, and 17, the Goldbach pairs are

$$\{(17,\ 23),\ (11,\ 29),\ (3,\ 37)\}.$$

An intriguing observation is that if Goldbach were not true for this number, then you could *not* reuse the values 3 or 9 or 17 in the construction of the upper primes (those greater than n), thereby falling short of the prerequisite number of odd primes less than 40, getting 8 instead of the actual value of $\pi(40) - 1 = 11$.

On the other hand, if you are given $2n$ where n is odd, just look at all of the even numbers r less than n. Any r that makes both $n - r$ and $n + r$ prime gives you a Goldbach pair.

If Goldbach were to be proved false, what would happen to Theorems 1 and 2 or any of the other AGTs discussed here? Nothing, because they were all proved independent of Goldbach. What would happen if the Goldbach conjecture were to be actually proved true? All of the AGTs would still be true. Also, it would make some other prime number theorems easier to prove, and we would know that Goldbach is not one of Gödel's undecidable problems.

Summary

In this Note we acknowledged that the weak Goldbach conjecture has been completely proved by Helfgott, but that the strong Goldbach has never been proved. We gave a list of theorems that are 'almost' the strong Goldbach. These serve as invitations to future researchers to continue to pare down the side-conditions in a manner similar to what has been done on them historically. We have also introduced a new AGT and a useful restatement of the strong conjecture that can be used as easy ways to find Goldbach pairs. We suggest the need to explore the idea that if some even number $2n$ does not have *any* Goldbach pairs, then the number of odd primes less than that number would be less than the normal $\pi(2n) - 1$.

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